# SEMI-CLASSICAL LIMIT OF SCHRÖDINGER–POISSON EQUATIONS IN SPACE DIMENSION $n \geqslant 3$

THOMAS ALAZARD AND RÉMI CARLES

Abstract. We prove the existence of solutions to the Schrödinger–Poisson system on a time interval independent of the Planck constant, when the doping profile does not necessarily decrease at infinity, in the presence of a subquadratic external potential. The lack of integrability of the doping profile is resolved by working in Zhidkov spaces, in space dimension at least three. We infer that the main quadratic quantities (position density and modified momentum density) converge strongly as the Planck constant goes to zero. When the doping profile is integrable, we prove pointwise convergence.

## 1. Introduction

We consider the semi-classical limit $\varepsilon \to 0$ of the Schrödinger–Poisson system:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = V_{\rm ext} u^\varepsilon + V_{\rm p}^\varepsilon u^\varepsilon, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

$$\Delta V_{\rm p}^\varepsilon = q\left(|u^\varepsilon|^2 - c\right), \tag{1.2}$$

$$u^\varepsilon_{|t=0} = a_0^\varepsilon(x) e^{i\Phi_0(x)/\varepsilon}, \tag{1.3}$$

where $V_{\rm ext} = V_{\rm ext}(t,x)$ is an external potential (harmonic potential for instance), $c = c(x)$ is a *doping profile* (or *impurity*, *background ions*), and $q \in \mathbb{R}$ represents an electric charge; $V_{\rm ext}$, $c$ and $q$ are data of the problem (see e.g. [17]). We consider the case where the space dimension is $n \geqslant 3$. This is due to a lack of control of low frequencies for the Poisson equation (1.2) when $n \leqslant 2$.

The conditions we impose to solve the Poisson equation (1.2) will be given according to the different cases we consider.

The doping profile $c$ is supposed to be bounded, and does not necessarily goes to zero at infinity (see Assumption 1 or Assumption 2 below). Suppose for instance that $c \equiv 1$. Then (1.1)–(1.2) is reminiscent of the Gross-Pitaevskii equation (see e.g. [15, 11] and references therein):

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \left(|u^\varepsilon|^2 - 1\right) u^\varepsilon. \tag{1.4}$$

For this equation, the Hamiltonian structure yields, at least formally:

$$\frac{d}{dt}\left(\|\varepsilon\nabla u^\varepsilon(t)\|^2_{L^2} + \left\||u^\varepsilon(t)|^2 - 1\right\|^2_{L^2}\right) = 0.$$

---

2000 *Mathematics Subject Classification.* 35B40, 35C20, 35Q40, 81Q05, 81Q20, 82D37.

*Key words and phrases.* Schrödinger-Poisson, doping profile, semi-classical analysis.

Supports by European network HYKE, funded by the EC as contract HPRN-CT-2002-00282, by Centro de Matemática e Aplicações Fundamentais (Lisbon), funded by FCT as contract POCTI-ISFL-1-209, and by the ANR project SCASEN, are acknowledged.

A natural space to study the Cauchy problem associated to (1.4) is therefore the energy space

$$E = \{u \in H^1_{\rm loc}(\mathbb{R}^n)\ ;\ \nabla u \in L^2(\mathbb{R}^n),\ |u|^2 - 1 \in L^2(\mathbb{R}^n)\}.$$

For this quantity to be well defined, one cannot assume that $u^\varepsilon$ is in $L^2(\mathbb{R}^n)$; morally, the modulus of $u^\varepsilon$ goes to one at infinity. To study solutions which are bounded, but not in $L^2(\mathbb{R}^n)$, P. E. Zhidkov introduced in the one-dimensional case in [22] (see also [23]):

$$X^s(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n)\ ;\ \nabla u \in H^{s-1}(\mathbb{R}^n)\}, \quad s > n/2.$$

The study of these spaces was generalized in the multidimensional case by C. Gallo [9]. They make it possible to consider solutions to (1.4) whose modulus has a non-zero limit as $|x| \to \infty$, but not necessarily satisfying $|u^\varepsilon(t,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$.

Recently, P. Gérard [11] solved the Cauchy problem for the Gross-Pitaevskii equation in the more natural space $E$, in space dimensions two and three. The main novelty consists in working with distances instead of norms, in order to apply a fixed point argument in $E$. In particular, the constraint $|u^\varepsilon(t,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$ is satisfied (and propagated).

We have to face a similar issue, when solving the Poisson equation. Mimicking the approach of [15, 11], it is natural to work with the property:

$$|u^\varepsilon(t,\cdot)|^2 - c(\cdot) \in L^2(\mathbb{R}^n).$$

We shall always assume that this holds at time $t = 0$. We prove that this property holds on $[0,T]$ for some $T > 0$ independent of $\varepsilon$, provided that we consider an external potential whose unbounded part is linear in $x$. However, our analysis shows that in the presence of a quadratic external potential, this property is not relevant off $t = 0$ (see Section 5).

Note that we make no assumption on the sign of $q$ (which models the charge of the element considered in a semiconductor device). This is in sharp contrast with the mathematical analysis of the semi-classical limit of the nonlinear Schrödinger equation. When the Poisson term $V_{\rm p}^\varepsilon(t,x)u^\varepsilon$ is replaced with the nonlinear term $f(|u^\varepsilon|^2)u^\varepsilon$, E. Grenier [12] proposed a strategy to obtain a phase/amplitude representation of the solution $u^\varepsilon$. This leads to study a quasi-linear system whose principal part writes:

$$\Box_{f'} := \partial_t^2 - \operatorname{div}\big(f'(|u^\varepsilon|^2)\nabla\cdot\big).$$

Hence, to prove that the Cauchy problem is well-posed, one has to assume that the nonlinearity is defocusing and cubic at the origin ($f' > 0$), except for analytic initial data [10], for which one can solve elliptic evolution equations. Here, we are not restricted to the case when $q > 0$. As will be clear below, the reason is that the quasi-linear operator $\Box_{f'}$ is replaced with the *semi-linear* operator $\partial_t^2 - q\Delta^{-1}\nabla\big((|u^\varepsilon|^2 - 1)\operatorname{div}\cdot\big)$.

**Notation.** Recall that for $s > n/2$, Zhidkov spaces are defined by[1]:

$$X^s(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n)\ ;\ \nabla u \in H^{s-1}(\mathbb{R}^n)\}\cdot$$

We denote

$$\|u\|_{X^s} := \|u\|_{L^\infty} + \|\nabla u\|_{H^{s-1}}.$$

We write $H^s = H^s(\mathbb{R}^n)$, $X^s = X^s(\mathbb{R}^n)$, $H^\infty := \cap_{s\in\mathbb{N}} H^s$, $X^\infty := \cap_{s\in\mathbb{N}} X^s$. We do not use specific notations for vector-valued functions: for instance, we write abusively $\nabla^2 f \in H^\infty$ when $\partial^2_{jk} f \in H^\infty$ for every $1 \leqslant j,k \leqslant n$.

[1] For general $s > 0$, another definition is used, see [9].

*Remark* 1.1. Zhidkov spaces contain all the functions of the form

$$\gamma + v, \text{ with } \gamma = \text{Const.} \in \mathbb{C} \text{ and } v \in H^s(\mathbb{R}^n).$$

The converse is not true, as shown by the following example:

$$u(x) = \frac{x_1}{1+|x|^2}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

On the other hand, if $n \geqslant 3$ and $u \in X^s$ for some $s > n/2$, then there exists $\gamma \in \mathbb{C}$ such that $u - \gamma \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ (see Lemma 2.1 below).

In this paper, we consider the system (1.1)–(1.3) in three cases:

- The external potential and the initial phase are sub-linear in $x$, and the mobility $c$ is in Zhidkov spaces (Part 1).
- The external potential and the initial phase are sub-quadratic in $x$, and $c$ is a short range perturbation of a non-zero constant (Part 2).
- The mobility is integrable, and the external potential and the initial phase are sub-quadratic in $x$ (Part 3).

In the first two cases, we construct a solution to (1.1)–(1.3) in Zhidkov spaces, and describe the asymptotic behavior of the main quadratic observables as $\varepsilon \to 0$. In the last case, we construct a solution in Sobolev spaces, and give pointwise asymptotics of the solution as $\varepsilon \to 0$.

In this introduction, we describe more precisely the results corresponding to the first case. We emphasize the fact that if we simply assume $V_{\text{ext}} \in C(\mathbb{R}; H^\infty)$ and $\Phi_0 \in H^\infty$, then our analysis becomes much simpler. The unboundedness of $V_{\text{ext}}$ and $\Phi_0$ require some geometrical description that complicates the technical approach. Yet, this makes our assumptions more physically relevant (see e.g. [**?**] and references therein).

**Assumption 1.** *Recall that $n \geqslant 3$.*
• External potential: $V_{\text{ext}} \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ *writes*

$$V_{\text{ext}}(t,x) = E(t) \cdot x + V_{\text{pert}}(t,x), \text{ with } E \in C^\infty(\mathbb{R}) \text{ and } \nabla V_{\text{pert}} \in C(\mathbb{R}; H^\infty).$$

• Doping profile: $c \in X^\infty$.

• Initial amplitude: $a_0^\varepsilon(x) = a_0(x) + r^\varepsilon(x)$, *where* $a_0 \in X^\infty$ *is such that* $|a_0|^2 - c \in L^2(\mathbb{R}^n)$, *and* $r^\varepsilon \in H^\infty$, *with*

$$\|r^\varepsilon\|_{H^s} \underset{\varepsilon \to 0}{\longrightarrow} 0, \quad \forall s \geqslant 0.$$

• Initial phase: *we have* $\Phi_0 \in C^\infty(\mathbb{R}^n)$ *with*

$$\Phi_0(x) = \alpha_0 \cdot x + \phi_0(x), \text{ with } \alpha_0 \in \mathbb{R}^n \text{ and } \nabla\phi_0 \in H^\infty.$$

**Lemma 1.2.** *Under the Assumption 1, there exists a unique solution $\phi_{\text{eik}} \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ to:*

$$\partial_t \phi_{\text{eik}} + \frac{1}{2}|\nabla\phi_{\text{eik}}|^2 + E(t) \cdot x = 0 \quad ; \quad \phi_{\text{eik}}(0,x) = \alpha_0 \cdot x + \beta_0 \,. \tag{1.5}$$

*This solution is given explicitly by $\phi_{\text{eik}}(t,x) = \alpha(t) \cdot x + \beta(t)$, where:*

$$\alpha(t) = \alpha_0 - \int_0^t E(\tau)d\tau \quad ; \quad \beta(t) = \beta_0 - \frac{1}{2}\int_0^t \alpha(\tau)^2 d\tau.$$

We skip the proof of this lemma; a more general result is proved in Section 5. We will see that if $V_{\rm ext}$ and/or $\Phi_0$ have a quadratic dependence on $x$, then we have to consider an eikonal phase $\phi_{\rm eik}$ which is quadratic in $x$.

**Theorem 1.3.** *Let Assumption 1 be satisfied. There exists $T > 0$ independent of $\varepsilon \in ]0,1]$ and a solution $u^\varepsilon \in L^\infty([0,T]\times\mathbb{R}^n)$ to (1.1)-(1.3), with*

$$\nabla V_{\rm p}^\varepsilon(t,x) \to 0 \quad \text{as } |x|\to\infty, \qquad V_{\rm p}^\varepsilon(t,0)=0,$$

*and such that $|u^\varepsilon|^2 - c \in L^\infty([0,T];L^2)$. Moreover, one can write $u^\varepsilon = a^\varepsilon e^{i(\phi_{\rm eik}+\phi^\varepsilon)/\varepsilon}$, where:*

- *$a^\varepsilon \in C^\infty([0,T]\times\mathbb{R}^n)\cap C([0,T];X^\infty)$, and $|a^\varepsilon|^2 - c\in C([0,T];L^2)$.*
- *$\phi^\varepsilon \in C^\infty([0,T]\times\mathbb{R}^n)$ and $\nabla\phi^\varepsilon \in C([0,T];X^\infty)$.*
- *We have the following uniform estimate: for every $s>n/2$, there exists $M_s$* independent of $\varepsilon \in ]0,1]$ *such that*

$$\|a^\varepsilon\|_{L^\infty(0,T;X^s)} + \left\||a^\varepsilon|^2 - c\right\|_{L^\infty(0,T;L^2)} + \|\nabla\phi^\varepsilon\|_{L^\infty(0,T;X^s)} \leqslant M_s.$$

*Remark* 1.4. We could not prove a uniqueness result for $u^\varepsilon$.

*Remark* 1.5. The above conditions to solve the Poisson equation are similar to those given in [21]. We explain at the end of Section 3.3 why in our framework, we cannot impose $V_{\rm p}^\varepsilon(t,x)\to 0$ as $|x|\to\infty$ (as in [1, 20] for instance).

Besides the uniform bounds, even the existence of such a solution $u^\varepsilon$ is new. First, the presence of the external potential seems to have never been studied rigorously before. As we already mentioned, this makes the proof more technically involved. Next, in most of the previous studies, $u^\varepsilon$ is supposed to be in $L^2$: see e.g. [4, 18]. In [20], the author considers the case $c\in L^1\cap H^s$. As we will see in Section 8, this case makes the analysis easier, and also makes it possible to have $u^\varepsilon\in L^2$. The main difficulty in the analysis lies in the fact that when $c\notin L^1(\mathbb{R}^n)$, the condition $|u^\varepsilon|^2 - c\in L^2(\mathbb{R}^n)$ is somehow "more nonlinear", as in [11].

The general idea to prove Theorem 1.3 consists in adapting the idea of [12]: with techniques from the hyperbolic theory, we construct a solution to

$$\begin{aligned}
&\partial_t\Phi^\varepsilon + \frac12|\nabla\Phi^\varepsilon|^2 + V_{\rm ext} + V_{\rm p}^\varepsilon = 0 \quad ; \quad \Phi^\varepsilon_{|t=0} = \Phi_0.\\
&\partial_t a^\varepsilon + \nabla\Phi^\varepsilon\cdot\nabla a^\varepsilon + \frac12 a^\varepsilon\Delta\Phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon_{|t=0} = a_0^\varepsilon.\\
&\Delta V_{\rm p}^\varepsilon = q\big(|a^\varepsilon|^2 - c\big) \quad ; \quad \nabla V_{\rm p}^\varepsilon(t,x) \underset{|x|\to\infty}{\longrightarrow} 0,\ V_{\rm p}^\varepsilon(t,0)=0.
\end{aligned} \tag{1.6}$$

Following [3], we write $\Phi^\varepsilon = \phi_{\rm eik}+\phi^\varepsilon$: with the unknown $(a^\varepsilon,\phi^\varepsilon)$, (1.6) becomes (we keep the term $\Delta\phi_{\rm eik}$ which is zero here, for future references):

$$\begin{aligned}
&\partial_t\phi^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla\phi^\varepsilon + \frac12|\nabla\phi^\varepsilon|^2 + V_{\rm pert} + V_{\rm p}^\varepsilon = 0\ ;\ \phi^\varepsilon_{|t=0} = \phi_0.\\
&\partial_t a^\varepsilon + \nabla\left(\phi^\varepsilon+\phi_{\rm eik}\right)\cdot\nabla a^\varepsilon + \frac12 a^\varepsilon\Delta\left(\phi^\varepsilon+\phi_{\rm eik}\right) = i\frac{\varepsilon}{2}\Delta a^\varepsilon\ ;\ a^\varepsilon_{|t=0} = a_0^\varepsilon.\\
&\Delta V_{\rm p}^\varepsilon = q\big(|a^\varepsilon|^2 - c\big) \quad ; \quad \nabla V_{\rm p}^\varepsilon(t,x) \underset{|x|\to\infty}{\longrightarrow} 0,\ V_{\rm p}^\varepsilon(t,0)=0.
\end{aligned} \tag{1.7}$$

Proving the existence and uniqueness of solution to (1.7) as we do in the proof of Theorem 1.3 is one of the main results of this paper. Because of the difficulties pointed out above, and the fact that one can easily be mistaken by using the usual approach, we give full details for the construction of the solution to (1.7). Passing formally to the limit, it is natural to consider:

$$
\begin{aligned}
&\partial_t\phi + \nabla\phi_{\rm eik}\cdot\nabla\phi + \frac{1}{2}|\nabla\phi|^2 + V_{\rm pert} + V_{\rm p} = 0\ ;\ \phi_{|t=0} = \phi_0.\\
&\partial_t a + \nabla\left(\phi+\phi_{\rm eik}\right)\cdot\nabla a + \frac{1}{2}a\Delta\left(\phi+\phi_{\rm eik}\right) = 0\ ;\ a_{|t=0} = a_0.\\
&\Delta V_{\rm p} = q\big(|a|^2 - c\big)\quad ;\quad \nabla V_{\rm p}(t,x)\underset{|x|\to\infty}{\longrightarrow} 0,\ V_{\rm p}(t,0) = 0.
\end{aligned}
\tag{1.8}
$$

**Notation.** The symbol $\lesssim$ stands for $\leqslant$ up to a positive, multiplicative constant which depends only on parameters that are considered fixed.
We shall also denote $L^\infty_T Y$ for $L^\infty([0,T];Y)$.

**Theorem 1.6.** *Under Assumption 1, there exists a smooth solution $(a,\phi)$ of (1.8) such that $a,\nabla\phi\in C([0,T],X^\infty)$, $|a|^2 - c\in C([0,T],L^2)$, and*

$$
\|a^\varepsilon - a\|_{L^\infty_T H^s} + \|\nabla(\phi^\varepsilon-\phi)\|_{L^\infty_T X^s}\underset{\varepsilon\to 0}{\longrightarrow} 0,\qquad \forall s>n/2.
$$

*In particular:*

$$
\begin{aligned}
&|u^\varepsilon|^2\underset{\varepsilon\to 0}{\longrightarrow}|a|^2\quad \text{in } L^\infty_T H^s,\ \text{and}\\
&\varepsilon\,\mathrm{Im}\left(\overline{u^\varepsilon}\nabla u^\varepsilon\right)\underset{\varepsilon\to 0}{\longrightarrow}|a|^2\nabla\left(\phi_{\rm eik}+\phi\right)\quad \text{in } L^\infty_T X^s,\ \forall s>n/2.
\end{aligned}
$$

Recall that in general, none of the terms $a$ or $a^\varepsilon$ is in $L^2(\mathbb{R}^n)$. Though, the difference $a^\varepsilon - a$ is in $L^2(\mathbb{R}^n)$, and asymptotically small as $\varepsilon\to 0$. Note that $(\rho,\mathbf{v}) := (|a|^2,\nabla(\phi+\phi_{\rm eik}))$ solves the Euler–Poisson system:

$$
\left\{
\begin{aligned}
&\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0,\\
&\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla V_{\rm ext} + \nabla V_{\rm p} = 0,\\
&\Delta V_{\rm p} = q\left(\rho - c\right)\quad ;\quad \nabla V_{\rm p}(t,x)\underset{|x|\to\infty}{\longrightarrow} 0,\ V_{\rm p}(t,0) = 0.
\end{aligned}
\right.
\tag{1.9}
$$

The existence of solutions to (1.9) under Assumption 1 is new.

This paper borrows several ideas from [3], [11] and [12]. As we have already mentioned, an important difference with [12] is that the underlying wave equation associated to (1.6) is semi-linear, and not quasi-linear. The reduction to (1.7) is similar to the approach in [3]. Several important differences should be pointed out. First, we work in Zhidkov spaces instead of Sobolev spaces, an aspect which requires some extra care. Integrating the Poisson equation, especially when we have $\Delta V^\varepsilon_{\rm p}\in L^2(\mathbb{R}^n)$ and not necessarily $\Delta V^\varepsilon_{\rm p}\in L^1(\mathbb{R}^n)$, is also a new problem. Finally, the propagation of the initial assumption $|a^\varepsilon_0|^2 - c\in L^2(\mathbb{R}^n)$ turns out to be different from the phenomenon studied in [11]. As we shall see in Section 5, the presence of quadratic "geometric" quantities (such as an external harmonic potential) requires a highly non-trivial adaptation of the approach in [3].

The rest of this paper is organized as follows. In Section 2, we collect various technical estimates, in order not to interrupt the proofs later on. In Section 3, we prove Theorem 1.3. Theorem 1.6 is proved in Section 4. In Part 2 (Sections 5–7), we consider the case when $c-1\in L^1\cap H^\infty$, and the external potential and the initial phase contain quadratic terms. In Part 3 (Section 8), we assume $c\in L^1(\mathbb{R}^n)$, and prove a refined convergence result.

*Remark* 1.7. Before leaving this introduction, let us explain why we concentrated on the whole space problem. Indeed, some problems require considering the periodic case (see [**?**] and the references therein), where the space variable belongs to the torus $\mathbb{T}^n$. As a matter of fact, the periodic case is easier. This follows from two observations: first, the computations below apply *mutatis mutandis* in the periodic setting; and second, for all $\sigma \in \mathbb{R}$, the operator $\Delta^{-1}\nabla$ is well-defined in $H^\sigma(\mathbb{T}^n)$.

## 2. Estimates in Sobolev and Zhidkov spaces

This section serves as the requested background for what follows. We first recall a consequence of the Hardy-Littlewood-Sobolev inequality, which can be found in [13, Th. 4.5.9] or [11, Lemma 7]:

**Lemma 2.1.** *If $\varphi \in \mathcal{D}'(\mathbb{R}^n)$ is such that $\nabla\varphi \in L^p(\mathbb{R}^n)$ for some $p \in ]1, n[$, then there exists a constant $\gamma$ such that $\varphi - \gamma \in L^q(\mathbb{R}^n)$, with $1/p = 1/q + 1/n$.*

This shows that under Assumption 1, the doping profile is of the form

$$c = \gamma + \widetilde{c}, \quad \text{where } \gamma \text{ is a constant, and } \widetilde{c} \in L^{\frac{2n}{n-2}}(\mathbb{R}^n), \nabla\widetilde{c} \in H^\infty.$$

Define the Fourier transform as

$$\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx.$$

**Lemma 2.2.** *Let $n \geqslant 3$. For every $s > n/2$, there exists $C_s$ such that*

$$\|\varphi\|_{L^\infty} \leqslant C_s \|\nabla\varphi\|_{H^{s-1}}, \quad \forall \varphi \in H^s(\mathbb{R}^n). \tag{2.1}$$

*Remark.* In space dimension $n \leqslant 2$, low frequencies rule out the above inequalities. For instance, in space dimension $n = 1$, the function

$$f(x) = \int_0^x \frac{dy}{\sqrt{1+y^2}} = \arg\sinh(x)$$

is not in $L^\infty(\mathbb{R})$, but its derivative is in $H^\infty$. In space dimension $n = 2$, consider the function

$$f(x_1, x_2) = \log\left|\log(x_1^2 + x_2^2)\right|.$$

One can check that $\nabla f \in H^\infty$, while clearly, $f \notin L^\infty(\mathbb{R}^2)$.

**Lemma 2.3.** *Let $n \geqslant 3$, $q \geqslant 2$ and $s > n/2 - 1$. There exists $C = C(n, q, s)$ such that for all $\varphi \in L^q(\mathbb{R}^n)$ with $\nabla\varphi \in H^s(\mathbb{R}^n)$,*

$$\|\varphi\|_{L^\infty} \leqslant C\left(\|\varphi\|_{L^q} + \|\nabla\varphi\|_{H^s}\right).$$

*Proof.* The usual Sobolev embedding yields, for any $\sigma > n/q$,

$$\|\varphi\|_{L^\infty} \lesssim \|\varphi\|_{L^q} + \||\nabla|^\sigma\varphi\|_{L^q}.$$

On the other hand, for $k = n(1/2 - 1/q)$,

$$\||\nabla|^\sigma\varphi\|_{L^q} \lesssim \||\nabla|^\sigma\varphi\|_{H^k} \lesssim \|\nabla\varphi\|_{H^{k+\sigma-1}},$$

provided that $\sigma \geqslant 1$. If $s > n/2 - 1$, $\sigma$ given by $s = n(1/2 - 1/q) + \sigma - 1$ is such that $\sigma > n/q$ and $\sigma \geqslant 1$. The above two estimates then yield the lemma. □

**Lemma 2.4.** *Let $n \geqslant 3$. For every $s \geqslant 0$, $\nabla\Delta^{-1}$ maps $L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ to $H^{s+1}(\mathbb{R}^n)$: there exists $C_s$ such that*

$$\|\nabla\Delta^{-1}\varphi\|_{H^{s+1}} \leqslant C_s \left(\|\varphi\|_{L^1} + \|\varphi\|_{H^s}\right), \quad \forall \varphi \in L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n).$$

The following variant of the classical Kato-Ponce estimates can be found in [14, Theorem 5]:

**Lemma 2.5.** *Let $n \geqslant 1$ and $s > n/2+1$. Denote $\Lambda = (I-\Delta)^{1/2}$. There exists a constant $C_s$ such that, for all $f \in X^{s+1}(\mathbb{R}^n)$ and all $u \in H^{s-1}(\mathbb{R}^n)$,*

$$\|f\Lambda^s u - \Lambda^s(fu)\|_{L^2} \leqslant C_s \left(\|\nabla f\|_{L^\infty} \|u\|_{H^{s-1}} + \|\nabla f\|_{H^{s-1}} \|u\|_{L^\infty}\right). \tag{2.2}$$

**Lemma 2.6.** *Let $s > n/2$. The Sobolev space $H^s(\mathbb{R}^n)$ and the Zhidkov space $X^s(\mathbb{R}^n)$ are algebras: there exists a constant $C_s$ such that, for all $u, v \in H^s(\mathbb{R}^n)$ and $a, b \in X^s(\mathbb{R}^n)$,*

$$\|uv\|_{H^s} \leqslant C_s \|u\|_{H^s} \|v\|_{H^s} \quad ; \quad \|ab\|_{X^s} \leqslant C_s \|a\|_{X^s} \|b\|_{X^s}.$$

*There exists $C_s$ such that for all $v \in H^s(\mathbb{R}^n)$ and $a \in X^s(\mathbb{R}^n)$,*

$$\|av\|_{H^s} + \|av\|_{X^s} \leqslant C_s \|v\|_{H^s} \|a\|_{X^s}.$$

*There exists $C_s$ such that for all $a \in X^s(\mathbb{R}^n)$ and $b \in X^{s+1}(\mathbb{R}^n)$,*

$$\|a\nabla b\|_{H^s} \leqslant C_s \|a\|_{X^s} \|b\|_{X^{s+1}}.$$

In order to use Arzela–Ascoli's theorem, we will invoke:

**Lemma 2.7.** *Let $\sigma > n/2$ and $(\varphi_j)_{j\in\mathbb{N}}$ be a bounded sequence in $X^\sigma(\mathbb{R}^n)$. For all $\sigma' < \sigma$, there exists a subsequence which converges in $H^{\sigma'}_{\rm loc}(\mathbb{R}^n)$.*

*Proof.* This follows from the fact that, for all test function $\chi \in C_0^\infty(\mathbb{R}^n)$, $(\chi\varphi_j)_{j\in\mathbb{N}}$ is a bounded sequence in $H^\sigma(\mathbb{R}^n)$. □

*Remark.* It might seem more natural to state a precompactness result in

$$X^{\sigma'}_{\rm loc}(\mathbb{R}^n) := \left\{\varphi \in L^\infty_{\rm loc}(\mathbb{R}^n)\ ;\ \nabla\varphi \in H^{\sigma'-1}_{\rm loc}(\mathbb{R}^n)\right\}.$$

Actually, one can check that for $\sigma' > n/2$, $X^{\sigma'}_{\rm loc}(\mathbb{R}^n) = H^{\sigma'}_{\rm loc}(\mathbb{R}^n)$.

**Lemma 2.8.** *Let $n \geqslant 3$ and $s > n/2$.*

- *For all $p > \frac{2n}{n-2}$, there exists $C = C(s,p,n)$ such that:*

$$\|\Delta^{-1}\nabla f\|_{L^p} \leqslant C\|f\|_{H^s}, \quad \forall f \in H^s.$$

- *There exists $C = C(s,n)$ such that:*

$$\left\|\mathcal{F}\left(\Delta^{-1}\nabla f\right)\right\|_{L^1} \leqslant C\|f\|_{H^s}, \quad \forall f \in H^s.$$

*Proof.* Essentially, we use the property $\widehat{f} \in L^2$ for low frequencies, and $\widehat{f} \in L^1$ for high frequencies ($\mathcal{F}(H^s) \subset L^1$ if $s > n/2$). For $p > 2n/(n-2)$,

$$\left\|\frac{\xi}{|\xi|^2}\widehat{f}\right\|_{L^{p'}} \lesssim \left\||\xi|^{-1}\right\|_{L^{\frac{2p'}{2-p'}}(|\xi|<1)} \left\|\widehat{f}\right\|_{L^2} + \left\||\xi|^{-1}\right\|_{L^\infty(|\xi|\geqslant 1)} \left\|\widehat{f}\right\|_{L^{p'}}.$$

The norms involving $|\xi|^{-1}$ are finite since $p > 2n/(n-2)$. For $s > n/2$,

$$\left\|\widehat{f}\right\|_{L^1} \lesssim \left\|\langle\xi\rangle^s \widehat{f}\right\|_{L^2} = \|f\|_{H^s}.$$

The first point follows from the Hausdorff–Young inequality:

$$\|\Delta^{-1}\nabla f\|_{L^p} \lesssim \left\| \frac{\xi}{|\xi|^2}\widehat{f}\right\|_{L^{p'}}$$

The second point is straightforward, with $p'=1$. □

**Part 1. Sublinear eikonal phase**

## 3. Proof of Theorem 1.3

Our first task is to construct a solution to (1.7). As explained in the introduction, it is convenient to introduce the "velocity" $v^\varepsilon = \nabla\phi^\varepsilon$. Denoting $v_{\rm eik} = \nabla\phi_{\rm eik}$, and recalling that $v_{\rm eik}$ is a function of time only, we infer from (1.7) that $(a^\varepsilon, v^\varepsilon)$ has to solve:

$$\begin{cases} \partial_t v^\varepsilon + (v_{\rm eik}+v^\varepsilon)\cdot\nabla v^\varepsilon + \nabla V_{\rm pert} + \nabla V_{\rm p}^\varepsilon = 0,\\ \partial_t a^\varepsilon + (v_{\rm eik}+v^\varepsilon)\cdot\nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon\nabla\cdot v^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon,\\ \Delta V_{\rm p}^\varepsilon = q\big(|a^\varepsilon|^2 - c\big), \end{cases} \tag{3.1}$$

together with

$$\nabla V_{\rm p}^\varepsilon(t,x) \underset{|x|\to\infty}{\longrightarrow} 0 \quad ; \quad V_{\rm p}^\varepsilon(t,0) = 0 \quad ; \quad v^\varepsilon_{|t=0} = \nabla\phi_0 \ ; \ a^\varepsilon_{|t=0} = a_0^\varepsilon. \tag{3.2}$$

In the context of Assumption 1, we show that the solutions of (3.1)–(3.2) exist and are uniformly bounded for a time interval independent of $\varepsilon$.

**Proposition 3.1.** *Let Assumption 1 be satisfied. Let $s > n/2$. For all $M > M_0 > 0$, there exists $T>0$ such that, if for all $\varepsilon\in[0,1]$,*

$$\|\nabla\phi_0\|_{H^{s+2}} + \big\||a_0^\varepsilon|^2 - c\big\|_{L^2} + \|a_0^\varepsilon\|_{X^{s+1}} \leqslant M_0, \tag{3.3}$$

*then the Cauchy problem (3.1)–(3.2) has a unique classical solution $(v^\varepsilon, a^\varepsilon)$ in $C^\infty([0,T]\times\mathbb{R}^n)$ such that*

$$\|v^\varepsilon\|_{L^\infty_T X^{s+2}} + \big\||a^\varepsilon|^2 - c\big\|_{L^\infty_T L^2} + \|a^\varepsilon\|_{L^\infty_T X^{s+1}} \leqslant M. \tag{3.4}$$

As suggested by the above statement, we construct $\nabla V_{\rm p}^\varepsilon$ (only the gradient of $V_{\rm p}^\varepsilon$ is present in (3.1)), and the condition $V_{\rm p}^\varepsilon(t,0)=0$ is given only to insure uniqueness for $V_{\rm p}^\varepsilon$ (even though it is not stated in the above result). Therefore, we shall neglect this condition for a while.

3.1. **Regularized equations.** Let $\jmath$ be a $C^\infty$ function of $\xi\in\mathbb{R}^n$, with

$$0\leqslant \jmath \leqslant 1,\quad \jmath(\xi)=1 \text{ for } |\xi|\leqslant 1,\quad \jmath(\xi)=0 \text{ for } |\xi|\geqslant 2,\quad \jmath(\xi)=\jmath(-\xi).$$

Set $\jmath_h(\xi) := \jmath(h\xi)$, for $h>0$ and $\xi\in\mathbb{R}^n$; $\jmath_h$ is supported in the ball of radius $2/h$ about the origin. Define $J_h$ as the Fourier multiplier with symbol $\jmath_h$:

$$J_h := \jmath(hD_x).$$

Also, for our purpose it is interesting to introduce a family of operators that cut the low frequency component of a function. Indeed, the Poisson term $\nabla V_{\rm p}^\varepsilon = q\Delta^{-1}\nabla(|a^\varepsilon|^2 - c)$, is not well defined in general. We replace the operator $q\Delta^{-1}\nabla$ by a family of operators $R_h\nabla$ well-defined on Sobolev spaces and prove that, in the end, there is no need to estimate the low frequency component of $\nabla V_{\rm p}^\varepsilon$. To do that, we set

$$G_h = I - J_{1/h},$$

that is, $G_h$ is the Fourier multiplier with symbol $1 - \jmath_{1/h}$, which is supported in $\{|\xi| \geqslant h\}$. Consequently, the operator

$$R_h := q\Delta^{-1}G_h,$$

is bounded in all Sobolev spaces (with operator norm going to $+\infty$ when $h$ tends to 0). More precisely, there exists a constant $C$ such that, for all $\sigma \geqslant 0$,

$$\left\|\Delta^{-1}G_h\right\|_{H^\sigma\to H^{\sigma+2}} \leqslant Ch^{-2}.$$

Consider the following approximation of (3.1):

$$\begin{cases} \partial_t v_h^\varepsilon + J_h\left((v_{\rm eik} + v_h^\varepsilon)\cdot\nabla J_h v_h^\varepsilon\right) + \nabla V_{\rm pert} = -R_h\nabla(|a_h^\varepsilon|^2 - c), \\ \partial_t a_h^\varepsilon + J_h\left((v_{\rm eik} + v_h^\varepsilon)\cdot\nabla J_h a_h^\varepsilon\right) + \dfrac{1}{2}a_h^\varepsilon\nabla\cdot v_h^\varepsilon = i\dfrac{\varepsilon}{2}\Delta J_h^2 a_h^\varepsilon. \end{cases} \tag{3.5}$$

We keep the same initial data:

$$v^\varepsilon_{h|t=0} = \nabla\phi_0 \ ; \ a^\varepsilon_{h|t=0} = a_0^\varepsilon. \tag{3.6}$$

Note that Assumption 1 implies that $v^\varepsilon_{h|t=0}$ is in $H^\infty$ and is independent of $\varepsilon \in [0,1]$ and $h \in ]0,1]$, while $a^\varepsilon_{h|t=0}$ is in $X^\infty$, and uniformly bounded in $X^s$ for any $s > n/2$, for $\varepsilon \in [0,1]$ and $h \in ]0,1]$.

The point is that the regularized equations (3.5)–(3.6) have been chosen so that the Cauchy problem can be solved as in the standard framework of Sobolev spaces:

**Lemma 3.2.** *Let $s > n/2$. For all $\varepsilon \in [0,1]$ and all $h \in ]0,1]$ there exists $T_h^\varepsilon > 0$ such that the Cauchy problem* (3.5)–(3.6) *has a unique solution $(v_h^\varepsilon, a_h^\varepsilon) \in C^1([0,T_h^\varepsilon]; H^{s+2}(\mathbb{R}^n) \times X^{s+1}(\mathbb{R}^n))$.*

*Proof.* The proof is based on the usual theorem for ordinary differential equations. Set $\mathbf{u}_h^\varepsilon = (v_h^\varepsilon, a_h^\varepsilon)$ and we rewrite (3.5) under the form

$$\partial_t\mathbf{u}_h^\varepsilon = F_1(\varepsilon, h, \mathbf{u}_h^\varepsilon) + F_2(t)\mathbf{u}_h^\varepsilon + F_3(t,x),$$

where $F_1(\varepsilon, h, \mathbf{u})$ is at most quadratic in $\mathbf{u}$, and we have used the property that $v_{\rm eik}$ is a function of time only. We have to verify that the functions $F$ are smooth. This follows from Lemmas 2.2 and 2.6, and the fact that the operators $R_h$ and $\Delta J_h$ are of order $-2$ and 0 respectively:

$$\begin{aligned}
&\|J_h(v_h^\varepsilon\cdot\nabla J_h v_h^\varepsilon)\|_{H^{s+2}} \lesssim \|v_h^\varepsilon\|_{H^{s+2}}\|\nabla J_h v_h^\varepsilon\|_{H^{s+2}} \lesssim h^{-1}\|v_h^\varepsilon\|^2_{H^{s+2}}, \\
&\|J_h(v_h^\varepsilon\cdot\nabla J_h a_h^\varepsilon)\|_{X^{s+1}} \lesssim \|v_h^\varepsilon\|_{H^{s+1}}\|\nabla J_h a_h^\varepsilon\|_{X^{s+1}} \lesssim h^{-1}\|v_h^\varepsilon\|_{H^{s+1}}\|a_h^\varepsilon\|_{X^{s+1}}, \\
&\|a_h^\varepsilon\nabla\cdot v_h^\varepsilon\|_{X^{s+1}} \lesssim \|v_h^\varepsilon\|_{H^{s+2}}\|a_h^\varepsilon\|_{X^{s+1}}, \\
&\left\|R_h\nabla\,|a_h^\varepsilon|^2\right\|_{H^{s+2}} \lesssim h^{-2}\left\|\nabla\,|a_h^\varepsilon|^2\right\|_{H^s} \lesssim h^{-2}\left\|a_h^\varepsilon\right\|^2_{X^{s+1}}, \\
&\left\|\Delta J_h^2 a_h^\varepsilon\right\|_{X^{s+1}} = \|J_h\Delta J_h a_h^\varepsilon\|_{X^{s+1}} \leqslant \|\Delta J_h a_h^\varepsilon\|_{H^{s+1}} \lesssim h^{-2}\|a_h^\varepsilon\|_{X^{s+1}}.
\end{aligned}$$

□

3.2. **Uniform bounds.** We first prove that the solutions $(v_h^\varepsilon, a_h^\varepsilon)$ exist and they are uniformly bounded for a time independent of the parameters $\varepsilon$ and $h$. Below, $T_h^{\varepsilon *}$ denotes the lifespan, that is the supremum of all the positive times $T_h^\varepsilon$ such that the Cauchy problem for (3.5)–(3.6) has a unique solution in $C^1([0,T_h^\varepsilon]; H^{s+2}(\mathbb{R}^n) \times X^{s+1}(\mathbb{R}^n))$.

**Proposition 3.3.** *Let $s > n/2$. There exists a continuous function $g\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that, for all $\varepsilon \in [0,1]$ and all $h \in ]0,1]$, the norm $M_h^\varepsilon\colon [0,T_h^{\varepsilon *}] \to \mathbb{R}_+^*$ defined by*

$$M_h^\varepsilon(T) := \|a_h^\varepsilon\|_{L_T^\infty X^{s+1}} + \big\||a_h^\varepsilon|^2 - c\big\|_{L_T^\infty L^2} + \|\nabla v_h^\varepsilon\|_{L_T^\infty H^{s+1}},$$

*satisfies the estimate: $M_h^\varepsilon(T) \leqslant M_h^\varepsilon(0)e^{Tg(M_h^\varepsilon(T))}$, $\forall T \in [0,T_h^{\varepsilon *}]$.*

*Proof.* Before we proceed, two comments are in order. Firstly, the functions $(v_h^\varepsilon, a_h^\varepsilon)$ are smooth ($C^1$ in time with values in Sobolev/Zhidkov spaces), so that it is easily verified that all the following computations are meaningful. Secondly, it is useful to note that, in view of Lemma 2.2, it suffices to prove that

$$m_h^\varepsilon(T) \leqslant m_h^\varepsilon(0)e^{Tg(\widetilde{M_h^\varepsilon}(T))}, \qquad \forall T \in [0,T_h^{\varepsilon *}], \tag{3.7}$$

where

$$m_h^\varepsilon(t) = \|\nabla a_h^\varepsilon\|_{L_T^\infty H^s} + \big\||a_h^\varepsilon|^2 - c\big\|_{L_T^\infty L^2} + \|\nabla v_h^\varepsilon\|_{L_T^\infty H^{s+1}},$$

and

$$\widetilde{M_h^\varepsilon}(T) := M_h^\varepsilon(T) + \|v_h^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^n)}\,.$$

Indeed, Lemma 2.2 provides us with a constant $C_s$ such that $\widetilde{M_h^\varepsilon} \leqslant C_s M_h^\varepsilon$, and we have:

**Lemma 3.4.** *Let $s > n/2$ and $c \in X^\infty$. There exists a constant $K$ such that, for all $T > 0$ and $\varphi \in X^\infty$,*

$$\|\varphi\|_{L^\infty(\mathbb{R}^n)}^2 \leqslant K\,\big\||\varphi|^2 - c\big\|_{L^2} + K\|\nabla\varphi\|_{H^s}^2 + K\|c\|_{X^{s+1}}.$$

*Proof.* By Lemma 2.3, we have:

$$\big\||\varphi|^2 - c\big\|_{L^\infty} \lesssim \big\||\varphi|^2 - c\big\|_{L^2} + \big\|\nabla\big(|\varphi|^2 - c\big)\big\|_{H^s}\,.$$

Since $s > n/2$,

$$\big\|\nabla\big(|\varphi|^2 - c\big)\big\|_{H^s} \lesssim \|\varphi\|_{L^\infty}\,\|\nabla\varphi\|_{H^s} + \|\nabla\varphi\|_{H^s}^2 + \|\nabla c\|_{H^s}\,.$$

Triangle inequality yields

$$\|\varphi\|_{L^\infty}^2 \lesssim \big\||\varphi|^2 - c\big\|_{L^2} + \|\varphi\|_{L^\infty}\,\|\nabla\varphi\|_{H^s} + \|\nabla\varphi\|_{H^s}^2 + \|c\|_{X^{s+1}}\,,$$

hence, the desired result follows by Young's inequality. □

With these preliminaries established, to prove (3.7), we begin by estimating the $L^2$ norm of $|a_h^\varepsilon|^2 - c$. To do that, we start from

$$\frac{d}{dt}\big\||a_h^\varepsilon|^2 - c\big\|_{L^2}^2 \leqslant 2\big\||a_h^\varepsilon|^2 - c\big\|_{L^2}\big\|\partial_t\big(|a_h^\varepsilon|^2 - c\big)\big\|_{L^2}.$$

The second factor in the right hand side is estimated by

$$\big\|\partial_t\big(|a_h^\varepsilon|^2 - c\big)\big\|_{L^2} \leqslant 2\,\|a_h^\varepsilon\|_{L^\infty}\,\|\partial_t a_h^\varepsilon\|_{L^2}\,.$$

Directly from the equations, we find that for bounded times,

$$\|\partial_t a_h^\varepsilon\|_{L^2} \lesssim (1 + \|v_h^\varepsilon\|_{L^\infty})\,\|\nabla a_h^\varepsilon\|_{L^2} + \|a_h^\varepsilon\|_{L^\infty}\,\|\nabla v_h^\varepsilon\|_{L^2} + \|\Delta a_h^\varepsilon\|_{L^2}\,.$$

Consequently, we obtain

$$\frac{d}{dt}\big\||a_h^\varepsilon|^2 - c\big\|_{L^2}^2 \lesssim M_h^\varepsilon\,(1 + M_h^\varepsilon)^2\,. \tag{3.8}$$

We now turn to the estimate of the $H^s$ norm of $\nabla a_h^\varepsilon$. Set $Q := \Lambda^s \nabla$, where $\Lambda = (I-\Delta)^{1/2}$. Since $[\nabla, Q] = 0 = [J_h, Q]$, by commuting $Q$ with the equation for $a_h^\varepsilon$, we find:

$$\partial_t Q a_h^\varepsilon + J_h\left((v_{\text{eik}} + v_h^\varepsilon)\cdot \nabla J_h Q a_h^\varepsilon\right) - i\frac{\varepsilon}{2}\Delta J_h^2 Q a_h^\varepsilon = f_h^\varepsilon,$$

with

$$f_h^\varepsilon := J_h\left([v_h^\varepsilon, Q]\cdot \nabla J_h a_h^\varepsilon\right) - \frac{1}{2}Q\big(a_h^\varepsilon \nabla\cdot v_h^\varepsilon\big).$$

Notice that $J_h$ is self-adjoint. We use the following convention for the scalar product in $L^2$:

$$\langle\, f\,,\, g\,\rangle := \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx.$$

We have, since $\nabla v_{\text{eik}} \equiv 0$:

$$\begin{aligned}
\operatorname{Re}\langle\, i\Delta J_h^2 Q a_h^\varepsilon\,,\, Q a_h^\varepsilon\,\rangle &= \operatorname{Re}\langle\, i\Delta J_h Q a_h^\varepsilon\,,\, J_h Q a_h^\varepsilon\,\rangle = 0,\\
2\langle\, J_h((v_{\text{eik}} + v_h^\varepsilon)\cdot\nabla J_h Q a_h^\varepsilon)\,,\, Q a_h^\varepsilon\,\rangle &= 2\langle\,(v_{\text{eik}} + v_h^\varepsilon)\cdot\nabla J_h Q a_h^\varepsilon\,,\, J_h Q a_h^\varepsilon\,\rangle\\
&= -\langle\,(\nabla\cdot v_h^\varepsilon) J_h Q a_h^\varepsilon\,,\, J_h Q a_h^\varepsilon\,\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\left\|Q a_h^\varepsilon\right\|_{L^2}^2 &= 2\operatorname{Re}\langle\,\partial_t Q a_h^\varepsilon\,,\, Q a_h^\varepsilon\,\rangle\\
&= \operatorname{Re}\langle\,(\nabla\cdot v_h^\varepsilon) J_h Q a_h^\varepsilon\,,\, J_h Q a_h^\varepsilon\,\rangle + 2\operatorname{Re}\langle\, f_h^\varepsilon\,,\, Q a_h^\varepsilon\,\rangle\\
&\leqslant \left\|\nabla v_h^\varepsilon\right\|_{L^\infty}\left\|J_h Q a_h^\varepsilon\right\|_{L^2}^2 + 2\left\|f_h^\varepsilon\right\|_{L^2}\left\|Q a_h^\varepsilon\right\|_{L^2}.
\end{aligned}$$

We now have to estimate the $L^2$ norm of $f_h^\varepsilon$. The first term is estimated by way of the commutator estimate (2.2) and the Sobolev embedding:

$$\begin{aligned}
\left\|J_h\left([v_h^\varepsilon, Q]\cdot\nabla J_h a_h^\varepsilon\right)\right\|_{L^2} &\lesssim \left\|[v_h^\varepsilon, Q]\cdot\nabla J_h a_h^\varepsilon\right\|_{L^2}\\
&\lesssim \left(\left\|\nabla v_h^\varepsilon\right\|_{L^\infty} + \left\|\nabla v_h^\varepsilon\right\|_{H^{s+1}}\right)\left\|\nabla J_h a_h^\varepsilon\right\|_{H^s}\\
&\lesssim \left\|\nabla v_h^\varepsilon\right\|_{H^{s+1}}\left\|\nabla a_h^\varepsilon\right\|_{H^s}.
\end{aligned}$$

To estimate the last term, we use Lemma 2.6, to obtain

$$\left\|Q\big(a_h^\varepsilon\nabla\cdot v_h^\varepsilon\big)\right\|_{L^2} \lesssim \left\|a_h^\varepsilon\nabla\cdot v_h^\varepsilon\right\|_{H^{s+1}} \lesssim \left\|a_h^\varepsilon\right\|_{X^{s+1}}\left\|\nabla v_h^\varepsilon\right\|_{H^{s+1}}.$$

We infer that

$$\left\|f_h^\varepsilon\right\|_{L^2} \lesssim \left\|\nabla v_h^\varepsilon\right\|_{H^{s+1}}\left\|a_h^\varepsilon\right\|_{X^{s+1}}.$$

Therefore, we end up with

$$\frac{d}{dt}\|\nabla a_h^\varepsilon\|_{H^s}^2 \lesssim \|\nabla v_h^\varepsilon\|_{H^{s+1}}\|a_h^\varepsilon\|_{X^{s+1}}^2. \tag{3.9}$$

The technique for estimating $\nabla v_h^\varepsilon$ in $H^{s+1}$ is similar. Indeed, the analysis establishing the previous estimate also yields

$$\begin{aligned}
\frac{d}{dt}\|\nabla v_h^\varepsilon\|_{H^{s+1}}^2 &\lesssim \left(1 + \|\nabla v_h^\varepsilon\|_{H^{s+1}}\right)\|\nabla v_h^\varepsilon\|_{H^{s+1}}^2\\
&\quad + \|\nabla^2 V_{\text{pert}}\|_{H^{s+1}}^2 + \|\nabla R_h \nabla\big(|a_h^\varepsilon|^2 - c\big)\|_{H^{s+1}}^2.
\end{aligned}$$

Since $\nabla R_h \nabla$ is uniformly bounded from $H^{s+1}$ to itself, we obtain

$$\frac{d}{dt}\|\nabla v_h^\varepsilon\|_{H^{s+1}}^2 \lesssim \left(1 + \|\nabla v_h^\varepsilon\|_{H^{s+1}}\right)\|\nabla v_h^\varepsilon\|_{H^{s+1}}^2 + 1 + \left\||a_h^\varepsilon|^2 - c\right\|_{H^{s+1}}^2.$$

Next, noting that

$$\begin{aligned}\left\||a_h^\varepsilon|^2 - c\right\|_{H^{s+1}} &\lesssim \left\||a_h^\varepsilon|^2 - c\right\|_{L^2} + \|\nabla c\|_{H^s} + \|\nabla |a_h^\varepsilon|^2\|_{H^s} \\ &\lesssim \left\||a_h^\varepsilon|^2 - c\right\|_{L^2} + 1 + \|a_h^\varepsilon\|_{L^\infty}^2 + \|\nabla a_h^\varepsilon\|_{H^s}^2 \\ &\lesssim \left\||a_h^\varepsilon|^2 - c\right\|_{L^2} + 1 + \|a_h^\varepsilon\|_{X^{s+1}}^2,\end{aligned}$$

we conclude that

$$\frac{d}{dt}\|\nabla v_h^\varepsilon\|_{H^{s+1}}^2 \lesssim C\big(\widetilde{M_h^\varepsilon}\big)\|\nabla v_h^\varepsilon\|_{H^{s+1}}^2 + C(M_h^\varepsilon). \tag{3.10}$$

Summing over (3.8), (3.9) and (3.10), Gronwall lemma yields the uniform estimate (3.7). □

Lemma 3.2 and Proposition 3.3 yield the following result:

**Corollary 3.5.** *Let Assumption 1 be satisfied, and let $s > n/2$. For all $M > M_0 > 0$, there exists $T > 0$ such that, if for all $\varepsilon \in [0,1]$,*

$$\|\nabla\phi_0\|_{H^{s+2}(\mathbb{R}^n)} + \left\||a_0^\varepsilon|^2 - c(\cdot)\right\|_{L^2(\mathbb{R}^n)} + \|a_0^\varepsilon\|_{X^{s+1}(\mathbb{R}^n)} \leqslant M_0,$$

*then the Cauchy problem (3.5)–(3.6) has a unique solution $(v_h^\varepsilon, a_h^\varepsilon) \in C^1([0,T]; H^{s+2}(\mathbb{R}^n) \times X^{s+1}(\mathbb{R}^n))$ satisfying*

$$\|\nabla v_h^\varepsilon\|_{L^\infty_T H^{s+1}} + \left\||a_h^\varepsilon|^2 - c\right\|_{L^\infty_T L^2} + \|a_h^\varepsilon\|_{L^\infty_T X^{s+1}} \leqslant M.$$

*Remark* 3.6. Refining the above computations thanks to Moser's calculus and tame estimates, we can see that the above existence time $T$ can be taken independent of $s > n/2$ (see e.g. [16, Section 2.2] or [19, Section 16.1]). This explains why we did not emphasize its dependence upon $s$, and why we consider different values for $s$ below, without changing the notation $T$.

3.3. **Convergence of the scheme.** We first claim that $\partial_t v_h^\varepsilon$ and $\partial_t a_h^\varepsilon$ are bounded in $C([0,T]; X^{s-1})$, uniformly for $h \in ]0,1]$. To see this, by using Lemma 2.2, (3.5) and Corollary 3.5, the point is to verify that the term $R_h\nabla\left(|a_h^\varepsilon|^2 - c\right)$, in the equation for $\partial_t v_h^\varepsilon$, is uniformly bounded in $C([0,T]; L^\infty)$. Denote

$$W_h^\varepsilon := R_h\nabla\left(|a_h^\varepsilon|^2 - c\right).$$

From Corollary 3.5, $W_h^\varepsilon \in C([0,T]; H^{s+2})$, and $\nabla W_h^\varepsilon$ is bounded in $C([0,T]; H^{s+1})$. In particular, Lemma 2.2 shows that $W_h^\varepsilon$ is bounded in $C([0,T]; L^\infty)$.

From Lemma 2.7 and Arzela–Ascoli's Theorem, for a subsequence $h'$ of $h$,

$$v_{h'}^\varepsilon \to v^\varepsilon \text{ and } a_{h'}^\varepsilon \to a^\varepsilon \text{ in } C([0,T]; H^{s'}_{\mathrm{loc}}), \text{ as } h' \to 0, \tag{3.11}$$

for any $s' < s-1$. Moreover, we have $v^\varepsilon, a^\varepsilon \in C_w([0,T]; X^s)$. We can then pass to the limit in all the terms in (3.5), except possibly the Poisson term, that is, the right hand side in the equation for $v_h^\varepsilon$.

To claim that $(v^\varepsilon, a^\varepsilon)$ solves (3.1)–(3.2), we introduce the Poisson potential

$$V_h^\varepsilon := q\Delta^{-1}G_h\left(|a_h^\varepsilon|^2 - c\right).$$

Then (3.5) can be rewritten as:

$$\begin{cases}\partial_t v_h^\varepsilon + J_h\left((v_{\mathrm{eik}} + v_h^\varepsilon)\cdot\nabla J_h v_h^\varepsilon\right) + \nabla V_{\mathrm{pert}} + \nabla V_h^\varepsilon = 0,\\ \partial_t a_h^\varepsilon + J_h\left((v_{\mathrm{eik}} + v_h^\varepsilon)\cdot\nabla J_h a_h^\varepsilon\right) + \dfrac{1}{2}a_h^\varepsilon\nabla\cdot v_h^\varepsilon = i\dfrac{\varepsilon}{2}\Delta J_h^2 a_h^\varepsilon,\\ \Delta V_h^\varepsilon = qG_h\left(|a_h^\varepsilon|^2 - c\right).\end{cases} \tag{3.12}$$

A subsequence of $W_h^\varepsilon$ converges in $\mathcal{D}'$ to some $W^\varepsilon \in L^\infty([0,T]\times\mathbb{R}^n)$. Since $\nabla\times W_h^\varepsilon = 0$ for every $h\in]0,1]$, we deduce that $\nabla\times W^\varepsilon = 0$. We infer that there exists $V_{\rm p}^\varepsilon$ such that $W^\varepsilon = \nabla V_{\rm p}^\varepsilon$ (see e.g. [5, Prop. 1.2.1]), and we note that $\nabla^2 V_{\rm p}^\varepsilon \in C_w([0,T];L^2)$.

On the other hand, Corollary 3.5 and Fatou's lemma imply that $|a^\varepsilon|^2 - c \in L^\infty([0,T];L^2)$. To prove that $(v^\varepsilon, a^\varepsilon)$ solves (3.1)–(3.2), we now just have to check that $\Delta V_{\rm p}^\varepsilon - q\left(|a^\varepsilon|^2 - c\right) = 0$. We proceed in two steps: first, we prove that this quantity is a function of time only. Then, since it is in $L^\infty([0,T];L^2)$, we conclude that it is necessarily zero. We have

$$\left\|\nabla\left(\Delta V_{\rm p}^\varepsilon - q\left(|a^\varepsilon|^2-c\right)\right)\right\|_{L^2} \leqslant \liminf_{h\to 0}\left\|\nabla\left(\Delta V_h^\varepsilon - q\left(|a_h^\varepsilon|^2-c\right)\right)\right\|_{L^2}.$$

The last quantity is equal to:

$$|q|\left\|\nabla J_{1/h}\left(|a_h^\varepsilon|^2-c\right)\right\|_{L^2}.$$

This goes to zero with $h$, since $|a_h^\varepsilon|^2 - c$ is uniformly bounded in $L_T^\infty L^2$:

$$\left\|\nabla J_{1/h}\left(|a_h^\varepsilon|^2-c\right)\right\|_{L^2} \lesssim \left\|\xi\jmath\left(\frac{\xi}{h}\right)\mathcal{F}\left(|a_h^\varepsilon|^2-c\right)\right\|_{L^2} \lesssim h\left\|\mathcal{F}\left(|a_h^\varepsilon|^2-c\right)\right\|_{L^2} \lesssim h.$$

We infer that

$$\nabla\left(\Delta V_{\rm p}^\varepsilon - q\left(|a^\varepsilon|^2-c\right)\right) \equiv 0,$$

that is, $\Delta V_{\rm p}^\varepsilon - q\left(|a^\varepsilon|^2-c\right)$ is a function of time only. We conclude that $(v^\varepsilon,a^\varepsilon)$ solves (3.1)–(3.2).

We prove additional regularity for $(v^\varepsilon,a^\varepsilon)$ by showing that $(v_h^\varepsilon - v^\varepsilon, a_h^\varepsilon - a^\varepsilon)$ (and not a subsequence) goes to zero in $L^\infty([0,T];X^{s+2}\times X^{s+1})$. We will use:

**Lemma 3.7.** *Let $\varphi\in H^1$. Then:*

*(1)* $\left\|J_{1/h}\varphi\right\|_{L^2}\to 0$ *as* $h\to 0$.

*(2)* $\|(I-J_h)\varphi\|_{L^2} + \left\|(I-J_h^2)\varphi\right\|_{L^2} \leqslant 2h\|\nabla\varphi\|_{L^2}$.

*(3) There exists $C>0$ such that for all $h\in]0,1]$, $\|R_h\nabla^2\|_{L^2\to L^2}\leqslant C$.*

Denote $(w_h^\varepsilon, d_h^\varepsilon) := (v_h^\varepsilon - v^\varepsilon, a_h^\varepsilon - a^\varepsilon)$, and for $s>n/2+1$, introduce

$$\rho_h^\varepsilon(t) := \|w_h^\varepsilon(t)\|_{L^\infty} + \|\nabla w_h^\varepsilon(t)\|_{H^s} + \|d_h^\varepsilon(t)\|_{H^s}.$$

By construction, $\rho_h^\varepsilon(0)=0$. Write the equation for $(w_h^\varepsilon,\nabla w_h^\varepsilon, d_h^\varepsilon)$ as:

$$\left\{\begin{aligned}
&\partial_t w_h^\varepsilon + (v_{\rm eik}+v_h^\varepsilon)\cdot\nabla w_h^\varepsilon + w_h^\varepsilon\cdot\nabla v^\varepsilon = \\
&\qquad\qquad = -R_h\nabla(I-J_{1/h})\left(|a_h^\varepsilon|^2 - |a^\varepsilon|^2\right) + S_h^\varepsilon,\\
&\partial_t\nabla w_h^\varepsilon + (v_{\rm eik}+v_h^\varepsilon)\cdot\nabla^2 w_h^\varepsilon + w_h^\varepsilon\cdot\nabla^2 v^\varepsilon + \nabla w_h^\varepsilon\cdot\nabla v_h^\varepsilon\\
&\quad +\nabla v^\varepsilon\cdot\nabla w_h^\varepsilon = -R_h\nabla^2(I-J_{1/h})\left(|a_h^\varepsilon|^2-|a^\varepsilon|^2\right) + \nabla S_h^\varepsilon,\\
&\partial_t d_h^\varepsilon + w_h^\varepsilon\cdot\nabla a_h^\varepsilon + (v_{\rm eik}+v^\varepsilon)\cdot\nabla d_h^\varepsilon + \frac12\left(d_h^\varepsilon\nabla\cdot v_h^\varepsilon + a^\varepsilon\nabla\cdot w_h^\varepsilon\right)\\
&\qquad\qquad = i\frac{\varepsilon}{2}\Delta d_h^\varepsilon + \Sigma_h^\varepsilon,
\end{aligned}\right. \tag{3.13}$$

where the source terms are given by

$$\begin{aligned}
S_h^\varepsilon &= v_h^\varepsilon\cdot\nabla v_h^\varepsilon - J_h\left(v_h^\varepsilon\cdot\nabla J_h v_h^\varepsilon\right) + q\Delta^{-1}\nabla J_{1/h}\left(|a^\varepsilon|^2-c\right),\\
\Sigma_h^\varepsilon &= v_h^\varepsilon\cdot\nabla a_h^\varepsilon - J_h\left(v_h^\varepsilon\cdot\nabla J_h a_h^\varepsilon\right) - i\frac{\varepsilon}{2}\Delta(I-J_h^2)a_h^\varepsilon.
\end{aligned}$$

The error term $r_h^\varepsilon$ may seem to involve too many quantities (too much regularity), compared to the classical approach explained for instance in [16, 19]. The usual approach would consist in estimating $(w_h^\varepsilon, d_h^\varepsilon)$ in $L^2$ only. We cannot get such estimates because of the Poisson term in $S_h^\varepsilon$: we can prove it goes to zero in $X^s$, but not in $L^2$. We proceed in two steps:

(1) We show that we can apply Gronwall lemma for $r_h^\varepsilon(t)$, with sources terms $S_h^\varepsilon$ and $\Sigma_h^\varepsilon$.
(2) We show that these source terms go to zero with $h$ in the norms involved at the first step.

To estimate the first term of $r_h^\varepsilon$, integrate in time the first equation in (3.13), and use Corollary 3.5:

$$\begin{aligned}\|w_h^\varepsilon(t)\|_{L^\infty} &\lesssim \int_0^t \|\nabla w_h^\varepsilon(\tau)\|_{L^\infty} d\tau + \int_0^t \|w_h^\varepsilon(\tau)\|_{L^\infty} d\tau \\ &+ \int_0^t \left\| R_h \nabla (I - J_{1/h}) \left( |a_h^\varepsilon|^2 - |a^\varepsilon|^2 \right)(\tau) \right\|_{L^\infty} d\tau + \int_0^t \|S_h^\varepsilon(\tau)\|_{L^\infty} d\tau.\end{aligned}$$

Estimate the third term of the right hand side thanks to Lemma 2.2, Corollary 3.5 and the last point of Lemma 3.7:

$$\begin{aligned}&\left\| R_h \nabla (I - J_{1/h}) \left( |a_h^\varepsilon|^2 - |a^\varepsilon|^2 \right)(\tau) \right\|_{L^\infty} \\ &\quad\lesssim \left\| \nabla R_h \nabla (I - J_{1/h}) \left( |a_h^\varepsilon|^2 - |a^\varepsilon|^2 \right)(\tau) \right\|_{H^{s-1}} \\ &\quad\lesssim \left\| (I - J_{1/h}) \left( |a_h^\varepsilon|^2 - |a^\varepsilon|^2 \right)(\tau) \right\|_{H^s} \\ &\quad\lesssim \left\| \left( |a_h^\varepsilon|^2 - |a^\varepsilon|^2 \right)(\tau) \right\|_{H^s} \\ &\quad\lesssim \|(a_h^\varepsilon - a^\varepsilon)(\tau)\|_{H^s}.\end{aligned}$$

Using Sobolev embedding for the term in $\nabla w_h^\varepsilon$, we end up with:

$$\|w_h^\varepsilon(t)\|_{L^\infty} \lesssim \int_0^t r_h^\varepsilon(\tau) d\tau + \int_0^t \|S_h^\varepsilon(\tau)\|_{L^\infty} d\tau. \tag{3.14}$$

Now estimate the $H^s$ norm of $\nabla w_h^\varepsilon$. From the second equation in (3.13),

$$\begin{aligned}\frac{d}{dt}\|\Lambda^s \nabla w_h^\varepsilon\|_{L^2}^2 &= 2\operatorname{Re}\langle \Lambda^s \nabla w_h^\varepsilon, \partial_t \Lambda^s \nabla w_h^\varepsilon\rangle \\ &\lesssim \left|\operatorname{Re}\left\langle \Lambda^s \nabla w_h^\varepsilon, \Lambda^s (v_h^\varepsilon \cdot \nabla^2 w_h^\varepsilon)\right\rangle\right| + \left|\operatorname{Re}\left\langle \Lambda^s \nabla w_h^\varepsilon, \Lambda^s (w_h^\varepsilon \cdot \nabla^2 v^\varepsilon)\right\rangle\right| \\ &+ \rho_h^\varepsilon(t)^2 + \rho_h^\varepsilon(t) \left\| R_h \nabla^2 (I - J_{1/h}) \Lambda^s \left( |a_h^\varepsilon|^2 - |a^\varepsilon|^2 \right) \right\|_{L^2} \\ &+ \rho_h^\varepsilon(t) \left\|\nabla S_h^\varepsilon\right\|_{H^s}.\end{aligned}$$

Write the first term of the right hand side as:

$$\begin{aligned}\operatorname{Re}\left\langle \Lambda^s \nabla w_h^\varepsilon, \Lambda^s (v_h^\varepsilon \cdot \nabla^2 w_h^\varepsilon)\right\rangle &= \operatorname{Re}\left\langle \Lambda^s \nabla w_h^\varepsilon, v_h^\varepsilon \cdot \nabla^2 \Lambda^s w_h^\varepsilon\right\rangle \\ &+ \operatorname{Re}\left\langle \Lambda^s \nabla w_h^\varepsilon, \Lambda^s (v_h^\varepsilon \cdot \nabla^2 w_h^\varepsilon) - v_h^\varepsilon \cdot \nabla^2 \Lambda^s w_h^\varepsilon\right\rangle.\end{aligned}$$

Integration by parts and Kato-Ponce estimates (2.2) yield:

$$\begin{aligned}\left|\operatorname{Re}\left\langle \Lambda^s \nabla w_h^\varepsilon, \Lambda^s (v_h^\varepsilon \cdot \nabla^2 w_h^\varepsilon)\right\rangle\right| &\lesssim \rho_h^\varepsilon(t)^2 + \rho_h^\varepsilon(t) \|\nabla v_h^\varepsilon\|_{L^\infty} \|\nabla^2 w_h^\varepsilon\|_{H^{s-1}} \\ &+ \rho_h^\varepsilon(t) \|\nabla v_h^\varepsilon\|_{H^{s-1}} \|\nabla^2 w_h^\varepsilon\|_{L^\infty} \lesssim \rho_h^\varepsilon(t)^2,\end{aligned}$$

where we have used Corollary 3.5 and Sobolev embeddings. Similarly,

$$\begin{aligned}\mathrm{Re}\left\langle \Lambda^s\nabla w_h^\varepsilon, \Lambda^s(w_h^\varepsilon\cdot\nabla^2 v^\varepsilon)\right\rangle &= \mathrm{Re}\left\langle \Lambda^s\nabla w_h^\varepsilon, w_h^\varepsilon\cdot\nabla^2\Lambda^s v^\varepsilon\right\rangle\\ &\quad+\mathrm{Re}\left\langle \Lambda^s\nabla w_h^\varepsilon, \Lambda^s(w_h^\varepsilon\cdot\nabla^2 v^\varepsilon)-w_h^\varepsilon\cdot\nabla^2\Lambda^s v^\varepsilon\right\rangle,\end{aligned}$$

and:

$$\begin{aligned}\left|\mathrm{Re}\left\langle \Lambda^s\nabla w_h^\varepsilon, \Lambda^s(w_h^\varepsilon\cdot\nabla^2 v^\varepsilon)\right\rangle\right| &\lesssim \rho_h^\varepsilon(t)^2+\rho_h^\varepsilon(t)\|\nabla w_h^\varepsilon\|_{L^\infty}\|\nabla^2 v^\varepsilon\|_{H^{s-1}}\\ &\quad+\rho_h^\varepsilon(t)\|\nabla w_h^\varepsilon\|_{H^{s-1}}\|\nabla^2 v^\varepsilon\|_{L^\infty}\lesssim \rho_h^\varepsilon(t)^2.\end{aligned}$$

We also have

$$\left\|R_h\nabla^2(I-J_{1/h})\Lambda^s\left(|a_h^\varepsilon|^2-|a^\varepsilon|^2\right)\right\|_{L^2}\lesssim\left\|\Lambda^s\left(|a_h^\varepsilon|^2-|a^\varepsilon|^2\right)\right\|_{L^2}\lesssim\rho_h^\varepsilon(t),$$

and we infer:

$$\frac{d}{dt}\|\nabla w_h^\varepsilon\|_{H^s}^2\lesssim\rho_h^\varepsilon(t)^2+\rho_h^\varepsilon(t)\left\|\nabla S_h^\varepsilon\right\|_{H^s}. \tag{3.15}$$

Proceeding similarly for $d_h^\varepsilon$, we find:

$$\frac{d}{dt}\|d_h^\varepsilon\|_{H^s}^2\lesssim\rho_h^\varepsilon(t)^2+\rho_h^\varepsilon(t)\left\|\Sigma_h^\varepsilon\right\|_{H^s}. \tag{3.16}$$

Summing over (3.14) and the time integrated Equations (3.15) and (3.16), we complete the first task of the program announced above:

$$\rho_h^\varepsilon(t)\lesssim\int_0^t\rho_h^\varepsilon(\tau)d\tau+\int_0^t\left(\|S_h^\varepsilon(\tau)\|_{L^\infty}+\|\nabla S_h^\varepsilon(\tau)\|_{H^s}+\|\Sigma_h^\varepsilon(\tau)\|_{H^s}\right)d\tau. \tag{3.17}$$

Since $S_h^\varepsilon\in H^s$, Lemma 2.2 implies:

$$\rho_h^\varepsilon(t)\lesssim\int_0^t\rho_h^\varepsilon(\tau)d\tau+\int_0^t\left(\|\nabla S_h^\varepsilon(\tau)\|_{H^s}+\|\Sigma_h^\varepsilon(\tau)\|_{H^s}\right)d\tau.$$

It is an easy consequence of Corollary 3.5 and Lemma 3.7 that we have:

$$\|\nabla S_h^\varepsilon\|_{L^\infty_T H^s}+\|\Sigma_h^\varepsilon\|_{L^\infty_T H^s}\to 0\quad\text{as } h\to 0.$$

We infer from Gronwall lemma that $\rho_h^\varepsilon\to 0$ as $h\to 0$, uniformly on $[0,T]$. Therefore, we have:

$$(v^\varepsilon,a^\varepsilon)\in C([0,T];X^{s+1}\times X^s)\quad;\quad |a^\varepsilon|^2-c\in C([0,T];L^2),$$

and the existence part of Proposition 3.1 follows by a bootstrap argument (to prove the extra smoothness).

Uniqueness follows from the above computations: up to changing the notations, we have the same estimates as above, with now $S^\varepsilon=\Sigma^\varepsilon\equiv 0$. Uniqueness then follows from Gronwall lemma.

To see that there exists $\phi^\varepsilon$ such that $v^\varepsilon=\nabla\phi^\varepsilon$, apply the curl operator to the equation satisfied by $v^\varepsilon$ (3.1). Energy estimates then show that $\nabla\times v^\varepsilon\equiv 0$. We conclude thanks to [5, Prop. 1.2.1].

Before being more precise about the properties of $\phi^\varepsilon$ (we already know that $\nabla\phi^\varepsilon\in C([0,T];X^\infty)$), we examine the Poisson potential $V_{\rm p}^\varepsilon$. We have

$$\Delta V_{\rm p}^\varepsilon=q\left(|a^\varepsilon|^2-c\right)\in C([0,T];H^\infty).$$

We infer from Lemma 2.8 that

$$\mathcal{F}_{y\to\xi}\left(\nabla V_{\rm p}^\varepsilon\right)\in C([0,T];L^1).$$

We deduce $\nabla V_{\rm p}^\varepsilon \in C([0,T]\times\mathbb{R}^n)$, and Riemann-Lebesgue lemma implies that

$$\nabla V_{\rm p}^\varepsilon(t,x) \underset{|x|\to\infty}{\longrightarrow} 0.$$

So far, we have worked with $\nabla V_{\rm p}^\varepsilon$ only, and we know that it is smooth. At this stage, $V_{\rm p}^\varepsilon$ is determined up to a function of time only. The condition $V_{\rm p}^\varepsilon(t,0)=0$ fixes the value of that function, and yields a unique, smooth, Poisson potential (so far, only its gradient was unique). As announced in the introduction, we explain why we cannot (in general) impose the behavior

$$V_{\rm p}^\varepsilon(t,x) \underset{|x|\to\infty}{\longrightarrow} 0, \tag{3.18}$$

instead of $V_{\rm p}^\varepsilon(t,0)=0$. We know from Lemma 2.8 that for all $t\in[0,T]$, $\nabla V_{\rm p}^\varepsilon(t,\cdot)\in L^p(\mathbb{R}^n)$, for $p>2n/(n-2)$. We can then apply Lemma 2.1 only when $n\geqslant 5$. The following example shows that in space dimension $n=3$, we may have $\nabla f(x) \underset{|x|\to\infty}{\longrightarrow} 0$, $\Delta f\in H^\infty$, and $f(x) \underset{|x|\to\infty}{\longrightarrow} +\infty$:

$$f(x)=\log\left(1+|x|^2\right),\quad x\in\mathbb{R}^3.$$

Note also that in the case $c\in L^1(\mathbb{R}^n)$ discussed below, we have the additional property $\Delta V_{\rm p}^\varepsilon\in C([0,T];L^1\cap H^\infty)$, which makes it possible to impose (3.18). Back to $\phi^\varepsilon$, we have:

$$\nabla\left(\partial_t\phi^\varepsilon+\nabla\phi_{\rm eik}\cdot\nabla\phi^\varepsilon+\frac12|\nabla\phi^\varepsilon|^2+V_{\rm pert}+V_{\rm p}^\varepsilon\right)=0.$$

We infer:

$$\partial_t\phi^\varepsilon+\nabla\phi_{\rm eik}\cdot\nabla\phi^\varepsilon+\frac12|\nabla\phi^\varepsilon|^2+V_{\rm pert}+V_{\rm p}^\varepsilon=F,$$

where $F=F(t)$ is a function of time only. In the above equation, all the terms are uniquely determined, except $\partial_t\phi^\varepsilon$ and $F$. Imposing $\phi^\varepsilon_{|t=0}=\phi_0$, and replacing $\phi^\varepsilon$ with $\phi^\varepsilon+\int_0^t G(\tau)d\tau$ if necessary, we may assume that $F\equiv 0$. This condition fully determines $\phi^\varepsilon$. This completes the proof of Theorem 1.3.

## 4. Convergence as $\varepsilon\to0$: proof of Theorem 1.6

In this section, we prove Theorem 1.6. First, the existence of $(a,\phi)$ solving (1.8) follows from the proof of Theorem 1.3, since (1.8) is nothing but (1.7) with $\varepsilon=0$. Denote

$$w_v^\varepsilon := v^\varepsilon - v = \nabla\phi^\varepsilon-\nabla\phi \quad ; \quad w_a^\varepsilon := a^\varepsilon - a.$$

The pair $(w_v^\varepsilon,w_a^\varepsilon)$ solves a system similar to (3.13):

$$\begin{aligned}
&\partial_t w_v^\varepsilon + w_v^\varepsilon\cdot\nabla v+(v_{\rm eik}+v^\varepsilon)\cdot\nabla w_v^\varepsilon+\nabla\left(V_{\rm p}^\varepsilon-V_{\rm p}\right)=0.\\
&\partial_t w_a^\varepsilon + w_v^\varepsilon\cdot\nabla a+(v_{\rm eik}+v^\varepsilon)\cdot\nabla w_a^\varepsilon+\frac12\left(w_a^\varepsilon\nabla\cdot v^\varepsilon+a\nabla\cdot w_v^\varepsilon\right)=i\frac{\varepsilon}{2}\Delta a^\varepsilon.\\
&\Delta\left(V_{\rm p}^\varepsilon-V_{\rm p}\right)=q\left(|a^\varepsilon|^2-|a|^2\right).\\
&\nabla\left(V_{\rm p}^\varepsilon-V_{\rm p}\right)(t,x)\to0 \text{ as } |x|\to\infty \quad ; \quad w_v^\varepsilon\big|_{t=0}=0 \quad ; \quad w_a^\varepsilon\big|_{t=0}=r^\varepsilon.
\end{aligned}$$

Let $s>n/2+1$. Mimicking the computations made in Section 3.3, we find:

$$\begin{aligned}
\frac{d}{dt}\left(\|w_v^\varepsilon(t)\|_{X^{s+1}}^2+\|w_a^\varepsilon(t)\|_{H^s}^2\right)&\lesssim \|w_v^\varepsilon(t)\|_{X^{s+1}}^2+\|w_a^\varepsilon(t)\|_{H^s}^2\\
&+\varepsilon\|\Delta a^\varepsilon\|_{H^s}\|w_a^\varepsilon(t)\|_{H^s}+\|\nabla\left(V_{\rm p}^\varepsilon-V_{\rm p}\right)\|_{X^{s+1}}\|w_v^\varepsilon(t)\|_{X^{s+1}}.
\end{aligned}$$

From Theorem 1.3,

$$\|\Delta a^\varepsilon\|_{L^\infty_T H^s} \lesssim 1.$$

To estimate the term corresponding to the Poisson potentials, write:

$$\|\nabla \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{X^{s+1}} \lesssim \|\nabla \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{L^\infty} + \|\Delta \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{H^s}.$$

We estimate the first term thanks to Lemmas 2.1 and 2.3: we have

$$\Delta \left(V_{\rm p}^\varepsilon - V_{\rm p}\right) = q\left(|a^\varepsilon|^2 - |a|^2\right) = q\left(|a^\varepsilon|^2 - c + c - |a|^2\right).$$

Therefore, $\Delta \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)$ is bounded in $L^\infty_T H^s$. By definition $\partial^2_{jk}\left(V_{\rm p}^\varepsilon - V_{\rm p}\right)$ is bounded in $L^\infty_T H^s$ for every pair $(j,k)$. Lemma 2.1 (with $p=2$) shows that there exists a function $\gamma_j^\varepsilon(t)$ of time only such that

$$\partial_j \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)(t,\cdot) - \gamma_j^\varepsilon(t) \in L^{\frac{2n}{n-2}}(\mathbb{R}^n).$$

On the other hand, for all $t\in[0,T]$,

$$\partial_j \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)(t,x) \underset{|x|\to\infty}{\longrightarrow} 0.$$

Therefore, $\gamma_j^\varepsilon(t) \equiv 0$, and $\partial_j \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)(t,\cdot) \in L^{\frac{2n}{n-2}}(\mathbb{R}^n)$. The critical Sobolev embedding then shows that

$$\|\nabla \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{L^{\frac{2n}{n-2}}} \lesssim \|\Delta \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{L^2}.$$

Along with Lemma 2.2 (with $q = 2n/(n-2)$), this yields:

$$\|\nabla \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{L^\infty} \lesssim \|\Delta \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{H^s},$$

and we have:

$$\|\nabla \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{X^{s+1}} \lesssim \|\Delta \left(V_{\rm p}^\varepsilon - V_{\rm p}\right)\|_{H^s} \lesssim \|w_a^\varepsilon\|_{H^s}.$$

We infer:

$$\frac{d}{dt}\left(\|w_v^\varepsilon(t)\|^2_{X^{s+1}} + \|w_a^\varepsilon(t)\|^2_{H^s}\right) \lesssim \|w_v^\varepsilon(t)\|^2_{X^{s+1}} + \|w_a^\varepsilon(t)\|^2_{H^s} + \varepsilon\|w_a^\varepsilon(t)\|_{H^s}.$$

By assumption,

$$\|w_v^\varepsilon(0)\|_{X^{s+1}} + \|w_a^\varepsilon(0)\|_{H^s} = \|r^\varepsilon\|_{H^s} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

and we conclude with Gronwall lemma:

$$\|w_v^\varepsilon\|_{L^\infty_T X^{s+1}} + \|w_a^\varepsilon\|_{L^\infty_T H^s} \lesssim \varepsilon + \|r^\varepsilon\|_{H^s}.$$

The strong convergence of the quadratic quantities described in Theorem 1.6 follows easily. Note that a similar convergence has been obtained by P. Zhang [20], when $V_{\rm ext} \equiv 0 = \alpha_0$ (hence $\phi_{\rm eik} \equiv \beta_0$) and $c \in L^1(\mathbb{R}^n)$. The convergence in [20] is proved is a weaker sense though (in the sense of measures), due to a different technical approach based on the use of Wigner measures.

To conclude this section, we note that one must not expect $ae^{i(\phi+\phi_{\rm eik})/\varepsilon}$ to be a good *pointwise* approximation of $u^\varepsilon = a^\varepsilon e^{i(\phi^\varepsilon+\phi_{\rm eik})/\varepsilon}$. We have:

$$\begin{aligned} u^\varepsilon - ae^{i(\phi_{\rm eik}+\phi)/\varepsilon} &= a^\varepsilon e^{i(\phi_{\rm eik}+\phi^\varepsilon)/\varepsilon} - ae^{i(\phi_{\rm eik}+\phi)/\varepsilon} \\ &= (a^\varepsilon - a)\, e^{i(\phi_{\rm eik}+\phi^\varepsilon)/\varepsilon} + ae^{i\phi_{\rm eik}/\varepsilon}\left(e^{i\phi^\varepsilon/\varepsilon} - e^{i\phi/\varepsilon}\right). \end{aligned}$$

The first term is $\mathcal{O}(\varepsilon)$ in $L^2\cap L^\infty$ (we avoid differentiation because of rapid oscillations). The modulus of the last term is of order

$$|a|\left|\sin\left(\frac{\phi^\varepsilon-\phi}{\varepsilon}\right)\right|.$$

Note that our results do not allow us to estimate the argument of the sine function. Formally, it should not be smaller than $\mathcal{O}(1)$ in general, so we must not expect $ae^{i(\phi_{\rm eik}+\phi)/\varepsilon}$ to be a good approximation for $u^\varepsilon$. To have a good approximation, we would have to compute the next term in the asymptotic expansion for $(a^\varepsilon,\phi^\varepsilon)$ as $\varepsilon\to 0$. We leave out this question at this stage here, because we do not have completely satisfactory answers for that issue, and resume this discussion when $c\in L^1(\mathbb{R}^n)$ below, a case where we have more precise information at hand.

## Part 2. Subquadratic eikonal phase

We now allow the external potential and the initial phase to have quadratic components. After some geometrical reductions, the analysis boils down to the previous one. This reveals some differences though: for instance, even if $|a_0^\varepsilon|^2 - c\in L^2$, one must not expect $|u^\varepsilon(t)|^2 - c\in L^2$ for $t>0$.

**Assumption 2.** *Recall that* $n\geqslant 3$*.*
• External potential: $V_{\rm ext}\in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ *writes*

$$V_{\rm ext}(t,x) = V_{\rm quad}(t,x) + V_{\rm pert}(t,x),$$

*where* $V_{\rm quad}\in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ *is a polynomial of degree at most two in* $x$ *(*$\nabla^3 V_{\rm quad}\equiv 0$*), and* $\nabla V_{\rm pert}\in C(\mathbb{R};H^\infty)$*.*

• Doping profile: *it is a short range perturbation of a constant. For simplicity, we assume that this constant is* 1*:*

$$c = 1+\widetilde{c}, \quad \text{where } \widetilde{c}\in L^1\cap H^\infty.$$

• Initial amplitude: *it has the following expansion,*

$$a_0^\varepsilon(x) = a_0(x) + r^\varepsilon(x),$$

*where* $a_0\in X^\infty$ *is such that* $|a_0|^2-1\in L^2(\mathbb{R}^n)$*, and* $r^\varepsilon\in H^\infty$*, with*

$$\|r^\varepsilon\|_{H^s}\underset{\varepsilon\to 0}{\longrightarrow} 0,\quad \forall s\geqslant 0.$$

• Initial phase: *we have* $\Phi_0\in C^\infty(\mathbb{R}^n)$ *with*

$$\Phi_0(x) = \phi_{\rm quad}(x) + \phi_0(x),$$

*where* $\phi_{\rm quad}$ *is a polynomial of order at most two, and* $\nabla\phi_0\in H^\infty$*.*

*Example* (External potential). We may take

$$V_{\text{quad}}(t,x) = \sum_{j=1}^{n} \lambda_j(t) x_j^2,$$

an anisotropic harmonic potential with smooth time-dependent coefficients. Of course, we may take $V_{\text{pert}} \in C^\infty(\mathbb{R}; H^\infty)$.

## 5. The eikonal phase and the associated transport operator

The generalization of Lemma 1.2 is:

**Lemma 5.1.** *Under the Assumption 2, there exists $T^* > 0$ and a unique solution $\phi_{\text{eik}} \in C^\infty([0,T^*] \times \mathbb{R}^n)$ to:*

$$\partial_t \phi_{\text{eik}} + \frac{1}{2}|\nabla \phi_{\text{eik}}|^2 + V_{\text{quad}}(t,x) = 0 \quad ; \quad \phi_{\text{eik}|t=0} = \phi_{\text{quad}} . \tag{5.1}$$

*This solution is a polynomial of order at most two in $x$: $\nabla^3 \phi_{\text{eik}} \equiv 0$.*

*Proof.* The first part of the lemma was established in [3]. Consider the Hamiltonian flow associated to

$$\frac{1}{2}|\xi|^2 + V_{\text{quad}}(t,x),$$

which yields $x(t,y)$ and $\xi(t,y)$ solving:

$$\begin{cases} \partial_t x(t,y) = \xi\left(t,y\right) \quad ; \quad x(0,y) = y, \\ \partial_t \xi(t,y) = -\nabla_x V_{\text{quad}}\left(t, x(t,y)\right) \quad ; \quad \xi(0,y) = \nabla \phi_{\text{quad}}(y). \end{cases} \tag{5.2}$$

Following this flow and using a global inversion theorem (see [7] for these general results), we construct $\phi_{\text{eik}}$, locally in time, but globally in space. The idea for the global inversion is to notice that $\nabla_y x$ is the identity, plus a perturbation which is uniformly bounded in space, and continuous in time with initial value equal to zero: there exists $T^* > 0$ such that, for all $t \in [0,T^*]$, $y \mapsto x(t,y)$ is a global diffeomorphism. We denote by $y(t,x)$ its inverse. This yields $\phi_{\text{eik}} \in C^\infty([0,T^*] \times \mathbb{R}^n)$, with

$$\nabla \phi_{\text{eik}}(t,x) = \xi\left(t, y(t,x)\right). \tag{5.3}$$

As a byproduct, the function $\phi_{\text{eik}}$ is sub-quadratic: $\partial_x^\alpha \phi_{\text{eik}} \in L^\infty([0,T^*] \times \mathbb{R}^n)$ as soon as $|\alpha| \geqslant 2$.

Differentiating (5.1) three times with respect to any triplet of space variables, we see that $\Psi = \nabla^3 \phi_{\text{eik}}$ solves a system of the form:

$$\left(\partial_t + \nabla \phi_{\text{eik}} \cdot \nabla\right) \Psi = \mathcal{M} \Psi \quad ; \quad \Psi_{|t=0} = 0,$$

where $\mathcal{M} \in L^\infty([0,T^*] \times \mathbb{R}^n)$ ($\mathcal{M}$ is a linear combination of derivatives of order at least two of $\phi_{\text{eik}}$). Note that the absence of source term and initial datum follows from Assumption 2. Since $\nabla \phi_{\text{eik}}$ is given by (5.3), we can then use the method of characteristics: setting $\widetilde{\Psi}(t,y) = \Psi(t, x(t,y))$ (which makes sense since $x(t,\cdot)$ is a global diffeomorphism), the above equation becomes

$$\partial_t \widetilde{\Psi} = \widetilde{\mathcal{M}} \widetilde{\Psi} \quad ; \quad \widetilde{\Psi}_{|t=0} = 0 \quad ; \quad \widetilde{\mathcal{M}} \in L^\infty([0,T^*] \times \mathbb{R}^n).$$

We conclude with Gronwall lemma that $\widetilde{\Psi} \equiv \Psi \equiv 0$. $\square$

In view of the energy estimates performed in Section 3, we will not consider (5.1), but a nonlinear perturbation of this equation. Indeed, if we try to mimic the computations after Lemma 3.4, and after having changed variables to work on the characteristics, we have to estimate $D_t a_h^\varepsilon$ in $L^2$. From the equation,

$$\|D_t a_h^\varepsilon\|_{L^2} \lesssim \left(1+\|v_h^\varepsilon\|_{L^\infty}+\|\nabla v_h^\varepsilon\|_{L^2}\right)\left(\|a_h^\varepsilon\|_{L^\infty}+\|\nabla a_h^\varepsilon\|_{L^2}\right)+\|\Delta a_h^\varepsilon\|_{L^2} \\ +\|a_h^\varepsilon \Delta\phi_{\rm eik}\|_{L^2}.$$

The last term is new, since now $\Delta\phi_{\rm eik}$ is a non-trivial function (of time only). This means that we must not even expect the last term to be finite! To overcome this difficulty, we proceed as on the baby model

$$\partial_t a + \frac{1}{2} a\Delta\phi_{\rm eik} = 0,$$

where from Lemma 5.1, $\Delta\phi_{\rm eik}$ is a function of time only. It is convenient to introduce the auxiliary function

$$\widetilde{a}(t,x) = a(t,x)\exp\left(\frac{1}{2}\int_0^t \Delta\phi_{\rm eik}(\tau)d\tau\right).$$

Therefore, it is tempting to replace the condition $|u^\varepsilon|-1 \in L^\infty_T L^2$ with a condition of the form

$$\left|u^\varepsilon e^{\frac{1}{2}\int_0^t \Delta\phi_{\rm eik}(\tau)d\tau}\right|^2 - 1 \in L^\infty_T L^2.$$

Apparently, we have solved the issue mentioned above, but the price to pay is that we no longer consider the quantity which is natural in view of the Poisson equation. The idea is then to introduce a "ghost Poisson potential":

$$V_{\rm p}^\varepsilon = \widetilde{V_{\rm p}}^\varepsilon + V_{\rm g} + V_{\widetilde{c}}, \quad \text{where}$$

$$\begin{aligned} \Delta\widetilde{V_{\rm p}}^\varepsilon &= q e^{-\int_0^t \Delta\phi_{\rm eik}(\tau)d\tau}\left(\left|u^\varepsilon e^{\frac{1}{2}\int_0^t \Delta\phi_{\rm eik}(\tau)d\tau}\right|^2 - 1\right), \\ \Delta V_{\rm g} &= q\left(e^{-\int_0^t \Delta\phi_{\rm eik}(\tau)d\tau} - 1\right), \\ \Delta V_{\widetilde{c}} &= q\widetilde{c}. \end{aligned}$$

In particular, $\Delta V_{\rm g}$ is a function of time only: $V_{\rm g}$ is quadratic in $x$, and we may choose

$$V_{\rm g}(t,x) = q\frac{|x|^2}{2n}\left(e^{-\int_0^t \Delta\phi_{\rm eik}(\tau)d\tau} - 1\right). \tag{5.4}$$

Following the idea of [3], it is consistent to replace $V_{\rm quad}$ with $V_{\rm quad}+V_{\rm g}$ in (5.1), since $V_{\rm g}$ is quadratic and cannot be considered as a perturbation or a source term. Even though $V_{\rm g}$ depends on $\phi_{\rm eik}$, it is reasonable to try to extend Lemma 5.1. Indeed, if we consider the iterative scheme

$$\partial_t\phi_{\rm eik}^{(j+1)} + \frac{1}{2}\left|\nabla\phi_{\rm eik}^{(j+1)}\right|^2 + V_{\rm quad}(t,x) = -q\frac{|x|^2}{2n}\left(e^{-\int_0^t \Delta\phi_{\rm eik}^{(j)}(\tau,x)d\tau} - 1\right), \\ \phi^{j+1}_{{\rm eik}|t=0} = \phi_{\rm quad},$$

with $\phi_{\rm eik}^{(0)} = \phi_{\rm quad}$, we see that applying Lemma 5.1 inductively shows that every iterate is a smooth, sub-quadratic function. We have precisely:

**Proposition 5.2.** *Under Assumption 2, there exists $T^* > 0$ and a unique solution $\phi_{\rm eik} \in C^\infty([0,T^*]\times\mathbb{R}^n)$, polynomial of order at most two in $x$ ($\nabla^3\phi_{\rm eik}\equiv 0$), to:*

$$\partial_t\phi_{\rm eik} + \frac{1}{2}|\nabla\phi_{\rm eik}|^2 + V_{\rm quad}(t,x) = -q\frac{|x|^2}{2n}\left(e^{-\int_0^t \Delta\phi_{\rm eik}(\tau)d\tau} - 1\right), \quad \phi_{{\rm eik}|t=0} = \phi_{\rm quad}\,. \tag{5.5}$$

*We denote:*

$$g(t) := \frac{1}{2}\int_0^t \Delta\phi_{\rm eik}(\tau)d\tau. \tag{5.6}$$

*Proof.* Inspired by Lemma 5.1, we seek directly $\phi_{\rm eik}$ of the form

$$\phi_{\rm eik}(t,x) = {}^t xM(t)x + \alpha(t)\cdot x + \beta(t),$$

where $M\in\mathcal{M}_{n\times n}(\mathbb{R})$, $\alpha\in\mathbb{R}^n$ and $\beta\in\mathbb{R}$. Plugging this expression into (5.5) and identifying the coefficients of the polynomials in $x$, we find:

$$\begin{aligned}
\dot M(t) + 2M(t)^2 + Q(t) &= -\frac{q}{2n}\left(e^{-2\int_0^t {\rm Tr}M(\tau)d\tau} - 1\right)I_n \quad ; \quad M(0) = M_0,\\
\dot\alpha(t) + 2M(t)\alpha(t) + E(t) &= 0 \quad ; \quad \alpha(0) = \alpha_0,\\
\dot\beta(t) + \frac{1}{2}|\alpha(t)|^2 + \gamma(t) &= 0 \quad ; \quad \beta(0) = \beta_0,
\end{aligned}$$

where

$$V_{\rm quad}(t,x) = {}^t xQ(t)x + E(t)\cdot x + \gamma(t) \quad ; \quad \phi_{\rm quad}(x) = {}^t xM_0x + \alpha_0\cdot x + \beta_0.$$

Introducing the unknown function $R(t) = \int_0^t M(\tau)d\tau$, we see that the equation in $M$ can be solved thanks to Cauchy-Lipschitz Theorem applied to $(M(t),R(t))$. Then $\alpha(t)$ and $\beta(t)$ follow by simple integration. □

The above proof shows that unless $Q(t)\equiv 0 = M_0$ (a case which boils down to Part 1), $g$ is a non-trivial function of time.

The previous result implies that the characteristics associated to the transport operator $\partial_t + \nabla\phi_{\rm eik}\cdot\nabla$ present in (1.7) can be described very easily.

**Corollary 5.3.** *Let $x(t,y)$ be as defined in* (5.2)*. There exist $\alpha\in C^\infty([0,T^*];\mathbb{R}^n)$, and $A\in C^\infty([0,T^*];\mathcal{M}_{n\times n}(\mathbb{R}))$, symmetric, such that, for all $t\in[0,T^*]$,*

$$x(t,y) = e^{A(t)}y + \alpha(t).$$

*Proof.* By Proposition 5.2, $\nabla\phi_{\rm eik}(t,x) = M(t)x + \alpha(t)$ for some symmetric matrix $M$. Since

$$\partial_t x(t,y) = \nabla\phi_{\rm eik}(t,x(t,y)),$$

the result follows by integration. □

*Remark* 5.4. Under Assumption 1, $\nabla\phi_{\rm eik}$ is a function of time only, and the transport operator $\partial_t + \nabla\phi_{\rm eik}\cdot\nabla$ is trivial. In the above proof, $M\equiv 0$, and we have $x(t,y) = y + \int_0^t \alpha(\tau)d\tau$. This relation is reminiscent of Avron–Herbst formula (see e.g. [6]).

## 6. Main results

The analogue of Theorem 1.3 is:

**Theorem 6.1.** *Let Assumption 2 be satisfied. There exists $T>0$ independent of $\varepsilon\in]0,1]$ and a solution $u^\varepsilon\in L^\infty([0,T]\times\mathbb{R}^n)$ to (1.1)-(1.3), with*

$$\nabla\widetilde{V_{\rm p}}^\varepsilon(t,x)=\nabla\left(V_{\rm p}^\varepsilon-V_{\rm g}-V_{\widetilde{c}}\right)(t,x)\underset{|x|\to\infty}{\longrightarrow}0,\quad \widetilde{V_{\rm p}}^\varepsilon(t,0)=0,$$

*and such that $|u^\varepsilon e^g|^2-1\in L^\infty([0,T];L^2)$, where $g$ is given by (5.6). Moreover, one can write $u^\varepsilon=a^\varepsilon e^{i(\phi_{\rm eik}+\phi^\varepsilon)/\varepsilon}$, where:*

- *$a^\varepsilon\in C^\infty([0,T]\times\mathbb{R}^n)\cap C([0,T];X^\infty)$, and $|a^\varepsilon e^g|^2-1\in C([0,T];L^2)$.*
- *$\phi_{\rm eik}$ is given by Proposition 5.2.*
- *$\phi^\varepsilon\in C^\infty([0,T]\times\mathbb{R}^n)$ and $\nabla\phi^\varepsilon\in C([0,T];X^\infty)$.*
- *We have the following uniform estimate: for every $s>n/2$, there exists $M_s$* indepen-dent of $\varepsilon\in]0,1]$ *such that*

$$\|a^\varepsilon\|_{L^\infty(0,T;X^s)}+\left\||a^\varepsilon e^g|^2-1\right\|_{L^\infty(0,T;L^2)}+\|\nabla\phi^\varepsilon\|_{L^\infty(0,T;X^s)}\leqslant M_s.$$

*Remark* 6.2. We impose conditions on $\widetilde{V_{\rm p}}^\varepsilon$, and not on $V_{\rm p}^\varepsilon$. This is related to the *arbitrary* choice (5.4) to integrate the "ghost Poisson equation" (this equation introduces additional degrees of freedom), since we will impose

$$\nabla V_{\widetilde{c}}(x)\underset{|x|\to\infty}{\longrightarrow}0\quad;\quad V_{\widetilde{c}}(x)\underset{|x|\to\infty}{\longrightarrow}0.$$

Note that since $g$ is non-trivial, the above result shows that one must not expect $|u^\varepsilon(t)|^2-1\in L^2$ for $t>0$.

Proceeding like before, we want $(a^\varepsilon,v^\varepsilon)$ to solve:

$$\begin{cases}\partial_t v^\varepsilon+(v_{\rm eik}+v^\varepsilon)\cdot\nabla v^\varepsilon+v^\varepsilon\cdot\nabla v_{\rm eik}+\nabla V_{\rm pert}+\nabla V_{\rm p}^\varepsilon=0,\\ \partial_t a^\varepsilon+(v_{\rm eik}+v^\varepsilon)\cdot\nabla a^\varepsilon+\dfrac{1}{2}a^\varepsilon\nabla\cdot(v_{\rm eik}+v^\varepsilon)=i\dfrac{\varepsilon}{2}\Delta a^\varepsilon,\\ \Delta V_{\rm p}^\varepsilon=q\big(|a^\varepsilon|^2-c\big),\end{cases}\tag{6.1}$$

together with

$$\nabla\widetilde{V_{\rm p}}^\varepsilon(t,x)\underset{|x|\to\infty}{\longrightarrow}0\quad;\quad\widetilde{V_{\rm p}}^\varepsilon(t,0)=0\quad;\quad v^\varepsilon_{|t=0}=\nabla\phi_0\ ;\ a^\varepsilon_{|t=0}=a_0^\varepsilon.\tag{6.2}$$

With this existence result, we can study the asymptotic behavior as $\varepsilon\to0$ of the solution we construct:

**Theorem 6.3.** *Under Assumption 2, there exists a smooth solution $(a,\phi)$ of (6.1) with $\varepsilon=0$, such that $a,\nabla\phi\in C([0,T],X^\infty)$, $|ae^g|^2-1\in C([0,T],L^2)$, and*

$$\|a^\varepsilon-a\|_{L^\infty_TH^s}+\|\nabla(\phi^\varepsilon-\phi)\|_{L^\infty_TX^s}\underset{\varepsilon\to0}{\longrightarrow}0,\qquad\forall s>n/2.$$

*In particular, the position density and the momentum density converge:*

$$|u^\varepsilon|^2\underset{\varepsilon\to0}{\longrightarrow}|a|^2\quad\textit{in } L^\infty_TH^s,\ \textit{and}$$

$$\varepsilon\,\mathrm{Im}\left(\overline{u^\varepsilon e^{-i\phi_{\rm eik}/\varepsilon}}\nabla\left(u^\varepsilon e^{-i\phi_{\rm eik}/\varepsilon}\right)\right)\underset{\varepsilon\to0}{\longrightarrow}|a|^2\nabla\phi\quad\textit{in } L^\infty_TX^s,\ \forall s>n/2.$$

*Remark* 6.4. We slightly altered the usual notion of momentum density, by removing first the eikonal phase $\phi_{\rm eik}$. Indeed, we do not prove that

$$|a^\varepsilon|^2\nabla\phi_{\rm eik}\underset{\varepsilon\to 0}{\longrightarrow}|a|^2\nabla\phi_{\rm eik}\quad\text{in } L^\infty_T X^s,$$

since $\nabla\phi_{\rm eik}$ may grow linearly in $x$, while $|a|^2$ morally goes to 1 as $|x|\to\infty$, hence $|a|^2\nabla\phi_{\rm eik}\notin L^\infty_T X^s$.

We first show that the solutions of (6.1) exist and are uniformly bounded for a time interval independent of $\varepsilon$.

**Proposition 6.5.** *Let Assumption 2 be satisfied. Let $s>n/2$. For all $M> M_0>0$, there exists $0<T\leqslant T^*$ such that, if for all $\varepsilon\in[0,1]$,*

$$\|\nabla\phi_0\|_{H^{s+2}}+\left\||a_0^\varepsilon|^2-1\right\|_{L^2}+\|a_0^\varepsilon\|_{X^{s+1}}\leqslant M_0, \tag{6.3}$$

*then the Cauchy problem (6.1) has a unique classical solution $(v^\varepsilon,a^\varepsilon)$ in $C^\infty([0,T]\times\mathbb{R}^n)$ such that*

$$\|v^\varepsilon\|_{L^\infty_T X^{s+2}}+\left\||a^\varepsilon e^g|^2-1\right\|_{L^\infty_T L^2}+\|a^\varepsilon\|_{L^\infty_T X^{s+1}}\leqslant M. \tag{6.4}$$

We perform some geometrical reductions so that the proofs of the above results follow from Section 3.

## 7. Reduction to the first case

We begin by proving that (6.1) is equivalent to a system which does not involve the operator $v_{\rm eik}\cdot\nabla$, thanks to Corollary 5.3. Resuming the notations of Section 5, define, for any function $f$ of time and space:

$$\widetilde{f}(t,y)=f\left(t,x(t,y)\right).$$

Working with $\widetilde{f}$ instead of $f$, the characteristics associated to $v_{\rm eik}\cdot\nabla$ are straightened so that:

$$\partial_t\widetilde{f}(t,y)=(\partial_t+v_{\rm eik}\cdot\nabla)f\left(t,x(t,y)\right).$$

The good news for us is the fact that the above change of variable does not change the structural properties of (6.1). Indeed, Corollary 5.3 implies that

$$\widetilde{\nabla f}(t,y)=e^{-A(t)}\nabla\widetilde{f}(t,y), \tag{7.1}$$

for some symmetric $n\times n$ matrix $A(t)$ which is independent of $y$.

We are now in position to make precise the fact that the change of variables does not change the structural properties.

**Lemma 7.1.** *Fix $t\in[0,T^*]$ and set $\delta_t:=e^{-A(t)}\nabla$, where $A$ is as in Corollary 5.3. The following properties hold:*
(1) *For all $u\in H^2(\mathbb{R}^n)$ and all $v\in W^{1,\infty}(\mathbb{R}^n)$ one has:*

$$\operatorname{Re}\langle\, i\delta_t^*\delta_t u\,,\,u\,\rangle=0\quad;\quad 2\langle\, v\cdot\delta_t u\,,\,u\,\rangle=-\langle\,(\delta_t v)u\,,\,u\,\rangle.$$

(2) *The Fourier multiplier $\nabla(-\delta_t^*\delta_t)^{-1}\delta_t$ is well defined and bounded on Sobolev spaces: for all $\sigma\geqslant 0$, there exists a constant $K_\sigma$ independent of $t\in[0,T^*]$ such that:*

$$\left\|\nabla(-\delta_t^*\delta_t)^{-1}\delta_t u\right\|_{H^\sigma}\leqslant K_\sigma\left\|u\right\|_{H^\sigma},\qquad\forall u\in H^\sigma(\mathbb{R}^n).$$

(3) *For all function* $u\colon \mathbb{R}^n \to \mathbb{R}$,

$$u(x) \underset{|x|\to\infty}{\longrightarrow} 0 \quad \Leftrightarrow \quad u(x(t,y)) \underset{|y|\to\infty}{\longrightarrow} 0.$$

*Proof.* By integrating by parts, the first property follows from the fact that $\delta_t$ is a linear combination of spatial derivatives whose coefficients are constant symmetric matrices. The property (2) is immediate using Fourier transform. The property (3) is obvious. □

**Notation.** Introduce the operator $\partial$ by, for all $u\colon [0,T^*] \to \mathcal{S}'(\mathbb{R}^n)$,

$$(\partial u)(t) := e^{-A(t)}\nabla u(t).$$

The difference between the above notation and Lemma 7.1 is that $\delta_t$ is defined for *fixed* $t \in [0,T^*]$. Following what we did in Section 5, introduce

$$\widetilde{a}^\varepsilon := a^\varepsilon e^g \quad ; \quad V_{\rm p}^\varepsilon = \widetilde{V_{\rm p}}^\varepsilon + V_{\rm g} + V_{\widetilde{c}}.$$

Since $\widetilde{c} \in L^1 \cap H^\infty$, $\Delta^{-1}\widetilde{c}$ is well defined as a temperate distribution:

$$\Delta^{-1}\widetilde{c} = -\mathcal{F}^{-1}\left(|\xi|^{-2}\mathcal{F}(\widetilde{c})\right).$$

Setting $\widetilde{V}_{\rm pert} := V_{\rm pert} + q\Delta^{-1}\widetilde{c}$, we still have $\nabla\widetilde{V}_{\rm pert} \in C(\mathbb{R};H^\infty)$, from Lemma 2.4. With these notations, (6.1) is equivalent to:

$$\begin{cases} \partial_t v^\varepsilon + v^\varepsilon\cdot\partial v^\varepsilon + v^\varepsilon\cdot\partial v_{\rm eik} + \partial\widetilde{V}_{\rm pert} + \partial\widetilde{V_{\rm p}}^\varepsilon = 0, \\ \partial_t\widetilde{a}^\varepsilon + v^\varepsilon\cdot\partial\widetilde{a}^\varepsilon + \frac{1}{2}\widetilde{a}^\varepsilon\partial\cdot v^\varepsilon = -i\frac{\varepsilon}{2}\partial^*\partial\widetilde{a}^\varepsilon, \\ \partial^*\partial\widetilde{V_{\rm p}}^\varepsilon = -qe^{-2g}\big(|\widetilde{a}^\varepsilon|^2 - 1\big). \end{cases} \tag{7.2}$$

Note that the fact that the right hand side of the equation for $a^\varepsilon$ is skew-symmetric remains, from the first point of Lemma 7.1: following the idea of E. Grenier [12], this is crucial in the proof of Theorem 1.3. This is so thanks to Corollary 5.3, and would not be if $\phi_{\rm eik}$ was not exactly polynomial.

Directly from Corollary 5.3, we verify that, for all $\sigma \geqslant 0$, there exists $C_\sigma$ such that, for all $t \in [0,T^*]$,

$$C_\sigma^{-1}\left\|u(x(t,\cdot))\right\|_{H^\sigma} \leqslant \left\|u\right\|_{H^\sigma} \leqslant C_\sigma\left\|u(x(t,\cdot))\right\|_{H^\sigma}, \quad \forall u \in H^\sigma(\mathbb{R}^n). \tag{7.3}$$

Similarly, there exists a constant $C$ such that

$$C^{-1}\left\|u(x(t,\cdot))\right\|_{L^\infty} \leqslant \left\|u\right\|_{L^\infty} \leqslant C\left\|u(x(t,\cdot)\right\|_{L^\infty}, \quad \forall u \in L^\infty(\mathbb{R}^n).$$

Note that (7.2) is very similar to (3.1). The transport operator is simplified, but we have two new features:

- The term $v^\varepsilon\cdot\partial v_{\rm eik}$ in the equation for $v^\varepsilon$.
- The factor $e^{-2g}$ in the Poisson equation.

The latter changes very little computations, since $g$ is a function of time only. One can check that the term $v^\varepsilon\cdot\partial v_{\rm eik}$ does not require a modification of the proof given in Section 3; unlike for $\partial_t a^\varepsilon$, we do not estimate $\partial_t v^\varepsilon$ in $L^2$. Therefore, Theorem 6.1 follows. Similarly, Theorem 6.3 follows like in Section 4, up to some slight modifications; like for the proof of Theorem 6.1, replace $(a^\varepsilon, V_{\rm p}^\varepsilon, V_{\rm pert})$ with $(\widetilde{a}^\varepsilon, \widetilde{V_{\rm p}}^\varepsilon, \widetilde{V}_{\rm pert})$.

# Part 3. Integrable doping profile

## 8. Integrable doping profile

There are many results concerning the case when the doping profile $c$ is decaying at spatial infinity, say $c \in L^1(\mathbb{R}^n)$. We refer for instance to [20] and references therein. We restrict our attention to the case $n = 3$ for simplicity.

**Assumption 3.** *We consider the case $n = 3$.*
• External potential: $V_{\rm ext} \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ *writes*

$$V_{\rm ext}(t,x) = V_{\rm quad}(t,x) + V_{\rm pert}(t,x),$$

*where $V_{\rm quad} \in C^\infty(\mathbb{R} \times \mathbb{R}^3)$ is a polynomial of degree at most two in $x$ ($\nabla^3 V_{\rm quad} \equiv 0$), and $V_{\rm pert} \in C(\mathbb{R}; H^\infty)$.*

• Doping profile: $c \in L^1(\mathbb{R}^3) \cap X^\infty$.

• Initial amplitude: $a_0^\varepsilon(x) = a_0(x) + \varepsilon a_1(x) + \varepsilon r_1^\varepsilon(x)$, *where $a_0, a_1, r_1^\varepsilon \in H^\infty$, with*

$$\|r_1^\varepsilon\|_{H^s} \underset{\varepsilon \to 0}{\longrightarrow} 0, \quad \forall s \geqslant 0.$$

• Initial phase: *we have $\Phi_0 \in C^\infty(\mathbb{R}^n)$ with*

$$\Phi_0(x) = \phi_{\rm quad}(x) + \phi_0(x),$$

*where $\phi_{\rm quad}$ is a polynomial of order at most two, and $\phi_0 \in X^\infty$.*

Our goal is to state a convergence result which is more precise than Theorem 1.6: we shall need some properties of $\phi^\varepsilon$, and not only $\nabla\phi^\varepsilon$. This is why we change the boundary conditions to solve the Poisson equation: we consider

$$\begin{aligned} &i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = V_{\rm ext} u^\varepsilon + V_{\rm p}^\varepsilon u^\varepsilon, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3,\\ &\Delta V_{\rm p}^\varepsilon = q\left(|u^\varepsilon|^2 - c\right), \quad \nabla V_{\rm p}^\varepsilon(t,x) \to 0 \text{ and } V_{\rm p}^\varepsilon(t,x) \to 0 \text{ as } |x| \to \infty,\\ &u^\varepsilon_{|t=0} = a_0^\varepsilon(x) e^{i\Phi_0(x)/\varepsilon}. \end{aligned} \tag{8.1}$$

With these boundary conditions (which are as in [4, 20] for instance), we can define $\Delta^{-1}$ as:

$$\Delta^{-1} f = -\frac{1}{4\pi|x|} * f. \tag{8.2}$$

**Theorem 8.1.** *Let $n = 3$. Under Assumption 2, assume furthermore that $V_{\rm pert} \in C(\mathbb{R}; X^\infty)$, $c \in L^1(\mathbb{R}^3)$, $a_0 \in L^2(\mathbb{R}^3)$ and $\phi_0 \in L^\infty(\mathbb{R}^3)$. There exists $0 < T \leqslant T^*$ independent of $\varepsilon \in ]0,1]$ and a unique solution $u^\varepsilon \in C^\infty([0,T]\times\mathbb{R}^3) \cap C([0,T]; H^\infty)$ to (8.1). Moreover, one can write $u^\varepsilon = a^\varepsilon e^{i(\phi_{\rm eik} + \phi^\varepsilon)/\varepsilon}$, where:*

- $a^\varepsilon \in C([0,T]; H^\infty)$.
- *$\phi_{\rm eik}$ is given by Lemma 5.1.*
- $\phi^\varepsilon \in C([0,T]; X^\infty)$.
- *We have the following uniform estimate: for every $s \geqslant 0$, there exists $M_s$* independent of $\varepsilon \in ]0,1]$ *such that*

$$\|a^\varepsilon\|_{L^\infty_T H^s} + \|\phi^\varepsilon\|_{L^\infty_T X^s} \leqslant M_s.$$

Note that existence and uniqueness for (8.1) can be established in a larger class of functions, thanks to Strichartz estimates. We refer for instance to [4] for the case with no external potential, and simply recall that similar Strichartz estimates are available in the presence of a smooth, subquadratic external potential ([8], see also [2]). Note also that the term $\Delta^{-1}c$ can be treated as a "nice" linear potential, thanks to Lemma 2.4 and the following:

**Lemma 8.2.** *The operator $\Delta^{-1}$ defined by (8.2) maps $L^1 \cap L^2(\mathbb{R}^3)$ to $\mathcal{F}(L^1(\mathbb{R}^3))$, where $\mathcal{F}$ denotes the Fourier transform. Moreover, there exists $C$ such that*

$$\left\|\mathcal{F}\left(\Delta^{-1}\varphi\right)\right\|_{L^1} \leqslant C\left(\|\varphi\|_{L^1}+\|\varphi\|_{L^2}\right), \quad \forall \varphi \in L^1 \cap L^2(\mathbb{R}^3).$$

Uniqueness for (8.1) follows easily:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = V_{\rm ext} u^\varepsilon + q\Delta^{-1}\left(|u^\varepsilon|^2 - c\right)u^\varepsilon.$$

Let $u^\varepsilon$ and $v^\varepsilon$ be two solutions in $C([0,T^\varepsilon]; H^\infty)$ of the above equation, with the same initial data, for some $T^\varepsilon > 0$. Note that the dependence upon $\varepsilon$ is irrelevant, since $\varepsilon > 0$ is fixed. The difference $w^\varepsilon = u^\varepsilon - v^\varepsilon$ solves

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = V_{\rm ext} w^\varepsilon + q\Delta^{-1}\left(|u^\varepsilon|^2 - c\right)w^\varepsilon + q\Delta^{-1}\left(|u^\varepsilon|^2 - |v^\varepsilon|^2\right)v^\varepsilon.$$

The basic energy estimate yields:

$$\begin{aligned}
\varepsilon\frac{d}{dt}\|w^\varepsilon\|_{L^2}^2 &\lesssim \left\|\Delta^{-1}\left(|u^\varepsilon|^2 - |v^\varepsilon|^2\right)v^\varepsilon\right\|_{L^2}\|w^\varepsilon\|_{L^2}\\
&\lesssim \left\|\Delta^{-1}\left(|u^\varepsilon|^2 - |v^\varepsilon|^2\right)\right\|_{L^\infty}\|v^\varepsilon\|_{L^2}\|w^\varepsilon\|_{L^2}\\
&\lesssim \left(\left\||u^\varepsilon|^2 - |v^\varepsilon|^2\right\|_{L^1} + \left\||u^\varepsilon|^2 - |v^\varepsilon|^2\right\|_{L^2}\right)\|v^\varepsilon\|_{L^2}\|w^\varepsilon\|_{L^2}\\
&\lesssim \left(\|u^\varepsilon\|_{L^2} + \|v^\varepsilon\|_{L^2} + \|u^\varepsilon\|_{L^\infty} + \|v^\varepsilon\|_{L^\infty}\right)\|w^\varepsilon\|_{L^2}^2\|v^\varepsilon\|_{L^2}.
\end{aligned}$$

Uniqueness then follows from the Gronwall lemma.

To prove the existence part of Theorem 8.1, we consider

$$\begin{aligned}
&\partial_t\phi^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla\phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + V_{\rm pert} + V_{\rm p}^\varepsilon = 0\ ;\ \phi^\varepsilon_{|t=0} = \phi_0.\\
&\partial_t a^\varepsilon + \nabla\left(\phi^\varepsilon + \phi_{\rm eik}\right)\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\left(\phi^\varepsilon + \phi_{\rm eik}\right) = i\frac{\varepsilon}{2}\Delta a^\varepsilon\ ;\ a^\varepsilon_{|t=0} = a_0^\varepsilon.\\
&\Delta V_{\rm p}^\varepsilon = q\big(|a^\varepsilon|^2 - c\big) \quad ; \quad \nabla V_{\rm p}^\varepsilon(t,x) \underset{|x|\to\infty}{\longrightarrow} 0 \text{ and } V_{\rm p}^\varepsilon(t,x) \underset{|x|\to\infty}{\longrightarrow} 0.
\end{aligned} \tag{8.3}$$

The geometrical reduction presented in Section 7 makes it possible to transform the transport operator $\partial_t + v_{\rm eik}\cdot\nabla$ into $\partial_t$. Unlike in Section 7, we may keep the term $\Delta\phi_{\rm eik}$. Since $\Delta\phi_{\rm eik}$ is a function of time only, and since we work with $a^\varepsilon \in C([0,T]; H^s)$, the term $a^\varepsilon\Delta\phi_{\rm eik}$ can be treated like a perturbative term.

Since the proof of Theorem 8.1 involves more classical arguments, we essentially skip it, so that we can focus our discussion on the semi-classical limit $\varepsilon \to 0$.

After the geometrical reduction, (8.3) becomes what we would have found directly in the case $V_{\rm quad} = 0 = \phi_{\rm quad}$, up to terms which can be treated by Gronwall lemma. We may for instance resume the approach of Section 3, and replace $X^s$ with $H^s$. This way, we construct $a^\varepsilon, v^\varepsilon \in C([0,T]; H^\infty)$.

To complete the proof of Theorem 8.1, we finally notice that $\phi^\varepsilon \in C([0,T]; L^\infty)$, from Lemma 8.2 and (8.3) integrated along the characteristics.

We can now establish the analogue of Theorem 1.6, with a pointwise description. To do so, we introduce the solution to

$$
\begin{aligned}
&\partial_t\phi + \nabla\phi_{\rm eik}\cdot\nabla\phi + \frac{1}{2}|\nabla\phi|^2 + V_{\rm pert} + V_{\rm p} = 0 \ ; \ \phi_{|t=0} = \phi_0.\\
&\partial_t a + \nabla\left(\phi+\phi_{\rm eik}\right)\cdot\nabla a + \frac{1}{2}a\Delta\left(\phi+\phi_{\rm eik}\right) = 0 \ ; \ a_{|t=0} = a_0.\\
&\Delta V_{\rm p} = q\big(|a|^2 - c\big) \quad ; \quad \nabla V_{\rm p}(t,x) \underset{|x|\to\infty}{\longrightarrow} 0 \text{ and } V_{\rm p}(t,x) \underset{|x|\to\infty}{\longrightarrow} 0.
\end{aligned}
\tag{8.4}
$$

This system has a unique solution $(\phi,a)\in C([0,T];X^\infty\times H^\infty)$. As pointed out at the end of Section 4, the triplet $(\phi_{\rm eik},\phi,a)$ does not suffice to describe the pointwise limit of $u^\varepsilon$ as $\varepsilon\to 0$. This is the reason why in Assumption 3, we want to know $a^\varepsilon$ up to $o(\varepsilon)$ instead of $o(1)$ only. Consider the linearized system:

$$
\begin{aligned}
&\partial_t\phi_1 + \nabla(\phi_{\rm eik}+\phi)\cdot\nabla\phi_1 + V = 0 \ ; \ \phi_{1|t=0} = 0.\\
&\partial_t b + \nabla\left(\phi_{\rm eik}+\phi\right)\cdot\nabla b + \frac{1}{2}b\Delta\left(\phi_{\rm eik}+\phi\right) +\\
&\qquad + \nabla\phi_1\cdot\nabla a + \frac{1}{2}a\Delta\phi_1 = \frac{i}{2}\Delta a \ ; \ b_{|t=0} = a_1.\\
&\Delta V = 2q\,\mathrm{Re}\left(\overline{a}b\right) \ ; \ \nabla V(t,x) \underset{|x|\to\infty}{\longrightarrow} 0, \text{ and } V(t,x) \underset{|x|\to\infty}{\longrightarrow} 0.
\end{aligned}
\tag{8.5}
$$

It has a unique solution $(\phi_1,b)\in C([0,T];X^\infty\times H^\infty)$.

**Theorem 8.3.** *Under the Assumption 3, the solution to* (8.1) *can be approximated at leading order by* $ae^{i\phi_1}e^{i(\phi_{\rm eik}+\phi)/\varepsilon}$*:*

$$
\left\| u^\varepsilon - ae^{i\phi_1}e^{i(\phi_{\rm eik}+\phi)/\varepsilon}\right\|_{L^\infty_T(L^2\cap L^\infty)} \to 0 \ \text{ as } \varepsilon\to 0.
$$

*Remark* 8.4. Note that in general, $\phi_1$ is not trivial provided that $a_1\not\equiv 0$, and that the amplitude of $u^\varepsilon$ is, at leading order, $ae^{i\phi_1}$. This phenomenon is due to the fact that from the point of view of geometric optics, (1.1)-(1.3) (or (8.1)) is supercritical: to describe the exact solution at leading order as in Theorem 8.3, it is necessary to know its initial data up to $o(\varepsilon)$. This phenomenon was called *ghost effect* in [**?**], and may lead to instability results as in [**?**]: modifying $a_0^\varepsilon$ at order $\sqrt{\varepsilon}$ for instance, affects the solution $u^\varepsilon$ at order $\mathcal{O}(1)$ for times of order $\sqrt{\varepsilon}$.

*Sketch of the proof.* The idea is to resume the approach of Section 4. Set

$$
\widetilde{w}_v^\varepsilon = \nabla\left(\phi^\varepsilon - \phi - \varepsilon\phi_1\right) \quad ; \quad \widetilde{w}_a^\varepsilon = a^\varepsilon - a - \varepsilon b.
$$

Proceeding as in Section 4, we find, for $s$ sufficiently large:

$$
\|\widetilde{w}_v^\varepsilon\|_{L^\infty_T X^{s+1}} + \|\widetilde{w}_a^\varepsilon\|_{L^\infty_T H^s} \lesssim \varepsilon^2 + \varepsilon\|r_1^\varepsilon\|_{H^s}.
$$

As above, we infer an $L^2$ estimate for $\widetilde{w}_v^\varepsilon$:

$$
\|\widetilde{w}_v^\varepsilon\|_{L^\infty_T H^{s+1}} + \|\widetilde{w}_a^\varepsilon\|_{L^\infty_T H^s} \lesssim \varepsilon^2 + \varepsilon\|r_1^\varepsilon\|_{H^s},
$$

and directly from the equation,

$$
\|\phi^\varepsilon - \phi - \varepsilon\phi_1\|_{L^\infty([0,T]\times\mathbb{R}^3)} \lesssim \varepsilon^2 + \varepsilon\|r_1^\varepsilon\|_{H^s} = o(\varepsilon).
$$

We conclude:

$$\begin{aligned}\left|u^\varepsilon - ae^{i\phi_1}e^{i(\phi_{\rm eik}+\phi)/\varepsilon}\right| &= \left|a^\varepsilon e^{i\phi^\varepsilon/\varepsilon} - ae^{i\phi_1}e^{i\phi/\varepsilon}\right| = \left|a^\varepsilon e^{i\phi^\varepsilon/\varepsilon} - ae^{i(\phi+\varepsilon\phi_1)/\varepsilon}\right| \\ &\lesssim \varepsilon|b| + |\widetilde{w}_a^\varepsilon| + |a|\left|\sin\left(\frac{\phi^\varepsilon-\phi-\varepsilon\phi_1}{\varepsilon}\right)\right| \\ &\lesssim \varepsilon|b| + |\widetilde{w}_a^\varepsilon| + |a|\times o(1).\end{aligned}$$

The result follows by taking the $L^2$ or the $L^\infty$ norm in space. □

## References


1. F. Brezzi and P. A. Markowich, *The three-dimensional Wigner-Poisson problem: existence, uniqueness and approximation*, Math. Methods Appl. Sci. **14** (1991), no. 1, 35–61.
2. R. Carles, *Linear vs. nonlinear effects for nonlinear Schrödinger equations with potential*, Commun. Contemp. Math. **7** (2005), no. 4, 483–508.
3. ———, *WKB analysis for nonlinear Schrödinger equations with potential*, Comm. Math. Phys. (2006), to appear.
4. F. Castella, *$L^2$ solutions to the Schrödinger-Poisson system: existence, uniqueness, time behaviour, and smoothing effects*, Math. Models Methods Appl. Sci. **7** (1997), no. 8, 1051–1083.
5. J.-Y. Chemin, *Perfect incompressible fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 14, The Clarendon Press Oxford University Press, New York, 1998, Translated from the 1995 French original by Isabelle Gallagher and Dragoş Iftimie.
6. H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, study ed., Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987. MR 88g:35003
7. J. Dereziński and C. Gérard, *Scattering theory of quantum and classical N-particle systems*, Texts and Monographs in Physics, Springer Verlag, Berlin Heidelberg, 1997.
8. D. Fujiwara, *Remarks on the convergence of the Feynman path integrals*, Duke Math. J. **47** (1980), no. 3, 559–600.
9. C. Gallo, *Schrödinger group on Zhidkov spaces*, Adv. Differential Equations **9** (2004), no. 5-6, 509–538.
10. P. Gérard, *Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire*, Séminaire sur les Équations aux Dérivées Partielles, 1992–1993, École Polytech., Palaiseau, 1993, pp. Exp. No. XIII, 13.
11. ———, *The Cauchy problem for the Gross-Pitaevskii equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2006), to appear.
12. E. Grenier, *Semiclassical limit of the nonlinear Schrödinger equation in small time*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 523–530.
13. L. Hörmander, *The analysis of linear partial differential operators. I*, second ed., Springer Study Edition, Springer-Verlag, Berlin, 1990, Distribution theory and Fourier analysis.
14. D. Lannes, *Sharp estimates for pseudo-differential operators with symbols of limited smoothness and commutators*, J. Funct. Anal. **232** (2006), no. 2, 495–539.
15. F. Lin and P. Zhang, *Semiclassical limit of the Gross-Pitaevskii equation in an exterior domain*, Arch. Rational Mech. Anal. **179** (2005), no. 1, 79–107.
16. A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984.
17. P. A. Markowich, C. A. Ringhofer, and C. Schmeiser, *Semiconductors equations*, Springer-Verlag, Wien, 1990.
18. C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation, self-focusing and wave collapse*, Springer-Verlag, New York, 1999.
19. M. Taylor, *Partial differential equations. III*, Applied Mathematical Sciences, vol. 117, Springer-Verlag, New York, 1997, Nonlinear equations.
20. P. Zhang, *Wigner measure and the semiclassical limit of Schrödinger-Poisson equations*, SIAM J. Math. Anal. **34** (2002), no. 3, 700–718.

21. P. Zhang, Y. Zheng, and N. J. Mauser, *The limit from the Schrödinger-Poisson to the Vlasov-Poisson equations with general data in one dimension*, Comm. Pure Appl. Math. **55** (2002), no. 5, 582–632.
22. P. E. Zhidkov, *The Cauchy problem for a nonlinear Schrödinger equation*, JINR Commun., P5-87-373, Dubna (1987), (in Russian).
23. ———, *Korteweg-de Vries and nonlinear Schrödinger equations: qualitative theory*, Lecture Notes in Mathematics, vol. 1756, Springer-Verlag, Berlin, 2001.

Université Paris-Sud, Mathématiques, Bât. 425, 91405 Orsay cedex, France
*E-mail address*: `Thomas.Alazard@math.u-psud.fr`

Wolfgang Pauli Institute, c/o Inst. f. Math., Universität Wien, Nordbergstr. 15, A-1090 Wien, Austria[2]
*E-mail address*: `Remi.Carles@math.cnrs.fr`

[2]On leave from MAB, UMR 5466 CNRS, Bordeaux, France.