INCOMPRESSIBLE LIMIT OF THE NON-ISENTROPIC EULER EQUATIONS

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Abstract. We study the incompressible limit of classical solutions to the compressible Euler equations for non-isentropic fluids in a domain $\Omega \subset \mathbb{R}^d$. We consider the case of general initial data. For a domain Ω, bounded or unbounded, we first prove the existence of classical solutions for a time independent of the small parameter. Then, in the exterior case, we prove that the solutions converge to the solution of the incompressible Euler equations.

1. Introduction

This work is devoted to the study of the so-called incompressible limit for classical solutions of the compressible Euler equations for non-isentropic fluids. We consider the case of a flow in a domain $\Omega \subset \mathbb{R}^d$ with the solid-wall boundary condition. After the usual rescalings and changes of variables, see [14, 18], we are led to analyze a quasilinear hyperbolic system depending on a small parameter ε, which is the Mach number,

$$
\begin{cases}
a\big(\partial_t p + v \cdot \nabla p\big) + \dfrac{\operatorname{div} v}{\varepsilon} = 0, \\
\rho\big(\partial_t v + v \cdot \nabla v\big) + \dfrac{\nabla p}{\varepsilon} = 0, \\
\partial_t S + v \cdot \nabla S = 0,
\end{cases}
\tag{1.1}
$$

where $x \in \Omega$, $p = p(t,x) \in \mathbb{R}$, $v = v(t,x) \in \mathbb{R}^d$ $(d \geqslant 1)$, $S = S(t,x) \in \mathbb{R}$. An important feature of (1.1) is that the coefficients a and r depend on S:

$$
a = \mathcal{A}(S, \varepsilon p), \quad r = \mathcal{R}(S, \varepsilon p),
$$

where $\mathcal{A}$, $\mathcal{R}$ are C^∞ positive functions given by the state law of the fluid. The equations are supplemented with initial and boundary values:

$$
\begin{cases}
v_{|\partial\Omega} \cdot \nu = 0, \\
(p, v, S)_{|t=0} = (p_0^\varepsilon, v_0^\varepsilon, S_0^\varepsilon),
\end{cases}
\tag{1.2}
$$

where ν denotes the unit outward normal on the boundary $\partial\Omega$.

The analysis of the system (1.1) depends on several factors: the flow may be *isentropic* ($S_0 = 0$) or *non-isentropic* ($S_0 = O(1)$). The initial data may be *prepared* (namely $(\operatorname{div} v_0^\varepsilon, \nabla p_0^\varepsilon)$, are $O(\varepsilon)$), or *general*, which means here that $\{(p_0^\varepsilon, v_0^\varepsilon, S_0^\varepsilon)\}_{\varepsilon>0}$ is a bounded family in the Sobolev space $H^s(\Omega)$, where $s > d/2 + 1$. Finally, the domain may be the torus, the whole space or a domain $\Omega \subset \mathbb{R}^d$.

First, to study this singular limit one has to prove an existence and uniform boundedness result for a time independent of ε. Solutions of (1.1) are known to exist for a time interval which is independent of the small parameter ε in the equations whenever the flow is isentropic (see [14, 15]), whenever the initial data is prepared (see [22]), and whenever the domain is the torus or the whole space (see [18]). These existence results cover all cases of the above-mentioned factors except for the non-isentropic equations with general initial data in the boundary case. The first result in this paper is a uniform existence result for this case. More precisely, we extend the result of G. Métivier and S. Schochet ([18]) by assuming that Ω is a general domain, bounded or unbounded, with smooth compact boundary. Their proof relies upon the fact that one can establish uniform bounds by applying some spatial operators with appropriate weights to the equations. Here, as in Isozaki [10], Schochet [22] and Secchi [23], the acoustic components of the solutions are estimated by taking time derivatives which have the advantage of commuting with the boundary condition, and next by using the special structure of the equations to estimate the spatial derivatives.

We suppose that $\Omega \subset \mathbb{R}^d$ ($d \geqslant 1$) is an open, connected set lying on one side of its compact, smooth boundary $\partial\Omega$, it can be bounded or unbounded. We can also assume that Ω is the whole space (we remove the boundary condition) or the torus (with periodic boundary conditions). In the boundary case, to consider classical solutions, we need some compatibility conditions. To clarify matters, recall the following definition:

Definition 1.1. *The matrix E is invertible, so $\partial_t^k v(0,x)$ can be expressed as $A_k(u(0,x),S(0,x))$, for some functions A_k. We say that the solutions satisfy compatibility conditions up to order $\alpha \in \mathbb{N}$ if, on the boundary $x \in \partial\Omega$, one has $\nu \cdot A_k(u(0,x),S(0,x)) = 0$ for all $k \leqslant \alpha$.*

As an example, note that the compatibility conditions are satisfied up to any order for any smooth initial data vanishing in a neighborhood of $\partial\Omega$.

Our main result asserts that the solutions of (2.1) exist and are uniformly bounded for a time interval which is independent of ε.

Theorem 1.2 (Ω is bounded or unbounded)**.** *Let $s > 1 + d/2$ be an integer. For all real M_0, there is a positive $T = T(M_0)$ such that for all $\varepsilon \in (0,1]$, and all initial data $(p_0^\varepsilon, v_0^\varepsilon, S_0^\varepsilon)$ in the Sobolev space $H^s(\Omega)$ satisfying*

$$\|(p_0^\varepsilon, v_0^\varepsilon, S_0^\varepsilon)\|_{H^s(\Omega)} \leqslant M_0, \tag{1.3}$$

and compatibility conditions up to order $s-1$, the Cauchy problem for (2.1) *has a unique solution $(p^\varepsilon, v^\varepsilon, S^\varepsilon) \in C^0([0,T], H^s(\Omega))$. Furthermore, there exists $\Gamma = \Gamma(M_0) < +\infty$ such that for all $t \in [0,T]$ and $\varepsilon \in (0,1]$,*

$$\big\|\big(p^\varepsilon, v^\varepsilon(t), S^\varepsilon(t)\big)\big\|_{H^s(\Omega)} \leqslant \Gamma, \quad \|\partial_t S^\varepsilon(t)\|_{H^{s-1}(\Omega)} \leqslant \Gamma, \tag{1.4}$$

$$\big\|\partial_t \operatorname{curl}\big(\mathcal{R}(S^\varepsilon(t),0)v^\varepsilon(t)\big)\big\|_{H^{s-2}(\Omega)} \leqslant \Gamma. \tag{1.5}$$

Remark 1.3. We prepare the initial data only on the boundary $\partial\Omega$ (which is necessary), and not in the interior, so we consider general initial data.

Remark 1.4. As explained in [18, 19], since the matrix $E(S, \varepsilon u)$ depends on the unknown (through the entropy S), the linearized equations are unstable. Hence, we cannot obtain in standard fashion the nonlinear energy estimates from the L^2 estimate by an elementary argument using differentiation of the equations.

Our next task is to analyze the limit of solutions of (1.1) as the Mach number ε tends to 0. The solutions are known to converge to the solution of the corresponding incompressible Euler equations with the limit initial data whenever the initial data are prepared (see [14, 15, 22]). For the isentropic equations with general initial data, the velocity is the sum of the limit flow, which is a solution of the incompressible equations whose initial data is the incompressible part of the original initial data, and a highly oscillatory term created by the sound waves (see [7]). In that case the solutions are known to converge, although this convergence is not uniform for time close to zero (see [2, 9, 10, 12, 24]). Here, for the non-isentropic Euler equations in an exterior domain with general initial data, we prove strong compactness in $L^2(0,T;H^{s'}_{loc}(\Omega))$ for all $s'<s$. Therefore, we can prove a convergence theorem which extends the previous result of G. Métivier and S. Schochet [18] in the free space $\Omega=\mathbb{R}^d$. Yet, our proof follows closely the analysis of [18] in making use of semiclassical defect measures. We mention that the case where the domain is bounded seems much more complicated. We refer the reader to [4, 19, 20] for recent advances in the case of the non-isentropic Euler equations with spatially periodic boundary conditions.

We restrict ourselves to the case where $\Omega\subset\mathbb{R}^d$ (with $d\geqslant 2$) is an exterior domain, which means that, in addition to the previous assumptions, Ω is the exterior of a bounded domain and Ω is a connected neighborhood of spatial infinity (which includes $\Omega=\mathbb{R}^d$). When the initial data are prepared, $(\operatorname{div} v_0^\varepsilon, \nabla p_0^\varepsilon)=O(\varepsilon)$, it is known that the limit system reads

$$\begin{cases} \operatorname{div} v = 0,\\ \mathcal{R}(S,0)\big(\partial_t v + v\cdot\nabla v\big) + \nabla \pi = 0,\\ \partial_t S + v\cdot\nabla S = 0,\\ v_{|\partial\Omega}\cdot\nu = 0, \end{cases} \tag{1.6}$$

for some π such that $\nabla\pi$ belongs to $C^0([0,T];H^{s-1}(\Omega))$. We consider the same problem when the initial data are not so constrained. We suppose only that S_0^ε decays sufficiently rapidly at infinity.

Theorem 1.5 (Exterior case)**.** *Assume that Ω is an exterior domain and that $(p^\varepsilon, v^\varepsilon, S^\varepsilon)$ satisfy (1.1) and are uniformly bounded in $C^0([0,T];H^s(\Omega))$, for some fixed $T>0$ and $s>2+d/2$. Suppose that the initial data $(v_0^\varepsilon, S_0^\varepsilon)$ satisfy compatibility conditions up to order $s-1$ and converge in $H^s(\Omega)$ to (v_0,S_0). Assume further that S_0^ε satisfies*

$$|S_0^\varepsilon(x)| \leqslant C\,|x|^{-1-\delta}, \quad |\nabla S_0^\varepsilon(x)| \leqslant C\,|x|^{-2-\delta}, \tag{1.7}$$

for all ε and some fixed C and $\delta>0$. Then

$$(v^\varepsilon, p^\varepsilon, S^\varepsilon) \xrightarrow[\varepsilon\to 0]{} (v,0,S)$$

weakly in $L^\infty(0,T;H^s(\Omega))$ and strongly in $L^2(0,T;H^{s'}_{loc}(\Omega))$ for all $s'<s$, where (v,S) is the unique solution in $C^0([0,T];H^s(\Omega))$ of (1.6) with initial data (w_0,S_0), with w_0 being the unique solution in $H^s(\Omega)$ of

$$w_{0|\partial\Omega}\cdot\nu = 0,\ \operatorname{div} w_0 = 0,\ \operatorname{curl}(r_0 w_0) = \operatorname{curl}(r_0 v_0),\ \text{where } r_0 = \mathcal{R}(S_0,0).$$

2. Uniform Stability

In this section, we prove Theorem 1.2. It is convenient to rewrite (1.1) under the short form:

$$\begin{cases} E(S,\varepsilon u)\big(\partial_t u + b(S,u)\cdot\nabla u\big) + \dfrac{1}{\varepsilon}L(\partial_x)u = 0, \\ \partial_t S + b(S,u)\cdot\nabla S = 0, \end{cases} \tag{2.1}$$

where $u=(p,v)$ and

$$E(S,\varepsilon u) = \begin{pmatrix} \mathcal{A}(S,\varepsilon u) & 0 \\ 0 & \mathcal{R}(S,\varepsilon u)\mathrm{I}_d \end{pmatrix}, \quad b(S,u) = v, \quad L(\partial_x) = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}.$$

Recall that $\mathcal{A}$ and $\mathcal{R}$ are C^∞ positive functions of $(S,u)\in\mathbb{R}^{2+d}$. Instead of imposing $b(S,p,v)=v$, one can also consider coefficients b of the form $b(S,p,v)=\beta(S,p,v)v$ where $\beta=(\beta_1,\dots,\beta_d)$ is a C^∞ real-valued function of $(S,u)\in\mathbb{R}^{2+d}$. Recall also that the boundary and initial conditions read:

$$v_{|\partial\Omega}\cdot\nu = 0, \qquad u_{|t=0} = u_0, \quad S_{|t=0} = S_0.$$

Consider the equations (2.1) for fixed $\varepsilon>0$, and assume that assumptions (A1)–(A4) are satisfied, and that the initial data satisfy compatibility conditions (A5) up to order $s-1$. The system is symmetric hyperbolic; therefore, if $\Omega=\mathbb{R}^d$ we know from [17] that (2.1) is well-posed for regular initial data. In the boundary case it is shown in [22] that the problem for non-isentropic fluids is well-posed locally in time. Such a theorem is in no way a trivial one. As usual, the essential part lies in establishing the well-posedness of the linearized problem. Yet the linearized equations of the boundary-value problem are not included in classical frameworks. Indeed, the boundary matrix is singular and fails to be of constant rank near the boundary. For a general theory we refer to [21], where J. Rauch has studied the symmetric positive systems with boundary of constant multiplicity. He has proved the well-posedness in L^2 (by means of a "weak=strong" Lemma based on his tangential regularization) and the conormal regularity. Here, we use the special structure of the equations so as to estimate the normal derivatives.

Given $s>1+d/2$, for all fixed $\varepsilon>0$, we let $T_\varepsilon = T(\varepsilon,M_0)>0$ denote the lifespan, that is the supremum of all the positive times T such that for all initial data which satisfies (1.3) and $s-1$ compatibility conditions, the Cauchy problem has a unique solution on $C^0([0,T],H^s(\Omega))$. It results from the above-mentioned theorem of Schochet that either $T_\varepsilon=+\infty$ or

$$\limsup_{t\to T_\varepsilon} \left\lVert (u^\varepsilon,S^\varepsilon)(t)\right\rVert_{H^s(\Omega)} = +\infty. \tag{2.2}$$

On account of this alternative the problem reduces to establishing the following *a priori* bounds.

Proposition 2.1. *Given $s>1+d/2$ and $M_0\in\mathbb{R}$, there is a constant C_0 and a nonnegative function $C(\cdot)$, such that for all $T\in(0,T_\varepsilon)$, $\varepsilon\in(0,1]$ and $(u^\varepsilon,S^\varepsilon)\in C^0([0,T],H^s(\Omega))$ a solution of* (2.1)*, with initial data satisfying* (1.3)*, the norm*

$$M_\varepsilon(T) := \sup_{t\in[0,T]} \left\lVert (u^\varepsilon,S^\varepsilon)(t)\right\rVert_{H^s(\Omega)} \tag{2.3}$$

satisfies the estimate

$$M_\varepsilon(T) \leqslant C_0 + (T+\varepsilon) C\big(M_\varepsilon(T)\big). \tag{2.4}$$

The proof of this proposition is completed only after Lemma 2.11. Let us explain why Proposition 2.1 implies that T_ε is bounded from below by a positive constant for all ε less than 1.

Proof of Theorem 1.2 given Proposition 2.1. Choose first $M_1 > C_0$ and next ε_1 and T_1 such that

$$C_0 + (T_1 + \varepsilon_1) C(M_1) < M_1. \tag{2.5}$$

For $t < \inf\{T_\varepsilon, T_1\}$, and $\varepsilon \leqslant \varepsilon_1$, the inequalities (2.4) and (2.5) imply that $M_\varepsilon(t) \neq M_1$, where $M_\varepsilon(t)$ is the norm defined by (2.3) on $[0,t]$. Besides we can assume without restriction that $M_\varepsilon(0) < M_1$. Using a continuity argument, we infer $M_\varepsilon(t) < M_1$. Consequently, the continuation principle (2.2) shows that $T_\varepsilon > T_1$ for all $\varepsilon \leqslant \varepsilon_1$.

On the other hand, we can extract from the proof that the problem is well posed locally in time (for fixed ε) that T_ε is bounded from below by $T_1' > 0$ for all ε in $[\varepsilon_1, 1]$, a compact subset of $(0,1]$. □

Before entering into the details, let us pause to set some notations, as well as to recall basic rules in Sobolev spaces.

Let $\|\cdot\|_s$ denote the norm in the Sobolev space $H^s(\Omega)$. For $k \geqslant 0$, $l \geqslant 0$, $k+l \leqslant \sigma$, and $\sigma > d/2$, the product maps continuously $H^{\sigma-k}(\Omega) \times H^{\sigma-l}(\Omega)$ to $H^{\sigma-k-l}(\Omega)$, and

$$\|uv\|_{\sigma-k-l} \leqslant K \|u\|_{\sigma-k} \|v\|_{\sigma-l}. \tag{2.6}$$

Similarly, if F is a smooth function such that $F(0) = 0$, and $u \in H^\sigma(\Omega)$ with $\sigma > d/2$, then $F(u) \in H^\sigma(\Omega)$ and

$$\|F(u)\|_\sigma \leqslant C(\|u\|_\sigma) \|u\|_\sigma. \tag{2.7}$$

To take care of the hypothesis $F(0) = 0$, we use the notation $\widetilde{F} = F - F(0)$. As a consequence, *for any* smooth function $F\colon \mathbb{R}^n \to \mathbb{R}^p$, suppose $u \in H^\sigma(\Omega)$ and $v \in H^m(\Omega)$, with $m \in \{0, \dots, \sigma\}$ and $\sigma > d/2$; then $F(u)v \in H^m(\Omega)$ and

$$\|F(u)v\|_m \leqslant \big(|F(0)| + C(\|u\|_\sigma)\big) \|v\|_m. \tag{2.8}$$

It appears easier to estimate M_ε (as defined by (2.3)) by a stronger norm of $(u^\varepsilon, S^\varepsilon)$, which we introduce here. In a standard way, given $\mu \in \mathbb{N}$, set

$$X^\mu([0,T] \times \Omega) := \bigcap_{k=0}^{\mu} C^k([0,T], H^{\mu-k}(\Omega)).$$

These spaces come equipped with the weighted norms

$$\begin{aligned} \|w(t)\|_{\mu,\varepsilon} &:= \sum_{k=0}^{\mu} \big\|(\varepsilon\partial_t)^k w(t)\big\|_{H^{\mu-k}(\Omega)}, \\ \|w\|_{\mu,\varepsilon,T} &:= \sup_{t\in[0,T]} \|w(t)\|_{\mu,\varepsilon}. \end{aligned} \tag{2.9}$$

We will use the following properties. For all $T > 0$, all $u \in X^\sigma([0,T]\times\Omega)$ with $\mathbb{N} \ni \sigma > d/2$, and all $v \in X^n([0,T]\times\Omega)$ with $\mathbb{N} \ni n \leqslant \sigma$, we have

$$\|v(t)\|_n \leqslant \|v(t)\|_{n,\varepsilon},$$
$$\|u(t)v(t)\|_{n,\varepsilon} \leqslant K\|u(t)\|_{\sigma,\varepsilon}\|v(t)\|_{n,\varepsilon}. \tag{R4}$$

Notation 2.2. Below, in all the proofs,

the symbols C_0 and $C(\cdot)$ stand for various constants and functions depending only on s and M_0 (given by (1.3)); K stands for constants depending only on s. They may vary from relation to relation.

We omit the indexation by ε, we write S, u for S^ε, u^ε. We denote by $b(t)$, $E(t)$, $\widetilde{b}(t)$, $\widetilde{E}(t)$ the functions $b(S^\varepsilon(t), u^\varepsilon(t))$, $E(S^\varepsilon(t), \varepsilon u^\varepsilon(t))$, $(b - b(0))(S^\varepsilon(t), u^\varepsilon(t))$, $(E - E(0))(S^\varepsilon(t), \varepsilon u^\varepsilon(t))$.

With these preliminaries established, we now turn to the proof of Proposition 2.1. We begin by giving an estimate for the entropy S^ε.

Lemma 2.3. *There are a constant C_0 and a function $C(\cdot)$ depending only on M_0, such that*

$$\forall \varepsilon \in (0,1], \quad \forall t \in [0, T_\varepsilon), \quad \|S^\varepsilon(t)\|_s \leqslant C_0 + tC(M_\varepsilon(t)). \tag{2.10}$$

Proof. For $\alpha \in \mathbb{N}^d$, $|\alpha| \leqslant s$, introduce $f_\alpha = \partial_x^\alpha S$, which satisfies

$$\partial_t f_\alpha + b(S,u)\cdot\nabla f_\alpha = [b(S,u)\cdot\partial_x, \partial_x^\alpha]S. \tag{2.11}$$

The source term $g_\alpha(\tau) = [b(\tau)\cdot\nabla, \partial_x^\alpha]S(\tau)$ is a sum of terms $\partial_x^\beta\widetilde{b}(\tau)\partial_x^\gamma S(\tau)$ with $|\beta| + |\gamma| \leqslant s+1$, $\beta > 0$, $\gamma > 0$, so the rules (2.6) and (2.7) imply that g_α belongs to $C^0([0,T_\varepsilon[, L^2(\Omega))$, together with the bound

$$\forall \tau \leqslant t < T_\varepsilon, \quad \|g_\alpha(\tau)\|_0 \leqslant K\|\widetilde{b}(\tau)\|_s\|S(\tau)\|_s \leqslant C(M_\varepsilon(t)). \tag{2.12}$$

Multiply the equation (2.11) by f_α and integrate over $[0,t]\times\Omega$, for $t < T_\varepsilon$. The assumption made on the function $b = \beta(S,p,v)v$ ensures that $b(S,u)\cdot\nu$ vanishes on $\partial\Omega$. The boundary terms thus vanish when we integrate by parts the term $\int b f_\alpha \nabla f_\alpha \, dx$. We obtain

$$\|f_\alpha(t)\|_0^2 \leqslant \|f_\alpha(0)\|_0^2 + \|\nabla b\|_{L^\infty([0,t]\times\Omega)} \int_0^t \|f_\alpha(\tau)\|_0^2 \, d\tau + 2\int_0^t \|g_\alpha(\tau)\|_0 \|f_\alpha(\tau)\|_0 \, d\tau.$$

Since $s - 1 > d/2$, we get

$$\|\nabla b\|_{L^\infty([0,t]\times\Omega)} \leqslant \sup_{\tau\in[0,t]} \|\nabla b(\tau)\|_{s-1} \leqslant C(M_\varepsilon(t)),$$

thus

$$\begin{aligned}\|f_\alpha(t)\|_0^2 &\leqslant \|f_\alpha(0)\|_0^2 + C(M_\varepsilon(t))\int_0^t M_\varepsilon(t)^2\, d\tau + \int_0^t C(M_\varepsilon(t))M_\varepsilon(t)\, d\tau,\\ &\leqslant C_0 + tC(M_\varepsilon(t)).\end{aligned}$$

We can assume that $C_0 \geqslant 1$, therefore $\|f_\alpha(t)\|_0 \leqslant C_0 + tC(M_\varepsilon(t))$. The claim then follows by adding up these estimates for $|\alpha| \leqslant s$. □

Our next and main task is to prove a bound analogous to (2.10) for u^ε. We cannot apply the previous works. Indeed, the operators $(E^{-1}L(\partial_x))^m$ which were used in [18] to estimate the acoustic components do not commute with the equations for the boundary condition is violated. And it is the only paper which deals with the non-isentropic equations and general initial data. Nevertheless, the operators $(\varepsilon\partial_t)^m$ are relevant to the boundary case. In this proof they supersede $(E^{-1}L(\partial_x))^m$.

First and foremost, we estimate $(u^\varepsilon, S^\varepsilon)$ in the big norm $\|\cdot\|_{s,\varepsilon,T}$.

Lemma 2.4. *Given $s > 1 + d/2$, there is a function $C(\cdot)$ from $[0,\infty)$ to $[0,\infty)$, such that*

$$\forall \varepsilon \in]0,1], \quad \forall T < T_\varepsilon, \quad \|(u^\varepsilon, S^\varepsilon)\|_{s,\varepsilon,T} \leqslant C(M_\varepsilon(T)). \tag{2.13}$$

Proof. We prove by induction on m that $(\varepsilon\partial_t)^m(u,S)(t) \in H^{s-m}(\Omega)$ with norm bounded by $C(M)$, where $t \leqslant T < T_\varepsilon$ and $M := M_\varepsilon(T)$.

For $m \geqslant 1$, we commute $(\varepsilon\partial_t)^m$ with the equations (2.1). This yields

$$\begin{aligned}
(\varepsilon\partial_t)^{m+1}u &= -\varepsilon b \cdot \nabla(\varepsilon\partial_t)^m u - E^{-1}\big(L(\partial_x)(\varepsilon\partial_t)^m u + f_m\big),\\
f_m &:= \big[(\varepsilon\partial_t)^m, E(S,\varepsilon u)(\varepsilon\partial_t + \varepsilon b(S,u)\cdot\nabla)\big]u,\\
\partial_t(\varepsilon\partial_t)^m S &= -b(S,u)\cdot\nabla(\varepsilon\partial_t)^m S - g_m,\\
g_m &:= \big[(\varepsilon\partial_t)^m, b(S,u)\cdot\partial_x\big]S.
\end{aligned}$$

Assume the result for $m = n$, where $n \in [0, s-1]$, *i.e.* for all $0 \leqslant p \leqslant n$, $(\varepsilon\partial_t)^p(u,S)(t)$ belongs to $H^{s-p}(\Omega)$ with norm bounded by $C(M)$. By (2.8), to prove the result at order $m = n+1$, it is sufficient to show that $f_n(t)$ and $g_n(t)$ belong to $H^{s-n-1}(\Omega)$ together with the inequality $\|f_n(t)\|_{s-n-1} + \|g_n(t)\|_{s-n-1} \leqslant C(M)$.

One has to study commutators of the form $\mathrm{Com}(A) = \big[(\varepsilon\partial_t)^n, A(\varepsilon\partial_t)\big]$ or $\mathrm{Com}(B) = \big[(\varepsilon\partial_t)^n, B\partial_{x_j}\big]$. We get,

$$\begin{aligned}
\|\mathrm{Com}(A)u(t)\|_{s-1-n} &\leqslant K\sum_{p=0}^{n-1}\big\|(\varepsilon\partial_t)^{n-p}A\big\|_{s-1-(n-p)}\big\|(\varepsilon\partial_t)^{p+1}u(t)\big\|_{s-1-p}\\
&\leqslant C(M)\sum_{p=0}^{n-1}\big\|(\varepsilon\partial_t)^{n-p}A\big\|_{s-1-(n-p)}.
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
\|\mathrm{Com}(B)\eta(t)\|_{s-1-n} &\leqslant K\sum_{p=0}^{n-1}\big\|(\varepsilon\partial_t)^{n-p}B\big\|_{s-1-(n-p)}\big\|(\varepsilon\partial_t)^{p}\partial_{x_j}\eta(t)\big\|_{s-1-p}\\
&\leqslant C(M)\sum_{p=0}^{n-1}\big\|(\varepsilon\partial_t)^{n-p}B\big\|_{s-1-(n-p)},
\end{aligned}\tag{2.15}$$

where $\eta \in \{u, S\}$. Here we have expanded the commutators by use of the Leibniz' formula, and estimated the terms by means of the rule (2.6) (applied with $\sigma = s-1 > d/2$) and the induction hypothesis.

It remains to estimate the sums which appear in (2.14) and (2.15) when $A = E(S(t), \varepsilon u(t))$, $B = b_j(S(t), u(t))$ or $B = \varepsilon b_j(S(t), u(t))E(S(t), \varepsilon u(t))$. For that purpose we prove that, for $k \in [1, n]$, both $(\varepsilon\partial_t)^k b(S(t), u(t))$ and $(\varepsilon\partial_t)^k E(S(t), \varepsilon u(t))$ are bounded in $H^{s-k}(\Omega)$ by $C(M)$.

With $U := (S,u) = (u_0, u_1, \dots, u_n)$, $(\varepsilon\partial_t)^k b(S,u)$ expands as

$$\sum_{\substack{\alpha=(q_0,\dots,q_n)\\ 1\leqslant|\alpha|\leqslant k}} \sum_{\beta_\alpha} C_{\alpha,\beta_\alpha} \Big\{ \prod_{\substack{p=0,\dots,n\\ q_p\neq 0}} \prod_{\substack{i_p=1,\dots,q_p\\ \beta_p^{i_p}\neq 0}} (\varepsilon\partial_t)^{\beta_p^{i_p}} u_p \Big\} \left(\partial_U^\alpha b\right)(U),$$

where $|\alpha| = q_0 + \cdots + q_n$, $C_{\alpha,\beta_\alpha} \in \mathbb{N}$ (possibly 0) and $\sum_{\beta_\alpha}$ is taken over all β such that $\sum_{p=0}^n \sum_{i_p=1}^{q_p} \beta_p^{i_p} = k$. The induction hypothesis and the rule (2.6) imply that, for all $k \in [1,n]$, $(\varepsilon\partial_t)^k b(U) \in H^{s-k}(\Omega)$ with norm bounded by $C(M)$. Similarly, $(\varepsilon\partial_t)^k E(S, \varepsilon u)$ expands in the same way and we have the same conclusion. These estimates conclude the proof of Lemma 2.4. □

Remark 2.5. We infer from the previous identities that, for all $T < T_\varepsilon$,

$$\partial_t S \in X^{s-1}([0,T]\times\Omega) \quad \text{with } \|\partial_t S^\varepsilon\|_{s-1,\varepsilon,T} \leqslant C(M_\varepsilon(T)), \tag{2.16}$$

$$\widetilde{b}(S^\varepsilon, u^\varepsilon) \in X^s([0,T]\times\Omega) \qquad \text{with } \|\widetilde{b}(S^\varepsilon, u^\varepsilon)\|_{s,\varepsilon,T} \leqslant C(M_\varepsilon(T)), \tag{2.17}$$

$$\widetilde{E}(S^\varepsilon, \varepsilon u^\varepsilon) \in X^s([0,T]\times\Omega) \qquad \text{with } \|\widetilde{E}(S^\varepsilon, \varepsilon u^\varepsilon)\|_{s,\varepsilon,T} \leqslant C(M_\varepsilon(T)). \tag{2.18}$$

Lemma 2.6. *We can improve* (2.18)*. Let $T < T_\varepsilon$, we have*

$$\partial_t E(S^\varepsilon, \varepsilon u^\varepsilon) \in X^{s-1}([0,T]\times\Omega) \quad \textit{with} \quad \|\partial_t E\|_{s-1,\varepsilon,T} \leqslant C(M_\varepsilon(T)). \tag{2.19}$$

Proof. With $U := (S,u) = (u_0, u_1, \dots, u_n)$, $\partial_t(\varepsilon\partial_t)^{k-1} E(S, \varepsilon u)$ expands as

$$\sum_{\substack{\alpha=(q_0,\dots,q_n)\\ 1\leqslant|\alpha|\leqslant k}} \sum_{\beta_\alpha} C_{\alpha,\beta_\alpha} \varepsilon^{\ell_{\alpha,\beta_\alpha}} \Big\{ \prod_{\substack{p=0,\dots,n\\ q_p\neq 0}} \prod_{\substack{i_p=1,\dots,q_p\\ \beta_p^{i_p}\neq 0}} (\varepsilon\partial_t)^{\beta_p^{i_p}} u_p \Big\} \left(\partial_U^\alpha E\right)(S, \varepsilon u),$$

with the previous notations and $\ell_{\alpha,\beta_\alpha} = (\sum_{p=1}^n \sum_{i_p=1}^{q_p} \beta_p^{i_p}) - 1$.

We prove that each term is bounded in $H^{s-k}(\Omega)$ by $C(M_\varepsilon(T))$.

Either $\ell_{\alpha,\beta_\alpha} \geqslant 0$, then the result follows from the rule (2.6), or $\ell_{\alpha,\beta_\alpha} = -1$, which implies that $\alpha = (q_0, 0, \dots, 0)$. Which in turn implies

$$\varepsilon^{-1} \prod_{\substack{p=0,\dots,n\\ q_p\neq 0}} \prod_{\substack{i_p=1,\dots,q_p\\ \beta_p^{i_p}\neq 0}} (\varepsilon\partial_t)^{\beta_p^{i_p}} u_p = \partial_t(\varepsilon\partial_t)^{\gamma_1} S (\varepsilon\partial_t)^{\gamma_2} S \cdots (\varepsilon\partial_t)^{\gamma_r} S,$$

where $\gamma_1 + \cdots + \gamma_r = k-1$. The proof is therefore completed by applying (2.16) and the rule (2.6) with $\sigma = s$. □

Now, we are looking for L^2 estimates for the partially linearized equations

$$E(\partial_t \dot{u} + b\cdot\nabla\dot{u}) + \frac{1}{\varepsilon} L(\partial_x)\dot{u} = F, \tag{2.20}$$

$$\dot{v}_{|\partial\Omega}\cdot\nu = 0, \tag{2.21}$$

where $\dot{u} = (\dot{q}, \dot{v})^t$, $E := E(S^\varepsilon, \varepsilon u^\varepsilon)$ and $b := b(S^\varepsilon, u^\varepsilon)$.

Lemma 2.7. *Given $s > 1 + d/2$ and M_0, there is a constant C_0 and a non-negative function $C(\cdot)$, such that for all $\varepsilon \in]0,1]$, all $\dot{u} \in C^0([0,T], H^1(\Omega))$ and all $F \in C^0([0,T], L^2(\Omega))$ satisfying* (2.20) *and* (2.21)*,*

$$\|\dot{u}(t)\|_0^2 \leqslant C_0 e^{tC(M_\varepsilon(t))} \Big(\|\dot{u}(0)\|_0^2 + \int_0^t \|F(\tau)\|_0^2 \, d\tau \Big). \tag{2.22}$$

Proof. Multiply the equations (2.20) by $\dot{u}$ and integrate over $[0,t]\times\Omega$. Since $L(\partial_x)$ is formally skew-adjoint and $\dot{v}_{|\partial\Omega}\cdot\nu = 0$, the Green's identity applied to $\dot{u}(t)\in H^1(\Omega)$ shows that the term in $1/\varepsilon$ cancel out. Since $\nu\cdot b(S,u)_{|\partial\Omega}=0$, the boundary terms vanish when we integrate by parts the term in $\partial_x\dot{u}$. The derivatives of the coefficients which appear are $\partial_t E$, $\partial_x E$ and $\partial_x b$, which are estimated in $L^\infty(\Omega)$ by $C(M_\varepsilon)$ according to (2.19), the rule (2.7) and the embedding of H^{s-1} into L^∞. Using the symmetry of the matrices $E(t)$, this implies that $\big(E(t)\dot{u}(t),\dot{u}(t)\big)_0$ is bounded by

$$\big(E(0)\dot{u}(0),\dot{u}(0)\big)_0 + 2\int_0^t \|F(\tau)\|_0\,\|\dot{u}(\tau)\|_0\,d\tau + \int_0^t C(M_\varepsilon(\tau))\,\|\dot{u}(\tau)\|_0^2\,d\tau,$$

where $(\cdot,\cdot)_0$ denotes the scalar product in $L^2(\Omega)$. Moreover, we have

$$\|\dot{u}(t)\|_0^2 \leqslant \|E^{-1}(t)\|_{L^\infty}\big(E(t)\dot{u}(t),\dot{u}(t)\big)_0.$$

The proof of Lemma 2.6, with E replaced by E^{-1}, applies, and we find that

$$\|E^{-1}(t)\|_{L^\infty} \leqslant \|E^{-1}(0)\|_{s-1} + t\sup_{\tau\in[0,t]}\|\partial_t E^{-1}(\tau)\|_{s-1} \leqslant C_0 + tC(M_\varepsilon(t)).$$

The estimate (2.22) results from these bounds and Gronwall's Lemma. □

As a consequence of the previous results, we prove the following Lemma.

Lemma 2.8. *There exist C_0 and $C(\cdot)$ depending only on M_0, such that for all $m\leqslant s$,*

$$\forall\varepsilon\in]0,1],\quad \forall t\in[0,T_\varepsilon[,\quad \|(\varepsilon\partial_t)^m u^\varepsilon(t)\|_0 \leqslant C_0 + tC(M_\varepsilon(t)). \tag{2.23}$$

Proof. Let $\tau\leqslant t<T_\varepsilon$ and $M:=M_\varepsilon(t)$. We begin by assuming that $m\leqslant s-1$. According to Lemma 2.4, $(\varepsilon\partial_t)^m u$ belongs to $C^0([0,t],H^1(\Omega))$. What is more $(\varepsilon\partial_t)^m u$ satisfies (2.21) and (2.20) with

$$F = G_1 + G_2 := \big[E\partial_t,(\varepsilon\partial_t)^m\big]u + \sum_{j=1}^d \big[b_j E,(\varepsilon\partial_t)^m\big]\partial_{x_j}u. \tag{2.24}$$

We estimate the L^2 norm of F according to (2.13), (2.17), (2.18), (2.19), the rules (2.6) and (R4). First, we have

$$G_1 = -\sum_{p=0}^{m-1}\binom{m}{p}\big(\partial_t(\varepsilon\partial_t)^{m-p-1}E\big)(\varepsilon\partial_t)^{p+1}u.$$

Using the rule (2.6) (applied with $\sigma=s-1$, $k=p$, $l=m-p-1$), we obtain

$$\|G_1(\tau)\|_{s-m} \leqslant K\sum_{p=0}^{m-1}\|\partial_t(\varepsilon\partial_t)^{m-p-1}E(\tau)\|_{s-m+p}\|(\varepsilon\partial_t)^{p+1}u(\tau)\|_{s-1-p},$$

hence

$$\|G_1(\tau)\|_{s-m} \leqslant K\|\partial_t E(\tau)\|_{s-1,\varepsilon}\|u(\tau)\|_{s,\varepsilon} \leqslant C(M_\varepsilon(t)). \tag{2.25}$$

The technique for estimating G_2 is similar. We get

$$\|G_2(\tau)\|_{s-m} \leqslant K\sum_{j=1}^d\sum_{p=0}^{m-1}\|b_j(\tau)E(\tau)\|_{s,\varepsilon}\left\|(\varepsilon\partial_t)^p\partial_{x_j}u(\tau)\right\|_{s-1-p}$$

$$\|G_2(\tau)\|_{s-m} \leqslant K\,\|E(\tau)\|_{s,\varepsilon}\,\|b(\tau)\|_{s,\varepsilon}\,\|u(\tau)\|_{s,\varepsilon} \leqslant C(M_\varepsilon(t)). \tag{2.26}$$

In view of the inequalities (2.25) and (2.26) we see that for all $\tau \leqslant t < T_\varepsilon$, we have $\|F(\tau)\|_0 \leqslant C(M_\varepsilon(t))$. Next, we apply the energy inequality (2.22) together with the previous L^2 bound on the source term and the elementary inequality $e^{tC(M)} \leqslant 1 + t\widetilde{C}(M)$ for nonnegative t in a compact set. We find that $\|(\varepsilon\partial_t)^m u^\varepsilon(t)\|_0 \leqslant C_0 + tC(M_\varepsilon(t))$.

When $m = s$, we use the proof given by S. Schochet of the existence theorem for fixed ε (see [22]). He has proved that one can approximate u in $C^0([0,T], H^s(\Omega))$ by a sequence u^n in $C^0([0,T], H^{s+1}(\Omega))$ such that $u^n(0)$ satisfies the compatibility conditions up to order s, and u^n is the unique solution of an approximating system. Then we deduce, from the previous computations applied to u^n, that

$$\|(\varepsilon\partial_t)^s u^n(t)\|_0 \leqslant C_0(\|u^n(0)\|_s) + t\, C\big(\sup_{\tau\in[0,t]} \|u^n(\tau)\|_s\big).$$

Taking the limit when n goes to $+\infty$, we get (2.23) at order $m = s$. □

The forthcoming computations make use of the special structure of (2.1). We denote curl w the matrix with coefficients $(\mathrm{curl}\ w)_{i,j} = \partial_{x_j} w_i - \partial_{x_i} w_j$. The idea of the two following estimates is to apply the curl operator to the equations so as to cancel the large term $\varepsilon^{-1}\nabla p^\varepsilon$, and then to estimate the incompressible components.

Introduce the function $r_0(S) = r(S,0)\colon \mathbb{R} \to \mathbb{R}$, which is smooth and positive (recall that $E(S,u) = \begin{pmatrix} a(S,u) & 0 \\ 0 & r(S,u)\mathrm{I}_d \end{pmatrix}$).

We define $f_1(S,u) = 1 - r_0(S)/r(S,u)$. Hereafter, to shorten notation, we denote $r_0(t) := r_0(S^\varepsilon(t))$ and $f_1(t) := f_1(S^\varepsilon(t), \varepsilon u^\varepsilon(t))$. Note that $r(S^\varepsilon, \varepsilon u^\varepsilon)$ is bounded away from 0 uniformly with respect to ε. Indeed, we have shown in the proof of Lemma 2.5 that the L^∞ norm of $E^{-1}(S^\varepsilon, \varepsilon u^\varepsilon)$ is well-estimated, and so is the L^∞ norm of $1/r(S^\varepsilon, \varepsilon u^\varepsilon)$.

We can factor out $\varepsilon u^\varepsilon$ in f_1. There exists a smooth function g such that

$$f_1(t) = \varepsilon g(t) := \varepsilon g(S^\varepsilon(t), \varepsilon u^\varepsilon(t)), \quad \text{with} \quad \|g^\varepsilon(t)\|_s \leqslant C(M_\varepsilon(t)). \tag{2.27}$$

Since $\partial_t S^\varepsilon + b(S^\varepsilon, u^\varepsilon)\cdot\nabla S^\varepsilon = 0$, the equation for v^ε (recall that $u^\varepsilon = (p^\varepsilon, v^\varepsilon)^t$) is equivalent to the transport equation

$$(\partial_t + b\cdot\nabla)(r_0 v^\varepsilon) + \frac{1}{\varepsilon}\nabla p^\varepsilon = g\nabla p^\varepsilon, \tag{2.28}$$

with the previous notation.

Using the fact that curl $\nabla = 0$, we get

$$(\partial_t + b\cdot\nabla)\big(\mathrm{curl}(r_0 v^\varepsilon)\big) = \big[b\cdot\nabla, \mathrm{curl}\big](r_0 v^\varepsilon) + \big[\mathrm{curl}, g\big]\nabla p^\varepsilon. \tag{2.29}$$

Lemma 2.9. *There are C_0 and $C(\cdot)$ depending only on M_0, such that*

$$\forall \varepsilon \in (0,1], \quad \forall t \in [0, T_\varepsilon), \quad \big\|\mathrm{curl}\big(r_0(t) v^\varepsilon(t)\big)\big\|_{s-1} \leqslant C_0 + tC(M_\varepsilon(t)). \tag{2.30}$$

Proof. Let $M := M_\varepsilon(t)$. Using the proof of Lemma 2.3, to prove (2.30) it is sufficient to estimate the source term $\big[b\cdot\nabla, \mathrm{curl}\big](r_0 v) + \big[\mathrm{curl}, g\big]\nabla p$ in $H^{s-1}(\Omega)$.

For $w \in H^1(\Omega)$, we have $([\mathrm{curl}, g]\, w)_{i,j} = w_i\partial_{x_j} g - w_j \partial_{x_i} g$. Therefore, by the rule (2.6) and the estimate (2.27), we get for all $\tau \leqslant t < T_\varepsilon$,

$$\|[\mathrm{curl}, g(\tau)]\,\nabla p(\tau)\|_{s-1} \leqslant K\,\|\nabla g(\tau)\|_{s-1}\,\|\nabla p(\tau)\|_{s-1} \leqslant C(M). \tag{2.31}$$

Similarly,

$$\left\|[b_j(\tau), \mathrm{curl}]\, \partial_{x_j}(r_0(\tau)v(\tau))\right\|_{s-1} \leqslant C(M),$$

which concludes the proof of Lemma 2.9. □

Another result relies upon similar computations.

Lemma 2.10. *Given $s > 1 + d/2$ and M_0, there exist a constant C_0 and a function $C(\cdot)$ from $[0, \infty)$ to $[0, \infty)$, such that if $\varepsilon \in (0, 1]$, $n \in \{1, \ldots, s-1\}$, and $(\dot u, F) \in C^0([0, T], H^{n+1}(\Omega))$ satisfies (2.20) and (2.21), then we have*

$$\left\|\partial_t \,\mathrm{curl}\big(r_0(S^\varepsilon(t))\dot v(t)\big)\right\|_{n-1} \leqslant C(M_\varepsilon(T))\big(\|F(t)\|_n + \|\dot u(t)\|_{n+1}\big). \tag{2.32}$$

Proof. Let us introduce the notation $F(t, x) = (F_1(t, x), F_2(t, x)) \in \mathbb{R} \times \mathbb{R}^d$. Similar arguments to those above show that $\partial_t \,\mathrm{curl}(r_0\dot v)$ satisfies

$$\partial_t \,\mathrm{curl}(r_0\dot v) = \mathrm{curl}\big((1 - f_1)F_2\big) + [\mathrm{curl}, g]\, \nabla\dot q - \mathrm{curl}(b \cdot \nabla(r_0\dot v)),$$

The claim then follows from the bounds

$$\begin{aligned}\left\|\mathrm{curl}\big((1 - f_1)F_2\big)\right\|_{n-1} &\leqslant K\|f_1\|_s\|F_2\|_n + \|F_2\|_n \leqslant C(M)\|F\|_n,\\ \|[\mathrm{curl}, g]\, \nabla\dot q\|_{n-1} &\leqslant K\|g\|_s\|\dot q\|_n \leqslant C(M)\|\dot u\|_n,\\ \|\mathrm{curl}(b \cdot \nabla(r_0\dot v))\|_{n-1} &\leqslant K\|b\|_s\|r_0\|_s\|\dot v\|_{n+1} \leqslant C(M)\|\dot u\|_{n+1}.\end{aligned}$$

□

The foremost estimates are given in Lemma 2.12. We prepare its proof in the following lemma.

Lemma 2.11. *Given $s > 1 + d/2$ and M_0, there exist a constant C_0 and a function $C(\cdot)$ from $[0, \infty)$ to $[0, \infty)$, such that if $\varepsilon \in (0, 1]$, $n \in \{0, \ldots, s-1\}$, and $(\dot u, F) \in X^{n+1}([0, T] \times \Omega)$ satisfy (2.20) and (2.21), then*

$$\|L(\partial_x)\dot u(t)\|_n \leqslant \widetilde C\, \|\varepsilon\partial_t\dot u(t)\|_n + \varepsilon C(M)\, \|\dot u(t)\|_{n+1} + \varepsilon\, \|F(t)\|_n \tag{2.33}$$

with $\widetilde C := C_0 + tC(M)$ and $M := M_\varepsilon(T)$.

Proof. We rewrite (2.20):

$$L(\partial_x)\dot u = -E(S, \varepsilon u)\big(\varepsilon\partial_t\dot u + \varepsilon b(S, u) \cdot \nabla\dot u\big) + \varepsilon F. \tag{2.34}$$

We estimate $\|E(t)\|_{s-1}$ by the bound of its time derivative, see (2.19), and the identity

$$\|E(t)\|_{s-1} \leqslant \|E(t = 0)\|_{s-1} + t \sup_{\tau\in[0,t]} \|\partial_t E(\tau)\|_{s-1}\,.$$

It follows that $\|E(t)\|_{s-1} \leqslant \widetilde C = C_0 + tC(M)$.

Therefore, the multiplicative property (2.6), applied with $\sigma = s-1$, yields

$$\|E(t)(\varepsilon\partial_t)\dot u(t)\|_n \leqslant (C_0 + tC(M))\, \|\varepsilon\partial_t\dot u\|_n\,.$$

Again, the rule (2.6) implies

$$\|\varepsilon E(t)b(t) \cdot \nabla\dot u(t)\|_n \leqslant \varepsilon C(M)\, \|\nabla\dot u(t)\|_n\,.$$

The identity (2.34) and these bounds imply (2.33). □

We now come to our main estimates. First, we recall the following useful elliptic estimate, for all $v \in H^{k+1}(\Omega)$:

$$\|v\|_{k+1} \leqslant K \left(\|\text{div } v\|_k + \|\text{curl } v\|_k + \|v\|_k + \|v \cdot \nu\|_{k+1/2,\partial\Omega} \right).$$

This estimate is proved in [3] when Ω is a bounded domain; it is clear when Ω is the whole space $\mathbb{R}^d$ using the Fourier transform. As a consequence we can easily extend such a result to the exterior case. Recall that we assume only that Ω has a compact, smooth boundary, possibly empty if $\Omega = \mathbb{R}^d$.

Lemma 2.12. *Given $s > 1 + d/2$ and M_0, there is a constant C_0 and a function $C(\cdot)$ from $[0,\infty)$ to $[0,\infty)$, such that if $\varepsilon \in (0,1)$, $m \in \{1,\dots,s\}$, and $\dot{u}, F \in X^m([0,T]\times\Omega)$ satisfy (2.20) and (2.21), then*

(2.35)
$$\|\dot{u}(t)\|_{m,\varepsilon} \leqslant \widetilde{C} \left(\|(\varepsilon\partial_t)^m \dot{u}(t)\|_0 + \|\text{curl}(r_0(t)\dot{v}(t))\|_{m-1} + \|\dot{u}(t)\|_{m-1,\varepsilon} \right) + \varepsilon C(M) \left(\|F(t)\|_{m,\varepsilon} + \|\dot{u}(t)\|_{m,\varepsilon} \right),$$

with $M = M_\varepsilon(T)$ and $\widetilde{C} \equiv C_0 + tC(M)$.

Remark 2.13. We will apply this lemma with $(\dot{u}, F) = (u^\varepsilon, 0)$, yet we introduce $(\dot{u}, F)$ so as to prove the desired estimates by induction on m.

Proof. We prove (2.35) by induction on $m \in \{1,\dots,s\}$. Assume the result at order $m = n < s$, and suppose that $(\dot{u}, F) \in X^{n+1}([0,T]\times\Omega)$ satisfy (2.20) and (2.21).

By definition (2.9) we have

$$\|\dot{u}(t)\|_{n+1,\varepsilon} = \|\varepsilon\partial_t\dot{u}(t)\|_{n,\varepsilon} + \|\dot{u}(t)\|_{n+1}. \tag{2.36}$$

We use Lemma 2.10 and the induction hypothesis to estimate $\|\varepsilon\partial_t\dot{u}(t)\|_{n,\varepsilon}$. As regards $\|\dot{u}(t)\|_{n+1}$, we use Lemma 2.11 and the estimate

$$\|v\|_{k+1} \leqslant K \left(\|\text{div}\, v\|_k + \|\text{curl } v\|_k + \|v\|_k + \|v \cdot \nu\|_{k+1/2,\partial\Omega} \right). \tag{2.37}$$

In view of the boundary condition (2.21), identities (2.36) and (2.37) result in

(2.38)
$$\|\dot{u}(t)\|_{n+1,\varepsilon} \leqslant \|\varepsilon\partial_t\dot{u}(t)\|_{n,\varepsilon} + K\big(\|L(\partial_x)\dot{u}(t)\|_n + \|\text{curl } \dot{u}(t)\|_n + \|\dot{u}(t)\|_n \big);$$

recall that $L(\partial_x)\dot{u} = (\text{div}\,\dot{v}, \nabla\dot{q})^t$.

By the multiplicative rule (2.6) we infer that

$$\|\text{curl } \dot{v}(t)\|_n \leqslant K \left\| r_0^{-1}(t) \right\|_{s-1} \|r_0(t)\,\text{curl}(\dot{v}(t))\|_n. \tag{2.39}$$

Furthermore, as in (2.31) we have

$$\|r_0(t)\,\text{curl } \dot{v}(t) - \text{curl}(r_0(t)\dot{v}(t))\|_n \leqslant K \|r_0(t)\|_s \|\dot{v}(t)\|_n. \tag{2.40}$$

Since $(\partial_t + b\cdot\nabla)r_0(S) = 0$, we infer from the proof of Lemma 2.3 that $\|r_0(t)\|_s \leqslant \widetilde{C}$; similarly, $\left\|r_0^{-1}(t)\right\|_s \leqslant \widetilde{C}$. Therefore, the inequalities (2.39) and (2.40) imply that

$$\|\text{curl } \dot{v}(t)\|_n \leqslant \widetilde{C}\big(\|\text{curl}(r_0(t)\dot{v}(t))\|_n + \|\dot{v}(t)\|_n \big).$$

Thus, the estimate (2.38) turns into

$$\|\dot u(t)\|_{n+1,\varepsilon} \leqslant \|\varepsilon\partial_t \dot u(t)\|_{n,\varepsilon} + K\,\|L(\partial_x)\dot u(t)\|_n + \widetilde C\,\|\mathrm{curl}(r_0(t)\dot v(t))\|_n + \widetilde C\,\|\dot u(t)\|_n . \tag{2.41}$$

From Lemma 2.11 and (2.41), we get

$$\|\dot u(t)\|_{n+1,\varepsilon} \leqslant \widetilde C\left(\|\varepsilon\partial_t\dot u(t)\|_{n,\varepsilon} + \|\mathrm{curl}(r_0(t)\dot v(t))\|_n + \|\dot u(t)\|_n\right) + \varepsilon C(M)\,\|\dot u(t)\|_{n+1,\varepsilon} + \varepsilon K\,\|F(t)\|_n . \tag{2.42}$$

Next, we want to apply the induction hypothesis with $\varepsilon\partial_t\dot u$. We commute $\varepsilon\partial_t$ with (2.20):

$$E\big(\partial_t(\varepsilon\partial_t)\dot u + b\cdot\nabla(\varepsilon\partial_t)\dot u\big) + \frac{1}{\varepsilon}L(\partial_x)(\varepsilon\partial_t)\dot u = G(t),$$
$$G(t) := \varepsilon\partial_t F - \varepsilon\partial_t E\partial_t\dot u - \sum_{j=1}^{d}\varepsilon\partial_t(Eb_j)\partial_{x_j}\dot u.$$

We apply the rule (R4) in order to estimate the source term $G(t)$; it yields

$$\begin{aligned}\|G(t)\|_{n,\varepsilon} &\leqslant \|\varepsilon\partial_t F(t)\|_{n,\varepsilon} + K\,\|\partial_t E(t)\|_{s-1,\varepsilon}\,\|\varepsilon\partial_t\dot u(t)\|_{n,\varepsilon}\\ &\quad + K\|\varepsilon\partial_t\sum_{j=1}^{d}E(t)b_j(t)\|_{s-1,\varepsilon}\|\nabla\dot u(t)\|_{n,\varepsilon}\\ &\leqslant \|F(t)\|_{n+1,\varepsilon} + K\big(\|\partial_t E(t)\|_{s-1,\varepsilon} + \|E(t)\|_{s,\varepsilon}\|b(t)\|_{s,\varepsilon}\big)\|\dot u(t)\|_{n+1,\varepsilon}\\ &\leqslant \|F(t)\|_{n+1,\varepsilon} + C(M)\,\|\dot u(t)\|_{n+1,\varepsilon} . \end{aligned} \tag{2.43}$$

We estimated $\|\partial_t E(t)\|_{s-1,\varepsilon}$ and $\|E(t)\|_{s,\varepsilon}\,\|b(t)\|_{s,\varepsilon}$ by $C(M)$ thanks to (2.17), (2.18) and (2.19).

Next, we apply the induction hypothesis with $(\dot u, F)$ replaced by the $(\varepsilon\partial_t\dot u, G)$, which belong to $X^n([0,T]\times\Omega)$. We obtain

$$\begin{aligned}\|\varepsilon\partial_t\dot u(t)\|_{n,\varepsilon} &\leqslant \widetilde C\Big(\|(\varepsilon\partial_t)^n\varepsilon\partial_t\dot u(t)\|_0 + \|\mathrm{curl}(r_0(t)\varepsilon\partial_t\dot v(t))\|_{n-1}\\ &\quad + \|\varepsilon\partial_t\dot u(t)\|_{n-1,\varepsilon}\Big) + \varepsilon C(M)\left(\|G(t)\|_{n,\varepsilon} + \|\varepsilon\partial_t\dot u(t)\|_{n,\varepsilon}\right).\end{aligned} \tag{2.44}$$

We have $\mathrm{curl}\big(r_0(\varepsilon\partial_t)\dot v\big) = \varepsilon\partial_t\,\mathrm{curl}(r_0\dot v) - \varepsilon\,\mathrm{curl}\big((\partial_t r_0)\dot v\big)$. Therefore, we infer that

$$\|\,\mathrm{curl}(r_0(\varepsilon\partial_t)\dot v)\|_{n-1} \leqslant \varepsilon\|\partial_t\,\mathrm{curl}(r_0\dot v)\|_{n-1} + \varepsilon K\|\partial_t r_0\|_{s-1}\|\dot v\|_n.$$

Making use of Lemma 2.10 to estimate $\|\partial_t\,\mathrm{curl}(r_0\dot v)\|_{n-1}$, and the bound

$$\|\partial_t r_0(t)\|_{s-1} = \|b\cdot\nabla r_0(t)\|_{s-1} \leqslant C(M),$$

we end up with

$$\big\|\mathrm{curl}\big(r_0(t)(\varepsilon\partial_t)\dot v(t)\big)\big\|_{n-1} \leqslant \varepsilon C(M)\big(\|F(t)\|_{n+1} + \|\dot u(t)\|_{n+1}\big). \tag{2.45}$$

Combining the inequalities (2.43) and (2.45) with (2.44), we obtain

$$\|\varepsilon\partial_t\dot u(t)\|_{n,\varepsilon} \leqslant \widetilde C\left(\big\|(\varepsilon\partial_t)^{n+1}\dot u(t)\big\|_0 + \|\dot u(t)\|_{n,\varepsilon}\right) + \varepsilon C(M)\left(\|F(t)\|_{n+1,\varepsilon} + \|\dot u(t)\|_{n+1,\varepsilon}\right). \tag{2.46}$$

Finally, using the estimates (2.42) and (2.46), we conclude that

$$\begin{aligned}\|\dot u(t)\|_{n+1,\varepsilon} \leqslant \widetilde{C}\left(\left\|(\varepsilon\partial_t)^{n+1}\dot u(t)\right\|_0 + \|\mathrm{curl}(r_0(t)\dot v(t))\|_n + \|\dot u(t)\|_{n,\varepsilon}\right) \\ + \varepsilon C(M)(\|F(t)\|_{n+1,\varepsilon} + \|\dot u(t)\|_{n+1,\varepsilon}),\end{aligned}$$

which is the lemma at order $m = n+1$.

It remains to prove the Lemma for $m = 1$. Notice that we derived (2.42) without the use of the induction hypothesis. So we can let $n = 0$ in (2.42), which gives the result at order $m = 1$ and completes the proof of Lemma 2.12. □

Recall that the upshot of all previous lemmas is to prove a bound analogous to (2.10) for u^ε.

Lemma 2.14. *Given $s > 1 + d/2$ and M_0, there is a constant C_0 and a function $C(\cdot)$ from $[0,+\infty)$ to $[0,+\infty)$ such that*

$$\forall \varepsilon \in (0,1], \quad \forall t \in [0, T_\varepsilon), \quad \|u^\varepsilon(t)\|_s \leqslant C_0 + (t+\varepsilon)C(M_\varepsilon(t)). \tag{2.47}$$

Proof. Let $t < T_\varepsilon$ and $M = M_\varepsilon(t)$. The L^2 norm of $u(t)$ is estimated by Lemma 2.7, taking $F = 0$. Next, we prove by induction on $\sigma \in \{0, \dots, s\}$ that there are C_0 and $C(\cdot)$ such that $\|u(t)\|_{\sigma,\varepsilon} \leqslant C_0 + (t+\varepsilon)C(M_\varepsilon(t))$. Assume the result at order $\sigma = n < s$. The inequality (2.35), with $(m, \dot u, F)$ replaced by $(n+1, u, 0)$, applies, and we find that

$$\begin{aligned}\|u(t)\|_{n+1,\varepsilon} \leqslant \widetilde{C}\left(\left\|(\varepsilon\partial_t)^{n+1}u(t)\right\|_0 + \|\mathrm{curl}(r_0(t)v(t))\|_n + \|u(t)\|_{n,\varepsilon}\right) \\ + \varepsilon C(M)\,\|u(t)\|_{n+1,\varepsilon}\,.\end{aligned} \tag{2.48}$$

We list our bounds, with $M := M_\varepsilon(t)$:

by Lemma 2.8	$\left\|(\varepsilon\partial_t)^{n+1}u(t)\right\|_0 \leqslant C_0 + tC(M)$,
by Lemma 2.9	$\|\mathrm{curl}(r_0(S)v(t))\|_n \leqslant C_0 + tC(M)$,
by induction hypothesis	$\|u(t)\|_{n,\varepsilon} \leqslant C_0 + (t+\varepsilon)C(M)$,
by Lemma 2.4	$\|u(t)\|_{n+1,\varepsilon} \leqslant C(M)$.

Thus, we deduce from (2.48) that

$$\begin{aligned}\|u(t)\|_{n+1,\varepsilon} &\leqslant \widetilde{C}(\widetilde{C} + \widetilde{C} + C_0 + (t+\varepsilon)C(M)) + \varepsilon C(M)C(M) \\ &\leqslant C_0 + (t+\varepsilon)C(M),\end{aligned}$$

which is the result at order $\sigma = n+1$. The last estimate, with $\sigma = s$, is (2.47) stated in sharper form. □

From the estimates (2.10) and (2.47) we easily prove Proposition 2.1, which in turn implies the existence of classical solutions for a time independent of the small parameter. The uniform bounds (1.4) and (1.5) are immediate consequences of the previous computations. For instance, (1.5) results from Lemma 2.10 applied with $(\dot u, F) = (u^\varepsilon, 0)$.

Theorem 1.2 is proved.

3. Convergence toward the limit system

In this section we assume that Ω is the exterior of a bounded domain lying on one side of its compact, smooth boundary (which includes the case of $\mathbb{R}^d$). Recall that, here, $s > d/2 + 2$ is an integer; in particular, $s \geqslant 2$.

3.1. **Computation of the limit system.** The uniform bounds (1.4) and (1.5) imply that, up to the extraction of subsequences, one has the following convergences:

$$(p^\varepsilon, v^\varepsilon) \rightharpoonup (q, v) \text{ weakly } \star \text{ in } L^\infty([0,T], H^s(\Omega)), \tag{3.1}$$

$$S^\varepsilon \to S \text{ in } C^0([0,T], H^{s'}_{\text{loc}}(\Omega)), \text{ for all } s' < s, \tag{3.2}$$

$$\operatorname{curl}(r_0^\varepsilon v^\varepsilon) \to \operatorname{curl}(r_0 v) \text{ in } C^0([0,T], H^{s'}_{\text{loc}}(\Omega)), \text{ for all } s' < s - 1. \tag{3.3}$$

We first prove that $q = 0$ and $\operatorname{div} v = 0$. Starting from

$$\varepsilon E(S^\varepsilon, \varepsilon u^\varepsilon)\partial_t u^\varepsilon + L(\partial_x)u^\varepsilon = -\varepsilon E(S^\varepsilon, \varepsilon u^\varepsilon) b(S^\varepsilon, u^\varepsilon) \cdot \nabla u^\varepsilon,$$

using the uniform bounds (1.4) and $E(S^\varepsilon, \varepsilon u^\varepsilon) - E_0(S^\varepsilon) = O(\varepsilon)$, we obtain

$$\varepsilon E_0(S^\varepsilon)\partial_t u^\varepsilon + L(\partial_x)u^\varepsilon = \varepsilon f^\varepsilon, \tag{3.4}$$

where $E_0(S) = E(S, 0)$ and $(f^\varepsilon)_{\varepsilon>0}$ is bounded in $C^0([0,T], H^{s-1})$. Passing to the weak limit shows that $\nabla p = 0$ and $\operatorname{div} v = 0$. Since $q \in L^\infty(H^s)$, and since the Lebesgue measure of the open, connected set Ω is $+\infty$, we end up with $q = 0$.

In order to prove Theorem 1.5, the main point is to prove that the convergence (3.1) holds in the strong topology of $L^2([0,T], H^{s'}_{\text{loc}}(\Omega))$. Indeed, once this is established the proof proceeds as in the whole-space case (see [18]). The only difference is that the Leray's projectors, which are useful to derive the term $\nabla\pi$ in (1.6) and the initial data w_0, are defined in $\mathbb{R}^d$ as Fourier multipliers. In an exterior domain Ω we use the definitions and properties of these operators given in [10] (in particular, they are bounded operators from $H^m(\Omega)$ to $H^m(\Omega)$). Notice that the uniqueness of the solution of the limit system implies that the whole family converges.

We have strong compactness for the incompressible components by (3.3), so it is sufficient to focus attention on the acoustic components.

Proposition 3.1. *Assume that the hypotheses of Theorem* 1.5 *are satisfied; then p^ε converges strongly to* 0 *in $L^2([0,T], H^{s'}_{\text{loc}}(\Omega))$ for all $s' < s$, and $\operatorname{div} v^\varepsilon$ converges strongly to* 0 *in $L^2([0,T], H^{s'}_{\text{loc}}(\Omega))$ for all $s' < s - 1$.*

The end of this section is devoted to the proof of Proposition 3.1.

Analogous strong compactness was proved in [10] via the spectral and scattering theory for the linearized equations of acoustics, and in [24] by Strichartz estimates. Both proofs are based on the fact that they have considered isentropic equations. Indeed, in this case, one is led to study a wave equations with constant coefficients. In the non-isentropic case, where the wave coefficients have variable coefficients, we follow the analysis of G. Métivier and S. Schochet ([18, 20]). Namely, we introduce some semiclassical measures and prove that they vanish, which imply the strong compactness in time. Together with the strong compactness in space proved in Theorem 1.2, this gives Proposition 3.1.

The semiclassical measures will be defined as defect measures of wave-packets transforms. The definitions rely upon the works of P. Gérard (see [6]) and P.-L. Lions and T. Paul (see [16]). In [6] microlocal defect measures are defined for bounded sequences in $L^2_{\mathrm{loc}}(\mathcal{V}, H)$, where $\mathcal{V}$ is an open set of $\mathbb{R}^d$ and H is a separable Hilbert space. It leads to positive measure on the cosphere bundle of $\mathcal{V}$ by use of Garding's inequality. And in [16] semiclassical measures are defined by means of Wigner transform. It leads to positive measures via the Husimi's transform. Garding's inequality has been known to be related to the wave-packets transform since the work of A. Córdoba and C. Fefferman [5]. Moreover, in [16], the authors point out the connection between Husimi's transform and the wave-packets transform.

3.2. **The wave-packets transform.** Here we introduce the wave-packets transform associated to the scale ε^{-1}. It appears as a nice tool to measure in the phase plane how much of a function oscillates at frequencies $O(\varepsilon^{-1})$.

Proposition 3.2. *Let $C^1_{t,0}(\mathbb{R} \times \mathcal{Q})$ denote the subspace of $C^1(\mathbb{R} \times \mathcal{Q})$ whose elements are compactly supported in the time variable t. Let v be a function in $C^1_{t,0}(\mathbb{R} \times \overline{\Omega}) \cap L^2(\mathbb{R} \times \Omega)$. Define*

$$W^\varepsilon v(t, \tau, x) = c\varepsilon^{-3/4} \int_{\mathbb{R}} e^{(i(t-s)\tau-(t-s)^2)/\varepsilon} v(s, x)\, ds.$$

Then $W^\varepsilon v \in C^1(\mathbb{R} \times \mathbb{R} \times \overline{\Omega}) \cap L^2(\mathbb{R} \times \mathbb{R} \times \Omega)$, and, with $c = (2\pi^3)^{-1/4}$, W^ε extends as an isometry from $L^2(\mathbb{R} \times \Omega)$ to $L^2(\mathbb{R} \times \mathbb{R} \times \Omega)$.

Proof. We start from

$$|W^\varepsilon v(t, \tau, x)|^2 = c^2\varepsilon^{-6/4} \iint e^{(i(u-s)\tau-(t-s)^2-(t-u)^2)/\varepsilon} v(s, x)\overline{v(u, x)}\, ds, du.$$

We want to integrate this integral with respect to $(t, \tau, x) \in \mathbb{R} \times \mathbb{R} \times \Omega$. In standard fashion, to deal with absolutely convergent integrals we use the Gauss summability method. More precisely, we introduce an additional term $e^{-\delta^2\tau^2}$ in the summand, and let δ go to 0. This gives a rigorous meaning to

$$\int_{\mathbb{R}} e^{i(u-s)\tau/\varepsilon}\, d\tau = 2\pi\varepsilon\delta_0(u - s), \tag{3.5}$$

where δ_0 is the Dirac function. The end of the proof is a matter of straightforward computations. □

To introduce the wave-packets transform of u^ε, we have to carefully extend the functions to $t \in \mathbb{R}$. Hereafter, a subscript zero indicates compact support. Let $\chi_\varepsilon \in C_0^\infty((0, T))$ be a family of functions such that $\chi_\varepsilon(t) = 1$ for $t \in [\varepsilon^{1/2}, T - \varepsilon^{1/2}]$ and $\|\varepsilon\partial_t\chi_\varepsilon\|_{L^\infty} \leqslant 2\varepsilon^{1/2}$. We set

$$\widetilde{u}^\varepsilon = \begin{pmatrix} \widetilde{q}^\varepsilon \\ \widetilde{v}^\varepsilon \end{pmatrix} = \chi_\varepsilon u^\varepsilon = \begin{pmatrix} \chi_\varepsilon p^\varepsilon \\ \chi_\varepsilon v^\varepsilon \end{pmatrix}.$$

Next, we choose extensions $\widetilde{S}^\varepsilon$ of S^ε, supported in $t \in [-1, T+1]$, uniformly bounded in $C_0(\mathbb{R}, H^s(\Omega))$, and converging to $\widetilde{S}$ in $C_0(\mathbb{R}, H^{s'}_{\mathrm{loc}}(\Omega))$. Note that, according to (3.4), $\widetilde{u}^\varepsilon$ satisfies

$$\varepsilon E_0(\widetilde{S}^\varepsilon)\partial_t\widetilde{u}^\varepsilon + L(\partial_x)\widetilde{u}^\varepsilon = \varepsilon\widetilde{f}^\varepsilon, \tag{3.6}$$

where $(\widetilde{f}^\varepsilon)_{\varepsilon>0}$ is a bounded family in $C_0(\mathbb{R}, H^{s-1}(\Omega))$. By (2.6), it is easily verified that $\varepsilon\partial_t\widetilde{u}^\varepsilon$ is uniformly bounded in $C_0(\mathbb{R}, H^{s-1}(\Omega))$.

Lemma 3.3. *Let $U^\varepsilon = W^\varepsilon\widetilde{u}^\varepsilon$. As ε tends to 0,*

$$F^\varepsilon := \big(i\tau E_0(\widetilde{S}^\varepsilon(t)) + L(\partial_x)\big)U^\varepsilon \to 0 \quad \text{in} \quad L^2(\mathbb{R}^2, H^1(\Omega)). \tag{3.7}$$

Proof. Since $[W^\varepsilon, L(\partial_x)] = 0$, it results from equation (3.6) that

$$F^\varepsilon - \varepsilon W^\varepsilon\widetilde{f}^\varepsilon = \big[E_0(\widetilde{S}^\varepsilon), W^\varepsilon\big](\varepsilon\partial_t)\widetilde{u}^\varepsilon + E_0(\widetilde{S}^\varepsilon)\big(i\tau W^\varepsilon\widetilde{u}^\varepsilon - W^\varepsilon(\varepsilon\partial_t\widetilde{u}^\varepsilon)\big). \tag{3.8}$$

Since $\varepsilon W^\varepsilon\widetilde{f}^\varepsilon$ obviously converges to 0 in $L^2(\mathbb{R}^2, H^1(\Omega))$, we aim to prove that the right-hand side of (3.8) converge to 0.

If $\varepsilon\partial_t v \in L^2(\mathbb{R}\times\Omega)$, then

$$W^\varepsilon(\varepsilon\partial_t v) - i\tau W^\varepsilon v = 2c\varepsilon^{-3/4}\int_{\mathbb{R}} e^{(i(t-s)\tau-(t-s)^2)/\varepsilon}(s-t)v(s,x)\,ds;$$

therefore, using (3.5) and $\big[W^\varepsilon, \partial_x\big] = 0$, we easily get

$$\|W^\varepsilon(\varepsilon\partial_t v) - i\tau W^\varepsilon v\|_{L^2(\mathbb{R}^2, H^1(\Omega))} \leqslant K\sqrt{\varepsilon}\,\|v\|_{L^2(\mathbb{R}, H^1(\Omega))}\,. \tag{3.9}$$

Now, let $a \in C_0^1(\mathbb{R}, H^s(\Omega))$; we have

$$aW^\varepsilon v - W^\varepsilon(av) = c\varepsilon^{-3/4}\int_{\mathbb{R}} e^{(i(t-s)\tau-(t-s)^2)/\varepsilon}\,(a(t,x) - a(s,x))\,v(s,x)\,ds.$$

Again, (3.5) implies that

$$\begin{aligned} &\|aW^\varepsilon v - W^\varepsilon(av)\|^2_{L^2(\mathbb{R}^2\times\Omega)} \\ &\qquad = c_2\varepsilon^{-1/2}\iiint e^{-2(t-s)^2/\varepsilon}\,|a(t,x) - a(s,x)|^2\,|v(s,x)|^2\;ds\,dt\,dx. \end{aligned} \tag{3.10}$$

Since $s > d/2 + 2$, the embedding of $H^{s-2}(\Omega)$ into $L^\infty(\Omega)$ implies that

$$\begin{aligned} &\|aW^\varepsilon v - W^\varepsilon(av)\|^2_{L^2(\mathbb{R}^2\times\Omega)} \\ &\qquad \leqslant c_2\varepsilon^{-1/2}\iint e^{-2(t-s)^2/\varepsilon}\,\|a(t) - a(s)\|^2_{H^{s-2}(\Omega)}\,\|v(s)\|^2_{L^2(\Omega)}\;dt\,ds. \end{aligned}$$

Hence, we easily obtain

$$\|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2\times\Omega)} \leqslant K\sqrt{\varepsilon}\,\|a\|_{C_0^1(\mathbb{R}, H^{s-2}(\Omega))}\,\|v\|_{L^2(\mathbb{R}, L^2(\Omega))}\,.$$

From the previous bound and $\big[W^\varepsilon, \partial_x\big] = 0$, we get

$$\|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2, H^1(\Omega))} \leqslant K\sqrt{\varepsilon}\,\|a\|_{C_0^1(\mathbb{R}, H^{s-1}(\Omega))}\,\|v\|_{L^2(\mathbb{R}, H^1(\Omega))}\,. \tag{3.11}$$

To complete the proof, recall that $E_0(\widetilde{S}^\varepsilon)$ is bounded in $C_0^1(\mathbb{R}, H^{s-1}(\Omega))$, $\widetilde{u}^\varepsilon$ is bounded in $L^2(\mathbb{R}, H^1(\Omega))$, and $(\varepsilon\partial_t)\widetilde{u}^\varepsilon$ is bounded in $L^2(\mathbb{R}, H^1(\Omega))$. From (3.8), (3.9) and (3.11), we infer that $(F^\varepsilon - \varepsilon W^\varepsilon\widetilde{f}^\varepsilon)$ converges to 0 in $L^2(\mathbb{R}^2, H^1(\Omega))$. □

Let us next introduce a series of notations so as to put into relief the relevant wave equation. First, we set

$$\begin{aligned} U^\varepsilon &= (\Psi^\varepsilon, m^\varepsilon) = (W^\varepsilon\widetilde{q}^\varepsilon, W^\varepsilon\widetilde{v}^\varepsilon) \in L^2(\mathbb{R}^2, H^s(\Omega)) \times L^2(\mathbb{R}^2, H^s(\Omega)^d) \\ F^\varepsilon &= (F_1^\varepsilon, F_2^\varepsilon) \in L^2(\mathbb{R}^2, H^1(\Omega)) \times L^2(\mathbb{R}^2, H^1(\Omega)^d). \end{aligned}$$

From the definition $F^\varepsilon := \big(i\tau E_0(S^\varepsilon(t)) + L(\partial_x)\big)U^\varepsilon$, we obtain

$$i\tau a_0(\widetilde{S}^\varepsilon(t))\Psi^\varepsilon + \operatorname{div}(m^\varepsilon) = F_1^\varepsilon, \tag{3.12}$$

$$i\tau r_0(\widetilde{S}^\varepsilon(t))m^\varepsilon + \nabla\Psi^\varepsilon = F_2^\varepsilon. \tag{3.13}$$

where, to shorten notation, we denote $a_0(S) := a(S,0)$, $r_0(S) := r(S,0)$. Set

$$P_\varepsilon(t,\tau,\nabla) := a_0\big(\widetilde{S}^\varepsilon(t)\big)\tau^2 + \operatorname{div}\Big(\frac{1}{r_0\big(\widetilde{S}^\varepsilon(t)\big)}\nabla\cdot\Big),$$

$$P_0(t,\tau,\partial_x) := a_0\big(\widetilde{S}(t)\big)\tau^2 + \operatorname{div}\Big(\frac{1}{r_0\big(\widetilde{S}(t)\big)}\nabla\cdot\Big).$$

From (3.12) and (3.13), we obtain

$$P_\varepsilon(t,\tau,\partial_x)\Psi^\varepsilon = -i\tau F_1^\varepsilon + i\ \operatorname{div}\Big(\frac{1}{r_0(\widetilde{S}^\varepsilon(t))}F_2^\varepsilon\Big). \tag{3.14}$$

The following equation is simple to derive but very important, because this where the boundary condition enters. Using $v^\varepsilon_{|\mathbb{R}\times\partial\Omega}\cdot\nu = 0$, we obtain that $m^\varepsilon\cdot\nu = 0$ on the boundary $\mathbb{R}\times\mathbb{R}\times\partial\Omega$. Taking the dot product of (3.13) with ν, we infer that

$$\partial_\nu\Psi^\varepsilon := \nabla\Psi^\varepsilon\cdot\nu = F_2^\varepsilon\cdot\nu \quad \text{on} \quad \mathbb{R}\times\mathbb{R}\times\partial\Omega, \tag{3.15}$$

which is meaningful since, as already seen, $F^\varepsilon \in L^2(\mathbb{R}^2, H^1(\Omega))$.

Finally it appears useful to introduce a new family in order to change the characteristic variety of the equation (3.14). We set

$$\Theta^\varepsilon := \Psi^\varepsilon - \Delta\Psi^\varepsilon = \Psi^\varepsilon - \sum_{j=1}^{d}\partial_{x_j}^2\Psi^\varepsilon.$$

Since $s \geqslant 2$, Θ^ε is bounded in $L^2(\mathbb{R}^2\times\Omega)$.

Conversely, we can use (3.15) to recover Ψ^ε from Θ^ε and F_2^ε. One has

$$\Psi^\varepsilon = (1-\Delta_N)^{-1}\Theta^\varepsilon + N(F_2^\varepsilon\cdot\nu),$$

where we used the following definitions: given $g \in L^2(\Omega)$, $\varphi \in H^{1/2}(\partial\Omega)$,

$$f = (1-\Delta_N)^{-1}g \text{ if and only if } (1-\Delta)f = g \text{ in } \Omega, \text{ and } \partial_\nu f = 0 \text{ on } \partial\Omega,$$
$$f = N(\varphi) \text{ if and only if } (1-\Delta)f = 0 \text{ in } \Omega, \text{ and } \partial_\nu f = \varphi \text{ on } \partial\Omega.$$

The operator $(1-\Delta_N)^{-1}$, respectively N, belongs to $\mathcal{L}(L^2(\Omega), H^2(\Omega))$, respectively $\mathcal{L}(H^{1/2}(\partial\Omega), H^2(\Omega))$.

3.3. **Defect measures.** We denote by $\mathcal{L}$, $\mathcal{K}$, and $\mathcal{L}^1$, respectively, the space of bounded, compact, and trace class operators, respectively, in $L^2(\Omega)$, and we denote by $\mathcal{K}_+$, respectively $\mathcal{L}^1_+$, the subspace of nonnegative self-adjoint operators in $\mathcal{K}$, respectively $\mathcal{L}^1$ (we refer to [13, Chapter Ten, Section 1.3] for details and definitions). If $A \in \mathcal{L}^1$, we denote by $\operatorname{tr}(A)$ the trace of A. Recall that the dual space of $(\mathcal{K}, \|\cdot\|_{\mathcal{L}})$ is $(\mathcal{L}^1, \|\cdot\|_{\mathcal{L}^1})$ with the duality bracket $\operatorname{tr}(AB)$. If $A \in C_0(\mathbb{R}^2, \mathcal{K})$, then A acts on $\Theta \in L^2(\mathbb{R}^2\times\Omega)$ following $(A\Theta)(t,\tau,x) = (A\Theta(t,\tau,\cdot))(x)$.

Proposition 3.4. *Let $\Theta^\varepsilon = \Psi^\varepsilon - \Delta\Psi^\varepsilon$. There are a subsequence Θ^{ε_n}, a finite nonnegative Borel measure μ on $\mathbb{R}^2$, and $M \in L^1(\mathbb{R}^2, \mathcal{L}^1_+, \mu)$, such that for all $A \in C_0(\mathbb{R}^2, \mathcal{K})$,*

(3.16)
$$\int_{\mathbb{R}^2} \big(A(t,\tau)\Theta^{\varepsilon_n}(t,\tau), \Theta^{\varepsilon_n}(t,\tau)\big)_0\, dt\, d\tau \to \int \operatorname{tr}\big(A(t,\tau)M(t,\tau)\big)\mu(dt, d\tau),$$

and

$$\Big(a_0\big(\widetilde{S}(t)\big)\tau^2 + \operatorname{div}\big(\frac{1}{r_0(\widetilde{S}(t))}\nabla\cdot\big)\Big)(1-\Delta_N)^{-1}M(t,\tau) = 0 \quad \mu\text{-a.e.} \tag{3.17}$$

Remark 3.5. If one considers the defect measure (M', μ') of $(\Psi^\varepsilon)_\varepsilon$, and proves the additional regularity $M'(t,\tau) \in \mathcal{L}(L^2, H^2)$, then arguing that $F_\varepsilon \to 0$ in $L^2(\mathbb{R}^2 \times \Omega)$, one infers $(a_0(\widetilde{S}(t))\tau^2 + \operatorname{div}(1/r_0(\widetilde{S}(t))\nabla))M'(t,\tau) = 0$, μ'-almost-everywhere . Yet, it appears easier to introduce Θ, so (M,μ), and to prove the convergence of F_ε to 0 in $L^2(\mathbb{R}^2, H^1(\Omega))$, so as to infer the same conclusion, namely (3.17).

Proof. For the proof of the existence part of the proposition and (3.16), we refer the reader to [6] or [18, Lemma 4.3]. We only prove (3.17).

Let $\varphi \in C_0(\mathbb{R}^2)$ and $K \in \mathcal{K}$. Since $(\widetilde{S}^\varepsilon)_\varepsilon$ is a bounded family in $C_0(H^s)$, the rule (2.8) and the convergence of F_ε to 0 proved in (3.7) imply that

$$\varphi P_\varepsilon(t,\tau,\partial_x)N(F_2^\varepsilon \cdot \nu) \underset{\varepsilon\to 0}{\longrightarrow} 0, \text{ in } L^2(\mathbb{R}^2 \times \Omega). \tag{3.18}$$

Moreover, the local strong convergence (3.2) of $\widetilde{S}^\varepsilon$ implies that

$$\varphi K\Big(P_\varepsilon(t,\tau,\partial_x)(1-\Delta_N)^{-1}\Theta^\varepsilon - P_0(t,\tau,\partial_x)(1-\Delta_N)^{-1}\Theta^\varepsilon\Big) \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.19}$$

Furthermore, the right-hand side in identity (3.14) multiplied by φ converges to 0 in $L^2(\mathbb{R}^2 \times \Omega)$ according to (3.7); thus, we infer

$$\varphi P_\varepsilon(t,\tau,\partial_x)\Psi^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0, \text{ in } L^2(\mathbb{R}^2 \times \Omega). \tag{3.20}$$

Recall that $\Psi^\varepsilon = (1-\Delta_N)^{-1}\Theta^\varepsilon + N(F_2^\varepsilon \cdot \nu)$. Combining this equality with (3.18), (3.19), and (3.20) we conclude that

$$\varphi K P_0(t,\tau,\partial_x)(1-\Delta_N)^{-1}\Theta^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0, \text{ in } L^2(\mathbb{R}^2 \times \Omega). \tag{3.21}$$

Using that $P_0(t,\tau,\partial_x)(1-\Delta_N)^{-1} \in C^0(\mathbb{R}^2, L^2(\Omega))$, it results from (3.21) and (3.16) that, for all $\varphi \in C_0(\mathbb{R}^2)$ and $K \in \mathcal{K}$,

$$\int_{\mathbb{R}^2} \operatorname{tr}\big(\varphi(t,\tau)KP_0(t,\tau,\partial_x)(1-\Delta_N)^{-1}M(t,\tau)\big)\,\mu(dt,d\tau) = 0.$$

Since φ and K are arbitrary, we obtain (3.17). □

Assume that the kernel of $(a_0(\widetilde{S}(t))\tau^2 + \operatorname{div}(1/r_0(\widetilde{S}(t))\nabla))(1-\Delta_N)^{-1}$ in $L^2(\Omega)$ is reduced to $\{0\}$. Then the identity (3.17) and (3.16) imply that for all $\varphi \in C_0(\mathbb{R}^2)$ and $K \in \mathcal{K}_+$,

$$\int_{\mathbb{R}^2} \varphi(t,\tau)\big(K\Theta^{\varepsilon_n}(t,\tau), K\Theta^{\varepsilon_n}(t,\tau)\big)_0\, dt\, d\tau \underset{n\to\infty}{\longrightarrow} 0. \tag{3.22}$$

We want to show that this convergence holds for $\varphi(t,\tau) = 1$. The idea is that, on the one hand, $\widetilde{q}^\varepsilon$ is compactly supported in time; so is Θ^ε in the sense given below by (3.23). And, on the other hand, $\widetilde{q}^\varepsilon$ oscillates in time at most at frequencies $O(\varepsilon^{-1})$.

First, let $\zeta \in C_0^\infty((-1, T+1))$ be such that $\zeta(t) = 1$ for $t \in [0,T]$. In view of (3.10), one infers that

$$\|\zeta\Theta^\varepsilon - W^\varepsilon\big((1-\Delta)\zeta\widetilde{q}^\varepsilon\big)\|_{L^2(\mathbb{R}^2\times\Omega)} \leqslant C\sqrt{\varepsilon}\|\partial_t\zeta\|_{L^\infty(\mathbb{R})}\|(1-\Delta)\widetilde{q}^\varepsilon\|_{L^2(\mathbb{R}\times\Omega)}.$$

Furthermore, from the definition of $\widetilde{q}^\varepsilon$ we have $\zeta\widetilde{q}^\varepsilon = \widetilde{q}^\varepsilon$. Therefore, the previous inequality implies that

$$\int_{\mathbb{R}^2} \big(1-\zeta(t)\big)^2 \|\Theta^\varepsilon(t,\tau)\|_{L^2(\Omega)}^2 \, dt\, d\tau \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.23}$$

Since $(\varepsilon\partial_t\widetilde{q}^\varepsilon)_\varepsilon$ is a bounded family in $C_0(\mathbb{R}, H^{s-1}(\Omega)$, it follows from (3.9) that $(\tau\Theta^\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^2\times\Omega)$. Thus with (3.22) and (3.23) we conclude that for all $K \in \mathcal{K}_+$,

$$\int_{\mathbb{R}^2} \|K\Theta^{\varepsilon_n}(t,\tau)\|_{L^2(\Omega)}^2 \, dt\, d\tau \underset{n\to\infty}{\longrightarrow} 0. \tag{3.24}$$

Recall that, by the definition of $\Theta^\varepsilon = (1-\Delta)\Psi^\varepsilon$, W^ε is an isometry from $L^2(\mathbb{R}\times\Omega)$ to $L^2(\mathbb{R}^2\times\Omega)$, and W^ε commutes with $K(1-\Delta)$. So (3.24) implies that

$$\forall K \in \mathcal{K}_+, \quad \|K(1-\Delta)\widetilde{q}^{\varepsilon_n}\|_{L^2(\mathbb{R}\times\Omega)} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.25}$$

Given that $\widetilde{q}^\varepsilon$ is bounded in $L^2(\mathbb{R}, H^s(\Omega))$, the convergence (3.25) implies the convergence of $\widetilde{q}^{\varepsilon_n}$ to 0 in $L^2(\mathbb{R}, H^{s'}_{\mathrm{loc}}(\Omega))$ for all $s' < s$. Since the limit is 0 the convergence holds for the given family $\widetilde{q}^\varepsilon$. We end up with

$$p^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ in } L^2([0,T], H^{s'}_{\mathrm{loc}}(\Omega)), \text{ for all } s' < s.$$

Arguments similar to those above show that, from (3.12), the previous convergence of $\widetilde{q}^\varepsilon$, and (3.7), we have

$$\operatorname{div} v^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ in } L^2([0,T], H^{s'}_{\mathrm{loc}}(\Omega)), \text{ for all } s' < s-1. \tag{3.26}$$

Hence, we have seen that in order to prove Proposition 3.1 it is sufficient to prove that $H_0(t,\tau) = \{0\}$ for all $(t,\tau) \in \mathbb{R}^2$, where

$$H_0(t,\tau) = \left\{ u \in L^2 \,:\, \Big(a_0\big(\widetilde{S}(t)\big)\tau^2 + \operatorname{div}\big(\frac{1}{r_0(\widetilde{S}(t))}\nabla\cdot\big)\Big)(1-\Delta_N)^{-1}u = 0 \right\}.$$

Since the speed of propagation $b(S^\varepsilon, u^\varepsilon)$ is uniformly bounded in $H^s \hookrightarrow C^1$, the decay assumption (1.7) is propagated by the transport equation $\partial_t S^\varepsilon + b(S^\varepsilon, u^\varepsilon)\cdot\nabla S^\varepsilon = 0$. Thus the operator $\operatorname{div}((1/r_0(\widetilde{S}(t)))\nabla\cdot)$ is a perturbation of the Laplacean at infinity. Since Ω is a connected neighborhood of ∞, using the unique continuation principle for a second-order equation ([8, Theorem 17.2.8]), the proof thus reduces to establishing that if u belongs to H_0, then u vanishes in the sense that $|x|^n\, u \in L^2(\Omega)$ and $|x|^n\, \nabla u \in L^2(\Omega)$ for all $n \in \mathbb{N}$. The proof of this fact proceeds as in the whole-space case (see [18, Lemma 5.1]). Then we conclude $H_0(t,\tau) = \{0\}$ for all $(t,\tau) \in \mathbb{R}^2$. This completes the proof of Proposition 3.1.

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