About global existence and asymptotic behavior for two dimensional gravity water waves

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(joint work with Jean-Marc Delort)

Abstract

The main result of this talk is a global existence theorem for the water waves equation with smooth, small, and decaying at infinity Cauchy data. We obtain moreover an asymptotic description in physical coordinates of the solution, which shows that modified scattering holds.

The proof is based on a bootstrap argument involving L^2 and L^∞ estimates. The L^2 bounds are proved in the paper [5]. They rely on a normal forms paradifferential method allowing one to obtain energy estimates on the Eulerian formulation of the water waves equation. The uniform bounds, and the proof of the global existence result, are presented in [4]. These uniform bounds are proved interpreting the equation in a semi-classical way, and combining Klainerman vector fields with the description of the solution in terms of semi-classical lagrangian distributions.

1 Main result

We consider the initial value problem for the motion of a two-dimensional incompressible fluid under the influence of gravity. At time t, the fluid domain, denoted by $\Omega(t)$, has a free boundary described by the equation $y = \eta(t,x)$, so that

$$\Omega(t) = \left\{ (x,y) \in \mathbb{R}^2 \,;\, y < \eta(t,x) \right\}.$$

The equations by which the motion is to be determined are well-known. Firstly, the velocity field $v\colon \Omega \to \mathbb{R}^2$ is assumed to be irrotational and to satisfy the incompressible Euler equations. It follows that $v = \nabla_{x,y}\phi$ for some velocity potential $\phi\colon \Omega \to \mathbb{R}$ satisfying

$$\Delta_{x,y}\phi = 0, \quad \partial_t \phi + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0, \tag{1.1}$$

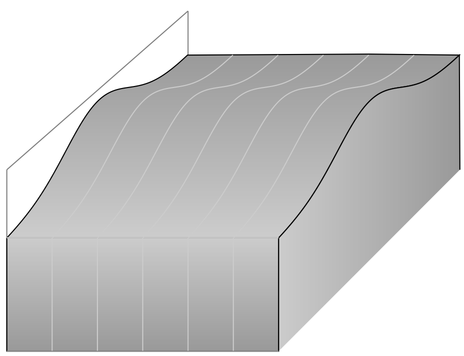
where $g > 0$ is the acceleration of gravity, P is the pressure term, $\nabla_{x,y} = (\partial_x, \partial_y)$ and $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. Hereafter, the units of length and time are chosen so that $g = 1$.

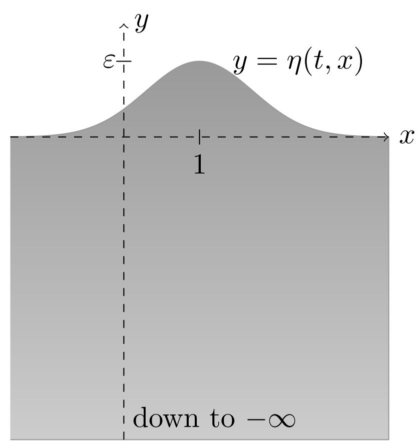
The problem is then given by two boundary conditions on the free surface:

$$\begin{cases} \partial_t \eta = \sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi & \text{on } \partial\Omega, \\ P = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where ∂_n is the outward normal derivative of Ω, so that $\sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi = \partial_y\phi - (\partial_x\eta)\partial_x\phi$. The former condition is a kinematic condition (which states that the free surface moves with the fluid). The condition $P = 0$ is a dynamic condition that expresses a balance of forces across the free surface.

Below are two pictures of the two-dimensional waves we consider. It is important to emphasize that these are small amplitude waves (of size $\varepsilon \ll 1$), localized in x (by contrast with periodic solutions) and that the fluid domain is infinitely deep.





Two different approaches were used in the analysis of the water waves equations: the Lagrangean formulation with a more geometrical point of view and the Eulerian formulation in relation with microlocal analysis. Our analysis is entirely based on the Eulerian formulation of the water waves equations: we shall work on the so-called Craig–Sulem–Zakharov system which we introduce now. Following Zakharov [48] and Craig and Sulem [19], we work with the trace of ϕ at the free boundary

$$\psi(t,x) := \phi(t,x,\eta(t,x)).$$

One also introduces the Dirichlet-Neumann operator $G(\eta)$ that relates ψ to the normal derivative $\partial_n\phi$ by

$$(G(\eta)\psi)(t,x) = \sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta(t,x)}.$$

Then (η,ψ) solves (see [19]) the so-called Craig–Sulem–Zakharov system

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + \eta + \dfrac{1}{2}(\partial_x\psi)^2 - \dfrac{1}{2(1+(\partial_x\eta)^2)}\big(G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)\big)^2 = 0. \end{cases} \tag{1.3}$$

In [3], it is proved that starting from a classical solution (η,ψ) of (1.3) (such that (η,ψ) belongs to $C^0([0,T];H^s(\mathbb{R}))$ for some $T>0$ and $s>3/2$), then one can define ϕ and P such that (1.1) and (1.2) hold. Thus it is sufficient to solve the Craig–Sulem–Zakharov formulation of the water waves equations.

Our main result is stated in full generality in [4]. A simplified statement asserts the following:

Main result. *For small enough initial data of size $\varepsilon \ll 1$, sufficiently decaying at infinity, the Cauchy problem for (1.3) is globally in time well-posed. Moreover, $u = |D_x|^{\frac{1}{2}}\psi + i\eta$ admits the following asymptotic expansion as t goes to $+\infty$: There is a continuous function $\underline{\alpha}\colon \mathbb{R} \to \mathbb{C}$, depending of ε but bounded uniformly in ε, such that*

$$u(t,x) = \frac{\varepsilon}{\sqrt{t}}\underline{\alpha}\Big(\frac{x}{t}\Big)\exp\Big(\frac{it}{4|x/t|} + \frac{i\varepsilon^2}{64}\frac{|\underline{\alpha}(x/t)|^2}{|x/t|^5}\log(t)\Big) + \varepsilon t^{-\frac{1}{2}-\kappa}\rho(t,x) \tag{1.4}$$

where κ is some positive number and ρ is a function uniformly bounded for $t \geq 1$, $\varepsilon \in]0, \varepsilon_0]$.

As an example of small enough initial data sufficiently decaying at infinity, consider

$$\eta|_{t=1} = \varepsilon\eta_0, \quad \psi|_{t=1} = \varepsilon\psi_0, \tag{1.5}$$

with η_0, ψ_0 in $C_0^\infty(\mathbb{R})$. Then there exists a unique solution (η, ψ) in $C^\infty([1, +\infty[; H^\infty(\mathbb{R}))$ of (1.3). In fact, one may allow ψ to be merely in some homogeneous Sobolev space.

The first term in the oscillatory exponential which appears in (1.4) is the phase of oscillation of the solutions of the linearized equation (this computation was performed by Cauchy in his memoir [11]). The second term in this oscillatory exponential, that is the logarithmic correction, shows that the solutions do not scatter (only a modified scattering holds). That the behavior of the solutions when time goes to infinity is not the same as the behavior of the solutions of the linearized equation comes from the fact that we are considering a critical problem. This is the main obstacle to prove global well-posedness.

We discuss in the end of this introduction some related previous works. There is quite a lot of results which are known and we shall recall only some of them. The study of the Cauchy problem for the water waves equations began with the pioneering work of Nalimov [37] who proved that the Cauchy problem is well-posed locally in time, in the framework of Sobolev spaces, under an additional smallness assumption on the data. We also refer to Shinbrot [41], Yoshihara [47] and Craig [15]. Without smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Wu for the case without surface tension (see [43, 44]) and by Beyer-Günther in [10] in the case with surface tension. Several extensions of their results have been obtained and we refer to Córdoba, Córdoba and Gancedo [13], Coutand-Shkoller [14], Lannes [32, 33, 34], Linblad [35], Masmoudi-Rousset [36] and Shatah-Zeng [39, 40] for recent results on the Cauchy problem for the gravity water waves equation.

Given $\varepsilon \geq 0$, consider the solutions to the water waves system (1.3) with initial data satisfying (1.5). In her breakthrough result [45], Wu proved that the maximal time of existence T_ε is larger or equal to $e^{c/\varepsilon}$ for $d = 1$. Then Germain–Masmoudi–Shatah [22] and Wu [46] have shown that the Cauchy problem for three-dimensional waves is globally in time well-posed for ε small enough (with linear scattering in Germain-Masmoudi-Shatah and no assumption about the decay to 0 at spatial infinity of $|D_x|^{\frac{1}{2}}\psi$ in Wu). Germain–Masmoudi–Shatah recently proved global existence for pure capillary waves in dimension $d = 2$ in [21].

It is worth recalling that the only known coercive quantity for (1.3) is the hamiltonian, which reads (see [48, 19])

$$\mathcal{H} = \frac{1}{2}\int \eta^2\, dx + \frac{1}{2}\int \psi G(\eta)\psi\, dx. \tag{1.6}$$

We refer to the paper by Benjamin and Olver [9] for considerations on the conservation laws of the water waves equations. One can compare the hamiltonian with the critical threshold given by the scaling invariance of the equations. Recall (see [9, 12]) that if (η, ψ) solves (1.3), then the functions $(\eta_\lambda, \psi_\lambda)$ defined by

$$\eta_\lambda(t,x) = \lambda^{-2}\eta\left(\lambda t, \lambda^2 x\right), \quad \psi_\lambda(t,x) = \lambda^{-3}\psi\left(\lambda t, \lambda^2 x\right) \qquad (\lambda > 0) \tag{1.7}$$

are also solutions of (1.3). In particular, one notices that the critical space for the scaling corresponds to η_0 in $\dot{H}^{3/2}(\mathbb{R})$. Since the hamiltonian (1.6) only controls the $L^2(\mathbb{R})$-norm of η, one sees that the hamiltonian is highly supercritical for the water waves equation and hence one cannot use it directly to prove global well-posedness of the Cauchy problem.

Our approach follows a variant of the vector fields method introduced by Klainerman in [30, 29] to study the wave and Klein-Gordon equations (see the book by Hörmander in [24] or the Bourbaki seminar by Lannes [31] for an introduction to this method). More precisely, we shall follow the approach introduced in [20] for the analysis of the Klein-Gordon equation in space dimension one, to cope with the fact that solutions of the equation do not scatter. Results for one dimensional Schrödinger equations, that display the same non scattering behavior, have been proved by Hayashi and Naumkin [23], and global existence for a simplified model of the water waves equation studied by Ionescu and Pusateri in [26].

Our proof of global existence is entirely based on the analysis of the Eulerian formulation of the water waves equations by means of microlocal analysis. In this direction, it is influenced by the papers by Lannes [32] and Iooss-Plotnikov [28]. More precisely, to prove Sobolev energy inequalities, we follow the paradifferential analysis introduced in [6] and further developed in [2, 1].

Finally, let us mention that Ionescu and Pusateri [25] independently obtained a global existence result very similar to the one we get here. The main difference is that they assume less decay on the initial data, and get asymptotics not for the solution in physical space, with control of the remainders in L^∞, but for its space Fourier transform, with remainders in L^2. These asymptotics, as well as ours, show that solutions do not scatter. To get asymptotics with remainders estimated in L^∞, we shall commute iterated vector field $Z = t\partial_t + 2x\partial_x$ to the water waves equations. This introduces several new difficulties and requires that the initial data be sufficiently decaying at infinity.

2 General strategy of proof

The water waves equations are fully nonlinear and contains quadratic terms. However, to explain the strategy of the proof we consider as a (much) simplified model an equation of the form

$$(D_t - P(D_x))u = N(u) \tag{2.1}$$
$$u|_{t=1} = \varepsilon u_0,$$

where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $P(\xi)$ is a real valued symbol (for the linearized water waves equation, $P(\xi)$ would be $|\xi|^{1/2}$), and $N(u)$ is a nonlinearity vanishing at order at least two at zero.

The general framework we use is the one of Klainerman vector fields. Recall that a Klainerman vector field for $D_t - P(D_x)$ is a space-time vector field Z such that $[Z, D_t - P(D_x)]$ is zero (or a multiple $\lambda(D_t - P(D_x))$ of $D_t - P(D_x)$). For the water waves system, Z will be $t\partial_t + 2x\partial_x$ or D_x. In that way, $(D_t - P(D_x))Z^k u = Z^k N(u)$ for any k (or $(D_t - P(D_x))(Z+\lambda)^k u = Z^k N(u)$), and since $P(\xi)$ is real valued, an easy energy inequality shows that

$$\|Z^k u(t,\cdot)\|_{L^2} \leq \|Z^k u(1,\cdot)\|_{L^2} + \int_1^t \|Z^k N(u)(\tau,\cdot)\|_{L^2}\, d\tau, \tag{2.2}$$

for any $t \geq 1$. Assume first that $N(u)$ is cubic, so that

$$\|Z^k N(u)\|_{L^2} \leq C\|u\|_{L^\infty}^2 \|Z^k u\|_{L^2} + C \sum_{\substack{k_1+k_2+k_3 \leq k \\ k_1,k_2 \leq k_3 \leq k-1}} \|Z^{k_1} u\|_{L^\infty} \|Z^{k_2} u\|_{L^\infty} \|Z^{k_3} u\|_{L^2}. \tag{2.3}$$

Assuming an a priori L^∞ bound, one can deduce from (2.2) an L^2 estimate. More precisely, introduce the following property, where s is a large even integer:

$$\text{For } t \text{ in some interval } [1, T[,\ \|u(t,\cdot)\|_{L^\infty} = O(\varepsilon/\sqrt{t}) \tag{A}$$
$$\text{and for } k = 0, \dots, s/2,\ \|Z^k u(t,\cdot)\|_{L^\infty} = O\big(\varepsilon t^{-\frac{1}{2}+\tilde{\delta}'_k}\big),$$

where $\tilde{\delta}'_k$ are small positive numbers. Plugging these a priori bounds in (2.2), (2.3), we get

$$\begin{aligned} \|Z^k u(t,\cdot)\|_{L^2} \leq \|Z^k u(1,\cdot)\|_{L^2} &+ C\varepsilon^2 \int_1^t \|Z^k u(\tau,\cdot)\|_{L^2} \frac{d\tau}{\tau} \\ &+ C\varepsilon^2 \int_1^t \|Z^{k-1} u(\tau,\cdot)\|_{L^2} \tau^{2\tilde{\delta}'_{k/2}-1}\, d\tau. \end{aligned} \tag{2.4}$$

Gronwall inequality implies then that

$$\|Z^k u(t,\cdot)\|_{L^2} = O(\varepsilon t^{\delta_k}),\ k \leq s, \tag{B}$$

for some small $\delta_k > 0$ ($\delta_k > C\varepsilon^2$ and $\delta_k > 2\tilde{\delta}'_{k/2}$).

The proof of global existence is done classically using a bootstrap argument allowing one to to show that if (A) and (B) are assumed to hold on some interval, they actually hold on the same interval with smaller constants in the estimates. We have outlined above the way of obtaining (B), assuming (A) for a solution of the model equation (2.1). In the next section, we shall explain the new difficulties that have to be solved to prove (B) for the water waves equation. In the last two sections, we shall explain the way of obtaining (A), assuming (B).

3 Sobolev estimates

The proof of a long time energy inequality of the form (B) for the water waves system (1.3) faces two serious obstacles. The first obstacle is an apparent loss of derivatives in energy inequalities. This difficulty already arises for local existence results, and was solved initially by Nalimov [37] and Wu [43, 44]. For long time existence problems, Wu [46] uses arguments combining the Eulerian and Lagrangian formulations of the system. Our approach in [5] is purely Eulerian.

We begin by explaining the idea on the model obtained from (1.3) paralinearizing the equations and keeping only the quadratic terms. If we denote $u = \begin{bmatrix} \eta \\ |D_x|^{1/2}\psi \end{bmatrix}$, such a model may be written as

$$\partial_t u = T_A u$$

where T_A is the paradifferential operator with symbol A, and where $A(u, x, \xi)$ is a matrix of symbols $A(u, x, \xi) = A_0(u, x, \xi) + A_1(u, x, \xi)$, with

$$A_0(u, x, \xi) = \begin{bmatrix} -i(\partial_x \psi)\xi & |\xi|^{1/2} \\ -|\xi|^{1/2} & -i(\partial_x\psi)\xi \end{bmatrix},\ A_1(u, x, \xi) = (|D_x|\psi)\,|\xi| \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Because of the A_1 contribution, which is self-adjoint, the eigenvalues of $A(u, x, \xi)$ are not purely imaginary. For large $|\xi|$, there is one eigenvalue with positive real part, which shows that one cannot expect for the solution of $\partial_t u = T_A u$ energy inequalities without derivative losses (see also the discussions in Craig [16, Section 4] or Lannes [32, Section 4.1]). A way to circumvent this difficulty is well known, and consists in using the "good unknown" of Alinhac [7]. For our quadratic model, this just means introducing as a new unknown $\tilde{u} = \begin{bmatrix} \eta \\ |D_x|^{1/2}\omega \end{bmatrix}$, where $\omega = \psi - T_{|D_x|\psi}\eta$ is the (quadratic approximation of the) good unknown. In that way, ignoring again remainders and terms which are at least cubic, one gets for $\tilde{u}$ an evolution equation $\partial_t \tilde{u} = T_{A_0}\tilde{u}$. Since A_0 is anti-self-adjoint, one gets L^2 or Sobolev energy inequalities for $\tilde{u}$. In particular, if for some s, $\||D_x|^{1/2}\omega\|_{H^s} + \|\eta\|_{H^s}$ is under control, and if one has also an auxiliary bound for $\||D_x|\,\psi\|_{L^\infty}$, one gets an estimate for $\||D_x|^{1/2}\psi\|_{H^{s-1/2}} + \|\eta\|_{H^s}$.

For the real water waves problem, one cannot discard cubic or higher order contributions to the nonlinearity, as we did in the above quadratic approximation. One has to express the equations in terms of a new unknown $\tilde{u} = \begin{bmatrix} \eta \\ |D_x|^{1/2}\omega \end{bmatrix}$ where ω, the good unknown, is given by $\omega = \psi - T_B\eta$, where B is is expressed in terms of the potential ϕ defined in (1.1) by $B = (\partial_y\phi)_{y=\eta(t,x)}$, the vertical derivative of ϕ at the boundary. If one denotes analogously by V the horizontal derivative $(\partial_x\phi)|_{y=\eta(t,x)}$, one gets explicit expressions for these quantities in terms of (ψ, η)

$$B = \frac{G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)}{1 + (\partial_x\eta)^2},\ V = \partial_x\psi - B\partial_x\eta.$$

Following the analysis in [1, 2, 6], one may derive (see [5]) an expression for $G(\eta)\psi$ in terms of ω,

$$G(\eta)\psi = |D_x|\omega - \partial_x(T_V\eta) + F(\eta)\psi,$$

where $F(\eta)\psi$ is a quadratic *smoothing term*, that belongs to $H^{s+\gamma-4}$ if η is in $C^\gamma \cap H^s$ and $|D_x|^{1/2}\psi$ belongs to $C^{\gamma-1/2} \cap H^{s-1/2}$. This gives a quite explicit expression for the main contributions to $G(\eta)\psi$. Moreover, one may prove as well tame estimates, that complement similar results due to Craig, Schanz and Sulem (see [18] and [42, Chapter 11], and [8, 28]), and establish bounds for the approximation of $G(\eta)\psi$ (resp. $F(\eta)\psi$) by its Taylor expansion at order two $G_{\leq 2}(\eta)\psi$ (resp. $F_{\leq 2}(\eta)\psi$). Putting together such estimates, one derives an evolution equation for $\tilde{u}$ and proves energy inequalities similar to the ones outlined above in the case of the quadratic approximation.

More precisely, we prove (see [5]) that

$$U = \begin{pmatrix} T_{\sqrt{a}}\eta \\ |D_x|^{\frac{1}{2}}\left(\psi - T_B\eta\right)\end{pmatrix} \quad \text{with} \quad \begin{cases} a = -\partial_n P|_{y=\eta} \\ B = \partial_y \phi|_{y=\eta}\end{cases}$$

satisfies

$$\partial_t U + DU + Q(U) + S(U) + C(U) = G, \tag{3.1}$$

where $D = \begin{pmatrix} 0 & -|D_x|^{\frac{1}{2}} \\ |D_x|^{\frac{1}{2}} & 0 \end{pmatrix} \equiv i\,|D_x|^{\frac{1}{2}}$ and

$$\begin{array}{lll}
\|Q(U)\|_{H^{s-1}} \leq K\,\|U\|_{C^\rho}\,\|U\|_{H^s}\,, & Q(U) \text{ is of order } 1, & \text{quadratic} \\
\|S(U)\|_{H^{s+1}} \leq K\,\|U\|_{C^\rho}\,\|U\|_{H^s}\,, & S(U) \text{ is of order } -1, & \text{quadratic} \\
\|C(U)\|_{H^{s-1}} \leq C(\|U\|_{C^\rho})\,\|U\|_{C^\rho}^2\,\|U\|_{H^s}\,, & C(U) \text{ is of order } 1, & \text{cubic} \\
\|G\|_{H^s} \leq C(\|U\|_{C^\rho})\,\|U\|_{C^\rho}^2\,\|U\|_{H^s}\,, & G \text{ is of order } 0, & \text{cubic}
\end{array}$$

for some fixed $\rho > 0$ and s large enough (these are tame estimates).

• Paradifferential normal forms method

In the model equation (2.1) discussed above, we considered a cubic nonlinearity: this played an essential role to make appear in the first integral in the right hand side of (2.4) the almost integrable factor $1/\tau$. For a quadratic nonlinearity, we would have had instead a $1/\sqrt{\tau}$-factor, which would have given in (B), through Gronwall, a $O(e^{\varepsilon\sqrt{t}})$-bound, instead of $O(\varepsilon t^{\delta_k})$. The way to overcome such a difficulty is well known since the work of Shatah [38] devoted to the non-linear Klein-Gordon equation: it is to use a normal forms method to eliminate the quadratic part of the nonlinearity.

In practice, one looks for a local diffeomorphism at 0 in H^s, for s large enough, so that the Sobolev energy inequality written for the equation obtained by conjugation by this diffeomorphim be of the form (2.4). Nonlinear changes of unknowns, reducing the water waves system to a cubic equation, have been known for quite a time (see Craig [17] or Iooss and Plotnikov [27, Lemma 1]). However, these transformations were losing derivatives, as a consequence of the quasi-linear character of the problem.

The idea is to seek $\Phi = U + E(U,U)$ such that E is bilinear and chosen in such a way that the equation on Φ is cubic, of the form $\partial_t \Phi + D\Phi = N_{(\geq 3)}(\Phi)$. Since

$$\partial_t \Phi + D\Phi = DE(U,U) - Q(U) - S(U) - E(DU,U) - E(U,DU) \quad + \text{cubic}$$

it is thus tempting to seek $E = E_1 + E_2$ such that

$$S(U) + E_1(DU,U) + E_1(U,DU) = DE_1(U,U)$$
$$Q(U) + E_2(DU,U) + E_2(U,DU) = DE_2(U,U).$$

One finds bilinear Fourier multipliers with (matrix valued) symbols

$$M_1(\xi_1,\xi_2) = \big(\cdots\big)\frac{\xi_1}{|\xi_1|}, \qquad M_2(\xi_1,\xi_2) = \big(\cdots\big)\xi_2.$$

The problem presents itself: ξ_2 (resp. $\xi_1/|\xi_1|$) is the symbol of the D_x (resp. the Hilbert transform), which is not bounded on L^2 (resp. L^∞).

Nevertheless, one can construct a bona fide change of unknown, without derivatives losses, if one notices that it is not necessary to eliminate the whole quadratic part of the nonlinearity, but only the part of it that would bring non zero contributions in a Sobolev energy inequality. It is proved in [5] that there is a bilinear mapping $(U_1,U_2) \mapsto E(U_1,U_2)$ such that $U_2 \mapsto E(U_1,U_2)$ is of order 0 (which means that it is bounded from H^μ to H^μ) and

$$B(U) := E(DU,U) + E(U,DU) - DE(U,U)$$

satisfies

$$\operatorname{Re}\big\langle\, Q(U) + S(U) + B(U)\,,\, U\,\big\rangle_{H^s\times H^s} = 0.$$

Moreover, as a byproduct of the paradifferential analysis, one can prove that $E(U_1,U_2)$ depends tamely on U_1, in a sense which is made precise in the following proposition.

Proposition. *There exist $\gamma > 0$ and a bounded bilinear mapping $(v,f) \mapsto E(v,f)$ satisfying*

$$\|E(v,f)\|_{H^\mu} \leq K\,\|v\|_{C^3}\,\|f\|_{H^\mu} \qquad (\forall \mu \geq -1)$$

and such that, for any s large enough, $\Phi = U + E(U,U)$ satisfies

$$\partial_t \Phi + D\Phi + L(U)\Phi + C(U)\Phi = \Gamma$$

where the operators D and $C(u)$ are as in (3.1), the source term satisfies

$$\|\Gamma\|_{H^s} \leq C(\|U\|_{C^\gamma})\,\|U\|_{C^\gamma}^2\,\|\Phi\|_{H^s}$$

and the quadratic term $L(U)\Phi$ satisfies

$$\operatorname{Re}\langle L(U)\Phi, \Phi\rangle_{H^s\times H^s} = 0 \tag{3.2}$$

where $\langle\cdot,\cdot\rangle_{H^s\times H^s}$ denotes the H^s-scalar product.

The key point is that the quadratic term $L(U)\Phi$ do not contribute to the energy estimate in view of (3.2). As a corollary, we obtain that there exists $\gamma > 0$ such that, for any $s \geq \gamma + 1/2$, if $N_\gamma(t) = \|\eta(t,\cdot)\|_{C^\gamma} + \left\| |D_x|^{\frac{1}{2}} \psi(t,\cdot) \right\|_{C^{\gamma-\frac{1}{2}}}$ is small enough, then the H^s-Sobolev energy M_s defined by $M_s(t) = \|U + E(U,U)\|_{H^s}^2$ satisfies

$$M_s(t) \sim \|\eta(t,\cdot)\|_{H^s(\mathbb{R})}^2 + \left\| |D_x|^{\frac{1}{2}} \psi \right\|_{H^{s-\frac{1}{2}}(\mathbb{R})}^2 + \left\| (\nabla_{x,y}\phi)|_{y=\eta}(t,\cdot) \right\|_{H^{s-\frac{1}{2}}(\mathbb{R})}^2,$$

and

$$M_s(t) \leq M_s(0) + \int_0^t C(N_\gamma(\tau)) N_\gamma(\tau)^2 M_s(\tau)\, d\tau.$$

Then one obtains, through Gronwall inequality, an estimate of the form (B) (when $Z = D_x$) assuming an a priori decay in Hölder norms of the type (A). Finally, let us mention that one has to combine the preceding ideas with the commutation of $(t\partial_t + 2x\partial_x)$-vector fields to the equations. We refer the reader to the last two chapters of [5] for a full description of the technical issues that arise then.

4 Klainerman-Sobolev inequalities

As previously mentioned, the proof of global existence relies on a bootstrap argument on properties (A) and (B). We have indicated in the preceding section how (B) may be deduced from (A). On the other hand, one has to prove that conversely, (A) and (B) imply that (A) holds with smaller constants in the inequalities. In the last two sections of this paper, we describe the general strategy of proof, the difficulties one has to cope with, and the ideas used to overcome them.

The first step is to show that if the L^2-estimate (B) holds for $k \leq s$, then bounds of the form

(A') $$\|Z^k u(t,\cdot)\|_{L^\infty} = O(\varepsilon t^{\delta'_k}),\ k \leq s - 100$$

are true, for small positive δ'_k. This is not (A), since the δ'_k may be larger that the $\tilde{\delta}'_k$ of (A), and since this does not give a *uniform* bound for $\|u(t,\cdot)\|_{L^\infty}$. But this first information will allow us to deduce, in the last step of the proof, estimates of the form (A) from (A') and the equation.

Let us make a change of variables $x \to x/t$ in the water waves system. If $u(t,x)$ is given by $u(t,x) = (|D_x|^{1/2}\psi + i\eta)(t,x)$, we define v by $u(t,x) = \frac{1}{\sqrt{t}} v(t, x/t)$. We set $h = 1/t$ and eventually consider v as a family of functions of x depending on the semi-classical parameter h. Moreover, for $a(x,\xi)$ a function satisfying convenient symbol estimates, and $(v_h)_h$ a family of functions on $\mathbb{R}$, we define

$$\mathrm{Op}_h(a) v_h = a(x, hD) v_h = \frac{1}{2\pi} \int e^{ix\xi} a(x, h\xi) \hat{v}_h(\xi)\, d\xi.$$

Then the water waves system is equivalent to the equation

$$(4.1)\qquad \big(D_t - \mathrm{Op}_h\big(x\xi + |\xi|^{1/2}\big)\big)v = \sqrt{h}Q_0(V) + h\left[C_0(V) - \frac{i}{2}v\right] + h^{1+\kappa}R(V),$$

where we used the following notations

• Q_0 (resp. C_0) is a nonlocal quadratic (resp. cubic) form of $V = (v, \bar{v})$ that may be written as a linear combination of expressions $\mathrm{Op}_h(b_0)[\prod_{j=1}^{\ell}\mathrm{Op}_h(b_j)v_\pm]$, $\ell = 2$ (resp. $\ell = 3$), where $b_\ell(\xi)$ are homogeneous functions of degree $d_\ell \geq 0$ with $\sum_0^2 d_\ell = 3/2$ (resp. $\sum_0^3 d_\ell = 5/2$) and $v_+ = v, v_- = \bar{v}$.

• $R(V)$ is a remainder, made of the contributions vanishing at least at order four at $V = 0$.

To simplify the exposition, we shall assume that v satisfies $\varphi(hD)v = v$ for some $C_0^\infty(\mathbb{R}-\{0\})$-function φ, equal to one on a large enough compact subset of of $\mathbb{R}-\{0\}$. Such a property is not satisfied by solutions of (4.1), but one can essentially reduce to such a situation performing a dyadic decomposition $v = \sum_{j\in\mathbb{Z}}\varphi(2^{-j}hD)v$.

The Klainerman vector field associated to the linearization of the water waves equation may be written, in the new coordinates that we are using, as $Z = t\partial_t + x\partial_x$. Remembering $h = 1/t$ and expressing ∂_t from Z in equation (4.1), we get

$$(4.2)\qquad \mathrm{Op}_h\big(2x\xi + |\xi|^{1/2}\big)v = -\sqrt{h}Q_0(V) + h\left[\frac{i}{2}v - iZv - C_0(V)\right] - h^{1+\kappa}R(V).$$

Since we factored out the expected decay in $1/\sqrt{t}$, our goal is to deduce from assumptions (A) and (B) estimates of the form $\|Z^k v\|_{L^\infty} = O(\varepsilon h^{-\delta'_k})$ for $k \leq s - 100$.

Proposition. *Assume that for t in some interval $[T_0, T[$ (i.e. for h in some interval $]h', h_0]$), one has estimates (A) and (B):*

$$(4.3)\qquad \|Z^k v\|_{L^\infty} = O\big(\varepsilon h^{-\tilde{\delta}'_k}\big),\ k \leq s/2,\ \|Z^k v\|_{L^2} = O\big(\varepsilon h^{-\delta_k}\big),\ k \leq s.$$

Denote $\Lambda = \{(x, d\omega(x)); x \in \mathbb{R}^*\}$ where $\omega(x) = 1/(4|x|)$. Then, if γ_Λ is smooth, supported close to Λ and equal to one on a neighborhood of Λ, and if $\gamma_\Lambda^c = 1-\gamma_\Lambda$, we have for $k \leq s-100$

$$(4.4)\qquad \|Z^k\,\mathrm{Op}_h(\gamma_\Lambda^c)v\|_{L^2} = O\big(\varepsilon h^{\frac{1}{2}-\delta'_k}\big),$$

$$(4.5)\qquad \|(hD_x - d\omega)Z^k\,\mathrm{Op}_h(\gamma_\Lambda)v\|_{L^2} = O\big(\varepsilon h^{1-\delta'_k}\big),$$

$$(4.6)\qquad \|Z^k v\|_{L^\infty} = O\big(\varepsilon h^{-\delta'_k}\big).$$

Idea of proof. One applies k vector fields Z to (4.2) and uses their commutation properties to the linearized equation. In that way, taking into account the assumptions, one gets

$$(4.7)\qquad \mathrm{Op}_h\left(2x\xi + |\xi|^{1/2}\right)Z^k v = O_{L^2}\big(\varepsilon h^{\frac{1}{2}-\delta'_k}\big).$$

One remarks that $2x\xi + |\xi|^{1/2}$ vanishes exactly on Λ. Consequently, this symbol is elliptic on the support of γ_Λ^c, and this allows one to get (4.4) by ellipticity.

To prove the second inequality, one uses the fact that,

$$\operatorname{Op}_h\big(2x\xi+|\xi|^{1/2}\big)Z^k\operatorname{Op}_h(\gamma_\Lambda)v=-\sqrt{h}\operatorname{Op}_h(\gamma_\Lambda)Z^kQ_0(V)+O\big(\varepsilon h^{1-\delta_k'}\big). \tag{4.8}$$

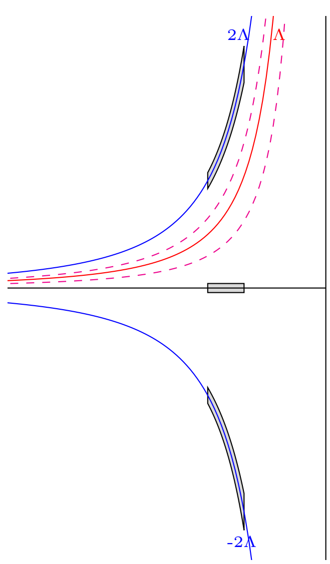
We may decompose $v=v_\Lambda+v_{\Lambda^c}$ where $v_\Lambda=\operatorname{Op}_h(\gamma_\Lambda)v$ and $v_{\Lambda^c}=\operatorname{Op}_h(\gamma_\Lambda^c)v$. We write

$$Z^kQ_0(V)-Z^kQ_0(v_\Lambda,\bar v_\Lambda)=B(v_\Lambda,Z^kv_{\Lambda^c})+\cdots$$

where B is the polar form of Q_0. By (4.4), $\|Z^kv_{\Lambda^c}\|_{L^2}=O(\varepsilon h^{\frac12-\delta_k'})$, and by assumption $\|v_\Lambda\|_{L^\infty}=O(\varepsilon)$. It follows that $\big\|B(v_\Lambda,Z^kv_{\Lambda^c})\big\|_{L^2}=O\big(\varepsilon h^{\frac12-\delta_k'}\big)$. The other contributions to $Z^kQ_0(V)-Z^kQ_0(v_\Lambda,\bar v_\Lambda)$ may be estimated in a similar way, up to extra contributions, that we do not write explicitly in this outline, and that may be absorbed in the left hand side of (4.6) at the end of the reasoning. The right hand side of (4.8) may thus be written

$$-\sqrt{h}\operatorname{Op}_h(\gamma_\Lambda)Z^kQ_0(V_\Lambda)+O_{L^2}\big(\varepsilon h^{1-\delta_k'}\big), \tag{4.9}$$

where $V_\Lambda=(v_\Lambda,\bar v_\Lambda)$. One notices then that since v_Λ (resp. $\bar v_\Lambda$) is microlocally supported close to Λ (resp. $-\Lambda$), $Q_0(V_\Lambda)$ is microlocally supported close to the union of 2Λ, 0Λ and -2Λ, so far away from the support of the cut-off γ_Λ (where $\ell\Lambda=\{(x,\ell d\omega(x));x\in\mathbb{R}^*\}$) – see picture below.



Consequently, the first term in (4.9) vanishes, and we get

$$\operatorname{Op}_h\big(2x\xi+|\xi|^{1/2}\big)Z^k\operatorname{Op}_h(\gamma_\Lambda)v=O_{L^2}\big(\varepsilon h^{1-\delta_k'}\big). \tag{4.10}$$

Since $2x\xi+|\xi|^{1/2}$ and $\xi-d\omega(x)$ have the same zero set, namely Λ, one may write

$$\xi-d\omega(x)=e_0(x,\xi)\big(2x\xi+|\xi|^{1/2}\big)+he_1(x,\xi)$$

for some symbols e_0,e_1. Using symbolic calculus, one deduces (4.5) from (4.10).

Finally, to obtain (4.6), we write

$$\|Z^k v_\Lambda\|_{L^\infty} = \|e^{-i\omega/h} Z^k v_\Lambda\|_{L^\infty} \le C\|e^{-i\omega/h} Z^k v_\Lambda\|_{L^2}^{1/2} \|D_x(e^{-i\omega/h} Z^k v_\Lambda)\|_{L^2}^{1/2}.$$

The last factor is $h^{-1/2}\|(hD_x - d\omega)Z^k v_\Lambda\|_{L^2}^{1/2}$, which is $O(\sqrt{\varepsilon}h^{-\delta'_k/2})$ by (4.5). Moreover, (4.4) and Sobolev inequality imply that $\|Z^k v_{\Lambda^c}\|_{L^\infty} = O\big(\varepsilon h^{-\delta'_k}\big)$, since we have assumed that v is spectrally localized for $|\xi| \sim 1/h$. This gives (4.6).

5 Optimal L^∞ bounds

As seen in the preceding section, one can deduce from the L^2-estimates (B) some L^∞-estimates (4.6), which are *not* the optimal estimates of the form (A) that we need (because the exponents δ'_k are larger than $\tilde{\delta}'_k$, and because δ'_0 is positive, while we need a uniform estimate when no Z field acts on v). In order to get (A), we deduce from the PDE (4.1) an ODE satisfied by v.

Proposition. *Under the conclusions of the preceding proposition, we may write*

$$v = v_\Lambda + \sqrt{h}(v_{2\Lambda} + v_{-2\Lambda}) + h(v_{3\Lambda} + v_{-\Lambda} + v_{-3\Lambda}) + h^{1+\kappa}g, \tag{5.1}$$

where $\kappa > 0$, g satisfies bounds of the form $\|Z^k g\|_{L^\infty} = O(\varepsilon h^{-\delta'_k})$, and $v_{\ell\Lambda}$ is microlocally supported close to $\ell\Lambda$ and is a semi-classical lagrangian distribution along $\ell\Lambda$, as well as $Z^k w_{\ell\Lambda}$ for $k \le s/2$, in the following sense

$$\|Z^k v_{\ell\Lambda}\|_{L^\infty} = O(\varepsilon h^{-\delta'_k}), \tag{5.2}$$

$$\|\mathrm{Op}_h(e_\ell(x,\xi))Z^k v_{\ell\Lambda}\|_{L^\infty} = O(\varepsilon h^{1-\delta'_k}),\ \ \ell \in \{1,-2,2\}, \tag{5.3}$$

$$\|\mathrm{Op}_h(e_\ell(x,\xi))Z^k v_{\ell\Lambda}\|_{L^\infty} = O(\varepsilon h^{\frac{1}{2}-\delta'_k}),\ \ \ell \in \{-3,-1,3\}, \tag{5.4}$$

if e_ℓ vanishes on $\ell\Lambda$.

Remark. Consider a function $w = \alpha(x)\exp(i\omega(x)/h)$. If α is smooth and bounded as well as its derivatives, we see that $(hD_x - d\omega(x))w = O_{L^\infty}(h)$ i.e. w satisfies the second of the above conditions with $\ell = 1$, where $e_1(x,\xi) = \xi - d\omega(x)$ is an equation of Λ. The conclusion of the proposition thus means that $v_{\ell\Lambda}$ enjoys a weak form of such an oscillatory behavior.

The proposition is proved using equation (4.2). For instance, the bound (5.3) for $v_\Lambda = \mathrm{Op}_h(\gamma_\Lambda)v$ is proved in the same way as (4.5), with L^2-norms replaced by L^∞ ones, using (4.6) to estimate the right hand side. In the same way, one defines $v_{\pm 2\Lambda}$ as the cut-off of v close to $\pm 2\Lambda$. As in the proof of (4.4), one shows an $O_{L^\infty}(h^{\frac{1}{2}-\delta'_k})$ bound for $Z^k v_{\Lambda^c}$, which implies that the main contribution to $Q_0(v,\bar{v})$ is $Q_0(v_\Lambda, \bar{v}_\Lambda)$. Localizing (4.2) close to $\pm 2\Lambda$, one gets an elliptic equation that allows to determine $v_{\pm 2\Lambda}$ as a quadratic function of $v_\Lambda, \bar{v}_\Lambda$. Iterating the argument, one gets the expansion of the proposition. One does not get in the $\sqrt{h}$-terms of the expansion a contribution associated to 0Λ because $Q_0(V)$ may be factored out by a Fourier multiplier vanishing on the zero section. Consequently, non oscillating terms form part of the $O(h^{1+\kappa})$ remainder.

Let us use the result of the preceding proposition to obtain an ODE satisfied by v:

Proposition. *The function v satisfies an ODE of the form*

$$
\begin{aligned}
D_t v = {} & \frac{1}{2}(1-\chi(h^{-\beta}x))|d\omega|^{1/2}v - i\sqrt{h}(1-\chi(h^{-\beta}x))\Big[\Phi_2(x)v^2+\Phi_{-2}(x)\bar{v}^2\Big] \\
& + h(1-\chi(h^{-\beta}x))\Big[\Phi_3(x)v^3+\Phi_1(x)|v|^2v+\Phi_{-1}(x)|v|^2\bar{v}+\Phi_{-3}(x)\bar{v}^3\Big] \\
& + O(\varepsilon h^{1+\kappa}),
\end{aligned}
\tag{5.5}
$$

where $\kappa>0, \beta>0$ are small, Φ_ℓ are real valued functions of x defined on $\mathbb{R}^*$ and χ is in $C_0^\infty(\mathbb{R})$, equal to one close to zero.

To prove the proposition, one plugs expansion (5.1) in equation (4.1). The key point is to use (5.3), (5.4) to express all (pseudo-)differential terms from multiplication operators and remainders. For instance, if $b(\xi)$ is some symbol, one may write $b(\xi)=b|_{\ell\Lambda}+e_\ell$ where e_ℓ vanishes on $\ell\Lambda=\{\xi=\ell d\omega\}$. Consequently

$$
\mathrm{Op}_h(b)v_{\ell\Lambda} = b(\ell d\omega)v_{\ell\Lambda}+\mathrm{Op}_h(e_\ell)v_{\ell\Lambda},
$$

and by (5.3), when $\ell=-2,1,2$, one gets $\|\mathrm{Op}_h(e_\ell)v_{\ell\Lambda}\|_{L^\infty}=O(\varepsilon h^{1-\delta_0'})$. Since $Q_0(v_\Lambda,\bar{v}_\Lambda)$ is made of expressions of type

$$
S=\mathrm{Op}_h(b_0)[(\mathrm{Op}_h(b_1)v_\Lambda)(\mathrm{Op}_h(b_2)v_\Lambda)]
$$

(and similar ones replacing v_Λ by $\bar{v}_\Lambda$), one gets, using that v_Λ^2 is lagrangian along 2Λ,

$$
S=b_0(2d\omega)b_1(d\omega)b_2(d\omega)v_\Lambda^2+O_{L^\infty}(h^{1-\delta_0'}).
$$

One applies a similar procedure to the other pseudo-differential terms of equation (4.1), namely $\mathrm{Op}_h(x\xi+|\xi|^{1/2})v$ and $C_0(V)$, where v is expressed using (5.1) in which the $v_{\ell\Lambda}$ are written as explicit quadratic or cubic forms in $(v_\Lambda,\bar{v}_\Lambda)$. This permits to write all those terms as polynomial expressions in $(v_\Lambda,\bar{v}_\Lambda)$ with x-depending coefficients, up to a remainder vanishing like $h^{1+\kappa}$ when h goes to zero. Expressing back v_Λ from v, one gets the ODE (5.5).

As soon as the preceding proposition has been established, the proof of optimal L^∞-estimates for v is straightforward. Applying a Poincaré normal forms method to (5.5), one is reduced to an equivalent ODE of the form

$$
D_t f = \frac{1}{2}(1-\chi(h^{-\beta}x))|d\omega|^{1/2}\left[1+\frac{|d\omega|^2}{t}|f|^2\right]f+O(\varepsilon t^{-1-\kappa}).
$$

This implies that $\partial_t|f|^2$ is integrable in time, whence a uniform bound for f and explicit asymptotics when t goes to infinity. Expressing v in terms of f, and writing $u(t,x)=\frac{1}{\sqrt{t}}v(t,x/t)$, one obtains the uniform $O(t^{-1/2})$ bound for u given in (A) as well as the asymptotics of the statement of the main theorem. Estimates for $Z^k u$ are proved in the same way.

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