ON THE WATER WAVES EQUATIONS WITH SURFACE TENSION

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ABSTRACT. The purpose of this article is to clarify the Cauchy theory of the water waves equations as well in terms of regularity indexes for the initial conditions as for the smoothness of the bottom of the domain (namely no regularity assumption is assumed on the bottom). Our main result is that, following the approach developed in [1], after suitable paralinearizations, the system can be arranged into an explicit symmetric system of Schrödinger type. We then show that the smoothing effect for the (one dimensional) surface tension water waves is in fact a rather direct consequence of this reduction, and following this approach, we are able to obtain a sharp result in terms of regularity of the indexes of the initial data, and weights in the estimates.

1. INTRODUCTION

We consider a solution of the incompressible Euler equations for a potential flow in a domain with free boundary, of the form

$$\{ (t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} \, : \, (x, y) \in \Omega_t \},$$

where Ω_t is the domain located between a free surface

$$\Sigma_t = \{ (x, y) \in \mathbf{R}^d \times \mathbf{R} \, : \, y = \eta(t, x) \},$$

and a given bottom denoted by $\Gamma = \partial\Omega_t \setminus \Sigma_t$. The only assumption we shall make on the domain is that the top boundary, Σ_t, and the bottom boundary, Γ are separated by a "strip" of fixed length.

More precisely, we assume that the initial domain satisfies (for $t = 0$) the following assumption.

$\mathbf{H_t}$ The domain Ω_t is the intersection of the half space, denoted by $\Omega_{1,t}$, located below the free surface Σ_t,

$$\Omega_{1,t} = \{(x, y) \in \mathbf{R}^d \times \mathbf{R} \, : \, y < \eta(t, x)\}$$

and an open set $\Omega_2 \subset \mathbf{R}^{d+1}$ such that Ω_2 contains a fixed strip around Σ_t, which means that there exists $h > 0$ such that,

$$\{(x, y) \in \mathbf{R}^d \times \mathbf{R} \, : \, \eta(t, x) - h \leq y \leq \eta(t, x)\} \subset \Omega_2.$$

We shall also assume that the domain Ω_2 (and hence the domain $\Omega_t = \Omega_{1,t} \cap \Omega_2$) is connected.

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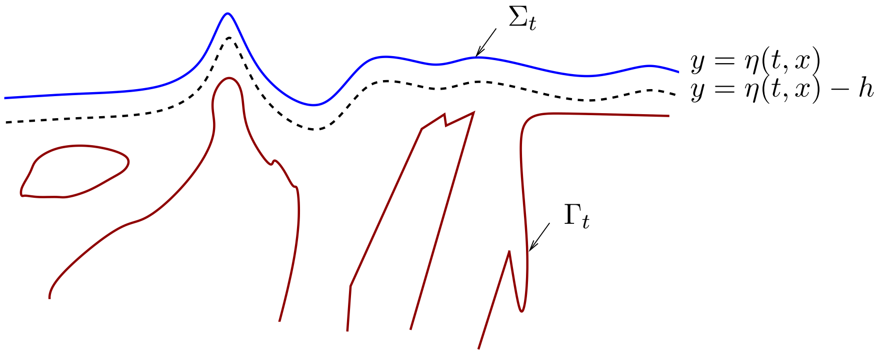
We emphasize that no regularity assumption is made on the domain (apart from the regularity of the top boundary Σ_t). Notice that our setting contains both cases of infinite depth and bounded depth bottoms (and all cases in-between).



The domain

A key feature of the water waves equations is that there are two boundary conditions on the free surface $\Sigma_t = \{y = \eta(t,x)\}$. Namely, we consider a potential flow so that the velocity field is the gradient of a potential $\phi = \phi(t,x,y)$ which is a harmonic function. The water waves equations are then given by the Neumann boundary condition on the bottom Γ, and the classical kinematic and dynamic boundary conditions on the free surface Σ_t. The system reads

$$
\left\{
\begin{aligned}
&\Delta\phi + \partial_y^2\phi = 0 && \text{in } \Omega_t,\\
&\partial_t\eta = \partial_y\phi - \nabla\eta\cdot\nabla\phi && \text{on } \Sigma_t,\\
&\partial_t\phi = -g\eta + \kappa H(\eta) - \frac{1}{2}\left|\nabla\phi\right|^2 - \frac{1}{2}(\partial_y\phi)^2 && \text{on } \Sigma_t,\\
&\partial_n\phi = 0 && \text{on } \Gamma,
\end{aligned}
\right.
\tag{1.1}
$$

where $\nabla = (\partial_{x_i})_{1\le i\le d}$, $\Delta = \sum_{i=1}^d \partial_{x_i}^2$, n is the normal to the boundary Γ, $g > 0$ denotes the acceleration of gravity, $\kappa \ge 0$ is the coefficient of surface tension and $H(\eta)$ is the mean curvature of the free surface:

$$
H(\eta) = \operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right).
$$

We are concerned with the problem with surface tension and then we set $\kappa = 1$. Since we make no regularity assumption on the bottom, giving sense to the system (1.1) requires some care (see Section 2).

Following Zakharov we shall reduce (1.1) to a system on the free surface $\Sigma_t = \{y = \eta(t,x)\}$. If $\psi = \psi(t,x) \in \mathbf{R}$ is defined by

$$
\psi(t,x) = \phi(t,x,\eta(t,x)),
$$

then $\phi(t,x,y)$ is the unique variational solution of

$$
\Delta\phi = 0 \quad \text{in } \Omega_t, \qquad \phi(t,x,\eta(t,x)) = \psi(t,x),
$$

and the Dirichlet-Neumann operator is defined by

$$\begin{aligned}(G(\eta)\psi)(t,x) &= \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta(t,x)}\\ &= (\partial_y\phi)(t,x,\eta(t,x)) - \nabla\eta(t,x)\cdot(\nabla\phi)(t,x,\eta(t,x)).\end{aligned}$$

(We refer to Section 2 for a precise construction). Now (η,ψ) solves

$$\text{(1.2)}\qquad \begin{cases}\partial_t\eta - G(\eta)\psi = 0,\\ \partial_t\psi + g\eta - H(\eta) + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{\big(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\big)^2}{1+|\nabla\eta|^2} = 0.\end{cases}$$

The study of the Cauchy problem was iniciated by Kano-Nishida [18] and Yosihara [37, 38]. In the framework of Sobolev spaces and without smallness assumptions on the data, the well-posedness of the Cauchy problem was first proved by Beyer-Günther in [8] in the case with surface tension and by Wu [36, 35] without surface tension. Several extensions of their results have been obtained by different proofs by Ambrose-Masmoudi [6], Schneider-Wayne [28], Schweizer [30], Iguchi [17], Shatah-Zeng [29], Ming-Zhang [26], Coutand-Shkoller [13], Rousset-Tzvetkov [27] and also Christianson-Hur-Staffilani [12].

Using the paralinearization approach developed by Alazard-Métivier [1] we prove first the well-posedness of the Cauchy problem (in any dimension) for rougher data, without any assumption on the bottom. Previous results required the bottom to be the graph of a smooth function (at least $W^{13,\infty}$ [17, 26]). Secondly, under the same conditions, we prove the smoothing effect when $d=1$ with the natural weights in the estimate.

Our first result (Cauchy theory) is the following

Theorem 1.1. *Let $d\geq 1$, $s>2+d/2$ and $(\eta_0,\psi_0)\in H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d)$ be such that the assumption $H_{t=0}$ is satisfied. Then there exists $T>0$ such that the Cauchy problem for* (1.2) *with initial data (η_0,ψ_0) has a unique solution*

$$(\eta,\psi)\in C^0\big([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d)\big)$$

such that the assumption H_t is satisfied for $t\in[0,T]$.

Concerning the dependence of the solutions on the data, we have the following result.

Theorem 1.2. *Consider $(\eta,\psi)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$ a solution of* (1.2) *and a sequence $(\eta_{n,0},\psi_{n,0})_{n\in\mathbb{N}^*}$ converging in $H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d)$ to $(\eta,\psi)\mid_{t=0}$. Then, for n sufficiently large, the solutions $(\eta_n,\psi_n)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$ with data $(\eta_{n,0},\psi_{n,0})$ are defined on the time interval $[0,T]$ and satisfy*

$$\text{(1.3)}\qquad \lim_{n\to+\infty}\|(\eta_n,\psi_n)-(\eta,\psi)\|_{C^0([0,T];H^{s+\frac{1}{2}}\times H^s)} = 0.$$

Remark 1.3. Notice that our threshold of regularities appears to be the natural one (as long as dispersive effects are not taken into account). Indeed, $\psi\in H^s(\mathbf{R}^d)$ is equivalent to $\phi\in H^{s+\frac{1}{2}}(\Omega_h)$; hence the velocity $u=\nabla_x\phi$ belongs to $H^{s-\frac{1}{2}}(\Omega_h)$ and therefore $u\in\mathrm{Lip}(\Omega_h)$ as soon as $s-\frac{1}{2}-1>\frac{d+1}{2}$, that is $s>2+\frac{d}{2}$. Consequently our assumption is the minimal one (in terms of L^2 based Sobolev spaces) which ensures the Lipschitz regularity of the initial velocity field (and the Lipschitz regularity assumption is well known to be required for the local well posedness of differential equations). As a consequence, solving this quasilinear system without using further dispersion properties requires working a least at this level of regularity. However, working with such rough data gives rise to many technical difficulties, which would be avoided (to a large extent) by choosing $s > 3 + \frac{d}{2}$. On the other hand, the dispersive properties enjoyed by the solutions of the water-waves system (as the Strichartz estimates proved in [2]) should precisely enable us to go below this threshold. This will be the purpose of our forthcoming work [3].

Remark 1.4. Notice also that the natural assumptions on the water-waves system would be to assume $(\eta, \nabla_x \psi) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^{s-1}(\mathbf{R}^d)$. The methods developped in this paper apply with this assumption only. However it would require more developments, and for the sake of conciseness, we prefered to keep the (less natural but simpler) assumption $(\eta, \psi) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)$

Our second result is the following 1/4-smoothing effect for 2D-water waves.

Theorem 1.5. *Assume that $d = 1$ and let $s > 5/2$ and $T > 0$. Consider a solution (η, ψ) of (1.2) on the time interval $[0, T]$, such that Ω_t satisfies the assumption H_t. If*

$$(\eta, \psi) \in C^0\big([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})\big),$$

then for any $\delta > 0$

$$\langle x \rangle^{-\frac{1}{2}-\delta} (\eta, \psi) \in L^2\big(0, T; H^{s+\frac{3}{4}}(\mathbf{R}) \times H^{s+\frac{1}{4}}(\mathbf{R})\big).$$

Notice that in [12], Christianson-Hur-Staffilani iniciated the study of the dispersive properties of the solutions of the water-waves system and proved Strichartz type estimates, for smooth enough initial data (in the semi-classical regime, and consequently with loss of derivatives).

Many variations are possible concerning the fluid domain. Our method would apply to the case where the free surface is not a graph over the hyperplane $\mathbf{R}^d \times \{0\}$, but rather a graph over a fixed hypersurface. Notice also that our proof might apply to the radial case in dimension 3. However, the non radial case would certainly require some non trapping assumption on the initial geometry.

Remark 1.6. Notice that our method would apply to the case where the bottom is time-dependent, under an additional Lipschitz regularity assumption on the bottom. In this case, the system reads

$$\begin{cases} \Delta \phi + \partial_y^2 \phi = 0 & \text{in } \Omega_t, \\ \partial_t \eta = \partial_y \phi - \nabla \eta \cdot \nabla \phi & \text{on } \Sigma_t, \\ \partial_t \phi = -g\eta + \kappa H(\eta) - \dfrac{1}{2} \left| \nabla \phi \right|^2 - \dfrac{1}{2} (\partial_y \phi)^2 & \text{on } \Sigma_t, \\ \partial_n \phi(m) = \dfrac{dm}{dt} \cdot n(m) & \text{for } m \in \Gamma_t, \end{cases} \tag{1.4}$$

where here $\frac{dm}{dt}$ is the time derivative of the point m on the boundary Γ_t.

As will appear clearly in the proof of Theorems 1.1 and 1.5, the only difference between (1.1) and (1.4) which is the boundary condition at the bottom has an influence which is negligible (in the high frequency regime) as soon as the Dirichlet-Neumann operator is well defined (which is the case by a variational approach as soon as the bottom is Lipschitz).

To prove Theorem 1.5, we start in §2 by defining and proving regularity properties of the Dirichlet-Neumann operator. Then in §3 we perform several reductions to a paradifferential system on the boundary by means of the analysis in [1]. The key technical lemma in this paper in a reduction of the system (1.2) to a simple hyperbolic form. To perform this reduction, we prove in §4 the existence of a paradifferential symmetrizer. We deduce Theorem 1.1 from this symmetrization in §5. Theorem 1.5 is then proved in §7 by means of Doi's approach [14, 15]. Note that our strategy is based on a direct analysis in Eulerian coordinates. In this direction it is influenced by the important paper by Lannes ([21]). It can be remarked that in [21], the Nash-Moser scheme is applied due to a loss of regularity in the estimates obtained while symmetrizing the system. It happens very often that in such situations, this scheme can be avoided by applying several derivatives to the equation (see for example [8, 17, 27]). Here, the loss of derivatives encountered in Lannes' work is avoided by the systematic use of the paradifferential calculus which enable a very precise analysis of the Dirichlet-Neumann operator, and consequently give a better symetrization method.

As it was shown by Zakharov (see [39] and references therein), the system (1.2) is a Hamiltonian one, of the form

$$\frac{\partial \eta}{\partial t}=\frac{\delta \mathcal{H}}{\delta \psi}, \quad \frac{\partial \psi}{\partial t}=-\frac{\delta \mathcal{H}}{\delta \eta},$$

$$\mathcal{H}=\frac{1}{2}\int G(\eta)\psi\cdot\psi\,dx+\frac{g}{2}\int \eta^2\,dx+\kappa\int\frac{|\nabla\eta|^2}{1+\sqrt{1+|\nabla\eta|^2}}\,dx$$

Denoting by $\mathcal{H}_0$ the Hamiltonian associated to the linearized system at the origin (in the case of an infinite bottom), we have

$$\mathcal{H}_0=\frac{1}{2}\int\Big[|\xi|\,|\widehat{\psi}|^2+(g+|\xi|^2)|\widehat{\eta}|^2\Big]\,d\xi,$$

where $\widehat{f}$ denotes the Fourier transform, $\widehat{f}(\xi)=\int e^{-ix\cdot\xi}f(x)\,dx$. An important observation is that the canonical transformation $(\eta,\psi)\mapsto a$ with

$$\widehat{a}=\frac{1}{\sqrt{2}}\Big\{\Big(\frac{g+|\xi|^2}{|\xi|}\Big)^{1/4}\widehat{\eta}-i\Big(\frac{|\xi|}{g+|\xi|^2}\Big)^{1/4}\widehat{\psi}\Big\},$$

diagonalizes the Hamiltonian H_0 and reduces the analysis of the linearized system to one complex equation (see [39]). We shall show that there exists a similar diagonalization for the nonlinear equation, by using paradifferential calculus instead of Fourier transform. As already mentioned, this is the main technical result in this paper. In fact, we strongly believe that all dispersive estimates on the water waves system with surface tension could be obtained by using our reduction.

2. The Dirichlet-Neumann operator

2.1. **Definition of the operator.** The purpose of this section is to define the Dirichlet-Neumann operator and prove some basic regularity properties. Let us recall that we assume that Ω_t is the intersection of the half space located below the free surface

$$\Omega_{1,t} = \{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, y < \eta(t,x)\}$$

and an open set $\Omega_2 \subset \mathbf{R}^{d+1}$ and that Ω_2 contains a fixed strip around Σ_t, which means that there exists $h > 0$ such that

$$\{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, \eta(t,x) - h \le y \le \eta(t,x)\} \subset \Omega_2.$$

We shall also assume that the domain Ω_2 (and hence the domain Ω_t) is connected. In the remainder of this subsection, we will drop the time dependence of the domain, and it will appear clearly from the proofs that all estimates are uniform as long as $\eta(t,x)$ remains bounded in the set of functions such that $\|\eta(t,\cdot)\|_{H^s(\mathbb{R}^d)}$ remains bounded.

Below we use the following notations

$$\nabla = (\partial_{x_i})_{1\le i\le d}, \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \sum_{1\le i\le d} \partial_{x_i}^2, \quad \Delta_{x,y} = \Delta + \partial_y^2.$$

Notation 2.1. Denote by $\mathscr{D}$ the space of functions $u \in C^\infty(\Omega)$ such that $\nabla_{x,y} u \in L^2(\Omega)$. We then define $\mathscr{D}_0$ as the subspace of functions $u \in \mathscr{D}$ such that u is equal to 0 near the top boundary Σ.

Proposition 2.2. *There exists a positive weight $g \in L^\infty_{loc}(\Omega)$, equal to 1 near the top boundary of Ω and a positive constant C such that for all $u \in \mathscr{D}_0$,*

$$\int_\Omega g(x,y)|u(x,y)|^2 \,dxdy \le C \int_\Omega |\nabla_{x,y} u(x,y)|^2 \,dxdy. \tag{2.1}$$

Let us set

$$\begin{aligned} \mathcal{O}_1 &= \Big\{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, \eta(x) - h < y < \eta(x)\Big\}, \\ \mathcal{O}_2 &= \Big\{(x,y) \in \Omega \,:\, y < \eta(x) - h\Big\}. \end{aligned} \tag{2.2}$$

To prove Proposition 2.2, the starting point is the following Poincaré inequality on $\mathcal{O}_1$.

Lemma 2.3. *For all $u \in \mathscr{D}_0$ we have*

$$\int_{\mathcal{O}_1} |u|^2 \,dxdy \le h^2 \int_\Omega |\nabla_{x,y} u|^2 \,dxdy.$$

Proof. For $(x,y) \in \mathcal{O}_1$ we can write $u(x,y) = -\int_y^{\eta(x)} (\partial_y u)(x,z)\,dz$, so using the Cauchy-Schwarz inequality we obtain

$$|u(x,y)|^2 \le h \int_{\eta(x)-h}^{\eta(x)} |(\partial_y u)(x,z)|^2 \,dz.$$

Integrating on $\mathcal{O}_1$ we obtain the desired conclusion. □

Lemma 2.4. *Let $m_0 \in \Omega$ and $\delta > 0$ such that*

$$B(m_0, 2\delta) = \{m \in \mathbf{R}^d \times \mathbf{R} : |m - m_0| < 2\delta\} \subset \Omega.$$

Then for any $m_1 \in B(m_0, \delta)$ and any $u \in \mathscr{D}$,

(2.3)

$$\int_{B(m_0,\delta)} |u|^2 \, dxdy \leq 2 \int_{B(m_1,\delta)} |u|^2 \, dxdy + 2\delta^2 \int_{B(m_0,2\delta)} |\nabla_{x,y} u|^2 \, dxdy.$$

Proof. Denote by $v = m_0 - m_1$ and write

$$u(m + v) = u(m) + \int_0^1 v \cdot \nabla_{x,y} u(m + tv) dt.$$

As a consequence, we get

$$|u(m + v)|^2 \leq 2|u(m)|^2 + 2|v|^2 \int_0^1 |\nabla_{x,y} u(m + tv)|^2 \, dt,$$

and integrating this last inequality on $B(m_1, \delta) \subset B(m_0, 2\delta) \subset \Omega$, we obtain (2.3). □

Corollary 2.5. *For any compact $K \subset \mathcal{O}_2$, there exists a constant $C(K) > 0$ such that, for all $u \in \mathscr{D}_0$, we have*

$$\int_K |u|^2 \, dxdy \leq C(K) \int_\Omega |\nabla_{x,y} u|^2 \, dxdy.$$

Proof. Consider now an arbitrary point $m_0 \in \mathcal{O}_2$. Since Ω is open and connected, there exists a continuous map $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = m_0$ and $\gamma(1) \in \mathcal{O}_1$. By compactness, there exists $\delta > 0$ such that for any $t \in [0, 1]$ $B(\gamma(t), 2\delta) \subset \Omega$. Taking smaller δ if necessary, we can also assume that $B(\gamma(1), \delta) \subset \mathcal{O}_1$ so that by Lemma 2.3

$$\int_{B(\gamma(1),\delta)} |u|^2 \, dxdy \leq C \int_\Omega |\nabla_{x,y} u|^2 \, dxdy.$$

We now can find a sequence $t_0 = 0, t_1, \cdots, t_N = 1$ such that the points $m_n = \gamma(t_n)$ satisfy $m_{n+1} \in B(m_n, \delta)$. Applying Lemma 2.4 successively, we obtain

$$\int_{B(m_0,\delta)} |u|^2 \, dxdy \leq C' \int_\Omega |\nabla_{x,y} u|^2 \, dxdy.$$

Then Corollary 2.5 follows by compactness. □

Proof of Proposition 2.2. Writing $\mathcal{O}_2 = \cup_{n=1}^\infty K_n$, and taking a partition of unity (χ_n) such that $0 \leq \chi_n \leq 1$ and $\operatorname{supp} \chi_n \subset K_n$, we can define the continuous function

$$\widetilde{g}(x, y) = \sum_{n=1}^\infty \frac{\chi_n(x, y)}{(1 + C(K_n))n^2},$$

which is clearly positive. Then by Corollary 2.5,

$$\begin{aligned} \int_{\mathcal{O}_2} \widetilde{g}(x, y) \, |u|^2 \, dxdy &\leq \sum_{n=1}^\infty \frac{1}{(1 + C(K_n))n^2} \int_{K_n} |u|^2 \, dxdy \\ &\leq 2 \int_{\mathcal{O}_2} |\nabla_{x,y} u|^2 \, dxdy. \end{aligned} \tag{2.4}$$

Finally, let us set

$$g(x,y)=1 \quad \text{for } (x,y)\in\mathcal{O}_1, \quad g(x,y)=\widetilde{g}(x,y) \quad \text{for } (x,y)\in\mathcal{O}_2.$$

Then Proposition 2.2 follows from Lemma 2.3 and (2.4). □

We now introduce the space in which we shall solve the variational formulation of our Dirichlet problem.

Definition 2.6. *Denote by $H^{1,0}(\Omega)$ the space of functions u on Ω such that there exists a sequence $(u_n)\in\mathscr{D}_0$ such that,*

$$\nabla_{x,y}u_n\to\nabla_{x,y}u \text{ in } L^2(\Omega,dxdy), \qquad u_n\to u \text{ in } L^2(\Omega,g(x,y)dxdy).$$

We endow the space $H^{1,0}$ with the norm

$$\|u\|=\|\nabla_{x,y}u\|_{L^2(\Omega)}.$$

The key point is that the space $H^{1,0}(\Omega)$ is a Hilbert space. Indeed, passing to the limit in (2.1), we obtain first that by definition, the norm on $H^{1,0}(\Omega)$ is equivalent to

$$\|\nabla_{x,y}u\|_{L^2(\Omega,dxdy)}+\|u\|_{L^2(\Omega,g(x,y)dxdy)}.$$

As a consequence, if (u_n) is a Cauchy sequence in $H^{1,0}(\Omega)$, we obtain easily from the completeness of L^2 spaces that there exists $u\in L^2(\Omega,g(x,y)dxdy)$ and $v\in L^2(\Omega,dxdy)$ such that

$$u_n\to u \text{ in } L^2(\Omega,g(x,y)dxdy), \qquad \nabla_{x,y}u_n\to v \text{ in } L^2(\Omega,dxdy).$$

But the convergence in $L^2(\Omega,g(x,y)dxdy)$ implies the convergence in $\mathcal{D}'(\Omega)$ and consequently $v=\nabla_{x,y}u$ in $\mathcal{D}'(\Omega)$ and it is easy to see that $u\in H^{1,0}(\Omega)$.

We are now in position to define the Dirichlet-Neumann operator. Let $\psi(x)\in H^1(\mathbf{R}^d)$. For $\chi\in C_0^\infty(]-1,1[)$ equal to 1 near 0, we first define

$$\widetilde{\psi}(x,y)=\chi\Big(\frac{y-\eta(x)}{h}\Big)\psi(x)\in H^1(\mathbf{R}^{d+1}),$$

which is the most simple lifting of ψ. Then the map

$$v\mapsto\langle\Delta_{x,y}\widetilde{\psi},v\rangle=-\int_\Omega\nabla_{x,y}\widetilde{\psi}\cdot\nabla_{x,y}v\,dxdy$$

is a bounded linear form on $H^{1,0}(\Omega)$. It follows from the Riesz theorem that there exists a unique $\widetilde{\phi}\in H^{1,0}(\Omega)$ such that

$$\forall v\in H^{1,0}(\Omega), \qquad \int_\Omega\nabla_{x,y}\widetilde{\phi}\cdot\nabla_{x,y}v\,dxdy=\langle\Delta_{x,y}\widetilde{\psi},v\rangle. \tag{2.5}$$

Then $\widetilde{\phi}$ is the variational solution to the problem

$$-\Delta_{x,y}\widetilde{\phi}=\Delta_{x,y}\widetilde{\psi} \quad \text{in } \mathcal{D}'(\Omega), \qquad \widetilde{\phi}\,|_\Sigma=0, \qquad \partial_n\widetilde{\phi}\,|_\Gamma=0,$$

the latter condition being justified as soon as the bottom Γ is regular enough.

We now set $\phi=\widetilde{\phi}+\widetilde{\psi}$ and define the Dirichlet-Neumann operator by

$$\begin{aligned}G(\eta)\psi(x)&=\sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta(x)},\\&=(\partial_y\phi)(x,\eta(x))-\nabla\eta(x)\cdot(\nabla\phi)(x,\eta(x)),\end{aligned}$$

Notice that a simple calculation shows that this definition is independent on the choice of the lifting function $\widetilde{\psi}$ as long as it remains bounded in $H^1(\Omega)$ and vanishes near the bottom.

2.2. **Boundedness on Sobolev spaces.**

Proposition 2.7. *Let $d \geq 1$, $s > 2 + \frac{d}{2}$ and $1 \leq \sigma \leq s$. Consider $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$. Then $G(\eta)$ maps $H^\sigma(\mathbf{R}^d)$ to $H^{\sigma-1}(\mathbf{R}^d)$. Moreover, there exists a function C such that, for all $\psi \in H^\sigma(\mathbf{R}^d)$ and $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$,*

$$\|G(\eta)\psi\|_{H^{\sigma-1}(\mathbf{R}^d)} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}}\right)\|\nabla\psi\|_{H^{\sigma-1}}.$$

Proof. The proof is in two steps.

First step: A localization argument. Let us define (by regularizing η), a smooth function $\widetilde{\eta} \in H^\infty(\mathbf{R}^d)$ such that $\|\widetilde{\eta} - \eta\|_{L^\infty} \leq h/100$ and $\|\widetilde{\eta} - \eta\|_{H^{s+1/2}} \leq h/100$. We now set $\eta_1 = \widetilde{\eta} - \frac{9h}{20}$. Then η_1 satisfies

$$\eta(x) - \frac{h}{4} < \eta_1(x) \leq \eta(x) - \frac{h}{5}. \tag{2.6}$$

Lemma 2.8. *Consider for $-3h/4 < a < b < h/5$, the strip*

$$S_{a,b} = \{(x,y) \in \mathbb{R}^{d+1}; a < y - \eta_1(x) < b\},$$

which is included in Ω. Let $k \geq 1$ and assume that $\|\phi\|_{H^k(S_{a,b})} < +\infty$. Then for any $a < a' < b' < b$ there exists $C > 0$ such that

$$\|\phi\|_{H^{k+1}(S_{a',b'})} \leq C\|\phi\|_{H^k(S_{a,b})}.$$

Proof. Choose a function $\chi \in C_0^\infty(a,b)$ equal to 1 on (a', b'). The function $w = \chi(y - \eta_1(x))\phi(x,y)$ is solution to

$$\Delta_{x,y} w = [\Delta_{x,y}, \chi(y - \eta_1(x))]\phi,$$

and since the assumption implies that the right hand side is bounded in H^{k-1}, the result follows from the (explicit) elliptic regularity of the operator $\Delta_{x,y}$ in $\mathbf{R}^{d+1}$. □

Lemma 2.9. *Assume that $-3h/4 < a < b < h/5$ then the strip $S_{a,b} = \{(x,y) \in \mathbb{R}^{d+1} : a < y - \eta_1(x) < b\}$ is included in Ω and for any $k \geq 1$, there exists $C > 0$ such that*

$$\|\phi\|_{H^k(S_{a,b})} \leq C\|\psi\|_{H^1(\mathbf{R}^d)}.$$

Proof. It follows from (2.5) and the definition of $\phi = \widetilde{\phi} + \widetilde{\psi}$, that

$$\|\nabla_{x,y}\phi\|_{L^2(\Omega)} \leq c\,\|\psi\|_{H^1(\mathbf{R}^d)}.$$

Noticing that $S_{a,b} \subset \mathcal{O}_1$ (cf (2.2)) and applying Lemma 2.3 we obtain the *a priori* H^1 bound

$$\|\phi\|_{H^1(S_{a,b})} \leq \|\phi\|_{H^1(\mathcal{O}_1)} \leq (1+h)\,\|\nabla_{x,y}\phi\|_{L^2(\Omega)} \leq c(1+h)\,\|\psi\|_{H^1(\mathbf{R}^d)}.$$

Since it is always possible to chose $a < a_2 < \cdots < a_k = a' < b' = b_k < \cdots < b_2 < b$, we deduce Lemma 2.9 from Lemma 2.8. □

We next introduce $\chi_0 \in C^\infty(\mathbf{R})$ such that $0 \le \chi_0 \le 1$,

$$\chi_0(z) = 1 \quad \text{for } z \ge 0, \quad \chi_0(z) = 0 \quad \text{for } z \le -\frac{1}{4}.$$

Then the function

$$\Phi(x, y) = \chi_0\Big(\frac{y - \eta_1(x)}{h}\Big)\phi(x, y)$$

is solution to

$$\Delta_{x,y}\Phi = f := \Big[\Delta_{x,y}, \chi_0\Big(\frac{y - \eta_1(x)}{h}\Big)\Big]\phi.$$

In view of (2.6), notice that f is supported in a set where ϕ is H^∞ according to Lemma 2.9, we find that

$$f \in H^\infty(\mathcal{O}_1) \quad \text{where } \mathcal{O}_1 = \big\{(x, y) \in \mathbf{R}^d \times \mathbf{R} \,:\, \eta(x) - h < y < \eta(x)\big\}.$$

In addition, using that $\chi_0(0) = 1$ and that $\Phi(x, y)$ is identically equal to 0 near the set $\{y = \eta - h\}$, we immediately verify that Φ satisfies the boundary conditions

$$\Phi \mid_{y=\eta(x)} = \psi(x), \quad \partial_y \Phi \mid_{y=\eta(x)-h} = 0, \quad \Phi \mid_{y=\eta(x)-h} = 0.$$

The fact that the strip $\mathcal{O}_1$ depends on η and not on η_1 is not a typographical error. Indeed, with this choice, the strip $\mathcal{O}_1$ is made of two parallel curves. As a result, a very simple (affine) change of variables will flatten both the top surface $\{y = \eta(x)\}$ and the bottom surface $\{y = \eta(x) - h\}$.

Second step: Elliptic estimates. To prove elliptic estimates, we shall first flatten the boundary. To do so we shall consider the simplest change of variables. Namely, introduce

$$\rho(x, z) = hz + \eta(x).$$

Then the map $(x, z) \mapsto (x, \rho(x, z))$, is a diffeomorphism from the strip $\mathbf{R}^d \times [-1, 0]$ to the set

$$\big\{\, (x, y) \in \mathbf{R}^d \times \mathbf{R} \,:\, \eta(x) - h \le y \le \eta(x) \,\big\}.$$

Let us define the function $v \colon \mathbf{R}^d \times [-1, 0] \to \mathbf{R}$ by

$$v(x, z) = \Phi(x, \rho(x, z)). \tag{2.7}$$

From $\Delta_{x,y}\Phi = f$ with $f \in H^\infty(\Pi_\eta)$, we deduce that v satisfies the elliptic equation

$$\Big(\frac{1}{\partial_z \rho}\partial_z\Big)^2 v + \Big(\nabla - \frac{\nabla \rho}{\partial_z \rho}\partial_z\Big)^2 v = g, \tag{2.8}$$

where $g(x, z) = f(x, hz + \eta(x))$ is in $C^2_z([-1, 0]; H^{s+\frac{1}{2}}(\mathbf{R}^d_x))$. This yields

$$\alpha \partial_z^2 v + \Delta v + \beta \cdot \nabla \partial_z v - \gamma \partial_z v = g, \tag{2.9}$$

where

$$\alpha := \frac{1 + |\nabla \eta|^2}{h^2}, \quad \beta := -\frac{2\nabla \eta}{h}, \quad \gamma := \frac{\Delta \eta}{h}. \tag{2.10}$$

Also v satisfies the boundary conditions

$$v \,\big|_{z=0} = \psi, \qquad \partial_z v|_{z=-1} = 0, \quad v|_{z=-1} = 0. \tag{2.11}$$

We are now in position to apply elliptic regularity results obtained by Alvarez-Samaniego and Lannes in [5, Section 2.2] to deduce the following result.

Lemma 2.10. *Suppose that v satisfies the elliptic equation* (2.9) *with the boundary condtions* (2.11) *with $\psi \in H^{\sigma}(\mathbf{R}^d)$ and $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ where $1 \le \sigma \le s$, $s > 2 + \frac{d}{2}$,* $\operatorname{dist}(\Sigma, \Gamma) > 0$. *Then*

$$\nabla v, \partial_z v \in L^2_z\big([-1,0]; H_x^{\sigma-\frac{1}{2}}(\mathbf{R}^d)\big).$$

It follows from Lemma 2.10 and a classical interpolation argument that $(\nabla v, \partial_z v)$ are continuous in $z \in [-1,0]$ with values in $H^{\sigma-1}(\mathbf{R}^d)$. Now note that, by definition,

$$G(\eta)\psi = \frac{1+|\nabla \eta|^2}{h}\partial_z v - \nabla \eta \cdot \nabla v \,\Big|_{z=0}.$$

Therefore, $G(\eta)\psi \in H^{\sigma-1}(\mathbf{R}^d)$ and we have the desired estimate.

This completes the proof of Proposition 2.7. □

2.3. **Linearization of the Dirichlet-Neumann operator.** The next proposition gives an explicit expression of the shape derivative of the Dirichlet-Neumann operator, that is, of its derivative with respect to the surface parametrization.

Proposition 2.11. *Let $\psi \in H^{\sigma}(\mathbf{R}^d)$ and $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ with $1 \le \sigma \le s$, $s > 2 + \frac{d}{2}$ be such that* $\operatorname{dist}(\Sigma,\Gamma) > 0$. *Then there exists a neighborhood $\mathcal{U}_\eta \subset H^{s+\frac{1}{2}}(\mathbf{R}^d)$ of η such that the mapping*

$$\sigma \in \mathcal{U}_\eta \subset H^{s+\frac{1}{2}}(\mathbf{R}^d) \mapsto G(\sigma)\psi \in H^{\sigma-1}(\mathbf{R}^d)$$

is differentiable. Moreover, for all $h \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$, we have

$$dG(\eta)\psi \cdot h := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big\{G(\eta+\varepsilon h)\psi - G(\eta)\psi\big\} = -G(\eta)(\mathfrak{B}h) - \operatorname{div}(Vh),$$

where

$$\mathfrak{B} = \frac{\nabla \eta \cdot \nabla \psi + G(\eta)\psi}{1+|\nabla \eta|^2}, \quad V = \nabla \psi - \mathfrak{B}\nabla \eta.$$

The above result goes back to Zakharov [39]. Notice that in the previous paragraph we reduced the analysis to studying an elliptic equation in a flat strip $\mathbf{R}^d \times [-1,0]$. As a consequence, the proof of this result by Lannes [21] applies (see also [9, 20, 1]).

Let us mention a key cancellation in the previous formula, which is proved in [9, Lemma 1] (see also [21]).

Lemma 2.12. *We have $G(\eta)\mathfrak{B} = -\operatorname{div} V + R$ where $R \in H^{s-1}(\mathbf{R}^d)$.*

Remark 2.13. As we shall see (and as can be easily derived from the definition) we have $\mathfrak{B}, V \in H^{s-1}(\mathbf{R}^d)$ and hence $G(\eta)\mathfrak{B}$ and $\operatorname{div} V$ belong to $H^{s-2}(\mathbf{R}^d)$. The previous lemma shows that up to a smoother remainder, these two terms are equal. In fact the following proof establishes that the equality $G(\eta)\mathfrak{B} = -\operatorname{div} V$ holds in the case without bottom boundary ($\Gamma = \emptyset$) .

Proof. Recalling that, by definition,

$$G(\eta)\psi = (\partial_y \phi - \nabla\eta \cdot \nabla\phi)\big|_{y=\eta},$$

and using the chain rule to write

$$\nabla\psi = \nabla(\phi|_{y=\eta}) = (\nabla\phi + \partial_y\phi\nabla\eta)\big|_{y=\eta},$$

we obtain

$$\begin{aligned}\mathfrak{B} &:= \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}\\ &= \frac{\{\nabla\eta\cdot(\nabla\phi+\partial_y\phi\nabla\eta)+\partial_y\phi-\nabla\eta\cdot\nabla\phi\}}{1+|\nabla\eta|^2}\big|_{y=\eta} = (\partial_y\phi)\big|_{y=\eta}.\end{aligned}$$

Therefore the function Φ defined by $\Phi(x,y) = \partial_y\phi(x,y)$ satisfies

$$\Delta_{x,y}\Phi = 0, \quad \Phi|_{y=\eta} = \mathfrak{B}.$$

Now introduce the variational solution $\tilde{\Phi}$ to the system where we add the bottom boundary condition:

$$\Delta_{x,y}\tilde{\Phi} = 0, \quad \tilde{\Phi}|_{y=\eta} = \mathfrak{B}, \quad \partial_n\tilde{\Phi}|_{\Gamma} = 0.$$

Then it follows from elliptic regularity (see Lemma 2.10) that $\Phi - \tilde{\Phi}$ belongs to $H^{s+1/2}(\Omega_h)$ (recall that Ω_h is an h-neighborhood of the free surface). Directly from the definition of the Dirichlet-Neumann operator, we have

$$G(\eta)\mathfrak{B} = \partial_y\tilde{\Phi} - \nabla\eta\cdot\nabla\tilde{\Phi}\big|_{y=\eta}. = \partial_y\Phi - \nabla\eta\cdot\nabla\Phi\big|_{y=\eta} + R,$$

where

$$R = \partial_y(\tilde{\Phi}-\Phi) - \nabla\eta\cdot\nabla(\tilde{\Phi}-\Phi)\big|_{y=\eta} \in H^{s-1}(\mathbf{R}^d).$$

It remains to show that $\partial_y\Phi - \nabla\eta\cdot\nabla\Phi\big|_{y=\eta} = -\operatorname{div} V$. To do that we first write that $\partial_y\Phi = \partial_y^2\phi = -\Delta\phi$ to obtain

$$\partial_y\Phi - \nabla\eta\cdot\nabla\Phi\big|_{y=\eta} = -\Delta\phi - \nabla\eta\cdot\nabla\Phi\big|_{y=\eta}.$$

On the other hand, directly from the definition of V, we have

$$\operatorname{div} V = \operatorname{div}(\nabla\psi - \mathfrak{B}\nabla\eta) = \Delta\psi - \operatorname{div}(\mathfrak{B}\nabla\eta).$$

Using that $\psi(x) = \phi(x,\eta(x))$, we check that

$$\begin{aligned}\Delta\psi = \operatorname{div}\nabla\psi &= \operatorname{div}\left(\nabla\phi\big|_{y=\eta} + \partial_y\phi\big|_{y=\eta}\nabla\eta\right)\\ &= (\Delta\phi + \nabla\partial_y\phi\cdot\nabla\eta)\big|_{y=\eta} + \operatorname{div}\left(\partial_y\phi\big|_{y=\eta}\nabla\eta\right)\\ &= (\Delta\phi + \nabla\partial_y\phi\cdot\nabla\eta)\big|_{y=\eta} + \operatorname{div}(\mathfrak{B}\nabla\eta)\end{aligned}$$

so that

$$\begin{aligned}\operatorname{div} V &= \Delta\psi - \operatorname{div}(\mathfrak{B}\nabla\eta) = (\Delta\phi + \nabla\partial_y\phi\cdot\nabla\eta)\big|_{y=\eta}\\ &= (\Delta\phi + \nabla\Phi\cdot\nabla\eta)\big|_{y=\eta} = -G(\eta)\mathfrak{B},\end{aligned}$$

which is the desired identity. □

3. Paralinearization

3.1. **Paradifferential calculus.** In this paragraph we review notations and results about Bony's paradifferential calculus. We refer to [10, 16, 22, 25, 32] for the general theory. Here we follow the presentation by Métivier in [22].

For $\rho \in \mathbf{N}$, according to the usual definition, we denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the Sobolev spaces of L^∞ functions whose derivatives of order ρ are in L^∞. For $\rho \in]0,+\infty[\backslash \mathbf{N}$, we denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the space of bounded functions whose derivatives of order $[\rho]$ are uniformly Hölder continuous with exponent $\rho - [\rho]$. Recall also that, for all C^∞ function F, if $u \in W^{\rho,\infty}(\mathbf{R}^d)$ for some $\rho \geq 0$ then $F(u) \in W^{\rho,\infty}(\mathbf{R}^d)$.

Definition 3.1. *Given $\rho \geq 0$ and $m \in \mathbf{R}$, $\Gamma^m_\rho(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \backslash 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbf{R}^d)$ and there exists a constant C_α such that,*

$$\forall \left|\xi\right| \geq \frac{1}{2}, \quad \left\|\partial_\xi^\alpha a(\cdot,\xi)\right\|_{W^{\rho,\infty}} \leq C_\alpha (1+\left|\xi\right|)^{m-\left|\alpha\right|}. \tag{3.1}$$

We next introduce the spaces of (poly)homogeneous symbols.

Definition 3.2. *i) $\dot{\Gamma}^m_\rho(\mathbf{R}^d)$ denotes the subspace of $\Gamma^m_\rho(\mathbf{R}^d)$ which consists of symbols $a(x,\xi)$ which are homogeneous of degree m with respect to ξ.*
ii) If

$$a = \sum_{0 \leq j < \rho} a^{(m-j)} \qquad (j \in \mathbf{N}),$$

where $a^{(m-j)} \in \dot{\Gamma}^{m-j}_{\rho-j}(\mathbf{R}^d)$, then we say that $a^{(m)}$ is the principal symbol of a.

Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\psi(\eta)\widehat{u}(\eta)\, d\eta, \tag{3.2}$$

where $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta} a(x,\xi)\, dx$ is the Fourier transform of a with respect to the first variable; χ and ψ are two fixed C^∞ functions such that:

$$\psi(\eta) = 0 \quad \text{for } \left|\eta\right| \leq 1, \qquad \psi(\eta) = 1 \quad \text{for } \left|\eta\right| \geq 2,$$

and $\chi(\theta,\eta)$ is homogeneous of degree 0 and satisfies, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta,\eta) = 1 \quad \text{if} \quad \left|\theta\right| \leq \varepsilon_1 \left|\eta\right|, \qquad \chi(\theta,\eta) = 0 \quad \text{if} \quad \left|\theta\right| \geq \varepsilon_2 \left|\eta\right|.$$

We shall use quantitative results from [22] about operator norms estimates in symbolic calculus. To do so, introduce the following semi-norms.

Definition 3.3. *For $m \in \mathbf{R}$, $\rho \geq 0$ and $a \in \Gamma^m_\rho(\mathbf{R}^d)$, we set*

$$M^m_\rho(a) = \sup_{\left|\alpha\right| \leq \frac{d}{2}+1+\rho} \; \sup_{\left|\xi\right| \geq 1/2} \left\| (1+\left|\xi\right|)^{\left|\alpha\right|-m} \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)}. \tag{3.3}$$

Remark 3.4. If a is homogeneous of degree m in ξ, then

$$M^m_\rho(a) \leq K_{d,m} \sup_{\left|\alpha\right| \leq \frac{d}{2}+1+\rho} \; \sup_{\left|\xi\right| = 1} \left\|\partial_\xi^\alpha a(\cdot,\xi)\right\|_{W^{\rho,\infty}(\mathbf{R}^d)}.$$

The main features of symbolic calculus for paradifferential operators are given by the following theorems.

Definition 3.5. *Let $m \in \mathbf{R}$. An operator T is said of order m if, for all $\mu \in \mathbf{R}$, it is bounded from H^{μ} to $H^{\mu-m}$.*

Theorem 3.6. *Let $m \in \mathbf{R}$. If $a \in \Gamma_0^m(\mathbf{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that*

$$\|T_a\|_{H^{\mu}\to H^{\mu-m}} \le K M_0^m(a). \tag{3.4}$$

Theorem 3.7 (Composition)**.** *Let $m \in \mathbf{R}$ and $\rho > 0$. If $a \in \Gamma_{\rho}^m(\mathbf{R}^d), b \in \Gamma_{\rho}^{m'}(\mathbf{R}^d)$ then $T_a T_b - T_{a\#b}$ is of order $m + m' - \rho$ where*

$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_{\xi}^{\alpha} a \partial_x^{\alpha} b.$$

Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_a T_b - T_{a\#b}\|_{H^{\mu}\to H^{\mu-m-m'+\rho}} \le K M_{\rho}^m(a) M_{\rho}^{m'}(b). \tag{3.5}$$

Remark 3.8. We have the following corollary for poly-homogeneous symbols: if

$$a = \sum_{0\le j<\rho} a^{(m-j)} \in \sum_{0\le j<\rho} \Gamma_{\rho-j}^{m-j}(\mathbf{R}^d), \quad b = \sum_{0\le k<\rho} b^{(m-k)} \in \sum_{0\le k<\rho} \Gamma_{\rho-k}^{m'-k}(\mathbf{R}^d),$$

with $m, m' \in \mathbf{R}$ and $\rho > 0$, then $T_a T_b - T_c$ is of order $m + m' - \rho$ with

$$c = \sum_{|\alpha|+j+k<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_{\xi}^{\alpha} a^{(m-j)} \partial_x^{\alpha} b^{(m'-k)}.$$

Remark 3.9. Clearly a paradifferential operator is not invertible ($T_a u = 0$ for any function u whose spectrum is included in the ball $|\xi| \le 1/2$). However, the previous result implies that there are left and right parametrix for elliptic symbols. Namely, assume that $a \in \Gamma_{\rho}^m$ is an elliptic symbol (such that $|a| \ge K |\xi|^m$ for some $K > 0$), then there exists $b, b' \in \Gamma_{\rho}^{-m}$ such that

$$T_b T_a - I \text{ and } T_a T_{b'} - I \quad \text{are of order } -\rho.$$

Consequently, if $u \in H^s$ and $T_a u \in H^{\mu}$ then $u \in H^r$ with $r = \min\{\mu + m, s + \rho\}$.

Theorem 3.10 (Adjoint)**.** *Let $m \in \mathbf{R}$, $\rho > 0$ and $a \in \Gamma_{\rho}^m(\mathbf{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of T_a and by $\overline{a}$ the complex-conjugated of a. Then $(T_a)^* - T_{a^*}$ is of order $m - \rho$ where*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_{\xi}^{\alpha} \partial_x^{\alpha} \overline{a}.$$

Moreover, for all μ there exists a constant K such that

$$\|(T_a)^* - T_{a^*}\|_{H^{\mu}\to H^{\mu-m+\rho}} \le K M_{\rho}^m(a). \tag{3.6}$$

If $a = a(x)$ is a function of x only, the paradifferential operator T_a is a called a paraproduct. It follows from Theorem 3.7 and Theorem 3.10 that:

(i) If $a \in H^\alpha(\mathbf{R}^d)$ and $b \in H^\beta(\mathbf{R}^d)$ with $\alpha > \frac{d}{2}$, $\beta > \frac{d}{2}$, then

$$T_a T_b - T_{ab} \text{ is of order } -\left(\min\{\alpha,\beta\} - \frac{d}{2}\right). \tag{3.7}$$

(ii) If $a \in H^\alpha(\mathbf{R}^d)$ with $\alpha > \frac{d}{2}$, then

$$(T_a)^* - T_{\overline{a}} \text{ is of order } -\left(\alpha - \frac{d}{2}\right). \tag{3.8}$$

We also have operator norm estimates in terms of the Sobolev norms of the functions.

A first nice feature of paraproducts is that they are well defined for functions $a = a(x)$ which are not in L^∞ but merely in some Sobolev spaces H^r with $r < d/2$.

Lemma 3.11. *Let $m > 0$. If $a \in H^{\frac{d}{2}-m}(\mathbf{R}^d)$ and $u \in H^\mu(\mathbf{R}^d)$ then $T_a u \in H^{\mu-m}(\mathbf{R}^d)$. Moreover,*

$$\|T_a u\|_{H^{\mu-m}} \le K \|a\|_{H^{\frac{d}{2}-m}} \|u\|_{H^\mu},$$

for some positive constant K independent of a and u.

On the other hand, a key feature of paraproducts is that one can replace nonlinear expressions by paradifferential expressions, to the price of error terms which are smoother than the main terms.

Theorem 3.12. *Let $\alpha, \beta \in \mathbf{R}$ be such that $\alpha > \frac{d}{2}$, $\beta > \frac{d}{2}$, then*

(i) For all C^∞ function F, if $a \in H^\alpha(\mathbf{R}^d)$ then

$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{d}{2}}(\mathbf{R}^d).$$

(ii) If $a \in H^\alpha(\mathbf{R}^d)$ and $b \in H^\beta(\mathbf{R}^d)$, then $ab - T_a b - T_b a \in H^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)$. Moreover,

$$\|ab - T_a b - T_b a\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)} \le K \|a\|_{H^\alpha(\mathbf{R}^d)} \|b\|_{H^\beta(\mathbf{R}^d)},$$

for some positive constant K independent of a, b.

We also recall the usual nonlinear estimates in Sobolev spaces (see chapter 8 in [16]):

- If $u_j \in H^{s_j}(\mathbf{R}^d)$, $j = 1, 2$, and $s_1 + s_2 > 0$ then $u_1 u_2 \in H^{s_0}(\mathbf{R}^d)$ and if

$$s_0 \le s_j, \quad j = 1, 2, \quad \text{and} \quad s_0 \le s_1 + s_2 - d/2, \quad \text{then}$$

$$\|u_1 u_2\|_{H^{s_0}} \le K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}}, \tag{3.9}$$

where the last inequality is strict if s_1 or s_2 or $-s_0$ is equal to $d/2$.

- For all C^∞ function F vanishing at the origin, if $u \in H^s(\mathbf{R}^d)$ with $s > d/2$ then

$$\|F(u)\|_{H^s} \le C(\|u\|_{H^s}), \tag{3.10}$$

for some non-decreasing function C depending only on F.

3.2. **Symbol of the Dirichlet-Neumann operator.** Given $\eta \in C^\infty(\mathbf{R}^d)$, consider the domain (without bottom)

$$\Omega = \{(x,y) \in \mathbf{R}^d \times \mathbf{R} \,:\, y < \eta(x)\}.$$

It is well known that the Dirichlet-Neumann operator associated to Ω is a classical elliptic pseudo-differential operator of order 1, whose symbol has an asymptotic expansion of the form

$$\lambda^{(1)}(x,\xi) + \lambda^{(0)}(x,\xi) + \lambda^{(-1)}(x,\xi) + \cdots$$

where $\lambda^{(k)}$ are homogeneous of degree k in ξ, and the principal symbol $\lambda^{(1)}$ and the sub-principal symbol $\lambda^{(0)}$ are given by (cf [20])

$$\begin{aligned} \lambda^{(1)} &= \sqrt{(1+|\nabla\eta|^2)\,|\xi|^2 - (\nabla\eta\cdot\xi)^2}, \\ \lambda^{(0)} &= \frac{1+|\nabla\eta|^2}{2\lambda^{(1)}}\Big\{\operatorname{div}\Big(\alpha^{(1)}\nabla\eta\Big) + i\partial_\xi\lambda^{(1)}\cdot\nabla\alpha^{(1)}\Big\}, \end{aligned} \tag{3.11}$$

with

$$\alpha^{(1)} = \frac{1}{1+|\nabla\eta|^2}\Big(\lambda^{(1)} + i\nabla\eta\cdot\xi\Big).$$

The symbols $\lambda^{(-1)},\ldots$ are defined by induction and we can prove that $\lambda^{(k)}$ involves only derivatives of η of order $|k|+2$.

In our case the function η will not be C^∞ but only at least C^2, so we shall set

$$\lambda = \lambda^{(1)} + \lambda^{(0)}, \tag{3.12}$$

which will be well-defined in the C^2 case.

The following observation contains one of the key dichotomy between 2D waves and 3D waves which can be checked by a direct computation using (3.11).

Proposition 3.13. *If $d=1$ then λ simplifies to $\lambda^{(1)}(x,\xi) = |\xi|\,, \quad \lambda^{(0)}(x,\xi) = 0$.*

Also, directly from (3.11), one can check the following formula (which holds for all $d \geq 1$)

$$\operatorname{Im}\lambda^{(0)} = -\frac{1}{2}(\partial_\xi\cdot\partial_x)\lambda^{(1)}, \tag{3.13}$$

which reflects the fact that the Dirichlet-Neumann operator is a symmetric operator.

3.3. **Paralinearization of the Dirichlet-Neumann operator.** Here is the main result of this section. Following the analysis in [1], we shall paralinearize the Dirichlet-Neumann operator. The main novelties are that we consider the case of finite depth (with a general bottom) and that we lower the regularity assumptions.

Proposition 3.14. *Let $d \geq 1$ and $s > 2 + d/2$. Assume that*

$$(\eta,\psi) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d),$$

and that η is such that $\operatorname{dist}(\Sigma,\Gamma) > 0$. *Then*

$$G(\eta)\psi = T_\lambda\big(\psi - T_{\mathfrak{B}}\eta\big) - T_V\cdot\nabla\eta + f(\eta,\psi), \tag{3.14}$$

where λ is given by (3.11) *and* (3.12),

$$\mathfrak{B} := \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \quad V := \nabla\psi - \mathfrak{B}\nabla\eta,$$

and $f(\eta,\psi) \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$*. Moreover, we have the estimate*

$$\|f(\eta,\psi)\|_{H^{s+\frac{1}{2}}} \le C\left(\|\eta\|_{H^{s+\frac{1}{2}}}\right)\|\nabla\psi\|_{H^{s-1}},$$

for some non-decreasing function C depending only on $\operatorname{dist}(\Sigma,\Gamma)>0$*.*

Remark 3.15. (i) It is well known that $\mathfrak{B}$ and V play a key role in the study of the water waves (these are simply the projection of the velocity field on the vertical and horizontal directions).

(ii) We would like to make a comment on the unknown $\psi - T_{\mathfrak{B}}\eta$. This unknown is related to the so-called 'good unknown' of Alinhac, as it is explained in [1] (see also [4, 21, 33]). It comes from the paracomposition theory of Alinhac which associates to a low regular diffeomorphism χ an operator denoted by χ^* acting on every Sobolev space. Starting with the equation $\Delta_{x,y}\phi = 0$, and making a low regular change of variables χ (which flattens the boundary) leads to the equation $0 = \chi^*\Delta_{x,y}\phi = T_p\chi^*\phi$ (modulo a good remainder) where p is a computable symbol. Then $\psi - T_{\mathfrak{B}}\eta$ is the trace of $\chi^*\phi$ on the boundary.

3.4. **Proof of Proposition 3.14.** Let v be given by (2.7). According to (2.9), v solves

$$\alpha\partial_z^2 v + \Delta v + \beta\cdot\nabla\partial_z v - \gamma\partial_z v = g,$$

where $g \in C_z^2([-1,0];H^{s+\frac{1}{2}}(\mathbf{R}^d))$ is given by (2.8) and

$$\alpha := \frac{1+|\nabla\eta|^2}{h^2}, \quad \beta := -2\frac{\nabla\eta}{h}, \quad \gamma := \frac{\Delta\eta}{h}. \tag{3.15}$$

Also v satisfies the boundary conditions

$$v\,\big|_{z=0} = \psi, \qquad v|_{z=-1} = 0, \quad \partial_z v|_{z=-1} = 0.$$

Henceforth we make intensive use of the following notations.

Notation 3.16. $C_z^0\big(H_x^r\big)$ denotes the space of continuous functions in $z \in [-1,0]$ with values in $H^r(\mathbf{R}^d)$.

It follows from Proposition 2.10 and a classical interpolation argument that

$$(\nabla v, \partial_z v) \in C_z^0\big(H_x^{s-1}\big).$$

In addition, directly from the equation (2.9) and the usual product rule in Sobolev spaces (cf (3.9)), we easily obtain

$$\partial_z^2 v \in C_z^0\big(H_x^{s-2}\big), \quad \partial_z^3 v \in C_z^0\big(H_x^{s-3}\big).$$

3.4.1. *The good unknown of Alinhac.* Below, we use the tangential paradifferential calculus, that is the paradifferential quantization T_a of symbols $a(z;x,\xi)$ depending on the phase space variables $(x,\xi)\in T^*\mathbf{R}^d$ and the parameter $z\in[-1,0]$. In particular, denote by $T_a u$ the operator acting on functions $u=u(z;x)$ so that for each fixed z, $(T_a u)(z)=T_{a(z)}u(z)$.

Note that a simple computation shows

$$G(\eta)\psi=\frac{1+|\nabla\rho|^2}{h}\partial_z v-\nabla\eta\cdot\nabla v\Big|_{z=0}.$$

Our purpose is to express $\partial_z v|_{z=0}$ in terms of tangential derivatives. To do this, the key technical point is to obtain an equation for $\psi-T_{\mathfrak{B}}\eta$.

Note that

$$\psi-T_{\mathfrak{B}}\eta=v-T_{\frac{\partial_z v}{h}}\rho\Big|_{z=0}.$$

We thus introduce

$$\mathfrak{b}:=\frac{\partial_z v}{h}\quad\text{and}\quad u:=v-T_{\mathfrak{b}}\rho=v-T_{\mathfrak{b}}\eta,$$

since $T_{\mathfrak{b}}(hz)=0$, so that $\psi-T_{\mathfrak{B}}\eta=u|_{z=0}$.

Lemma 3.17. *Set*

$$\delta=\min\Big\{\frac{1}{2},s-2-\frac{d}{2}\Big\}>0.$$

The good unknown $u=v-T_{\mathfrak{b}}\rho$ satisfies the paradifferential equation

$$T_\alpha\partial_z^2 u+\Delta u+T_\beta\cdot\nabla\partial_z u-T_\gamma\partial_z u=g+f, \tag{3.16}$$

where α,β,γ are as defined in (3.15), *$g\in C^1_z(H_x^{s+\frac{1}{2}})$ is given by* (2.8) *and*

$$f\in C^0_z\big(H_x^{s-\frac{1}{2}+\delta}\big).$$

Proof. We shall use the notation $f_1\sim f_2$ to say that $f_1-f_2\in C^0_z\big(H_x^{s-\frac{1}{2}+\delta}\big)$.

Introduce the operators

$$E:=\alpha\partial_z^2+\Delta+\beta\cdot\nabla\partial_z-\gamma\partial_z,$$
$$P:=T_\alpha\partial_z^2+\Delta+T_\beta\cdot\nabla\partial_z-T_\gamma\partial_z.$$

We shall prove that $Pu\sim g_1$, where $g_1\in C^0_z\big(H_x^{s+\frac{1}{2}}\big)$. To do so, we begin with the paralinearization formula for products. Recall that

$$\eta\in H^{s+\frac{1}{2}}(\mathbf{R}^d)\quad\text{and}\quad\partial_z^k v\in C^0_z\big(H_x^{s-k}\big)\text{ for }k\in\{1,2\}.$$

According to Theorem 3.12, *ii*), we have

$$Ev\sim Pv+T_{\partial_z^2 v}\alpha+T_{\nabla\partial_z v}\cdot\beta-T_{\partial_z v}\gamma.$$

Since $Ev=g\in C^0_z\big(H_x^{s+\frac{1}{2}}\big)$ and since $v=u+T_{\mathfrak{b}}\eta$, this yields

$$Pu+PT_{\mathfrak{b}}\eta+T_{\partial_z^2 v}\alpha+T_{\nabla\partial_z v}\cdot\beta-T_{\partial_z v}\gamma\sim g\sim 0.$$

Hence, we need only prove that

$$PT_{\mathfrak{b}}\eta+T_{\partial_z^2 v}\alpha+T_{\nabla\partial_z v}\cdot\beta-T_{\partial_z v}\gamma\sim 0. \tag{3.17}$$

By using the Leibniz rule we have

$$PT_{\mathfrak{b}}\eta=T_\alpha T_{\partial_z^2\mathfrak{b}}\eta+T_{\mathfrak{b}}\Delta\eta+2T_{\nabla\mathfrak{b}}\cdot\nabla\eta+T_{\Delta\mathfrak{b}}\eta$$
$$+T_\beta\cdot T_{\nabla\partial_z\mathfrak{b}}\eta+T_\beta\cdot T_{\partial_z\mathfrak{b}}\nabla\eta-T_\gamma T_{\partial_z\mathfrak{b}}\eta.$$

We claim that

$$T_\alpha T_{\partial_z^2 \mathfrak{b}}\eta \sim 0, \quad T_{\Delta \mathfrak{b}}\eta \sim 0, \quad T_\beta \cdot T_{\nabla \partial_z \mathfrak{b}}\eta \sim 0, \quad T_\gamma T_{\partial_z \mathfrak{b}}\eta \sim 0.$$

Since α, β and γ are bounded functions, the operators T_α, T_β and T_γ are of order 0 and hence to prove this claim it is enough to prove that

$$T_{\partial_z^2 \mathfrak{b}}\eta \sim 0, \quad T_{\Delta \mathfrak{b}}\eta \sim 0, \quad T_{\nabla \partial_z \mathfrak{b}}\eta \sim 0, \quad T_{\partial_z \mathfrak{b}}\eta \sim 0. \tag{3.18}$$

To prove these results, we shall use the rule given in Lemma 3.11 for paraproducts whose symbols belong to a Sobolev space of order less than $d/2$. Set $m = 1 - \delta$. Then by definition of δ (and assumption on s) we have

$$m > 0, \quad s - \frac{1}{2} + \delta = s + \frac{1}{2} - m \quad \text{and } s - 3 \geq \frac{d}{2} - m.$$

Therefore Lemma 3.11 implies that

$$\left\| T_a u \right\|_{H^{s-\frac{1}{2}+\delta}} \leq K \left\| a \right\|_{H^{s-3}} \left\| u \right\|_{H^{s+\frac{1}{2}}} . \tag{3.19}$$

Since $\mathfrak{b} = h^{-1}\partial_z v$ and since $\partial_z^k v \in C_z^0\big(H_x^{s-k}\big)$ for $k = 1, 2, 3$, we have

$$\partial_z^2 \mathfrak{b}, \Delta \mathfrak{b}, \nabla \partial_z \mathfrak{b}, \partial_z \mathfrak{b} \in C_z^0(H_x^{s-3}).$$

By applying the estimate (3.19) we end up with the desired results (3.18).

We have proved that

$$P T_{\mathfrak{b}}\eta \sim 2 T_{\nabla \mathfrak{b}} \cdot \nabla \eta + T_{\beta \partial_z \mathfrak{b}} \cdot \nabla \eta + T_{\mathfrak{b}} \Delta \eta.$$

On the other hand, according to (3.15), we have

$$T_{\partial_z v}\gamma = T_{\mathfrak{b}}\Delta \eta, \quad T_{\nabla \partial_z v}\beta = -2 T_{\nabla \mathfrak{b}} \nabla \eta, \quad T_{\beta \partial_z b} \nabla \eta = -\frac{2}{h^2} T_{\partial_z^2 v \nabla \eta} \nabla \eta \sim -T_{\partial_z^2 v}\alpha,$$

where the last equivalence is a consequence of (i) in Theorem 3.12 and (3.7).

Consequently, we end up with the key cancelation

$$T_{\partial_z^2 v}\alpha + T_{\nabla \partial_z v} \cdot \beta - T_{\partial_z v}\gamma + 2 T_{\nabla b} \cdot \nabla \eta + T_{\beta \partial_z \mathfrak{b}} \cdot \nabla \eta + T_{\mathfrak{b}} \Delta \eta \sim 0.$$

This concludes the proof of (3.17) and hence of the lemma. □

3.4.2. *Reduction to the boundary.* Our next task is to perform a decoupling into forward and backward elliptic evolution equations.

Lemma 3.18. *Assume that* $\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ *and recall that*

$$\delta = \min\Big\{ \frac{1}{2}, s - 2 - \frac{d}{2} \Big\} > 0.$$

There exist two symbols $a = a(x, \xi)$, $A = A(x, \xi)$ *(independent of* z*) with*

$$a = a^{(1)} + a^{(0)} \in \dot{\Gamma}^1_{3/2+\delta}(\mathbf{R}^d) + \dot{\Gamma}^0_{1/2+\delta}(\mathbf{R}^d),$$
$$A = A^{(1)} + A^{(0)} \in \dot{\Gamma}^1_{3/2+\delta}(\mathbf{R}^d) + \dot{\Gamma}^0_{1/2+\delta}(\mathbf{R}^d),$$

such that,

$$T_\alpha \partial_z^2 + \Delta + T_\beta \cdot \nabla \partial_z - T_\gamma \partial_z = T_\alpha(\partial_z - T_a)(\partial_z - T_A)u + R_0 + R_1 \partial_z, \tag{3.20}$$

where R_0 *is of order* $1/2 - \delta$ *and* R_1 *is of order* $-1/2 - \delta$.

Proof. We seek a and A such that

$$(3.21)\qquad \begin{aligned} &a^{(1)}A^{(1)} + \frac{1}{i}\partial_\xi a^{(1)}\cdot\partial_x A^{(1)} + a^{(1)}A^{(0)} + a^{(0)}A^{(1)} = -\frac{|\xi|^2}{\alpha},\\ &a + A = \frac{1}{\alpha}\left(-i\beta\cdot\xi+\gamma\right). \end{aligned}$$

According to Theorem 3.7 and (3.7),

$$R_0 := T_\alpha T_a T_A - \Delta \quad \text{is of order } 2-\frac{3}{2}-\delta = \frac{1}{2}-\delta,$$

while the second equation gives

$$R_1 := -T_\alpha(T_a+T_A) + \big(T_\beta\cdot\nabla - T_\gamma\big) \quad \text{is of order } 1-\frac{3}{2}-\delta = -\frac{1}{2}-\delta.$$

We thus obtain the desired result (3.20) from (3.16).

To solve (3.21), we first solve the principal system:

$$a^{(1)}A^{(1)} = -\frac{|\xi|^2}{\alpha}, \qquad a^{(1)}+A^{(1)} = -\frac{i\beta\cdot\xi}{\alpha},$$

by setting

$$\begin{aligned} a^{(1)}(x,\xi) &= \frac{1}{2\alpha}\Big(-i\beta\cdot\xi - \sqrt{4\alpha\,|\xi|^2-(\beta\cdot\xi)^2}\Big),\\ A^{(1)}(x,\xi) &= \frac{1}{2\alpha}\Big(-i\beta\cdot\xi + \sqrt{4\alpha\,|\xi|^2-(\beta\cdot\xi)^2}\Big). \end{aligned}$$

Directly from the definition of α and β note that

$$\sqrt{4\alpha\,|\xi|^2-(\beta\cdot\xi)^2} \geq \frac{2}{h}\,|\xi|\,,$$

so that the symbols $a^{(1)}, A^{(1)}$ belong to $\dot\Gamma^1_{3/2+\delta}(\mathbf{R}^d)$ (actually $a^{(1)}, A^{(1)}$ belong to $\dot\Gamma^1_{s-(d+1)/2}(\mathbf{R}^d)$ provided that $s-(d+1)/2$ is not an integer).

We next solve the system

$$a^{(0)}A^{(1)} + a^{(1)}A^{(0)} + \frac{1}{i}\partial_\xi a^{(1)}\partial_x A^{(1)} = 0, \qquad a^{(0)}+A^{(0)} = \frac{\gamma}{\alpha}.$$

It is found that

$$\begin{aligned} a^{(0)} &= \frac{1}{A^{(1)}-a^{(1)}}\left(i\partial_\xi a^{(1)}\cdot\partial_x A^{(1)} - \frac{\gamma}{\alpha}a^{(1)}\right),\\ A^{(0)} &= \frac{1}{a^{(1)}-A^{(1)}}\left(i\partial_\xi a^{(1)}\cdot\partial_x A^{(1)} - \frac{\gamma}{\alpha}A^{(1)}\right), \end{aligned}$$

so that the symbols $a^{(0)}, A^{(0)}$ belong to $\dot\Gamma^0_{1/2+\delta}(\mathbf{R}^d)$. □

We shall need the following elliptic regularity result.

Proposition 3.19. *Let $a\in\Gamma^1_1(\mathbf{R}^d)$ and $b\in\Gamma^0_0(\mathbf{R}^d)$, with the assumption that*

$$\operatorname{Re} a(x,\xi) \geq c\,|\xi|\,,$$

for some positive constant c. If $w\in C^1_z(H^{-\infty}_x)$ solves the elliptic evolution equation

$$\partial_z w + T_a w = T_b w + f,$$

with $f \in C^0_z\big(H^r_x\big)$ for some $r \in \mathbf{R}$, then for all $\varepsilon > 0$ we have

$$w(0) \in H^{r+1-\varepsilon}(\mathbf{R}^d). \tag{3.22}$$

Remark 3.20. This is a local result which means that the conclusion (3.22) remains true if we only assume that, for some $\delta > 0$,

$$f|_{-1\le z\le -\delta} \in C^0([-1,-\delta]; H^{-\infty}(\mathbf{R}^d)), \quad f|_{-\delta\le z\le 0} \in C^0([-\delta,0]; H^r(\mathbf{R}^d)).$$

In addition, the result still holds true for symbols $a \in C^0_z(\Gamma^1_1)$ and $b \in C^0_z(\Gamma^0_0)$, with the assumption that $\operatorname{Re} a \ge c\,|\xi|$, for some positive constant c.

Proof. The following proof gives the stronger conclusion that w is continuous in $z \in]-1,0]$ with values in $H^{r+1-\varepsilon}(\mathbf{R}^d)$. Therefore, by an elementary induction argument, we can assume without loss of generality that $b = 0$ and $w \in C^0_z\big(H^r_x\big)$. In addition one can assume that there exists $\delta > 0$ such that $w(x,z) = 0$ for $z \le -1/2$.

For $z \in [-1,0]$, introduce the symbol

$$e(z; x, \xi) := \exp\left(za(x,\xi)\right),$$

so that $e|_{z=0} = 1$ and $\partial_z e = ea$. Since $\operatorname{Re} a \ge c\,|\xi|$, we have the simple estimates

$$(|z|\,|\xi|)^m e(z;x,\xi) \le C_m.$$

Write

$$\partial_z\left(T_e w\right) = T_e f + (T_{\partial_z e} - T_e T_a) w,$$

and integrate on $[-1,0]$ to obtain

$$T_1 w(0) = \int_{-1}^0 (T_{\partial_z e} - T_e T_a) w(y)\,dy + \int_{-1}^0 (T_e f)(y)\,dy.$$

Since $w(0) - T_1 w(0) \in H^{+\infty}(\mathbf{R}^d)$ it remains only to prove that the right-hand side belongs to $H^{r+1-\varepsilon}(\mathbf{R}^d)$. Set

$$w_1(0) = \int_{-1}^0 (T_{\partial_z e} - T_e T_a) w(y)\,dy, \qquad w_2(0) = \int_{-1}^0 (T_e f)(y)\,dy.$$

To prove that $w_2(0)$ belongs to $H^{r+1-\varepsilon}(\mathbf{R}^d)$, the key observation is that, since $\operatorname{Re} a \ge c\,|\xi|$, the family

$$\left\{ (|y|\,|\xi|)^{1-\varepsilon} e(y;x,\xi) \;:\; -1 \le y \le 0 \right\}$$

is bounded in $\Gamma^0_1(\mathbf{R}^d)$. According to the operator norm estimate (3.4), we thus obtain that there is a constant K such that, for all $-1 \le y \le 0$ and all $v \in H^r(\mathbf{R}^d)$,

$$\left\| (|y|\,|D_x|)^{1-\varepsilon} (T_e v) \right\|_{H^r} \le K \left\| v \right\|_{H^r}.$$

Consequently, there is a constant K such that, for all $y \in [-1,0[$,

$$\left\| (T_e f)(y) \right\|_{H^{r+1-\varepsilon}} \le \frac{K}{|y|^{1-\varepsilon}} \left\| f(y) \right\|_{H^r}.$$

Since $|y|^{-(1-\varepsilon)} \in L^1(]-1,0[)$, this implies that $w_2(0) \in H^{r+1-\varepsilon}(\mathbf{R}^d)$.

With regards to the first term, we claim that, similarly,

$$\left\| (T_{\partial_z e} - T_e T_a)(y) \right\|_{H^r \to H^{r+1-\varepsilon}} \le \frac{K}{|y|^{1-\varepsilon}}.$$

Indeed, since $\partial_z e = ea$, this follows from (3.5) applied with $(m, m', r) = (-1+\varepsilon, 1, 1)$ and the fact that $M_1^{-1+\varepsilon}((|y|^{1-\varepsilon} e(y;\cdot,\cdot))$ is uniformly bounded for $-1 \le y \le 0$. This yields the desired result. □

We are now in position to describe the boundary value of $\partial_z u$ up to an error in $H^{s+\frac{1}{2}}(\mathbf{R}^d)$.

Corollary 3.21. *Let A be as given by Lemma* 3.18*. Then, on the boundary $\{z=0\}$, there holds*

$$(\partial_z u - T_A u)|_{z=0} \in H^{s+\frac{1}{2}}(\mathbf{R}^d).$$

Proof. Introduce $w := (\partial_z - T_A)u$ and write

$$\partial_z w - T_{a^{(1)}} w = T_{a^{(0)}} w + f',$$

with $f' \in C^0_z\big(H_x^{s-\frac{1}{2}+\delta}\big)$. Since $\operatorname{Re} a^{(1)} < -c|\xi|$, the previous proposition applied with $a = -a^{(1)}$, $b = a^{(0)}$ and $\varepsilon = \delta > 0$ implies that $w|_{z=0} \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$. □

By definition

$$G(\eta)\psi = \frac{1+|\nabla\eta|^2}{h}\partial_z v - \nabla\eta\cdot\nabla v\,\Big|_{z=0}.$$

As before, we find that

$$\begin{aligned}\frac{1+|\nabla\eta|^2}{h}\partial_z v - \nabla\eta\cdot\nabla v = T_{\frac{1+|\nabla\eta|^2}{h}}\partial_z v + 2T_{\mathfrak{b}\nabla\eta}\cdot\nabla\eta - T_{\mathfrak{b}\frac{1+|\nabla\eta|^2}{h}} h \\ - (T_{\nabla\eta}\cdot\nabla v + T_{\nabla v}\cdot\nabla\eta) + R,\end{aligned}$$

where $R \in C^0_z\big(H_x^{2s-\frac{3+d}{2}}\big)$. We next replace $\partial_z v$ and ∇v by $\partial_z(u + T_{\mathfrak{b}}\rho)$ and $\nabla(u + T_{\mathfrak{b}}\rho)$ in the right hand-side to obtain, after a few computations,

$$\frac{1+|\nabla\eta|^2}{h}\partial_z v - \nabla\eta\cdot\nabla v = T_{\frac{1+|\nabla\eta|^2}{h}}\partial_z u - T_{\nabla\eta}\cdot\nabla u - T_{\nabla v - \mathfrak{b}\nabla\eta}\cdot\nabla\rho - T_{\operatorname{div}(\nabla v - \mathfrak{b}\nabla\eta)}\rho + R',$$

with $R' \in C^0_z\big(H_x^{2s-\frac{3+d}{2}}\big)$. Furthermore, Corollary 3.21 implies that

$$T_{\frac{1+|\nabla\eta|^2}{h}}\partial_z u - T_{\nabla\eta}\cdot\nabla u\,\Big|_{z=0} = T_\lambda U + r, \tag{3.23}$$

with $U = u|_{z=0} = v - T_{\mathfrak{b}}\rho|_{z=0} = \psi - T_{\mathfrak{B}}\eta$, $r \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$ and

$$\lambda = \frac{1+|\nabla\eta|^2}{h}A - i\nabla\eta\cdot\xi\,\Big|_{z=0}. \tag{3.24}$$

After a few computations, we check that λ is as given by (3.11)–(3.12).

This concludes the analysis of the Dirichlet-Neumann operator. Indeed, we have obtained

$$G(\eta)\psi = T_\lambda U - T_{\nabla v - \mathfrak{b}\nabla\eta}\cdot\nabla\eta - T_{\operatorname{div}(\nabla v - \mathfrak{b}\nabla\eta)}\rho + f(\eta,\psi),$$

with $f(\eta,\psi) \in H^{s+\frac{1}{2}}(\mathbf{R}^d)$. This yields the first equation in (3.14) since

$$V = \nabla v - \mathfrak{b}\nabla\eta|_{z=0}, \quad \nabla\eta|_{z=0} = \nabla\eta,$$

and since

$$T_{\operatorname{div} V}\eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d).$$

3.5. **A simpler case.** Let us remark that if $(\eta,\psi)\in H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{s-1}(\mathbf{R}^d)$, the expressions above can be simplified and we have the following result that we shall use in Section 6.2.

Proposition 3.22. *Let $d\geq 1$, $s>2+d/2$ and $1\leq\sigma\leq s-1$. Assume that*

$$(\eta,\psi)\in H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^{\sigma}(\mathbf{R}^d),$$

and that η is such that $\operatorname{dist}(\Sigma,\Gamma)>0$. *Then*

$$G(\eta)\psi=T_{\lambda^{(1)}}\psi+F(\eta,\psi),$$

where $F(\eta,\psi)\in H^{\sigma}(\mathbf{R}^d)$ (and recall that $\lambda^{(1)}$ denotes the principal symbol of the Dirichlet-Neumann operator). Moreover,

$$\left\lVert F(\eta,\psi)\right\rVert_{H^{\sigma}}\leq C\left(\left\lVert\eta\right\rVert_{H^{s+\frac{1}{2}}}\right)\left\lVert\nabla\psi\right\rVert_{H^{\sigma-1}},$$

for some non-decreasing function C depending only on $\operatorname{dist}(\Sigma,\Gamma)>0$.

Remark 3.23. Notice that the proof below would still work assuming only

$$\eta\in H^{s+\varepsilon}(\mathbf{R}^d),\quad v\in C^0_z\big(H^{\sigma}_x\big),$$

with the same conclusion. A more involved proof (using regularized lifting for the function η following Lannes [21]) would give the result assuming only

$$(\eta,\psi)\in H^{s}(\mathbf{R}^d)\times H^{\sigma}(\mathbf{R}^d).$$

Proof. We follow the proof of Proposition 3.14. Let v be as given by (2.7): v solves

$$\alpha\partial_z^2 v+\Delta v+\beta\cdot\nabla\partial_z v-\gamma\partial_z v=g,$$

where $g\in C^0([-1,0];H^{s+\frac{1}{2}}(\mathbf{R}^d))$ is given by (2.8) and

$$\alpha:=\frac{(1+|\nabla\eta|^2)}{h^2},\quad \beta:=-2\frac{\nabla\eta}{h},\quad \gamma:=\frac{\Delta\eta}{h}.$$

Comparing with the proof of Proposition 3.14, an important simplification is that we need only in this proof to paralinearize with respect to v. In this direction, we claim that

$$T_\alpha\partial_z^2 v+\Delta v+T_\beta\cdot\nabla\partial_z v-T_\gamma\partial_z v\in C^0_z\big(H^{\sigma-\frac{1}{2}}_x\big). \tag{3.25}$$

To see this we first apply point (ii) in Theorem 3.12 to obtain

$$\alpha\partial_z^2 v-T_\alpha\partial_z^2 v-T_{\partial_z^2 v}\alpha\in C^0_z\big(H^{(s-\frac{1}{2})+\sigma-2-d/2}_x\big)\subset C^0_z\big(H^{\sigma-\frac{1}{2}}_x\big),$$

and similarly

$$\beta\cdot\nabla\partial_z v-T_\beta\cdot\nabla\partial_z v-T_{\nabla\partial_z v}\cdot\beta\in C^0_z\big(H^{\sigma-\frac{1}{2}}_x\big),$$

$$\gamma\partial_z v-T_\gamma\partial_z v-T_{\partial_z v}\gamma\in C^0_z\big(H^{\sigma-\frac{1}{2}}_x\big).$$

Moreover, writing $\sigma-2=d/2-(d/2+2-\sigma)$, using Lemma 3.11 with $m=d/2+2-\sigma$, we obtain

$$T_{\partial_z^2 v}\alpha\in C^0_z\big(H^{s-\frac{1}{2}-(d/2+2-\sigma)}_x\big)\subset C^0_z\big(H^{\sigma-\frac{1}{2}}_x\big),$$

and

$$T_{\nabla\partial_z v}\cdot\beta\in C^0_z\big(H^{\sigma-\frac{1}{2}}_x\big).$$

Similarly, we have

$$T_{\partial_z v}\gamma \in C^0_z\big(H^{\sigma-\frac{1}{2}}_x\big).$$

Therefore, summing up directly gives the desired result (3.25).

Now, by applying Lemma 3.18, we obtain that

$$T_\alpha \partial_z^2 + \Delta v + T_\beta \cdot \nabla \partial_z v - T_\gamma \partial_z v = T_\alpha(\partial_z - T_a)(\partial_z - T_A)v + f$$

with $f = R_0 v + R_1 \partial_z v \in C^0_z\big(H^{\sigma-1+\delta}_x\big)$ where $\delta = \min\left\{\frac{1}{2}, s-2-\frac{d}{2}\right\} > 0$.

Then, as in Corollary 3.21, we deduce that

$$(\partial_z v - T_A v)|_{z=0} \in H^\sigma(\mathbf{R}^d).$$

Since $v(0) \in H^{s-1}(\mathbf{R}^d)$ we deduce $T_{A^{(0)}} v|_{z=0} \in H^{s-1}(\mathbf{R}^d) \subset H^\sigma(\mathbf{R}^d)$ ($A^{(0)}$ is the sub-principal symbol of A, which is of order 0) and hence

$$(\partial_z v - T_{A^{(1)}} v)|_{z=0} \in H^\sigma(\mathbf{R}^d).$$

The rest of the proof is as in the proof of Proposition 3.14 □

3.6. **Paralinearization of the full system.** Consider a given solution (η, ψ) of system (1.2) on the time interval $[0, T]$ with $0 < T < +\infty$, such that

$$(\eta, \psi) \in C^0\big([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)\big),$$

for some $s > 2 + d/2$, with $d \geq 1$.

In the sequel we consider functions of (t, x), considered as functions of t with values in various spaces of functions of x. In particular, denote by $T_a u$ the operator acting on u so that for each fixed t, $(T_a u)(t) = T_{a(t)} u(t)$.

The main result of this paragraph is a paralinearization of the water-waves system (1.2).

Proposition 3.24. *Introduce $U := \psi - T_{\mathfrak{B}}\eta$. Then (η, U) satisfies a system of the form*

$$\left\{\begin{aligned} &\partial_t \eta + T_V \cdot \nabla \eta - T_\lambda U = f_1, \\ &\partial_t U + T_V \cdot \nabla U + T_\ell \eta = f_2, \end{aligned}\right. \tag{3.26}$$

with

$$f_1 \in L^\infty\big(0, T; H^{s+\frac{1}{2}}(\mathbf{R}^d)\big), \quad f_2 \in L^\infty\big(0, T; H^s(\mathbf{R}^d)\big).$$

Moreover,

$$\left\|(f_1, f_2)\right\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)} \leq C\left(\left\|(\eta, \psi)\right\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}\right),$$

for some function C depending only on $\operatorname{dist}(\Sigma_0, \Gamma)$.

At this point, we have already performed the paralinearization of the Dirichlet-Neumann operator. We now paralinearize the nonlinear terms which appear in the dynamic boundary condition. This step is much easier.

Lemma 3.25. *There holds $H(\eta) = -T_\ell \eta + f$, where $\ell = \ell^{(2)} + \ell^{(1)}$ with*

$$\begin{aligned} &\ell^{(2)} = \big(1 + |\nabla\eta|^2\big)^{-\frac{1}{2}}\left(|\xi|^2 - \frac{(\nabla\eta\cdot\xi)^2}{1 + |\nabla\eta|^2}\right), \\ &\ell^{(1)} = -\frac{i}{2}(\partial_x \cdot \partial_\xi)\ell^{(2)}, \end{aligned} \tag{3.27}$$

and $f \in L^\infty(0,T;H^{2s-2-d/2})$ *is such that*

$$\|f\|_{L^\infty(0,T;H^{2s-2-\frac{d}{2}})} \le C(\|\eta\|_{L^\infty(0,T;H^{s+1/2})}), \tag{3.28}$$

for some non-decreasing function C.

Proof. Theorem 3.12 applied with $\alpha = s - 1/2$ implies that

$$\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}} = T_M\nabla\eta + \widetilde{f}, \qquad M = \frac{1}{\sqrt{1+|\nabla\eta|^2}}I - \frac{\nabla\eta\otimes\nabla\eta}{(1+|\nabla\eta|^2)^{3/2}},$$

where $\tilde{f} \in L^\infty(0,T;H^{2s-1-\frac{d}{2}})$ is such that

$$\left\|\widetilde{f}\right\|_{L^\infty(0,T;H^{2s-1-\frac{d}{2}})} \le C(\|\eta\|_{L^\infty(0,T;H^{s+\frac{1}{2}})}),$$

for some non-decreasing function C. Since

$$\operatorname{div}(T_M\nabla\eta) = T_{-M\xi\cdot\xi+i\operatorname{div}M\xi}\eta,$$

we obtain the desired result with $\ell^{(2)} = M\xi\cdot\xi$, $\ell^{(1)} = -i\operatorname{div}M\xi$ and $f = \operatorname{div}\widetilde{f}$. □

Recall the notations

$$\mathfrak{B} = \frac{\nabla\eta\cdot\nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \quad V = \nabla\psi - \mathfrak{B}\nabla\eta. \tag{3.29}$$

Lemma 3.26. *We have*

$$\frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = T_V\cdot\nabla\psi - T_{\mathfrak{B}}T_V\cdot\nabla\eta - T_{\mathfrak{B}}G(\eta)\psi + f',$$

where $f' \in L^\infty(0,T;H^{2s-2-\frac{d}{2}}(\mathbf{R}^d))$ *satisfies*

$$\|f'\|_{L^\infty(0,T;H^{2s-2-\frac{d}{2}})} \le C\left(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}\right),$$

for some non-decreasing function C.

Proof. Again, we shall use the paralinearization lemma. Note that for

$$F(a,b,c) = \frac{1}{2}\frac{(a\cdot b+c)^2}{1+|a|^2} \quad (a\in\mathbf{R}^d, b\in\mathbf{R}^d, c\in\mathbf{R})$$

there holds

$$\partial_a F = \frac{(a\cdot b+c)}{1+|a|^2}\Big(b - \frac{(a\cdot b+c)}{1+|a|^2}a\Big),\ \partial_b F = \frac{(a\cdot b+c)}{1+|a|^2}a,\ \partial_c F = \frac{(a\cdot b+c)}{1+|a|^2}.$$

Using these identities for $a = \nabla\eta$, $b = \nabla\psi$ and $c = G(\eta)\psi$, the paralinearization lemma (cf (i) in Theorem 3.12) implies that

$$\frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = \{T_{V\mathfrak{B}}\cdot\nabla\eta + T_{\mathfrak{B}\nabla\eta}\nabla\psi + T_{\mathfrak{B}}G(\eta)\psi\} + r,$$

with $r \in L^\infty(0,T;H^{2s-2-\frac{d}{2}}(\mathbf{R}^d))$ satisfies the desired estimate. Since $V = \nabla\psi - \mathfrak{B}\nabla\eta$, this yields

$$\frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = \{T_V\cdot\nabla\psi - T_{V\mathfrak{B}}\cdot\nabla\eta - T_{\mathfrak{B}}G(\eta)\psi\} + r'$$

with $r' \in L^\infty(0,T;H^{2s-2-\frac{d}{2}}(\mathbf{R}^d))$. Since by (3.7)

$$T_{\mathfrak{B}V} - T_{\mathfrak{B}}T_V \quad \text{is of order } -\left(s-1-\frac{d}{2}\right),$$

this completes the proof. □

Lemma 3.27. *There exists a function C such that,*

$$\left\|T_{\partial_t \mathfrak{B}}\eta\right\|_{H^s} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

Proof. **a)** We claim that

(3.30)

$$\|\partial_t\eta\|_{H^{s-1}} + \|\partial_t\psi\|_{H^{s-\frac32}} + \|\mathfrak{B}\|_{H^{s-1}} + \|V\|_{H^{s-1}} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

The proof of this claim is straightforward (using the definition of $\mathfrak{B}$ (3.29)). It follows from Proposition 2.7 that we have the estimate

$$\|G(\eta)\psi\|_{H^{s-1}} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

Using that H^{s-1} is an algebra since $s-1 > d/2$, we thus get the desired estimate for $\mathfrak{B}$. This in turn implies that $V = \nabla\psi - \mathfrak{B}\nabla\eta$ satisfies the desired estimate. In addition, since $\partial_t\eta = G(\eta)\psi$, this gives the estimate of $\|\partial_t\eta\|_{H^{s-1}}$. To estimate $\partial_t\psi$ we simply write that

$$\partial_t\psi = F(\nabla\psi,\nabla\eta,\nabla^2\eta),$$

for some C^∞ function F vanishing at the origin. Consequently, since $s - 3/2 > d/2$, the usual nonlinear rule in Sobolev space implies that

$$\|\partial_t\psi\|_{H^{s-3/2}} \le C\left(\left\|(\nabla\psi,\nabla\eta,\nabla^2\eta)\right\|_{H^{s-3/2}}\right) \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

b) We are now in position to estimate $\partial_t\mathfrak{B}$. We claim that

$$\|\partial_t\mathfrak{B}\|_{H^{s-\frac52}} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right). \tag{3.31}$$

In view of (3.30) and the product rule (3.9), the only non trivial point is to estimate $\partial_t[G(\eta)\psi]$. To do so, we use the identity for the shape derivative of the Dirichlet-Neumann (see §2.3) to obtain

$$\partial_t[G(\eta)\psi] = G(\eta)\left(\partial_t\psi - \mathfrak{B}\partial_t\eta\right) - \operatorname{div}(V\partial_t\eta).$$

Therefore (3.30) and the boundedness of $G(\eta)$ on Sobolev spaces (cf Proposition 2.7) imply that

$$\|\partial_t[G(\eta)\psi]\|_{H^{s-\frac52}} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

This proves (3.31).

c) Next we use Lemma 3.11 with $m = 1/2$ (which asserts that if $a \in H^{\frac{d}{2}-\frac12}(\mathbf{R}^d)$ then the paraproduct T_a is of order $1/2$). Therefore, since by assumption $s - 5/2 > d/2 - 1/2$ for all $d \ge 1$, we conclude

$$\left\|T_{\partial_t\mathfrak{B}}\eta\right\|_{H^s} \le \|T_{\partial_t\mathfrak{B}}\|_{H^{s+\frac12}\to H^s}\|\eta\|_{H^{s+\frac12}} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

This completes the proof. □

End of the proof of Proposition 3.24. Using the equation satisfied by ψ and Lemmas 3.25-3.26, we obtain

$$\partial_t\psi + T_\ell\eta + T_V\cdot\nabla\psi - T_{\mathfrak{B}}T_V\cdot\nabla\eta - T_{\mathfrak{B}}G(\eta)\psi = F \in L^\infty\big(0,T;H^s(\mathbf{R}^d)\big).$$

Since $U = \psi - T_{\mathfrak{B}}\eta$, we get

$$\partial_t U = \partial_t\psi - T_{\mathfrak{B}}\partial_t\eta - T_{\partial_t\mathfrak{B}}\eta.$$

Now we have $G(\eta)\psi = \partial_t\eta$ and

$$T_V\cdot\nabla\psi - T_{\mathfrak{B}}T_V\cdot\nabla\eta - T_V\cdot\nabla U \in L^\infty\big(0,T;H^s(\mathbf{R}^d)\big).$$

So using Lemma 3.27 we obtain the desired result. □

4. Symmetrization

Consider a solution (η,ψ) of (1.2) on the time interval $[0,T]$ with $0 < T < +\infty$, such that

$$(\eta,\psi)\in C^0\big([0,T];H^{s+\frac12}(\mathbf{R}^d)\times H^s(\mathbf{R}^d)\big),$$

for some $s > 2 + d/2$, with $d \geq 1$. We proved in Proposition 3.24 that η and $U = \psi - T_{\mathfrak{B}}\eta$ satisfy the system

$$(\partial_t + T_V\cdot\nabla)\begin{pmatrix}\eta\\ U\end{pmatrix} + \begin{pmatrix}0 & -T_\lambda\\ T_\ell & 0\end{pmatrix}\begin{pmatrix}\eta\\ U\end{pmatrix} = f, \tag{4.1}$$

where $f\in L^\infty(0,T;H^{s+\frac12}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$. The main result of this section is that there exists a symmetrizer S of the form

$$S = \begin{pmatrix}T_p & 0\\ 0 & T_q\end{pmatrix},$$

which conjugates $\left(\begin{smallmatrix}0 & -T_\lambda\\ T_\ell & 0\end{smallmatrix}\right)$ to a skew-symmetric operator. Indeed we shall prove that there exists S such that, modulo admissible remainders,

$$S\begin{pmatrix}0 & -T_\lambda\\ T_\ell & 0\end{pmatrix} \simeq \begin{pmatrix}0 & -T_\gamma\\ (T_\gamma)^* & 0\end{pmatrix} S.$$

In addition, we shall obtain that the new unknown

$$\Phi = S\begin{pmatrix}\eta\\ U\end{pmatrix}$$

satisfies a system of the form

$$\partial_t\Phi + T_V\cdot\nabla\Phi + \begin{pmatrix}0 & -T_\gamma\\ (T_\gamma)^* & 0\end{pmatrix}\Phi = F, \tag{4.2}$$

with $F\in L^\infty(0,T;H^s(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$; moreover $\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}$ is controlled by means of $\|\Phi\|_{H^s}$.

This symmetrization has many consequences. In particular, in the following sections, we shall deduce our two main results from this symmetrization.

4.1. **Symbolic calculus with low regularity.** All the symbols which we consider below are of the form

$$a = a^{(m)} + a^{(m-1)}$$

where

(i) $a^{(m)}$ is a real-valued elliptic symbol, homogenous of degree m in ξ and depends only on the first order-derivatives of η;

(ii) $a^{(m-1)}$ is homogenous of degree $m-1$ in ξ and depends also, but only linearly, on the second order-derivatives of η.

Recall that in this section $\eta \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}^d))$ is a fixed given function.

Definition 4.1. *Given $m \in \mathbf{R}$, Σ^m denotes the class of symbols a of the form*

$$a = a^{(m)} + a^{(m-1)}$$

with

$$a^{(m)}(t,x,\xi) = F(\nabla\eta(t,x),\xi),$$
$$a^{(m-1)}(t,x,\xi) = \sum_{|\alpha|=2} G_\alpha(\nabla\eta(t,x),\xi)\partial_x^\alpha \eta(t,x),$$

such that

(i) T_a maps real-valued functions to real-valued functions;

(ii) F is a C^∞ real-valued function of $(\zeta,\xi) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, homogeneous of order m in ξ; and such that there exists a continuous function $K = K(\zeta) > 0$ such that

$$F(\zeta,\xi) \geq K(\zeta)\,|\xi|^m\,,$$

for all $(\zeta,\xi) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$;

(iii) G_α is a C^∞ complex-valued function of $(\zeta,\xi) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, homogeneous of order $m-1$ in ξ.

Notice that, as we only assume $s > 2 + d/2$, some technical difficulties appear. To overcome these problems, the observation that for all our symbols, the sub-principal terms have only a linear dependence on the second order derivative of η will play a crucial role.

Our first result contains the important observation that the previous class of symbols is stable by the standard rules of symbolic calculus (this explains why all the symbols which we shall introduce below are of this form). We shall state a symbolic calculus result modulo admissible remainders. To clarify the meaning of admissible remainder, we introduce the following notation.

Definition 4.2. *Let $m \in \mathbf{R}$ and consider two families of operators order m,*

$$\{A(t) \,:\, t \in [0,T]\}, \quad \{B(t) \,:\, t \in [0,T]\}.$$

We shall say that $A \sim B$ if $A - B$ is of order $m - 3/2$ (see Definition 3.5) and satisfies the following estimate: for all $\mu \in \mathbf{R}$, there exists a continuous function C such that for all $t \in [0,T]$,

$$\left\| A(t) - B(t) \right\|_{H^\mu \to H^{\mu-m+\frac{3}{2}}} \leq C\left(\left\| \eta(t) \right\|_{H^{s+\frac{1}{2}}} \right).$$

Proposition 4.3. *Let $m, m' \in \mathbf{R}$. Then*

(1) If $a \in \Sigma^m$ and $b \in \Sigma^{m'}$ then $T_a T_b \sim T_{a\sharp b}$ where $a\sharp b \in \Sigma^{m+m'}$ is given by

$$a\sharp b = a^{(m)}b^{(m')} + a^{(m-1)}b^{(m')} + a^{(m)}b^{(m'-1)} + \frac{1}{i}\partial_\xi a^{(m)} \cdot \partial_x b^{(m')}.$$

(2) If $a \in \Sigma^m$ then $(T_a)^* \sim T_b$ where $b \in \Sigma^m$ is given by

$$b = a^{(m)} + \overline{a^{(m-1)}} + \frac{1}{i}(\partial_x \cdot \partial_\xi)a^{(m)}.$$

Proof. It follows from (3.5) applied with $\rho = 3/2$ that

$$\left\| T_{a^{(m)}}T_{b^{(m')}} - T_{a^{(m)}b^{(m')}+\frac{1}{i}\partial_\xi a^{(m)}\cdot\partial_x b^{(m')}} \right\|_{H^\mu \to H^{\mu-m-m'+3/2}} \le C(\|\nabla\eta\|_{W^{3/2,\infty}}).$$

On the other hand, (3.5) applied with $\rho = 1/2$ implies that

$$\begin{aligned}
&\left\|T_{a^{(m)}}T_{b^{(m'-1)}} - T_{a^{(m)}b^{(m'-1)}}\right\|_{H^\mu \to H^{\mu-m-m'+3/2}} \le C(\|\nabla\eta\|_{W^{3/2,\infty}}),\\
&\left\|T_{a^{(m-1)}}T_{b^{(m')}} - T_{a^{(m-1)}b^{(m')}}\right\|_{H^\mu \to H^{\mu-m-m'+3/2}} \le C(\|\nabla\eta\|_{W^{3/2,\infty}}).
\end{aligned}$$

Eventually (3.4) implies that

$$\left\|T_{a^{(m-1)}}T_{b^{(m'-1)}}\right\|_{H^\mu \to H^{\mu-m-m'+2}} \le C(\|\nabla\eta\|_{W^{1,\infty}}).$$

The first point in the proposition then follows from the embedding $H^{s+\frac{1}{2}}(\mathbf{R}^d) \subset W^{\frac{5}{2},\infty}(\mathbf{R}^d)$. Furthermore, we easily verify that $a\sharp b \in \Sigma^{m+m'}$.

Similarly, the second point is a straightforward consequence of Theorem 3.10 and the fact that $a^{(m)}$ is, by assumption, a real-valued symbol. □

Given that $a \in \Sigma^m$, since $a^{(m-1)}$ involves two derivatives of η, the usual boundedness result for paradifferential operators and the embedding $H^s(\mathbf{R}^d) \subset W^{2,\infty}(\mathbf{R}^d)$ implies that we have estimates of the form

(4.3)

$$\left\|T_{a(t)}\right\|_{H^\mu \to H^{\mu-m}} \lesssim \sup_{|\alpha|\le \frac{d}{2}+1} \sup_{|\xi|=1} |\xi|^{|\alpha|-r} \left\|\partial_\xi^\alpha a(t,\cdot,\xi)\right\|_{L^\infty} \le C\left(\|\eta(t)\|_{H^s}\right).$$

Our second observation concerning the class Σ^m is that one can prove a continuity result which requires only an estimate of $\|\eta\|_{H^{s-1}}$.

Proposition 4.4. *Let $m \in \mathbf{R}$ and $\mu \in \mathbf{R}$. Then there exists a function C such that for all symbol $a \in \Sigma^m$ and all $t \in [0,T]$,*

$$\left\|T_{a(t)}u\right\|_{H^{\mu-m}} \le C(\|\eta(t)\|_{H^{s-1}}) \|u\|_{H^\mu}.$$

Remark 4.5. This result is obvious for $s > 3 + d/2$ since the L^∞-norm of $a(t,\cdot,\xi)$ is controlled by $\|\eta(t)\|_{H^{s-1}}$ in this case. As alluded to above, this proposition solves the technical difficulty which appears since we only assume $s > 2 + d/2$.

Proof. By abuse of notations, we omit the dependence in time.

a) Consider a symbol $p = p(x,\xi)$ homogeneous of degree r in ξ such that

$$x \mapsto \partial_\xi^\alpha p(\cdot,\xi) \quad \text{belongs to } H^{s-3}(\mathbf{R}^d) \quad \forall \alpha \in \mathbf{N}^d.$$

Let q be defined by

$$\widehat{q}(\theta,\xi) = \frac{\chi_1(\theta,\xi)\psi_1(\xi)}{|\xi|}\widehat{p}(\theta,\xi)$$

where $\chi_1 = 1$ on $\operatorname{supp}\chi$, $\psi_1 = 1$ on $\operatorname{supp}\psi$ (see (3.2)), $\psi_1(\xi) = 0$ for $|\xi| \le \frac{1}{3}$, $\chi_1(\theta,\xi) = 0$ for $|\theta| \ge |\xi|$ and $\widehat{f}(\theta,\xi) = \int e^{-ix\cdot\theta} f(x,\xi)\,dx$. Then

$$T_q\,|D_x| = T_p, \tag{4.4}$$

and

$$\left|\partial_\xi^\alpha \widehat{q}(\theta,\xi)\right| \lesssim \langle\theta\rangle^{-1} \sum_{\beta\le\alpha} \left|\partial_\xi^\beta \widehat{p}(\theta,\xi)\right|.$$

Therefore we have

$$\left\|\partial_\xi^\alpha q(\cdot,\xi)\right\|_{H^{s-2}} \lesssim \sum_{\beta\le\alpha} \left\|\partial_\xi^\beta p(\cdot,\xi)\right\|_{H^{s-3}}. \tag{4.5}$$

Now, it follows from the above estimate and the embedding $H^{s-2}(\mathbf{R}^d) \subset L^\infty(\mathbf{R}^d)$ that q is L^∞ in x and hence $q \in \Gamma_0^{r-1} \subset \Gamma_0^r$. Then, according to (3.4) applied with $m = r$ (and not $m = r-1$), we have for all $\sigma \in \mathbf{R}$,

$$\left\|T_q v\right\|_{H^{\sigma-r}} \lesssim \sup_{|\alpha|\le\frac{d}{2}+1}\ \sup_{|\xi|\ge\frac{1}{2}} |\xi|^{|\alpha|-r} \left\|\partial_\xi^\alpha q(\cdot,\xi)\right\|_{L^\infty} \left\|v\right\|_{H^\sigma}.$$

Applying this inequality with $v = |D_x|\,u$, $\sigma = \mu - 1$ and using again the Sobolev embedding and (4.4), (4.5), we obtain

$$\left\|T_p u\right\|_{H^{\mu-r-1}} \lesssim \sup_{|\alpha|\le\frac{d}{2}+1}\ \sup_{|\xi|=1} \left\|\partial_\xi^\alpha p(\cdot,\xi)\right\|_{H^{s-3}} \left\|u\right\|_{H^\mu}. \tag{4.6}$$

b) Consider a symbol $a \in \Sigma^m$ of the form

$$a = a^{(m)} + a^{(m-1)} = F(\nabla\eta,\xi) + \sum_{|\alpha|=2} G_\alpha(\nabla\eta,\xi)\partial_x^\alpha\eta. \tag{4.7}$$

Up to substracting the symbol of a Fourier multiplier of order m, we can assume without loss of generality that $F(0,\xi) = 0$.

It follows from the previous estimates that

$$\left\|T_{a^{(m)}} u\right\|_{H^{\mu-m}} \lesssim \sup_{|\xi|=1} \|a^{(m)}(\cdot,\xi)\|_{H^{s-2}} \left\|u\right\|_{H^\mu},$$

$$\left\|T_{a^{(m-1)}} u\right\|_{H^{\mu-m}} \lesssim \sup_{|\xi|=1} \|a^{(m-1)}(\cdot,\xi)\|_{H^{s-3}} \left\|u\right\|_{H^\mu}.$$

Now since $s > 2 + d/2$ it follows from the usual nonlinear estimates in Sobolev spaces (see (3.10)) that

$$\sup_{|\xi|=1} \|a^{(m)}(\cdot,\xi)\|_{H^{s-2}} = \sup_{|\xi|=1} \|F(\nabla\eta,\xi)\|_{H^{s-2}} \le C\left(\|\eta\|_{H^{s-1}}\right).$$

On the other hand, by using the product rule (3.9) with $(s_0, s_1, s_2) = (s-3, s-2, s-3)$ we obtain

$$\begin{aligned}\|a^{(m-1)}(\cdot,\xi)\|_{H^{s-3}} &\le \sum_{|\alpha|=2} \|G_\alpha(\nabla\eta,\xi)\partial_x^\alpha\eta\|_{H^{s-3}} \\ &\lesssim \Big(|G_\alpha(0,\xi)| + \sum_{|\alpha|=2} \|G_\alpha(\nabla\eta,\xi) - G_\alpha(0,\xi)\|_{H^{s-2}}\Big) \|\partial_x^\alpha\eta\|_{H^{s-3}},\end{aligned}$$

for all $|\xi| \le 1$. Therefore, (3.10) implies that

$$\|a^{(m-1)}(\cdot,\xi)\|_{H^{s-3}} \le C(\|\eta\|_{H^{s-1}}).$$

This completes the proof. □

Similarly we have the following result about elliptic regularity where one controls the various constants by the H^{s-1}-norm of η only.

Proposition 4.6. *Let $m \in \mathbf{R}$ and $\mu \in \mathbf{R}$. Then there exists a function C such that for all $a \in \Sigma^m$ and all $t \in [0,T]$, we have*

$$\|u\|_{H^{\mu+m}} \le C(\|\eta(t)\|_{H^{s-1}}) \left\{ \left\|T_{a(t)} u\right\|_{H^\mu} + \|u\|_{L^2} \right\}.$$

Remark 4.7. As mentioned in Remark 3.9, the classical result is that, for all elliptic symbol $a \in \Gamma^m_\rho(\mathbf{R}^d)$ with $\rho > 0$, there holds

$$\|f\|_{H^m} \le K \left\{ \|T_a f\|_{L^2} + \|f\|_{L^2} \right\},$$

where K depends only on $M^m_\rho(a)$. Hence, if we use the natural estimate

$$M^{m-1}_\rho(a^{(m-1)}(t)) \le C(\|\eta(t)\|_{W^{2+\rho}}) \le C(\|\eta(t)\|_{H^s})$$

for $\rho > 0$ small enough, then we obtain an estimate which is worse than the one just stated for $2 + d/2 < s < 3 + d/2$.

Proof. Again, by abuse of notations, we omit the dependence in time.

Introduce $b = 1/a^{(m)}$ and consider ε such that

$$0 < \varepsilon < \min\{s - 2 - d/2, 1\}.$$

By applying (3.5) with $\rho = \varepsilon$ we find that $T_b T_{a^{(m)}} = I + r$ where r is of order $-\varepsilon$ and satisfies

$$\|ru\|_{H^{\mu+\varepsilon}} \le C(\|\nabla\eta\|_{W^{\varepsilon,\infty}}) \|u\|_{H^\mu} \le C(\|\eta\|_{H^{s-1}}) \|u\|_{H^\mu}.$$

Then

$$u = T_b T_a u - ru - T_b T_{a^{(m-1)}}.$$

Denoting by $R = -r - T_b T_{a^{(m-1)}}$, we have

$$(I - R)u = T_b T_a u.$$

We claim that there exists a function C such that

$$\|T_{a^{(m-1)}} u\|_{H^{\mu-m+\varepsilon}} \le C(\|\eta\|_{H^{s-1}}) \|u\|_{H^\mu}.$$

To see this, notice that the previous proof applies with the decomposition $T_p = T_q |D_x|^{1-\varepsilon}$ where

$$\widehat{q}(\theta,\xi) = \frac{\chi_1(\theta,\xi)\psi_1(\xi)}{|\xi|^{1-\varepsilon}} \widehat{p}(\theta,\xi).$$

Once this claim is granted, since T_b is of order $-m$, we find that R satisfies

$$\|Ru\|_{H^{\mu+\varepsilon}} \le C(\|\eta\|_{H^{s-1}}) \|u\|_{H^\mu}.$$

Writing

$$(I + R + \cdots + R^N)(I - R)u = (I + R + \cdots + R^N) T_b T_a u$$

we get

$$u = (I + R + \cdots + R^N) T_b T_a u + R^{N+1} u.$$

The first term in the right hand side is estimated by means of the obvious inequality

$$\begin{aligned} &\left\|(I + R + \cdots + R^N) T_b\right\|_{H^\mu \to H^{\mu+m}} \\ &\qquad \le \left\|(I + R + \cdots + R^N)\right\|_{H^{\mu+m} \to H^{\mu+m}} \|T_b\|_{H^\mu \to H^{\mu+m}}, \end{aligned}$$

so that

$$\left\|(I+R+\cdots+R^N)T_bT_au\right\|_{H^{\mu+m}}\le C(\|\eta\|_{H^{s-1}})\left\|T_au\right\|_{H^\mu}.$$

Choosing N so large that $(N+1)\varepsilon>\mu+m$, we obtain that

$$\left\|R^{N+1}\right\|_{H^\mu\to H^{\mu+m}}\lesssim\|R\|_{H^{\mu+m-\varepsilon}\to H^{\mu+m}}\cdots\|R\|_{H^\mu\to H^{\mu+\varepsilon}}\le C(\|\eta\|_{H^{s-1}}),$$

which yields the desired estimate for the second term. □

4.2. **Symmetrization.** The main result of this section is that one can symmetrize the equations. Namely, we shall prove that there exist three symbols p,q,γ such that

$$T_pT_\lambda\sim T_\gamma T_q,\qquad T_qT_\ell\sim T_\gamma T_p,\qquad T_\gamma\sim(T_\gamma)^*, \tag{4.8}$$

where recall that the notation $A\sim B$ was introduced in Definition 4.2.

We want to explain how we find p,q,γ by a systematic method. We first observe that if (4.8) holds true then γ is of order $3/2$. To be definite, we chose q of order 0, and then necessarily p is of order $1/2$. Therefore we seek p,q,γ under the form

$$p=p^{(1/2)}+p^{(-1/2)},\quad q=q^{(0)}+q^{(-1)},\quad \gamma=\gamma^{(3/2)}+\gamma^{(1/2)}, \tag{4.9}$$

where $a^{(m)}$ is a symbol homogeneous in ξ of order $m\in\mathbf{R}$.

Let us list some necessary constraints on these symbols. Firstly, we seek real elliptic symbols such that,

$$p^{(1/2)}\ge K\left|\xi\right|^{1/2},\quad q^{(0)}\ge K,\quad \gamma^{(3/2)}\ge K\left|\xi\right|^{3/2},$$

for some positive constant K. Secondly, in order for T_p,T_q,T_γ to map real valued functions to real valued functions, we must have

$$\overline{p(t,x,\xi)}=p(t,x,-\xi),\quad \overline{q(t,x,\xi)}=q(t,x,-\xi),\quad \overline{\gamma(t,x,\xi)}=\gamma(t,x,-\xi). \tag{4.10}$$

According to Proposition 4.3, in order for T_γ to satisfy the last identity in (4.8), $\gamma^{(1/2)}$ must satisfy

$$\operatorname{Im}\gamma^{(1/2)}=-\frac{1}{2}(\partial_\xi\cdot\partial_x)\gamma^{(3/2)}. \tag{4.11}$$

Our strategy is then to seek q and γ such that

$$T_qT_\ell T_\lambda\sim T_\gamma T_\gamma T_q. \tag{4.12}$$

The idea is that if this identity is satisfied then the first two equations in (4.8) are compatible; this means that if any of these two equations is satisfied, then the second one is automatically satisfied. Therefore, once q and γ are so chosen that (4.12) is satisfied, then one can define p by solving either one of the first two equations. The latter task being immediate.

Recall that the symbol $\lambda=\lambda^{(1)}+\lambda^{(0)}$ (resp. $\ell=\ell^{(2)}+\ell^{(1)}$) is defined by (3.11) (resp. (3.27)). In particular, by notation,

$$\begin{aligned}\lambda^{(1)}&=\sqrt{(1+|\nabla\eta|^2)\,|\xi|^2-(\nabla\eta\cdot\xi)^2},\\ \ell^{(2)}&=\left(1+|\nabla\eta|^2\right)^{-\frac{1}{2}}\left(|\xi|^2-\frac{(\nabla\eta\cdot\xi)^2}{1+|\nabla\eta|^2}\right).\end{aligned} \tag{4.13}$$

Introduce the notations

$$\ell\sharp\lambda = \ell^{(2)}\lambda^{(1)} + \ell^{(1)}\lambda^{(1)} + \ell^{(2)}\lambda^{(0)} + \frac{1}{i}\partial_\xi \ell^{(2)}\cdot\partial_x\lambda^{(1)},$$
$$\gamma\sharp\gamma = \left(\gamma^{(3/2)}\right)^2 + 2\gamma^{(1/2)}\gamma^{(3/2)} + \frac{1}{i}\partial_\xi\gamma^{(3/2)}\cdot\partial_x\gamma^{(3/2)}.$$

By symbolic calculus, to solve (4.12), it is enough to find q and γ such that

$$(4.14)\quad q^{(0)}(\ell\sharp\lambda) + q^{(-1)}\ell^{(2)}\lambda^{(1)} + \frac{1}{i}\partial_\xi q^{(0)}\cdot\partial_x(\ell^{(2)}\lambda^{(1)})$$
$$= (\gamma\sharp\gamma)q^{(0)} + \left(\gamma^{(3/2)}\right)^2 q^{(-1)} + \frac{1}{i}\partial_\xi(\gamma^{(3/2)}\gamma^{(3/2)})\cdot\partial_x q^{(0)}.$$

We set

$$\gamma^{(3/2)} = \sqrt{\ell^{(2)}\lambda^{(1)}},$$

so that the leading symbols of both sides of (4.14) are equal. Then $\operatorname{Im}\gamma^{(1/2)}$ has to be fixed by means of (4.11). We set

$$\operatorname{Im}\gamma^{(1/2)} = -\frac{1}{2}(\partial_x\cdot\partial_\xi)\gamma^{(3/2)}.$$

With these choices of $\gamma^{(3/2)}$ and $\operatorname{Im}\gamma^{(1/2)}$, (4.14) is equivalent to the following equation (where the unknowns are $q^{(0)}, q^{(-1)}$ and $\operatorname{Re}\gamma^{(1/2)}$):

$$(4.15)\quad \begin{aligned} q^{(0)}(\ell\sharp\lambda - \gamma\sharp\gamma) &= \frac{1}{i}\partial_\xi(\ell^{(2)}\lambda^{(1)})\cdot\partial_x q^{(0)} - \frac{1}{i}\partial_\xi q^{(0)}\cdot\partial_x(\ell^{(2)}\lambda^{(1)}) \\ &= \frac{1}{i}\{\ell^{(2)}\lambda^{(1)}, q^{(0)}\} \end{aligned}$$

where

$$(4.16)\quad \ell\sharp\lambda - \gamma\sharp\gamma := \tau$$
$$= \frac{1}{i}\partial_\xi\ell^{(2)}\cdot\partial_x\lambda^{(1)} + \ell^{(1)}\lambda^{(1)} + \ell^{(2)}\lambda^{(0)} - 2\gamma^{(1/2)}\gamma^{(3/2)} + i\partial_\xi\gamma^{(3/2)}\cdot\partial_x\gamma^{(3/2)}.$$

Since $q^{(-1)}$ does not appear in this equation, one can freely set $q^{(-1)} = 0$. Since $\ell^{(2)}, \lambda^{(1)}$ are real-valued symbols, we see easily that (4.15) will be satisfied (with $q^{(0)}$ real) as soon as

$$(4.17)\qquad \operatorname{Re}\tau = 0,\qquad q^{(0)}\operatorname{Im}\tau = -\left\{\ell^{(2)}\lambda^{(1)}, q^{(0)}\right\}.$$

The first condition is satisfied if $\operatorname{Re}\gamma^{(1/2)}$ solves the equation

$$\ell^{(2)}\operatorname{Re}\lambda^{(0)} = 2\gamma^{(3/2)}\operatorname{Re}\gamma^{(1/2)},$$

that is

$$\operatorname{Re}\gamma^{(1/2)} = \frac{\ell^{(2)}\operatorname{Re}\lambda^{(0)}}{2\gamma^{(3/2)}} = \sqrt{\frac{\ell^{(2)}}{\lambda^{(1)}}}\frac{\operatorname{Re}\lambda^{(0)}}{2}.$$

It remains to solve the second equation in (4.17). Let us first recall that

$$\ell^{(1)} = -\frac{i}{2}(\partial_x\cdot\partial_\xi)\ell^{(2)},\ \operatorname{Im}\lambda^{(0)} = -\frac{1}{2}(\partial_x\cdot\partial_\xi)\lambda^{(1)},\ \operatorname{Im}\gamma^{(1/2)} = -\frac{1}{2}(\partial_x\cdot\partial_\xi)\gamma^{(3/2)},$$

and consequently

$$\operatorname{Im}\tau = -\partial_\xi \ell^{(2)} \cdot \partial_x \lambda^{(1)} - \frac{1}{2}\lambda^{(1)}(\partial_\xi \cdot \partial_x)\ell^{(2)} - \frac{1}{2}\ell^{(2)}(\partial_\xi \cdot \partial_x)\lambda^{(1)} + \gamma^{(3/2)}(\partial_\xi \cdot \partial_x)\gamma^{(3/2)} + \partial_\xi \gamma^{(3/2)} \cdot \partial_x \gamma^{(3/2)}.$$

Writing

$$\gamma^{(3/2)}(\partial_\xi \cdot \partial_x)\gamma^{(3/2)} + \partial_\xi \gamma^{(3/2)} \cdot \partial_x \gamma^{(3/2)} = \frac{1}{2}\partial_x \cdot \partial_\xi \left(\gamma^{(3/2)}\right)^2 = \frac{1}{2}\partial_x \cdot \partial_\xi (\ell^{(2)}\lambda^{(1)}),$$

we thus obtain

$$\operatorname{Im}\tau = \frac{1}{2}\partial_\xi \lambda^{(1)} \cdot \partial_x \ell^{(2)} - \frac{1}{2}\partial_\xi \ell^{(2)} \cdot \partial_x \lambda^{(1)},$$

and hence the second equation in (4.17) simplifies to

$$\frac{1}{2}\left\{\ell^{(2)}, \lambda^{(1)}\right\} q^{(0)} + \left\{\ell^{(2)}\lambda^{(1)}, q^{(0)}\right\} = 0. \tag{4.18}$$

The key observation is the following relation between $\ell^{(2)}$ and $\lambda^{(1)}$ (see (4.13)):

$$\ell^{(2)} = \left(c\lambda^{(1)}\right)^2 \quad \text{with} \quad c = \left(1 + |\nabla\eta|^2\right)^{-\frac{3}{4}}.$$

Consequently (4.18) reduces to

$$-q^{(0)}(\lambda^{(1)})^2 \partial_x c^2 \cdot \partial_\xi \lambda^{(1)} + 3c^2(\lambda^{(1)})^2 \partial_\xi \lambda^{(1)} \cdot \partial_x q^{(0)} - \partial_\xi q^{(0)} \cdot \partial_x \left(c^2(\lambda^{(1)})^3\right) = 0.$$

Seeking a solution $q^{(0)}$ which does not depend on ξ, we are led to solve

$$\frac{\partial_\xi \lambda^{(1)} \cdot \partial_x q^{(0)}}{q^{(0)}} = \frac{1}{3}\frac{\partial_\xi \lambda^{(1)} \cdot \partial_x c}{c}.$$

We find the following explicit solution:

$$q^{(0)} = c^{\frac{1}{3}} = \left(1 + |\nabla\eta|^2\right)^{-\frac{1}{2}}.$$

Then, we define p by solving the equation

$$T_q T_\ell \sim T_\gamma T_p.$$

By symbolic calculus, this yields

$$q\ell^{(2)} + q\ell^{(1)} = \gamma^{(3/2)}p^{(1/2)} + \gamma^{(1/2)}p^{(1/2)} + \gamma^{(3/2)}p^{(-1/2)} + \frac{1}{i}\partial_\xi \gamma^{(3/2)} \cdot \partial_x p^{(1/2)}.$$

Therefore, by identifying terms with the same homogeneity in ξ, we successively find that

$$p^{(1/2)} = \frac{q^{(0)}\ell^{(2)}}{\gamma^{(3/2)}} = q^{(0)}\sqrt{\frac{\ell^{(2)}}{\lambda^{(1)}}} = \left(1 + |\nabla\eta|^2\right)^{-\frac{5}{4}}\sqrt{\lambda^{(1)}},$$

and

$$p^{(-1/2)} = \frac{1}{\gamma^{(3/2)}}\left\{q^{(0)}\ell^{(1)} - \gamma^{(1/2)}p^{(1/2)} + i\partial_\xi \gamma^{(3/2)} \cdot \partial_x p^{(1/2)}\right\}. \tag{4.19}$$

Note that the precise value of $p^{(-1/2)}$ is meaningless since we have freely imposed $q^{(-1)} = 0$.

Gathering the previous results, and noting that $\gamma^{(1/2)}$ and $p^{(-1/2)}$ depend only linearly on the second order derivatives of η, we have proved the following result.

Proposition 4.8. *Let $q \in \Sigma^0$, $p \in \Sigma^{1/2}$, $\gamma \in \Sigma^{3/2}$ be defined by*

$$q = \left(1 + |\nabla\eta|^2\right)^{-\frac{1}{2}},$$
$$p = \left(1 + |\nabla\eta|^2\right)^{-\frac{5}{4}} \sqrt{\lambda^{(1)}} + p^{(-1/2)},$$
$$\gamma = \sqrt{\ell^{(2)}\lambda^{(1)}} + \sqrt{\frac{\ell^{(2)}}{\lambda^{(1)}}} \frac{\operatorname{Re}\lambda^{(0)}}{2} - \frac{i}{2}(\partial_\xi \cdot \partial_x)\sqrt{\ell^{(2)}\lambda^{(1)}},$$

where $p^{(-1/2)}$ is given by (4.19)*. Then*

$$T_p T_\lambda \sim T_\gamma T_q, \qquad T_q T_\ell \sim T_\gamma T_p, \qquad T_\gamma \sim (T_\gamma)^*.$$

By combining this symmetrization with the paralinearization, we thus obtain the following symmetrization of the equations.

Corollary 4.9. *Introduce the new unknowns*

$$\Phi_1 = T_p \eta \quad \textit{and} \quad \Phi_2 = T_q U.$$

Then $\Phi_1, \Phi_2 \in C^0([0,T]; H^s(\mathbf{R}^d))$ and

$$\begin{cases} \partial_t \Phi_1 + T_V \cdot \nabla \Phi_1 - T_\gamma \Phi_2 = F_1, \\ \partial_t \Phi_2 + T_V \cdot \nabla \Phi_2 + T_\gamma \Phi_1 = F_2, \end{cases} \tag{4.20}$$

where $F_1, F_2 \in L^\infty(0,T; H^s(\mathbf{R}^d))$. Moreover

$$\|(F_1,F_2)\|_{L^\infty(0,T;H^s\times H^s)} \leq C\left(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}\right),$$

for some function C depending only on $\operatorname{dist}(\Sigma_0, \Gamma)$*.*

To prove Corollary 4.9, we first note that it follows from Proposition 4.8 and Proposition 3.24 that

$$\begin{cases} \partial_t \Phi_1 + T_V \cdot \nabla \Phi_1 - T_\gamma \Phi_2 = B_1 \eta + f_1, \\ \partial_t \Phi_2 + T_V \cdot \nabla \Phi_2 + T_\gamma \Phi_1 = B_2 U + f_2, \end{cases}$$

with $f_1, f_2 \in L^\infty(0,T; H^s(\mathbf{R}^d))$,

$$\|(f_1,f_2)\|_{L^\infty(0,T;H^s(\mathbf{R}^d))} \leq C\left(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))}\right),$$

and

$$B_1 := [\partial_t, T_p] + [T_V \cdot \nabla, T_p],$$
$$B_2 := [\partial_t, T_q] + [T_V \cdot \nabla, T_q].$$

Writing

$$\|B_1\eta\|_{H^s} \leq \|B_1\|_{H^{s+\frac{1}{2}}\to H^s} \|\eta\|_{H^{s+\frac{1}{2}}},$$
$$\|B_2 U\|_{H^s} \leq \|B_2\|_{H^s\to H^s} \|U\|_{H^s},$$

it remains only to estimate $\|B_1\|_{H^{s+\frac{1}{2}}\to H^s}$ and $\|B_2\|_{H^s\to H^s}$. To do so, the only non trivial point is to prove the following lemma.

Lemma 4.10. *For all $\mu \in \mathbf{R}$ there exists a non-decreasing function C such that, for all $t \in [0,T]$,*

$$\left\|T_{\partial_t p(t)}\right\|_{H^\mu\to H^{\mu-\frac{1}{2}}} + \left\|T_{\partial_t q(t)}\right\|_{H^\mu\to H^\mu} \leq C\left(\|(\eta(t),\psi(t))\|_{H^{s+\frac{1}{2}}\times H^s}\right).$$

Proof. It follows from the Sobolev embedding and (3.30) that

$$\|\partial_t \eta\|_{W^{1,\infty}} \lesssim \|\partial_t \eta\|_{H^{s-1}} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

This implies that

$$\|\partial_t q\|_{L^\infty} + M_0^1\left(\partial_t p^{(1/2)}\right) \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right),$$

where the semi-norm M_0^1 has been defined in (3.3). On applying Theorem 3.6, this bound implies that

$$\left\|T_{\partial_t p^{(1/2)}}\right\|_{H^\mu\to H^{\mu-\frac12}} + \|T_{\partial_t q}\|_{H^\mu\to H^\mu} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

It remains only to estimate $\left\|T_{\partial_t p^{(-1/2)}}\right\|_{H^\mu\to H^{\mu-\frac12}}$. Since we only assume $s>2+d/2$, a technical difficulty appears. Indeed, since ∂_t has the weight of $3/2$ spatial derivatives, and since the explicit definition of $p^{(-1/2)}$ involves 2 spatial derivatives of η, the symbol $\partial_t p^{(-1/2)}$ do not belong to L^∞ in general. To overcome this technical problem, write $p^{(-1/2)}$ under the form

$$p^{(-1/2)} = \sum_{|\alpha|=2} P_\alpha(\nabla\eta,\xi)\partial_x^\alpha \eta,$$

where the P_α are smooth functions of their arguments for $\xi\neq 0$, homogeneous of degree $-1/2$ in ξ. Now write

$$T_{\partial_t p^{(-1/2)}} = \sum_{|\alpha|=2} T_{(\partial_t P_\alpha(\nabla\eta,\xi))\partial_x^\alpha\eta} + \sum_{|\alpha|=2} T_{P_\alpha(\nabla\eta,\xi)\partial_t\partial_x^\alpha\eta}. \tag{4.21}$$

As above, we obtain

$$M_0^1\left(\partial_t P_\alpha(\nabla\eta,\xi)\right) \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

On the other hand for $|\alpha|=2$ we have the estimate $\|\partial_x^\alpha\eta\|_{L^\infty} \lesssim \|\eta\|_{H^{s+\frac12}}$. On applying Theorem 3.6, these bounds imply that the first term in the right hand side of (4.21) is uniformly of order $-1/2$.

The analysis of the second term in the right hand side of (4.21) is based on the operator norm estimate (4.6). By applying this estimate with $r=-1/2$, we obtain

$$\left\|T_{P_\alpha(\nabla\eta,\xi)\partial_t\partial_x^\alpha\eta}\right\|_{H^\mu\to H^{\mu-1/2}} \lesssim \|P_\alpha(\nabla\eta,\xi)\partial_t\partial_x^\alpha\eta\|_{H^{s-3}}.$$

Now the product rule (3.9) implies that

$$\begin{aligned}&\|P_\alpha(\nabla\eta,\xi)\partial_t\partial_x^\alpha\eta\|_{H^{s-3}}\\ &\qquad\lesssim \left\{|P_\alpha(0,\xi)| + \|P_\alpha(\nabla\eta,\xi)-P_\alpha(0,\xi)\|_{H^{s-1}}\right\}\|\partial_t\partial_x^\alpha\eta\|_{H^{s-3}},\end{aligned}$$

and hence

$$\left\|T_{P_\alpha(\nabla\eta,\xi)\partial_t\partial_x^\alpha\eta}\right\|_{H^\mu\to H^{\mu-m}} \le C(\|\eta\|_{H^s})\|\partial_t\eta\|_{H^{s-1}} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac12}\times H^s}\right).$$

This completes the proof. □

5. A priori estimates

Consider the Cauchy problem

$$
\begin{aligned}
&\partial_t \eta - G(\eta)\psi = 0,\\
&\partial_t \psi + g\eta - H(\eta) + \frac{1}{2}\left|\nabla\psi\right|^2 - \frac{1}{2}\frac{\big(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\big)^2}{1+|\nabla\eta|^2} = 0,
\end{aligned}
\tag{5.1}
$$

with initial data

$$\eta|_{t=0} = \eta_0, \quad \psi|_{t=0} = \psi_0.$$

In this section we prove a priori estimates for solutions to the system (5.1) and approximate systems. These estimates are crucial in the proof of existence and uniqueness of solutions to (5.1) .

5.1. **Reformulation.** The first step is the following reformulation, whose proof is an immediate computation.

Lemma 5.1. *(η,ψ) solves (5.1) if and only if*

$$\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I\end{pmatrix}(\partial_t + T_V\cdot\nabla)\begin{pmatrix}\eta\\ \psi\end{pmatrix} + \begin{pmatrix} 0 & -T_\lambda \\ T_\ell & 0\end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I\end{pmatrix}\begin{pmatrix}\eta\\ \psi\end{pmatrix} = \begin{pmatrix} f^1\\ f^2\end{pmatrix},$$

where

$$
\begin{aligned}
&f^1 = G(\eta)\psi - \big\{T_\lambda(\psi - T_{\mathfrak{B}}\eta) - T_V\cdot\nabla\eta\big\},\\
&f^2 = -\frac{1}{2}\left|\nabla\psi\right|^2 + \frac{1}{2}\frac{\left(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\right)^2}{1+\left|\nabla\eta\right|^2} + H(\eta)\\
&\qquad + T_V\nabla\psi - T_{\mathfrak{B}}T_V\cdot\nabla\eta - T_{\mathfrak{B}}G(\eta)\psi + T_\ell\eta - g\eta.
\end{aligned}
\tag{5.2}
$$

Since

$$\begin{pmatrix} I & 0 \\ T_{\mathfrak{B}} & I\end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix},$$

we thus find that (η,ψ) solves (5.1) if and only if

$$
\begin{cases}
(\partial_t + T_V\cdot\nabla + \mathcal{L})\begin{pmatrix}\eta\\ \psi\end{pmatrix} = f(\eta,\psi),\\
(\eta,\psi)|_{t=0} = (\eta_0,\psi_0),
\end{cases}
\tag{5.3}
$$

with

$$\mathcal{L} := \begin{pmatrix} I & 0 \\ T_{\mathfrak{B}} & I\end{pmatrix}\begin{pmatrix} 0 & -T_\lambda \\ T_\ell & 0\end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I\end{pmatrix}, \quad f(\eta,\psi) := \begin{pmatrix} I & 0 \\ T_{\mathfrak{B}} & I\end{pmatrix}\begin{pmatrix} f^1\\ f^2\end{pmatrix}.$$

5.2. **Approximate equations.** We shall seek solutions of the Cauchy problem (5.3) as limits of solutions of approximating systems. The definition depends on two operators. The first one is a well-chosen mollifier. The second one is an approximate right-parametrix for the symmetrizer $S = \left(\begin{smallmatrix} T_p & 0 \\ 0 & T_q\end{smallmatrix}\right)$ defined in Section 4.

Mollifiers. To regularize the equations, we cannot use usual mollifiers of the form $\chi(\varepsilon D_x)$. Instead we use the following variant. Given $\varepsilon\in[0,1]$, we define J_ε as the paradifferential operator with symbol $\jmath_\varepsilon = \jmath_\varepsilon(t,x,\xi)$ given by

$$\jmath_\varepsilon = \jmath_\varepsilon^{(0)} + \jmath_\varepsilon^{(-1)} = \exp\big(-\varepsilon\gamma^{(3/2)}\big) - \frac{i}{2}(\partial_x\cdot\partial_\xi)\exp\big(-\varepsilon\gamma^{(3/2)}\big).$$

The important facts are that

$$\jmath_\varepsilon \in C^0([0,T];\Gamma^0_{3/2}(\mathbf{R}^d)), \quad \{\jmath_\varepsilon^{(0)},\gamma^{(3/2)}\}=0, \quad \operatorname{Im} \jmath_\varepsilon^{(-1)} = -\frac{1}{2}(\partial_x\cdot\partial_\xi)\jmath_\varepsilon^{(0)}.$$

Of course, for any $\varepsilon>0$, $\jmath_\varepsilon \in C^0([0,T];\Gamma^m_{3/2}(\mathbf{R}^d))$ for all $m\le 0$. However, the important fact is that $\jmath_\varepsilon$ is uniformly bounded in $C^0([0,T];\Gamma^0_{3/2}(\mathbf{R}^d))$ for all $\varepsilon\in[0,1]$. Therefore, we have the following uniform estimates:

$$\begin{aligned} &\|J_\varepsilon T_\gamma - T_\gamma J_\varepsilon\|_{H^\mu\to H^\mu} \le C(\|\nabla\eta\|_{W^{3/2,\infty}}),\\ &\|(J_\varepsilon)^* - J_\varepsilon\|_{H^\mu\to H^{\mu+3/2}} \le C(\|\nabla\eta\|_{W^{3/2,\infty}}), \end{aligned}$$

for some non-decreasing function C independent of $\varepsilon\in[0,1]$. In other words, we have

$$J_\varepsilon T_\gamma \sim T_\gamma J_\varepsilon, \quad (J_\varepsilon)^*\sim J_\varepsilon,$$

uniformly in ε.

Parametrix for the symmetrizer. Recall that the class of symbols Σ^m have been defined in Definition 4.1. We seek

$$\wp = \wp^{(-1/2)} + \wp^{(-3/2)} \in \Sigma^{-1/2}$$

such that

$$p\sharp\wp = p^{(1/2)}\wp^{(-1/2)} + p^{(1/2)}\wp^{(-3/2)} + p^{(-1/2)}\wp^{(-1/2)} + \frac{1}{i}\partial_\xi p^{(1/2)}\cdot\partial_x\wp^{(-1/2)} = 1.$$

To solve this equation we explicitly set

$$\begin{aligned} &\wp^{(-1/2)} = \frac{1}{p^{(1/2)}},\\ &\wp^{(-3/2)} = -\frac{1}{p^{(1/2)}}\left(\wp^{(-1/2)}p^{(-1/2)} + \frac{1}{i}\partial_\xi\wp^{(-1/2)}\cdot\partial_x p^{(1/2)}\right). \end{aligned} \tag{5.4}$$

Therefore

$$T_pT_\wp \sim I,$$

where recall that the notation $A\sim B$ is as defined in Definition 4.2.

On the other hand, since $q=\left(1+|\nabla\eta|^2\right)^{-\frac{1}{2}}$ does not depend on ξ, it follows from (3.7) that we have

$$T_qT_{1/q}\sim I.$$

Hence, with $\wp$ and q as defined above, we have

$$\begin{pmatrix} T_p & 0\\ 0 & T_q\end{pmatrix}\begin{pmatrix} T_\wp & 0\\ 0 & T_{1/q}\end{pmatrix}\sim\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}.$$

Approximate system. We then define

$$\mathcal{L}^\varepsilon := \begin{pmatrix} I & 0\\ T_{\mathfrak{B}} & I\end{pmatrix}\begin{pmatrix} 0 & -T_\lambda\\ T_\ell & 0\end{pmatrix}\begin{pmatrix} T_\wp J_\varepsilon T_p & 0\\ 0 & T_{1/q}J_\varepsilon T_q\end{pmatrix}\begin{pmatrix} I & 0\\ -T_{\mathfrak{B}} & I\end{pmatrix}.$$

(At first one may not expect to have to introduce J_ε and $\mathcal{L}^\varepsilon$. We explain the reason to introduce these operators in §5.4 below.) We seek solutions (η,ψ) of (5.3) as limits of solutions of the following Cauchy problems

$$\begin{cases} \left(\partial_t + T_V\cdot\nabla J_\varepsilon + \mathcal{L}^\varepsilon\right)\begin{pmatrix}\eta\\ \psi\end{pmatrix} = f(J_\varepsilon\eta, J_\varepsilon\psi),\\ (\eta,\psi)|_{t=0} = (\eta_0,\psi_0). \end{cases} \tag{5.5}$$

5.3. **Uniform estimates.** Our main task will consist in proving uniform estimates for this system. Namely, we shall prove the following proposition.

Proposition 5.2. *Let $d \geq 1$ and $s > 2 + d/2$. Then there exist a non-decreasing function C such that, for all $\varepsilon \in [0,1]$, all $T \in]0,1]$ and all solution (η,ψ) of (5.5) such that*

$$(\eta,\psi) \in C^1([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)),$$

the norm

$$M(T) = \left\lVert (\eta,\psi) \right\rVert_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)},$$

satisfies the estimate

$$M(T) \leq C(M_0) + TC(M(T)),$$

with $M_0 := \left\lVert (\eta_0,\psi_0) \right\rVert_{H^{s+\frac{1}{2}}\times H^s}$.

Remark 5.3. Notice that the estimate holds for $\varepsilon = 0$. In particular this proposition contains *a priori* estimates for the water waves system itself.

5.4. **The key identities.** To ease the reading, we here explain what are the key identities in the proof of Proposition 5.2.

By definition of $\mathcal{L}^\varepsilon$, using that $\left(\begin{smallmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 \\ T_{\mathfrak{B}} & I \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$, we have

$$\begin{aligned}
&\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}\mathcal{L}^\varepsilon \\
&\qquad = \begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} 0 & -T_\lambda \\ T_\ell & 0 \end{pmatrix}\begin{pmatrix} T_\wp J_\varepsilon T_p & 0 \\ 0 & T_{1/q} J_\varepsilon T_q \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}.
\end{aligned}$$

Now recall that

$$\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} 0 & -T_\lambda \\ T_\ell & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -T_\gamma \\ (T_\gamma)^* & 0 \end{pmatrix}\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix},$$

so that

$$\begin{aligned}
&\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}\mathcal{L}^\varepsilon \\
&\qquad \sim \begin{pmatrix} 0 & -T_\gamma \\ (T_\gamma)^* & 0 \end{pmatrix}\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} T_\wp J_\varepsilon T_p & 0 \\ 0 & T_{1/q} J_\varepsilon T_q \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}
\end{aligned}$$

uniformly in ε (notice that the remainders associated to the notation $\sim$ are uniformly bounded). We next use

$$\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} T_\wp & 0 \\ 0 & T_{1/q} \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

to obtain that, uniformly in ε, we have the key identity

$$\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}\mathcal{L}^\varepsilon \sim \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ (T_\gamma)^* J_\varepsilon & 0 \end{pmatrix}\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}.$$

In other words, the symmetrizer

$$\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}$$

conjugates $\mathcal{L}^\varepsilon$ to a simple operator which is skew symmetric in the following sense:

$$\begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ (T_\gamma)^* J_\varepsilon & 0 \end{pmatrix}^* \sim - \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ (T_\gamma)^* J_\varepsilon & 0 \end{pmatrix}.$$

This is our second key identity, which comes from the fact that

$$(T_\gamma)^* \sim T_\gamma, \quad J_\varepsilon^* \sim J_\varepsilon, \quad T_\gamma J_\varepsilon \sim J_\varepsilon T_\gamma.$$

In particular, it is essential to chose a good mollifier so that the last two identities hold true.

We could mention that, in the proof of Proposition 5.2 below, the main argument is the fact that the term $F_{2,\varepsilon}$ in (5.9) is uniformly bounded in $L^\infty(0,T;H^s \times H^s)$. The other arguments are only technical arguments. However, since we only assume that $s > 2 + \frac{d}{2}$, this requires some care and we give a complete proof.

5.5. **Proof of Proposition 5.2.** We now prove Proposition 5.2.

a) Let us set

$$U = \psi - T_{\mathfrak{B}}\eta, \quad \Phi = \begin{pmatrix} T_p \eta \\ T_q U \end{pmatrix} = \begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix} \begin{pmatrix} \eta \\ \psi \end{pmatrix}. \tag{5.6}$$

We claim that Φ satisfies an equation of the form

$$(\partial_t + T_V \cdot \nabla J_\varepsilon)\,\Phi + \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ T_\gamma J_\varepsilon & 0 \end{pmatrix} \Phi = F_\varepsilon, \tag{5.7}$$

where the remainder satisfies

$$\left\Vert F_\varepsilon \right\Vert_{L^\infty(0,T;H^s\times H^s)} \le C\left(\left\Vert (\eta,\psi) \right\Vert_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}\right), \tag{5.8}$$

for some non-decreasing function C independent of ε. To prove this claim, we begin by commuting the equation (5.5) with the matrix

$$\begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix},$$

to obtain that Φ satisfies (5.7) with $F_\varepsilon = F_{1,\varepsilon} + F_{2,\varepsilon} + F_{3,\varepsilon}$ where (cf §5.4)

$$\begin{aligned} F_{1,\varepsilon} &= \begin{pmatrix} T_p f^1(J_\varepsilon \eta, J_\varepsilon \psi) \\ T_q f^2(J_\varepsilon \eta, J_\varepsilon \psi) \end{pmatrix}, \\ F_{2,\varepsilon} &= \begin{pmatrix} 0 & -(T_p T_\lambda T_{1/q} J_\varepsilon - T_\gamma J_\varepsilon) \\ (T_q T_\ell T_\wp J_\varepsilon - T_\gamma J_\varepsilon) & 0 \end{pmatrix} \Phi, \\ F_{3,\varepsilon} &= \left[\partial_t + T_V \cdot \nabla J_\varepsilon, \begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix} \right] \begin{pmatrix} \eta \\ \psi \end{pmatrix}. \end{aligned} \tag{5.9}$$

The estimate of the first term follows from Proposition 3.14, Lemma 3.25 and Lemma 3.26 (clearly, these results applies with (η,ψ) replaced by $(J_\varepsilon\eta, J_\varepsilon\psi)$). For the second term we use that,

$$T_p T_\lambda \sim T_\gamma T_q, \quad T_q T_\ell \sim T_\gamma T_p, \quad T_p T_\wp \sim I, \quad T_q T_{1/q} \sim I,$$

to obtain

$$T_p T_\lambda T_{1/q} \sim T_\gamma, \quad T_q T_\ell T_\wp \sim T_\gamma.$$

Eventually, we estimate the last term as in the proof of Corollary 4.9.

b) We next claim that

$$\|(\eta,\psi)\|_{L^\infty(0,T;H^{s-1}\times H^{s-\frac{3}{2}})} \le C(M_0) + TC(M(T)). \tag{5.10}$$

We prove the desired estimate for $\partial_t\eta$ only. To do so, using the obvious inequality

$$\begin{aligned}\|\eta(t)\|_{H^{s-1}} &\le \|\eta(0)\|_{H^{s-1}} + \int_0^t \|\partial_t\eta\|_{H^{s-1}}\\ &\le M_0 + T\,\|\partial_t\eta\|_{L^\infty(0,T;H^{s-1})},\end{aligned}$$

we see that it is enough to prove that

$$\|\partial_t\eta\|_{L^\infty(0,T;H^{s-1})} \le C(M(T)). \tag{5.11}$$

This in turn follows directly from the equation for η. Indeed, directly from (5.5), write

$$\partial_t\eta = -T_V\cdot\nabla J_\varepsilon\eta + T_\lambda T_{1/q}J_\varepsilon T_q(\psi - T_{\mathfrak{B}}\eta) + f_1(J_\varepsilon\eta, J_\varepsilon\psi).$$

The last term is estimated by means of Proposition 3.14. Moving to the first two terms, by the usual continuity estimate for paradifferential operators (3.4), we have

$$\|T_V\cdot\nabla J_\varepsilon\eta\|_{H^{s-1}} \le \|V\|_{L^\infty}\|J_\varepsilon\eta\|_{H^s},$$

and

$$\begin{aligned}&\left\|T_\lambda T_{1/q}J_\varepsilon T_q(\psi - T_{\mathfrak{B}}\eta)\right\|_{H^{s-1}}\\ &\qquad\le \left\|T_\lambda T_{1/q}J_\varepsilon T_q\right\|_{H^s\to H^{s-1}}\left\{\|\psi\|_{H^s} + \|\mathfrak{B}\|_{L^\infty}\|\eta\|_{H^s}\right\},\end{aligned}$$

and hence, since $H^{s-1}(\mathbf{R}^d) \subset L^\infty(\mathbf{R}^d)$, the estimates for $\mathfrak{B}$ and V in (3.30) imply that $\partial_t\eta$ satisfies the desired estimate (5.11). The estimate of $\|\psi\|_{H^{s-3/2}}$ is analogous. This completes the proof of the claim.

c) To obtain estimates in Sobolev, we shall commute the equation with an elliptic operator of order s and then use an L^2-energy estimate. Again, one has to chose carefully the elliptic operator. The most natural choice consists in introducing the paradifferential operator T_β with symbol

$$\beta := \left(\gamma^{(3/2)}\right)^{\frac{2s}{3}} \in \Sigma^s. \tag{5.12}$$

The key point is that, since β and $\jmath_\varepsilon^{(0)}$ are (nonlinear) functions of $\gamma^{(3/2)}$, we have

$$\begin{aligned}\partial_\xi\beta\cdot\partial_x\gamma^{(3/2)} &= \partial_\xi\gamma^{(3/2)}\cdot\partial_x\beta,\\ \partial_\xi\beta\cdot\partial_x\jmath_\varepsilon^{(0)} &= \partial_\xi\jmath_\varepsilon^{(0)}\cdot\partial_x\beta.\end{aligned}$$

Therefore, as above, we find that $[T_\beta, T_\gamma]$ is of order s, while $[T_\beta, J_\varepsilon]$ is of order $s-3/2$. Also the commutator $[T_\beta, T_V\cdot\nabla J_\varepsilon]$ is clearly of order s. With regards to the commutator $[T_\beta, T_{\partial_t}] = -T_{\partial_t\beta}$ notice that there is no difficulty. Indeed, since β is of the form $\beta = B(\nabla\eta,\xi)$, the most direct estimate shows that the $L^\infty_x(\mathbf{R}^d)$-norm of $\partial_t\beta$ is estimated by the $L^\infty_x(\mathbf{R}^d)$-norm of $(\nabla\eta, \partial_t\nabla\eta)$ and hence by $C(M(T))$ in view of (5.11) and the Sobolev embedding $H^{s-1}(\mathbf{R}^d) \subset W^{1,\infty}(\mathbf{R}^d)$. We thus end up with the following uniform estimates

$$\|[T_\beta, T_\gamma]\, J_\varepsilon\|_{H^s\to L^2} + \|[T_\beta, \partial_t]\|_{H^s\to L^2} + \|T_\gamma\,[T_\beta, J_\varepsilon]\|_{H^s\to L^2} + \|[T_\beta, T_V\cdot\nabla J_\varepsilon]\|_{H^s\to L^2} \le C(M(T)),$$

for some non-decreasing function C independent of $\varepsilon\in[0,1]$. Therefore, by commuting the equation (5.7) with T_β, we find that the function $\varphi := T_\beta\Phi$ satisfies

$$(\partial_t + T_V\cdot\nabla J_\varepsilon)\,\varphi + \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ T_\gamma J_\varepsilon & 0\end{pmatrix}\varphi = F'_\varepsilon, \tag{5.13}$$

with

$$\|F'_\varepsilon\|_{L^\infty(0,T;L^2\times L^2)} \le C(M(T)),$$

for some non-decreasing function C independent of $\varepsilon\in[0,1]$.

d) Since by assumption $(\eta,\psi)\in C^1(0,T); H^{s+\frac12}(\mathbf{R}^d)\times H^s(\mathbf{R}^d)$, we have

$$\varphi\in C^1([0,T]; L^2(\mathbf{R}^d)\times L^2(\mathbf{R}^d)),$$

and hence we can write

$$\frac{d}{dt}\langle\varphi,\varphi\rangle = 2\,\mathrm{Re}\,\langle\partial_t\varphi,\varphi\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbf{R}^d)\times L^2(\mathbf{R}^d)$. Therefore, (5.13) implies that

$$\frac{d}{dt}\langle\varphi,\varphi\rangle = 2\,\mathrm{Re}\left\langle -T_V\cdot\nabla J_\varepsilon\varphi - \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ T_\gamma J_\varepsilon & 0\end{pmatrix}\varphi + F'_\varepsilon,\varphi\right\rangle$$

and hence

$$\frac{d}{dt}\langle\varphi,\varphi\rangle = \langle\mathscr{R}^\varepsilon\varphi,\varphi\rangle + 2\,\mathrm{Re}\,\langle F'_\varepsilon,\varphi\rangle\,,$$

where $\mathscr{R}^\varepsilon$ is the matrix-valued operator

$$\mathscr{R}^\varepsilon := -\left\{(T_V\cdot\nabla J_\varepsilon)^* + T_V\cdot\nabla J_\varepsilon\right\} I + \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ T_\gamma J_\varepsilon & 0\end{pmatrix} + \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ T_\gamma J_\varepsilon & 0\end{pmatrix}^*.$$

Now recall that

$$(T_\gamma)^*\sim T_\gamma,\quad (J_\varepsilon)^*\sim J_\varepsilon,\quad T_\gamma J_\varepsilon\sim J_\varepsilon T_\gamma.$$

Moreover, we easily verify that

$$\sup_{\varepsilon\in[0,1]}\sup_{t\in[0,T]}\|\mathscr{R}^\varepsilon(t)\|_{L^2\times L^2\to L^2\times L^2}\le C(M(T)).$$

Therefore, integrating in time we conclude that for all $t\in[0,T]$,

$$\|\varphi(t)\|^2_{L^2\times L^2} - \|\varphi(0)\|^2_{L^2\times L^2} \le C(M(T))\int_0^T\left(\|\varphi\|^2_{L^2\times L^2} + \|F'_\varepsilon\|^2_{L^2\times L^2}\right)dt',$$

which immediately implies that

$$\|\varphi\|_{L^\infty(0,T;L^2\times L^2)}\le C(M_0) + TC(M(T)).$$

By definition of φ, this yields

$$\|T_\beta T_p\eta\|_{L^\infty(0,T:L^2)} + \|T_\beta T_q U\|_{L^\infty(0,T;L^2)} \le C(M_0) + TC(M(T)). \tag{5.14}$$

First of all, we use Proposition 4.6 to obtain

$$\|\eta\|_{L^\infty(0,T;H^{s+\frac{1}{2}})} \le K\left\{\|T_\beta T_p\eta\|_{L^\infty(0,T;L^2)} + \|\eta\|_{L^\infty(0,T;H^{\frac{1}{2}})}\right\}, \tag{5.15}$$

$$\|\psi\|_{L^\infty(0,T;H^s)} \le K\left\{\|T_\beta T_q\psi\|_{L^\infty(0,T;L^2)} + \|\psi\|_{L^\infty(0,T;L^2)}\right\}, \tag{5.16}$$

where K depends only on $\|\eta\|_{L^\infty(0,T;H^{s-1})}$.

Let us prove that the constant K satisfies an inequality of the form

$$K \le C(M_0) + TC(M(T)). \tag{5.17}$$

To see this, notice that one can assume without loss of generality that

$$K \le F(\|\eta\|^2_{L^\infty(0,T;H^{s-1})})$$

for some non-decreasing function $F \in C^1(\mathbf{R})$. Set $\mathscr{C}(t) = F(\|\eta(t)\|^2_{H^{s-1}})$. We then obtain the desired bound (5.17) from (5.11) and the inequality

$$K \le \mathscr{C}(0)+\int_0^T \left|\mathscr{C}'(t)\right| \, dt \le F(M_0)+\int_0^T 2F'(\|\eta\|^2_{H^{s-1}}) \left\|\partial_t\eta\right\|_{H^{s-1}} \|\eta\|_{H^{s-1}} \, dt.$$

Consequently, (5.14) and (5.15) imply that we have

$$\|\eta\|_{L^\infty(0,T;H^{s+\frac{1}{2}})} \le C(M_0) + TC(M(T)).$$

It remains to prove an estimate for ψ. To do this, we begin by noticing that, since $\psi = U + T_{\mathfrak{B}}\eta$, we have

$$\begin{aligned}&\|T_\beta T_q\psi\|_{L^\infty(0,T;L^2)}\\ &\qquad \le \|T_\beta T_q U\|_{L^\infty(0,T;L^2)} + \|T_\beta T_q T_{\mathfrak{B}}\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\to L^2)} \|\eta\|_{L^\infty(0,T;H^{s+\frac{1}{2}})}.\end{aligned}$$

Now we have by means of Lemma 3.11

$$\begin{aligned}&\|T_\beta T_q T_{\mathfrak{B}}\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\to L^2)}\\ &\qquad \le \sup_{t\in[0,T]}\sup_{|\xi|=1} \|\beta(t,\cdot,\xi)\|_{L^\infty_x} \|q\|_{L^\infty(0,T;L^\infty)} \|\mathfrak{B}\|_{L^\infty(0,T;H^{\frac{d-1}{2}})}\end{aligned}$$

and hence

$$\|\psi\|_{H^s} \le K'\left\{\|T_q U\|_{H^s} + \|\psi\|_{L^2} + \|\eta\|_{H^{s+\frac{1}{2}}}\right\}, \tag{5.18}$$

where K' depends only on $\|(\eta,\psi)\|_{L^\infty(0,T;H^{s-1}\times H^{s-3/2})}$. By using the inequality (5.14) for $\|T_\beta U\|_{L^2}$, the estimate (5.10) for $\|\psi\|_{L^2}$, the previous estimate for η, and the fact that K' satisfies the same estimate as K does, we conclude that

$$\|\psi\|_{L^\infty(0,T;H^s)} \le C(M_0) + TC(M(T)).$$

We end up with $M(T) \le C(M_0) + TC(M(T))$. This completes the proof of Proposition 5.2.

5.6. Consider $(\eta,\psi)\in C^0([0,T];H^{s+\frac12}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$ solution to the system

$$\begin{cases} (\partial_t + T_V\cdot\nabla J_\varepsilon + \mathcal{L}^\varepsilon)\begin{pmatrix}\eta\\ \psi\end{pmatrix} = f(J_\varepsilon\eta, J_\varepsilon\psi),\\ (\eta,\psi)|_{t=0} = (\eta_0,\psi_0).\end{cases}$$

We now prove uniform estimates for solutions $(\tilde\eta,\tilde\psi)$ to the linear system

$$\begin{cases} (\partial_t + T_V\cdot\nabla J_\varepsilon + \mathcal{L}^\varepsilon)\begin{pmatrix}\tilde\eta\\ \tilde\psi\end{pmatrix} = F,\\ (\tilde\eta,\tilde\psi)|_{t=0} = (\tilde\eta_0,\tilde\psi_0).\end{cases} \tag{5.19}$$

To clarify notations, write (5.5) in the compact form

$$E(\varepsilon,\eta,\psi)\begin{pmatrix}\eta\\ \psi\end{pmatrix} = f(J_\varepsilon\eta, J_\varepsilon\psi)$$

Then, with this notations, we shall prove estimates for the system

$$E(\varepsilon,\eta,\psi)\begin{pmatrix}\tilde\eta\\ \tilde\psi\end{pmatrix} = F.$$

We shall also use the following notation: given $r\ge 0$, $T>0$ and two real-valued functions u_1,u_2, we set

$$\left\|(u_1,u_2)\right\|_{X^r(T)} := \left\|(u_1,u_2)\right\|_{L^\infty(0,T;H^{r+\frac12}\times H^r)} \tag{5.20}$$

We shall prove the following extension of Proposition 5.2.

Proposition 5.4. *Let $d\ge 1$, $s>2+d/2$ and $0\le\sigma\le s$. Then there exists a non-decreasing function C such that, for all $\varepsilon\in[0,1]$, all $T\in]0,1]$ and all $\tilde\eta,\tilde\psi,\eta,\psi,F$ such that*

$$E(\varepsilon,\eta,\psi)\begin{pmatrix}\eta\\ \psi\end{pmatrix} = f(J_\varepsilon\eta, J_\varepsilon\psi),\quad E(\varepsilon,\eta,\psi)\begin{pmatrix}\tilde\eta\\ \tilde\psi\end{pmatrix} = F,$$

and such that

$$\begin{aligned}&(\eta,\psi)\in C^0([0,T];H^{s+\frac12}(\mathbf{R}^d)\times H^s(\mathbf{R}^d)),\\ &(\tilde\eta,\tilde\psi)\in C^1([0,T];H^{\sigma+\frac12}(\mathbf{R}^d)\times H^\sigma(\mathbf{R}^d)),\\ &F=(F_1,F_2)\in L^\infty([0,T];H^{\sigma+\frac12}(\mathbf{R}^d)\times H^\sigma(\mathbf{R}^d)),\end{aligned}$$

we have

$$\begin{aligned}\left\|(\tilde\eta,\tilde\psi)\right\|_{X^\sigma(T)} \le \widetilde{C}\left\|(\tilde\eta_0,\tilde\psi_0)\right\|_{H^{\sigma+\frac12}\times H^\sigma}\\ + TC\left(\left\|(\eta,\psi)\right\|_{X^s(T)}\right)\left\{\left\|(\tilde\eta,\tilde\psi)\right\|_{X^\sigma(T)} + \left\|F\right\|_{X^\sigma(T)}\right\},\end{aligned} \tag{5.21}$$

where $\widetilde{C} := C\left(\left\|(\eta_0,\psi_0)\right\|_{H^{s+\frac12}\times H^s}\right) + TC\left(\left\|(\eta,\psi)\right\|_{X^s(T)}\right)$.

Remark 5.5. By applying this proposition with $(\eta,\psi)=(\tilde\eta,\tilde\psi)$ we obtain Proposition 5.2.

Proof. We still denote by $p,q,\gamma,\wp$ the symbols already introduced above. They are functions of η only. Similarly, $\mathfrak{B}$ and V are functions of (η,ψ). We use tildes to indicate that the new unknowns that we shall introduce depend linearly on $(\tilde\eta,\tilde\psi)$, with some coefficients depending on (η,ψ).

i) Let us set

$$\tilde{U} = \tilde{\psi} - T_{\mathfrak{B}}\tilde{\eta}, \quad \tilde{\Phi} = \begin{pmatrix} T_p\tilde{\eta} \\ T_q\tilde{U} \end{pmatrix}.$$

As above, we begin by computing that $\tilde{\Phi}$ satisfies

$$\left(\partial_t + T_V \cdot \nabla J_\varepsilon\right)\tilde{\Phi} + \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ T_\gamma J_\varepsilon & 0 \end{pmatrix}\tilde{\Phi} = \tilde{F},$$

with $\tilde{F} = \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3$ where

$$\tilde{F}_1 = \begin{pmatrix} T_p F_1 \\ T_q F_2 \end{pmatrix},$$

$$\tilde{F}_2 = \begin{pmatrix} 0 & -(T_p T_\lambda T_{1/q} J_\varepsilon - T_\gamma J_\varepsilon) \\ (T_q T_\ell T_\wp J_\varepsilon - T_\gamma J_\varepsilon) & 0 \end{pmatrix}\tilde{\Phi},$$

$$\tilde{F}_3 = \left[\partial_t + T_V\cdot\nabla J_\varepsilon, \begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}\begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}\right]\begin{pmatrix} \tilde{\eta} \\ \tilde{\psi} \end{pmatrix}.$$

Then we find that

$$\left\|\tilde{F}\right\|_{L^\infty(0,T;H^\sigma\times H^\sigma)} \le C\left(\|(\eta,\psi)\|_{X^s(T)}\right)\left\{\left\|(\tilde{\eta},\tilde{\psi})\right\|_{X^\sigma(T)} + \|F\|_{X^\sigma(T)}\right\},$$

for some non-decreasing function C independent of ε.

ii) Next, we introduce the symbol

$$\beta := \left(\gamma^{(3/2)}\right)^{\frac{2\sigma}{3}} \in \Sigma^\sigma.$$

As above, we find that

$$\begin{aligned}
\left\|[T_\beta, T_\gamma]\, J_\varepsilon\right\|_{H^\sigma\to L^2} &\le C(\|(\eta,\psi)\|_{X^s(T)}),\\
\left\|T_\gamma\, [T_\beta, J_\varepsilon]\right\|_{H^\sigma\to L^2} &\le C(\|(\eta,\psi)\|_{X^s(T)}),\\
\left\|[T_\beta, T_V\cdot\nabla J_\varepsilon]\right\|_{H^\sigma\to L^2} &\le C(\|(\eta,\psi)\|_{X^s(T)}),\\
\left\|[T_\beta, \partial_t]\right\|_{H^\sigma\to L^2} &\le C(\|(\eta,\psi)\|_{X^s(T)}),
\end{aligned}$$

for some non-decreasing function C independent of $\varepsilon \in [0,1]$. Therefore, by commuting the equation (5.7) with T_β, we find that

$$\tilde{\varphi} := T_\beta\tilde{\Phi}$$

satisfies

$$\left(\partial_t + T_V \cdot \nabla J_\varepsilon\right)\tilde{\varphi} + \begin{pmatrix} 0 & -T_\gamma J_\varepsilon \\ T_\gamma J_\varepsilon & 0 \end{pmatrix}\tilde{\varphi} = \tilde{F}',$$

with

$$\left\|\tilde{F}'\right\|_{L^\infty(0,T;L^2\times L^2)} \le C(\|(\eta,\psi)\|_{X^s(T)})\left\{\|(\tilde{\eta},\tilde{\psi})\|_{X^\sigma(T)} + \|F\|_{X^\sigma(T)}\right\},$$

for some non-decreasing function C independent of $\varepsilon\in[0,1]$.

iii) Therefore, we obtain that for all $t\in[0,T]$, $\|\tilde{\varphi}(t)\|^2_{L^2\times L^2} - \|\tilde{\varphi}(0)\|^2_{L^2\times L^2}$ is bounded by

$$C(\|(\eta,\psi)\|_{X^s(T)})\int_0^T \left(\|\tilde{\varphi}(t')\|^2_{L^2\times L^2} + \left\|\tilde{F}'(t')\right\|^2_{L^2\times L^2}\right)dt'$$

which immediately implies that $\|\tilde{\varphi}\|_{L^\infty(0,T;L^2\times L^2)}$ is bounded by

$$\|\tilde{\varphi}(0)\|_{L^2\times L^2} + TC(\|(\eta,\psi)\|_{X^s(T)})\,\|\tilde{\varphi}\|_{L^\infty(0,T;L^2\times L^2)} + T\,\|F\|_{X^\sigma(T)}\,.$$

Once this is granted, we end the proof as above. □

6. Cauchy problem

In this section we conclude the proof of Theorem 1.1. We divide the proof into two independent parts: (a) Existence; (b) Uniqueness. We shall prove the uniqueness by an estimate for the difference of two solutions. With regards to the existence, as mentioned above, we shall obtain solutions to the system (1.2) as limits of solutions to the approximate systems (5.5) which were studied in the previous section. To do that, we shall begin by proving that:

(1) For any $\varepsilon > 0$, the approximate systems (5.5) are well-posed locally in time (ODE argument).
(2) The solutions $(\eta_\varepsilon, \psi_\varepsilon)$ of the approximate system (5.5) are uniformly bounded with respect to ε (by means of the uniform estimates in Proposition 5.2).

The next task is to show that the functions $\{(\eta_\varepsilon, \psi_\varepsilon)\}$ converge to a limit (η, ψ) which is a solution of the water-waves system (1.2). To do this, one cannot apply standard compactness results since the Dirichlet-Neumann operator is not a local operator. To overcome this difficulty we shall prove as in [21] that

(3) The solutions $(\eta_\varepsilon, \psi_\varepsilon)$ form a Cauchy sequence in an appropriate bigger space (by an estimate of the difference of two solutions $(\eta_\varepsilon, \psi_\varepsilon)$ and $(\eta_{\varepsilon'}, \psi_{\varepsilon'})$).
(4) (η, ψ) is a solution to (1.2).
(5) $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d))$.

Notice that, as usual once we know the uniqueness of the limit system, one can assert that the whole family $\{(\eta_\varepsilon, \psi_\varepsilon)\}$ converges to (η, ψ).

Clearly, to achieve these various goals, the main part of the work was already accomplished in the previous section.

6.1. Existence.

Lemma 6.1. *For all $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R})$, and any $\varepsilon > 0$, the Cauchy problem*

$$\begin{cases} (\partial_t + T_V \cdot \nabla J_\varepsilon + \mathcal{L}^\varepsilon) \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f(J_\varepsilon \eta, J_\varepsilon \psi), \\ (\eta, \psi)|_{t=0} = (\eta_0, \psi_0). \end{cases}$$

has a unique maximal solution $(\eta_\varepsilon, \psi_\varepsilon) \in C^0([0, T_\varepsilon[; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d))$

Proof. Write (5.5) in the compact form

$$\partial_t Y = \mathcal{F}_\varepsilon(Y), \quad Y|_{t=0} = Y_0. \tag{6.1}$$

Since J_ε is a smoothing operator, (6.1) is an ODE with values in a Banach space for any $\varepsilon > 0$. Indeed, it is easily checked that the function $\mathcal{F}_\varepsilon$ is C^1 from $H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)$ to itself (the only non trivial terms come from the Dirichlet-Neumann operator, whose regularity follows from Proposition 2.11). The Cauchy Lipschitz theorem then implies the desired result. □

Lemma 6.2. *There exists $T_0 > 0$ such that $T_\varepsilon \geq T_0$ for all $\varepsilon \in]0,1]$ and such that $\{(\eta_\varepsilon,\psi_\varepsilon)\}_{\varepsilon\in]0,1]}$ is bounded in $C^0([0,T_0];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$.*

Proof. The proof is standard. For $\varepsilon \in]0,1]$ and $T < T_\varepsilon$, set

$$M_\varepsilon(T) := \|(\eta_\varepsilon,\psi_\varepsilon)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}.$$

Notice that automatically $(\eta_\varepsilon,\psi_\varepsilon) \in C^1([0,T_\varepsilon[;H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$, so that one can apply Proposition 5.2 to obtain that there exists a continuous function C such that, for all $\varepsilon \in]0,1]$ and all $T < T_\varepsilon$

$$M_\varepsilon(T) \leq C(M_0) + TC(M_\varepsilon(T)), \tag{6.2}$$

where we recall that $M_0 = \|(\eta_0,\psi_0)\|_{H^{s+\frac{1}{2}}\times H^s}$. Let us set $M_1 = 2C(M_0)$ and choose $0 < T_0 \leq 1$ small enough such that $C(M_0) + T_0C(M_1) < M_1$. We claim that

$$M_\varepsilon(T) < M_1, \quad \forall T \in I := [0,\min\{T_0,T_\varepsilon\}[.$$

Indeed, since $M_\varepsilon(0) = M_0 < M_1$, assume that there exists $T \in I$ such that $M_\varepsilon(T) = M_1$ then

$$M_1 = M_\varepsilon(T) \leq C(M_0) + TC(M_\varepsilon(T)) \leq C(M_0) + T_0C(M_1) < M_1,$$

hence the contradiction.

The continuation principle for ordinary differential equations then implies that $T_\varepsilon \geqslant T_0$ for all $\varepsilon \in]0,1]$, and we have

$$\sup_{\varepsilon\in]0,1]}\ \sup_{T\in[0,T_0]} M_\varepsilon(T) \leq M_1.$$

This completes the proof. □

Lemma 6.3. *Let $s' < s - \frac{3}{2}$. Then there exists $0 < T_1 \leq T_0$ such that $\{(\eta_\varepsilon,\psi_\varepsilon)\}_{\varepsilon\in]0,1]}$ is a Cauchy sequence in $C^0([0,T_1];H^{s'+\frac{1}{2}}(\mathbf{R}^d)\times H^{s'}(\mathbf{R}^d))$.*

Proof. The proof is sketched in §6.3 below. □

Then, as explained in the introduction to this section, the existence of a classical solution follows from standard arguments.

6.2. Uniqueness. To complete the proof of Theorem 1.1, it remains to prove the uniqueness.

Proposition 6.4. *Let $T_0 > 0$, $d \geq 1$ and $s > 2 + \frac{d}{2}$. Let (η_j,ψ_j), $j = 1,2$, be two solutions of system (1.2) in $C^0([0,T_0];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$, such that the assumption H_t is satisfied for all $t \in [0,T_0]$. Then*

$$\begin{aligned}\|(\eta_1,\psi_1)-(\eta_2,\psi_2)\|&_{L^\infty([0,T_0];H^{s-1}(\mathbf{R}^d)\times H^{s-\frac{3}{2}}(\mathbf{R}^d))}\\ &\leq C\|(\eta_1,\psi_1)-(\eta_2,\psi_2)\mid_{t=0}\|_{H^{s-1}(\mathbf{R}^d)\times H^{s-\frac{3}{2}}(\mathbf{R}^d))}.\end{aligned} \tag{6.3}$$

As we shall see, the proof of Proposition 6.4 requires a lot of care.

Recall (see §5.1) that (η,ψ) solves (1.2) if and only if

$$(\partial_t + T_V\cdot\nabla + \mathcal{L})\begin{pmatrix}\eta\\ \psi\end{pmatrix} = f(\eta,\psi),$$

with
(6.4)

$$\mathcal{L} := \begin{pmatrix} I & 0 \\ T_{\mathfrak{B}} & I \end{pmatrix} \begin{pmatrix} 0 & -T_\lambda \\ T_\ell & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_{\mathfrak{B}} & I \end{pmatrix}, \quad f(\eta,\psi) := \begin{pmatrix} I & 0 \\ T_{\mathfrak{B}} & I \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}.$$

where

$$\begin{aligned} f^1 &= G(\eta)\psi - \big\{ T_\lambda(\psi - T_{\mathfrak{B}}\eta) - T_V \cdot \nabla\eta \big\}, \\ f^2 &= \frac{1}{2}\left|\nabla\psi\right|^2 + \frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+\left|\nabla\eta\right|^2} + H(\eta) \\ &\quad + T_V\nabla\psi - T_{\mathfrak{B}}T_V\cdot\nabla\eta - T_{\mathfrak{B}}G(\eta)\psi + T_\ell\eta - g\eta. \end{aligned}$$

Introduce the notations

$$\mathfrak{B}_j = \frac{\nabla\eta_j\cdot\nabla\psi_j + G(\eta_j)\psi_j}{1+\left|\nabla\eta_j\right|^2}, \quad V_j = \nabla\psi_j - \mathfrak{B}_j\nabla\eta_j, \tag{6.5}$$

and denote by λ_j, ℓ_j the symbols obtained by replacing η by η_j in (3.11), (3.27) respectively. Similarly, denote by $\mathcal{L}_1$ the operator obtained by replacing $(\mathfrak{B},\lambda,\ell)$ by $(\mathfrak{B}_1,\lambda_1,\ell_1)$ in (6.4). To prove the uniqueness, the main technical lemma is the following.

Lemma 6.5. *Let $0 < T \le T_0$. The differences $\delta\eta := \eta_1 - \eta_2$ and $\delta\psi := \psi_1 - \psi_2$ satisfy a system of the form*

$$(\partial_t + T_{V_1}\cdot\nabla + \mathcal{L}_1)\begin{pmatrix} \delta\eta \\ \delta\psi \end{pmatrix} = f,$$

for some remainder term such that

$$\left\|f\right\|_{L^\infty(0,T;H^{s-1}\times H^{s-\frac{3}{2}})} \le C(M_1,M_2)N,$$

where

$$M_j := \left\|(\eta_j,\psi_j)\right\|_{L^\infty(0,T_0;H^{s+\frac{1}{2}}\times H^s)}, \ N := \left\|(\delta\eta,\delta\psi)\right\|_{L^\infty(0,T;H^{s-1}\times H^{s-\frac{3}{2}})}.$$

Assume this technical lemma for a moment, and let us deduce the desired result: $(\eta_1,\psi_1) = (\eta_2,\psi_2)$. To see this we use our previous analysis. Introducing

$$\delta U := \delta\psi - T_{\mathfrak{B}_1}\delta\eta = \psi_1 - \psi_2 - T_{\mathfrak{B}_1}(\eta_1 - \eta_2),$$

and

$$\delta\Phi := \begin{pmatrix} T_{p_1}\delta\eta \\ T_{q_1}\delta U \end{pmatrix},$$

we obtain that $\delta\Phi$ solves a system of the form

$$\partial_t\delta\Phi + T_{V_1}\cdot\nabla\delta\Phi + \begin{pmatrix} 0 & -T_{\gamma_1} \\ T_{\gamma_1} & 0 \end{pmatrix}\delta\Phi = F$$

with

$$\left\|F\right\|_{L^\infty(0,T;H^{s-\frac{3}{2}}\times H^{s-\frac{3}{2}})} \le C(M_1,M_2)N.$$

Then it follows from the estimate (5.21) applied with

$$\varepsilon = 0, \quad \sigma = s - \frac{3}{2}, \quad \tilde{\eta} = \delta\eta, \quad \tilde{\psi} = \delta\psi,$$

that N satisfies and estimate of the form (with $N_0 = \left\|(\eta,\psi)\ |_{t=0}\right\|_{H^{s-1}\times H^{s-\frac{3}{2}}}$)

$$N \le TC(M_1,M_2)N + C(M_1,M_2)N_0.$$

By choosing T small enough, this implies $N \leq 2N_0$ which is the desired result, but for a possibly time interval $[0,T]$ smaller than $[0,T_0]$. Now we can clearly iterate this result (because the size of the time interval T here depends only on the a priori bounds $M_1; M_2$) to get Proposition 6.4.

It remains to prove Lemma 6.5. To do this, we begin with the following lemma.

Lemma 6.6. *We have*

$$\|V_1 - V_2\|_{H^{s-\frac{5}{2}}} \leq C \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac{3}{2}}},$$
$$\|\mathfrak{B}_1 - \mathfrak{B}_2\|_{H^{s-\frac{5}{2}}} \leq C \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac{3}{2}}},$$

$$\sum_{k=0}^{1} \sup_{|\xi|=1} \left\| \partial_\xi^\alpha \big(\lambda_1^{(k)}(\cdot,\xi) - \lambda_2^{(k)}(\cdot,\xi)\big)\right\|_{H^{s-3+k}} \leq C\|\delta\eta\|_{H^{s-1}},$$
$$\sum_{k=0}^{1} \sup_{|\xi|=1} \left\| \partial_\xi^\alpha \big(\ell_1^{(1+k)}(\cdot,\xi) - \ell_2^{(1+k)}(\cdot,\xi)\big)\right\|_{H^{s-3+k}} \leq C\|\delta\eta\|_{H^{s-1}},$$

for all $\alpha \in \mathbf{N}^d$ and some constant C depending only on M_1, M_2 and α.

Proof. The last two estimates are obtained from the product rule in Sobolev spaces (using similar arguments as in the end of the proof of Lemma 4.10). With regards to the first two estimates, notice that, by definition of $\mathfrak{B}_{\rm j}, V_j$ (see (6.5)), to prove them the only non trivial point is to prove that

$$\|G(\eta_1)\psi_1 - G(\eta_2)\psi_2\|_{H^{s-\frac{5}{2}}} \leq C \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac{3}{2}}}.$$

Indeed, setting $\eta_t = t\eta_1 + (1-t)\eta_2$ we have

$$G(\eta_1)\psi_1 - G(\eta_2)\psi_2 = G(\eta_1)\delta\psi + \int_0^1 dG(\eta_t)\psi_2 \cdot \delta\eta \, dt =: A + B.$$

It follows from Proposition 2.7 that

$$\|A\|_{H^{s-\frac{5}{2}}} \leq C(M_1) \|\delta\psi\|_{H^{s-\frac{3}{2}}}.$$

Now thanks to Proposition 2.11 we can write

$$B = -\int_0^1 \left[G(\eta_t)(\mathfrak{B}_t \delta\eta) + \operatorname{div}(V_t\delta\eta)\right] dt,$$

where $\mathfrak{B}_t = \mathfrak{B}(\eta_t, \psi_2)$, $V = V(\eta_t,\psi_2)$. Using again Proposition 2.7 we obtain

$$\|B\|_{H^{s-\frac{5}{2}}} \leq C(M_1, M_2) \|\delta\eta\|_{H^{s-\frac{3}{2}}}, \tag{6.6}$$

which completes the proof. □

Corollary 6.7. *We have*

$$\|T_{V_1-V_2}\cdot\nabla\eta_2\|_{H^{s-1}} \leq C \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac{3}{2}}},$$
$$\|T_{V_1-V_2}\cdot\nabla\psi_2\|_{H^{s-\frac{3}{2}}} \leq C \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac{3}{2}}},$$
$$\|T_{\lambda_1-\lambda_2}\psi_2\|_{H^{s-1}} \leq C \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac{3}{2}}},$$
$$\|T_{\ell_1-\ell_2}\eta_2\|_{H^{s-\frac{3}{2}}} \leq C \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac{3}{2}}},$$

for some constant C depending only on M_1 and M_2.

Proof. According to Lemma 3.11, we have

$$\|T_a u\|_{H^\mu} \lesssim \|a\|_{H^{\frac{d}{2}-\frac12}} \|u\|_{H^{\mu+\frac12}} .$$

so using the previous lemma we obtain the first two estimates. The last two estimates comes from the bounds for $\lambda_1 - \lambda_2$ and $\ell_1 - \ell_2$ and Proposition 4.4 (again it suffices to apply the usual operators norm estimate (3.4) for $s > 3 + d/2$). □

Similarly, we obtain that, for any $u \in H^{s+\frac12}$,

$$\|T_{\mathfrak{B}_1 - \mathfrak{B}_2} u\|_{H^s} \le C \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac32}} \|u\|_{H^{s+\frac12}} .$$

Therefore, to prove Lemma 6.5, it remains only to estimate the difference

$$f(\eta_1, \psi_1) - f(\eta_2, \psi_2),$$

where $f(\eta,\psi)$ is defined in (6.4). To do this, the most delicate part is to obtain an estimate for

$$f^1(\eta_1, \psi_1) - f^1(\eta_2, \psi_2),$$

where recall the notation

$$f^1(\eta,\psi) = G(\eta)\psi - \big\{ T_\lambda\big(\psi - T_{\mathfrak{B}}\eta\big) - T_V \cdot \nabla\eta \big\} . \tag{6.7}$$

We claim that

$$\big\| f^1(\eta_1,\psi_1) - f^1(\eta_2,\psi_2) \big\|_{H^{s-1}} \le C(M_1, M_2) \|(\delta\eta, \delta\psi)\|_{H^{s-1}\times H^{s-\frac32}} .$$

To prove this claim, we shall prove an estimate for the partial derivative of $f^1(\eta,\psi)$ with respect to η (since $f^1(\eta,\psi)$ is linear with respect to ψ, the corresponding result for the partial derivative with respect to ψ is easy). Let $(\eta,\psi) \in H^{s+\frac12}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)$ (again we forget the time dependence). Introduce the notation

$$d_\eta f^1(\eta,\psi) \cdot \dot\eta = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(f(\eta + \varepsilon\dot\eta, \psi) - f(\eta,\psi) \right).$$

Then, to complete the proof of the uniqueness, it remains only to prove the following technical lemma.

Lemma 6.8. *Let $s > 2 + d/2$. Then, for all $(\eta,\psi) \in H^{s+\frac12}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)$, and for all $\dot\eta \in H^{s+\frac12}(\mathbf{R}^d)$,*

$$\big\| d_\eta f^1(\eta,\psi) \cdot \dot\eta \big\|_{H^{s-1}} \le C \|\dot\eta\|_{H^{s-1}} ,$$

for some constant C which depends only on the $H^{s+\frac12}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)$-norm of (η,ψ).

Remark 6.9. The assumption $\dot\eta \in H^{s+\frac12}(\mathbf{R}^d)$ ensures that $d_\eta f^1(\eta,\psi)\dot\eta$ is well defined. However, of course, a key point is that we estimate the latter term in H^{s-1} by means of only the H^{s-1} norm of $\dot\eta$.

Proof. To prove this estimate we begin by computing $d_\eta f^1(\eta,\psi)\dot\eta$. Given a coefficient $c = c(\eta,\psi)$ we use the notation

$$\dot c = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(c(\eta + \varepsilon\dot\eta, \psi) - c(\eta,\psi) \right).$$

Using this notation for $\dot{\lambda}, \dot{\mathfrak{B}}, \dot{V}$, we have

(6.8)

$$d_\eta f^1(\eta,\psi)\cdot\dot\eta = -G(\eta)(\mathfrak{B}\dot\eta) - \operatorname{div}(V\dot\eta) \\ - \left\{T_{\dot\lambda}\big(\psi - T_{\mathfrak{B}}\eta\big) - T_\lambda T_{\dot{\mathfrak{B}}}\eta - T_\lambda T_{\mathfrak{B}}\dot\eta - T_{\dot V}\cdot\nabla\eta - T_V\cdot\nabla\dot\eta\right\},$$

We split the right-hand side into four terms (three of which are easy to estimate, whereas the last one requires some care): set

$$\begin{aligned} I_1 &= V\cdot\nabla\dot\eta - T_V\cdot\nabla\dot\eta,\\ I_2 &= -T_{\dot\lambda}\big(\psi - T_{\mathfrak{B}}\eta\big),\\ I_3 &= -T_\lambda T_{\dot{\mathfrak{B}}}\eta,\\ I_4 &= -G(\eta)(\mathfrak{B}\dot\eta) - (\operatorname{div} V)\dot\eta + T_\lambda T_{\mathfrak{B}}\dot\eta. \end{aligned}$$

To estimate I_1, we use that, for all function $a\in H^{s_0}(\mathbf{R}^d)$ with $s_0 > 1 + d/2$, we have

$$\left\lVert au - T_a u\right\rVert_{H^{\mu+1}} \le K \left\lVert a\right\rVert_{H^{s_0}} \left\lVert u\right\rVert_{H^\mu},$$

whenever $u\in H^\mu(\mathbf{R}^d)$ for some $0\le\mu\le s_0-1$. By applying this estimate with $s_0 = s-1$, we obtain

$$\left\lVert I_1\right\rVert_{H^{s-1}} = \left\lVert (V - T_V)\cdot\nabla\dot\eta\right\rVert_{H^{s-1}} \lesssim \left\lVert V\right\rVert_{H^{s-1}} \left\lVert\nabla\dot\eta\right\rVert_{H^{s-1-1}} \le C\left\lVert\dot\eta\right\rVert_{H^{s-1}}.$$

With regards to the second term, we use the arguments in the proof of Proposition 4.4 (notice that here, our symbol $\dot\lambda$ has not exactly the form (4.7), but rather

$$F(\nabla\eta,\xi)\nabla\dot\eta + G(\nabla\eta,\xi)\nabla^2\dot\eta + K(\nabla\eta,\xi)\nabla\dot\eta\nabla^2\eta$$

and the proof of Proposition 4.4 applies). We obtain

$$\left\lVert I_2\right\rVert_{H^{s-1}} \le C\left\lVert\dot\eta\right\rVert_{H^{s-1}}.$$

To estimate I_3, notice that (3.4) implies that

$$\left\lVert I_3\right\rVert_{H^{s-1}} \lesssim M_0^1(\lambda)\left\lVert T_{\dot{\mathfrak{B}}}\eta\right\rVert_{H^{s-1+1}} \le C\left\lVert T_{\dot{\mathfrak{B}}}\eta\right\rVert_{H^s}.$$

Next, using the general estimate

$$\left\lVert T_a u\right\rVert_{H^\mu} \le K\left\lVert a\right\rVert_{H^{\frac{d}{2}-m}}\left\lVert u\right\rVert_{H^{\mu+m}},$$

we conclude

$$\left\lVert I_3\right\rVert_{H^{s-1}} \le C\left\lVert\dot{\mathfrak{B}}\right\rVert_{H^{s-\frac52}}\left\lVert\eta\right\rVert_{H^{s+\frac12}}.$$

Therefore, the desired result for I_3 will follow from the claim

$$\left\lVert\dot{\mathfrak{B}}\right\rVert_{H^{s-\frac52}} \le C\left\lVert\dot\eta\right\rVert_{H^{s-1}}.$$

To see this, the only non-trivial point is to bound $dG(\eta)\psi\cdot\dot\eta$, which was precisely done above (cf (6.6)).

It remains to estimate I_4, which is the most delicate part. Indeed, one cannot estimate the terms separately, and we have to use a cancellation which comes from the identity $G(\eta)\mathfrak{B} = -\operatorname{div} V$ (see Lemma 2.12).

It follows from Proposition 3.22 that

$$G(\eta)(\mathfrak{B}\dot\eta) = T_{\lambda^{(1)}}(\mathfrak{B}\dot\eta) + F(\eta,\mathfrak{B}\dot\eta), \quad G(\eta)\mathfrak{B} = T_{\lambda^{(1)}}\mathfrak{B} + F(\eta,\mathfrak{B}),$$

where

$$\left\lVert F(\eta,\mathfrak{B}\dot\eta)\right\rVert_{H^{s-1}} \le C\left\lVert\dot\eta\right\rVert_{H^{s-1}}, \quad \left\lVert F(\eta,\mathfrak{B})\right\rVert_{H^{s-1}} \le C.$$

Therefore

$$\begin{aligned} I_4 &= -G(\eta)(\mathfrak{B}\dot{\eta}) - (\operatorname{div} V)\dot{\eta} + T_\lambda T_{\mathfrak{B}}\dot{\eta} \\ &= -T_{\lambda^{(1)}}(\mathfrak{B}\dot{\eta}) - F(\eta, \mathfrak{B}\dot{\eta}) - \dot{\eta}\operatorname{div} V + T_\lambda T_{\mathfrak{B}}\dot{\eta} \\ &= -T_{\lambda^{(1)}}(\mathfrak{B}\dot{\eta}) - F(\eta, \mathfrak{B}\dot{\eta}) - T_{\dot{\eta}}\operatorname{div} V - (\dot{\eta} - T_{\dot{\eta}})\operatorname{div} V + T_\lambda T_{\mathfrak{B}}\dot{\eta} \end{aligned}$$

and hence using $\operatorname{div} V = -G(\eta)\mathfrak{B} + R$ with $R \in H^{s-1}(\mathbf{R}^d)$ (see Lemma 2.12) we obtain

$$I_4 = -T_{\lambda^{(1)}}(\mathfrak{B}\dot{\eta}) - F(\eta, \mathfrak{B}\dot{\eta}) + T_{\dot{\eta}}(G(\eta)\mathfrak{B} - R) + (\dot{\eta} - T_{\dot{\eta}})\operatorname{div} V + T_\lambda T_{\mathfrak{B}}\dot{\eta}$$

and paralinearizing $G(\eta)\mathfrak{B}$ and gathering terms we conclude

$$\begin{aligned} I_4 = &-T_{\lambda^{(1)}}(\mathfrak{B}\dot{\eta}) - F(\eta, \mathfrak{B}\dot{\eta}) + T_{\dot{\eta}}\Big(T_{\lambda^{(1)}}\mathfrak{B} + F(\eta, \mathfrak{B})\Big) \\ &+ (\dot{\eta} - T_{\dot{\eta}})\operatorname{div} V + T_\lambda T_{\mathfrak{B}}\dot{\eta} - T_{\dot{\eta}}R \end{aligned}$$

then commuting $T_{\dot{\eta}}$ and $T_{\lambda^{(1)}}$ we conclude that

$$I_4 = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &= -T_{\lambda^{(1)}}\Big(\mathfrak{B}\dot{\eta} - T_{\dot{\eta}}\mathfrak{B} - T_{\mathfrak{B}}\dot{\eta}\Big) \\ J_2 &= T_{\lambda^{(0)}}T_{\mathfrak{B}}\dot{\eta} + [T_{\dot{\eta}}, T_{\lambda^{(1)}}]\mathfrak{B} + T_{\dot{\eta}}F(\eta, \mathfrak{B}) \\ &\quad + (\dot{\eta} - T_{\dot{\eta}})\operatorname{div} V - F(\eta, \mathfrak{B}\dot{\eta}) - T_{\dot{\eta}}R. \end{aligned}$$

Now both terms J_1 and J_2 are estimated using symbolic calculus (namely we estimate J_1 by means of (ii) in Theorem 3.12; and we estimate J_2 by means of (3.4), (3.5) and (ii) in Theorem 3.12). □

6.3. Sketch of the proof of Lemma 6.3. Let $0 < \varepsilon_1 < \varepsilon_2$ and consider two solutions $(\eta_{\varepsilon_j}, \psi_{\varepsilon_j}) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d))$ of (5.5). Introduce the notation

$$\mathfrak{B}_{\varepsilon_j} = \frac{\nabla\eta_{\varepsilon_j} \cdot \nabla\psi_{\varepsilon_j} + G(\eta_{\varepsilon_j})\psi_{\varepsilon_j}}{1 + \left|\nabla\eta_{\varepsilon_j}\right|^2}, \quad V_{\varepsilon_j} = \nabla\psi_{\varepsilon_j} - \mathfrak{B}_j\nabla\eta_{\varepsilon_j}, \tag{6.9}$$

and denote by λ_j, ℓ_j the symbols obtained by replacing η by η_{ε_j} in (3.11), (3.27) respectively. Here, the main technical lemma is the following.

Lemma 6.10. *Let $0 < \varepsilon_1 < \varepsilon_2$, consider s' such that*

$$\frac{1}{2} + \frac{d}{2} < s' < s - \frac{3}{2},$$

and set

$$a = s - \frac{3}{2} - s'.$$

Then the differences $\delta\eta := \eta_{\varepsilon_1} - \eta_{\varepsilon_2}$ and $\delta\psi := \psi_{\varepsilon_1} - \psi_{\varepsilon_2}$ satisfy a system of the form

$$\big(\partial_t + T_{V_{\varepsilon_1}} \cdot \nabla J_{\varepsilon_1} + \mathcal{L}^{\varepsilon_1}\big)\begin{pmatrix}\delta\eta \\ \delta\psi\end{pmatrix} = f, \tag{6.10}$$

for some remainder term such that

$$\|f\|_{X^{s'}(T)} \leq C\left\{\|(\delta\eta, \delta\psi)\|_{X^{s'}(T)} + \varepsilon_2^a\right\},$$

for some constant C depending only on $\sup_{\varepsilon\in]0,1]}\|(\eta_\varepsilon,\psi_\varepsilon)\|_{X^s(T)}$.

To prove Lemma 6.10, we proceed as in the previous paragraph. The only difference is that we use the fact that

$$\|J_{\varepsilon_2}-J_{\varepsilon_1}\|_{H^\mu\rightarrow H^{\mu-a}}\le C\varepsilon_2^a.$$

Now, since for $t=0$ we have $\delta\eta=0=\delta\psi$, it follows from Lemma 6.10 and (5.21) applied with

$$\sigma=s',\quad \varepsilon=\varepsilon_1,\quad \tilde{\eta}=\delta\eta,\quad \tilde{\psi}=\delta\psi,$$

that N satisfies and estimate of the form

$$N\le TC\left\{N+\varepsilon_2^a\right\}.$$

By choosing T and ε_2 small enough, this implies $N=O(\varepsilon_2^a)$. This proves Lemma 6.3.

6.4. Continuity in time. We now prove that the solution (η,ψ) constructed in the previous sections is continuous in time with values in $H^{s+\frac12}\times H^s$. To do so, it is enough to prove that (η,U) is continuous in time with values in $H^{s+\frac12}\times H^s$, which in turn will be clear if we prove that the complex-valued unknown Φ is continuous in time with values in H^s. Furthermore, by usual functional analysis arguments, (following the scheme of proof given for instance by Taylor in [32], see Proposition 5.1.D in [32]), it is enough to prove that the scalar function $t\mapsto\|\Phi(t)\|_{H^s}$ is continuous. To prove this, we shall prove that $\|J_\varepsilon(t)\Phi(t)\|_{H^s}^2$ is (uniformly with respect to ε) a Lipschitz function of $t\in[0,T]$, so that the desired continuity will be established provided that we prove that $J_\varepsilon(t)\Phi(t)$ converges to $\Phi(t)$ in H^s for all $t\in[0,T]$.

The fact that $\|J_\varepsilon(t)\Phi(t)\|_{H^s}^2$ is a Lipschitz function of $t\in[0,T]$ is a consequence of previous estimates. Indeed, the above analysis established that $\|J_\varepsilon\Phi\|_{H^s}^2$ satisfies an estimate of the form

$$\frac{d}{dt}\|J_\varepsilon(t)\Phi(t)\|_{H^s}^2\le C(\|(\eta,\psi)\|_{H^{s+1/2}\times H^s}).$$

The only technical point which remains to check is that, for fixed time $t\in[0,T]$, $J_\varepsilon(t)\Phi(t)$ converges to $\Phi(t)$ in $H^s(\mathbf{R}^d)$. In standard situation where the mollifiers J_ε are Fourier multipliers, this is an immediate consequence of the dominated convergence theorem. Here J_ε is a paradifferential operator whose symbol depends on x and this requires a verification.

Lemma 6.11. *For any $t\in[0,T]$ and any $v\in H^s(\mathbf{R}^d)$ whose spectrum is included in $|\xi|\ge 2$, $J_\varepsilon(t)v$ converges to v in $H^s(\mathbf{R}^d)$ when ε goes to* 0.

Proof. To simplify notations, we omit the time dependence, denote by $\|\cdot\|$ the H^s-norm and by $\langle\cdot,\cdot\rangle$ the scalar product in H^s. By symbolic calculus it is easy to prove that, for $\delta>0$,

$$\|I-J_\varepsilon\|_{H^s\rightarrow H^{s-\delta}}=O(\varepsilon^{\frac23\delta}),\tag{6.11}$$

which implies that $J_\varepsilon v$ converges to v in $H^{s-\delta}$. Consequently, using classical arguments, it is enough to prove that $\|J_\varepsilon v\|$ converges to $\|v\|$. Recall that, by definition,

$$\jmath_\varepsilon(x,\xi) = \jmath_\varepsilon^{(0)}(x,\xi) - -\frac{i}{2}(\partial_x\cdot\partial_\xi)\jmath_\varepsilon^{(0)}(x,\xi) \quad \text{with } \jmath_\varepsilon^{(0)} = \exp(-\varepsilon\gamma^{(3/2)}(x,\xi)).$$

Note that $\varepsilon^{-\frac{2}{3}}(\partial_x\cdot\partial_\xi)\jmath_\varepsilon^{(0)}(x,\xi)$ is uniformly bounded in Γ_0^0 and hence

$$\left\| T_{(\partial_x\cdot\partial_\xi)\jmath_\varepsilon^{(0)}} v \right\| = O(\varepsilon^{2/3}).$$

Consequently, it is enough to prove that $\|T_{\jmath_\varepsilon^{(0)}}v\|$ converge to $\|v\|$ when ε goes to 0. To avoid confusion of notations, introduce $p_\varepsilon(x,\xi) = (\jmath_\varepsilon^{(0)}(x,\xi))^2$. We have

$$\left\| T_{\jmath_\varepsilon^{(0)}} v \right\|^2 = \langle T_{p_\varepsilon}v, v\rangle + \langle R_\varepsilon v, v\rangle \quad \text{with } R_\varepsilon = (T_{\jmath_\varepsilon^{(0)}})^* T_{\jmath_\varepsilon^{(0)}} - T_{p_\varepsilon}.$$

Since $\varepsilon^{-a}\jmath_\varepsilon^{(0)}$ is uniformly bounded in $\Gamma_1^{3a/2}$ for any $a > 0$, by symbolic calculus we have

$$\|(T_{\jmath_\varepsilon^{(0)}})^* - T_{\jmath_\varepsilon^{(0)}}\|_{H^s\to H^{s-1}} \lesssim M_1^1(\jmath_\varepsilon^{(0)}) = O(\varepsilon^{2/3}),$$
$$\|(T_{\jmath_\varepsilon^{(0)}})^* T_{\jmath_\varepsilon^{(0)}} - T_{p_\varepsilon}\|_{H^s\to H^s} \lesssim M_1^{1/2}(\jmath_\varepsilon^{(0)})^2 = O(\varepsilon^{2/3})$$

and hence $\langle R_\varepsilon v, v\rangle = O(\varepsilon^{2/3})$. Consequently, it is enough to prove that $\langle T_{p_\varepsilon}v, v\rangle$ converges to $\|v\|^2$. To do that, we shall prove that

$$\|v\|^2 \geq \limsup\langle T_{p_\varepsilon}v, v\rangle \geq \liminf\langle T_{p_\varepsilon}v, v\rangle \geq \|v\|^2.$$

Note that there exist $c, C > 0$ such that $c\,|\xi|^{\frac{3}{2}} \leq \gamma^{(3/2)}(x,\xi) \leq C\,|\xi|^{\frac{3}{2}}$, and introduce the Fourier multipliers

$$a_\varepsilon(\xi) = \exp(-2\varepsilon C\,|\xi|^{\frac{3}{2}}), \quad b_\varepsilon(\xi) = \exp(-2\varepsilon c\,|\xi|^{\frac{3}{2}}),$$

so that $a_\varepsilon \leq p_\varepsilon \leq b_\varepsilon$. Introduce a positive constant $\delta > 0$. Then

$$q_\varepsilon := (p_\varepsilon(x,\xi) - a_\varepsilon(\xi) + \delta)^{1/2} \in \Gamma_{3/2}^0(\mathbf{R}^d).$$

We have

$$\langle T_{p_\varepsilon(x,\xi)-a_\varepsilon(\xi)+\delta}v, v\rangle = \langle T_{q_\varepsilon}v, T_{q_\varepsilon}v\rangle + \langle R_{\delta,\varepsilon}v, v\rangle,$$

where, using sharp operator norm estimates for symbolic calculus (see Theorems 2.16 and 2.18 in [24]), we have that

$$\langle R_{\delta,\varepsilon}v, v\rangle = O(\varepsilon^{1/3}).$$

The underlying constant depends on δ and even more blows up when δ goes to 0. However, the trick is that we shall let ε goes to 0 and then δ goes to 0, so that this large constant is harmless. Indeed, for fixed $\delta > 0$, we have

$$\liminf_{\varepsilon\to 0}\langle T_{p_\varepsilon(x,\xi)-a_\varepsilon(\xi)+\delta}v, v\rangle \geq 0.$$

Since the spectrum of v lies in the exterior of the ball of center 0 and radius 2, we have $T_{a_\varepsilon}v = a_\varepsilon(D_x)v$ and similarly $T_\delta v = \delta v$ (recall that we include a cut-off ψ in the definition of paradifferential operators). Now by the dominated convergence theorem, we have

$$\langle a_\varepsilon(D_x)v, v\rangle \to \|v\|^2.$$

Thefore we find

$$\liminf_{\varepsilon\to 0}\langle T_{p_\varepsilon(x,\xi)}v, v\rangle \geq (1-\delta)\|v\|^2.$$

Since this holds for any $\delta > 0$, we obtain $\liminf_{\varepsilon\to 0}\langle T_{p_\varepsilon(x,\xi)}v, v\rangle \geq \|v\|^2$. Similarly we show that $\limsup_{\varepsilon\to 0}\langle T_{p_\varepsilon(x,\xi)}v, v\rangle \leq \|v\|^2$. This completes the proof. $\square$

6.5. Continuity with respect to initial data. Notice that from the *a priori* bound in $L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)$ and the Lipschitz bound in $L^\infty(0,T;H^{s-1}\times H^{s-\frac{3}{2}})$, for any $\sigma < s$, the flow map

$$(\eta_0,\psi_0)\in H^{s+\frac{1}{2}}\times H^s \mapsto (\eta,\psi)\in C^0([0,T];H^{\sigma+\frac{1}{2}}\times H^\sigma) \tag{6.12}$$

is uniformly continuous. In this section we are going to prove Theorem 1.2 whose statement is recalled here.

Theorem 6.12. *Consider* $(\eta,\psi)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$ *a solution of* (1.2) *and a sequence* $(\eta_{n,0},\psi_{n,0})_{n\in\mathbb{N}^*}$ *converging in* $H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d)$ *to* $(\eta,\psi)\mid_{t=0}$. *Then, for* n *sufficiently large, the solutions* $(\eta_n,\psi_n)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$ *with data* $(\eta_{n,0},\psi_{n,0})$ *are defined on the time interval* $[0,T]$ *and satisfy*

$$\lim_{n\to+\infty}\|(\eta_n,\psi_n)-(\eta,\psi)\|_{C^0([0,T];H^{s+\frac{1}{2}}\times H^s)}=0. \tag{6.13}$$

In the context of quasilinear equations, this kind of results is rather standard (see for example [19]), and the methods used in this context can be (using the machinery we previously developped) adapted to the water-waves system. As a consequence, we shall only give the main steps. Here, we follow (an adaptation of) the Bona-Smith argument (see [7, 34]). The first part in Theorem 6.12 is a straightforward consequence of the proof we gave of the existence of solutions. To obtain the continuity, the main point is the following

Lemma 6.13. *Consider a sequence* $(\eta_n,\psi_n)_{n\in\mathbb{N}^*}$ *bounded in* $C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))$ *satisfying*

$$\lim_{n\to+\infty}\|(\eta_n,\psi_n)\mid_{t=0}-(\eta_0,\psi_0)\mid_{t=0}\|_{H^{s+\frac{1}{2}}\times H^s}=0.$$

Then

$$\lim_{\varepsilon\to 0}\sup_{n\in\mathbb{N}}\|(I-J_{\varepsilon,n})\Phi_n\|_{C^0([0,T];H^s(\mathbf{R}^d))}=0,$$

where Φ_n *is the function associated to* (η_n,ψ_n) *by* (5.6) *and* $J_{\varepsilon,n}$ *is the mollifier associated in section 5.2.*

Let us first show how we can prove Theorem 6.12 from Lemma 6.13. Denote by

$$\|(\eta,\psi)\|_{X^s_T}=\|(\eta,\psi)\|_{C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d))}.$$

We first deduce easily from Lemma 6.13

$$\lim_{\varepsilon\to 0}\sup_{n\in\mathbb{N}}\|(I-J_{\varepsilon,n})(\eta_n,\psi_n)\|_{X^s_T}=0.$$

Then we have (where J_ε is associated to (η,ψ))

$$\begin{aligned}&\|(\eta_n,\psi_n)-(\eta,\psi)\|_{X^s_T}\\&\quad\leq\|J_\varepsilon\big((\eta_n,\psi_n)-(\eta,\psi)\big)\|_{X^s_T}+\|(I-J_\varepsilon)\big((\eta_n,\psi_n)-(\eta,\psi)\big)\|_{X^s_T}\end{aligned} \tag{6.14}$$

The second term in the right hand side of (6.14) is bounded by

(6.15)
$$\begin{aligned}\|(I-J_\varepsilon)(\eta,\psi)\|_{X^s_T}+\|(I-J_{\varepsilon,n})(\eta_n,\psi_n)\|_{X^s_T}+\|(J_{\varepsilon,n}-J_\varepsilon)(\eta_n,\psi_n)\|_{X^s_T}\\ \leq o(1)_{\varepsilon\to 0}+\sup_{t\in[0,T]}\sup_{\sigma=s+\frac12,\sigma=s}\|J_{\varepsilon,n}-J_\varepsilon\|_{\mathcal{L}(H^\sigma(\mathbf{R}^d))}\end{aligned}$$

and by symbolic calculus (notice that γ is expressed in terms of $\nabla_x\eta$, and the norm on H^σ of a zeroth order paradifferential operator is bounded by the L^∞ norm of a finite number of ξ derivatives of the symbol and the norm of a paradifferential operator of order -1 on H^σ is bounded by a finite number of norms of ξ derivatives of the coefficients in $H^{\frac d2-1}$, see § 4.1), we can bound

$$\|J_{\varepsilon,n}-J_\varepsilon\|_{\mathcal{L}(H^\sigma(\mathbf{R}^d))}\leq C\|(\eta_n,\psi_n)-(\eta,\psi)\|_{X^\sigma_T}=o(1)_{\varepsilon\to+\infty}$$

as soon as $\sigma-\frac12>\frac d2$ (and we use here (6.13)). Now we can fix $\varepsilon>0$ small enough so that the second term in (6.14) is arbitrarily small (uniformly with respect to n). To bound the first term in (6.14) we use that, similar to (6.11),

$$\|J_\varepsilon\|_{X_T^{s-\frac32}\to X^s_T}\leq\frac{C}{\varepsilon},$$

and consequently, using again (6.13) (for $\sigma=s-\frac32$) this term gives a contribution $o(1)_{n\to+\infty}$ (ε is fixed).

Let us come back to the proof of Lemma 6.13. We already proved in § 6.4 that for any $n\in\mathbb{N}$,

$$\lim_{\varepsilon\to0}\|(I-J_{\varepsilon,n})\psi_n\mid_{t=0}\|_{H^s(\mathbf{R}^d)}=0$$

and consequently, as the family

$$\{(\eta_n,\psi_n)\mid_{t=0},n\in\mathbb{N}\}\cup\{(\eta,\psi)\mid_{t=0}\}$$

is compact, we deduce

$$\lim_{\varepsilon\to0}\sup_{n\in\mathbb{N}}\|(I-J_{\varepsilon,n})\psi_n\mid_{t=0}\|_{H^s(\mathbf{R}^d)}=0.$$

To conclude the proof of Lemma 6.13, it is enough to show that this estimate is propagated by the flow. For the sake of conciseness, we shall only show how to prove this fact for fixed n. Now, the function $(I-J_\varepsilon)\Phi$ satisfy the equation (with $\rho_0=\frac12\min(1,s-2-\frac d2)>0$)

$$(\partial_t+T_V\cdot\nabla+iT_\gamma)(I-J_\varepsilon)\Phi=F\in C^0([0,T];H^{s+\rho_0}(\mathbf{R}^d)).$$

Using that F is essentially of the form $(I-J_\varepsilon)G$, we deduce

$$\|F\|_{C^0([0,T];H^s(\mathbf{R}^d))}\leq C\varepsilon^{\frac23\rho_0},$$

and the energy estimates in §5 allow to conclude the proof of Lemma 6.13.

7. The smoothing effect

We consider a given solution (η,ψ) of (1.2) on the time interval $[0,T]$ with $0<T<+\infty$, such that the assumption H_t is satisfied for all $t\in[0,T]$ and such that

$$(\eta,\psi)\in C^0\big([0,T];H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R})\big),$$

for some $s > \frac{5}{2}$. In this section we prove Theorem 1.5. Namely, we shall prove that

$$\langle x\rangle^{-\frac{1}{2}-\delta}(\eta,\psi) \in L^2\big(0,T;H^{s+\frac{3}{4}}(\mathbf{R})\times H^{s+\frac{1}{4}}(\mathbf{R})\big),$$

for any $\delta > 0$.

7.1. Reduction to an L^2 estimate. Let Φ_1,Φ_2 be as defined in Corollary 4.9. Then the complex-valued unknown $\Phi = \Phi_1 + i\Phi_2$ satisfies a *scalar equation* of the form

$$\partial_t\Phi + T_V\partial_x\Phi + iT_\gamma\Phi = F, \tag{7.1}$$

with $F = F_1 + iF_2 \in L^\infty(0,T;H^s(\mathbf{R}^d))$. Recall from Proposition 3.13 and (3.27) that, if $d = 1$ then

$$\lambda^{(1)} = |\xi|\,, \quad \lambda^{(0)} = 0, \quad \ell^{(2)} = c^2\,|\xi|^2\,,$$

with

$$c = (1+|\partial_x\eta|^2)^{-\frac{3}{4}}.$$

Therefore, directly from the definition of γ (cf Proposition 4.8), notice that if $d = 1$ then γ simplifies to

$$\gamma = c\,|\xi|^{\frac{3}{2}} - \frac{3i}{4}\xi\,|\xi|^{-\frac{1}{2}}\,\partial_x c,$$

and hence modulo an error term of order 0, T_γ is given by $|D_x|^{\frac{3}{4}}\,T_c\,|D_x|^{\frac{3}{4}}$.

In this paragraph we shall prove that one can deduce Theorem 1.5 from the following proposition.

Proposition 7.1. *Assume that $\varphi \in C^0([0,T];L^2(\mathbf{R}))$ satisfies*

$$\partial_t\varphi + T_V\partial_x\varphi + iT_\gamma\varphi = f,$$

with $f \in L^1(0,T;L^2(\mathbf{R}))$. Then, for all $\delta > 0$,

$$\langle x\rangle^{-\frac{1}{2}-\delta}\varphi \in L^2(0,T;H^{\frac{1}{4}}(\mathbf{R})).$$

We postpone the proof of Proposition 7.1 to the next paragraph.

The fact that one can deduce Theorem 1.5 from the above proposition, though elementary, contains the idea that one simplify hardly all the nonlinear analysis by means of paradifferential calculus.

Proof of Theorem 1.5 given Proposition 7.1. Following the proof of Proposition 5.2 (cf §5.5), with

$$\beta := c^{\frac{2}{3}s}\,|\xi|^s\,. \tag{7.2}$$

we find that the commutators $[T_\beta,\partial_t]$, $[T_\beta,T_\gamma]$ and $[T_\beta,T_V\partial_x]$ are of order s. Consequently, (7.1) implies that

$$(\partial_t + T_V\partial_x + iT_\gamma)\,T_\beta\Phi \in L^\infty(0,T;L^2(\mathbf{R})),$$

and hence,

$$(\partial_t + T_V\partial_x + iT_\gamma)\,T_\beta\Phi \in L^1(0,T;L^2(\mathbf{R})).$$

Therefore it follows from Proposition 7.1 that

$$\langle x\rangle^{-\frac{1}{2}-\delta}T_\beta\Phi \in L^2(0,T;H^{\frac{1}{4}}(\mathbf{R})).$$

Since, by definition, $\Phi = T_p\eta + iT_qU$ where $T_p\eta$ and T_qU are real valued functions, this yields

$$\langle x\rangle^{-\frac{1}{2}-\delta} T_\beta T_p \eta \in L^2(0,T;H^{\frac{1}{4}}(\mathbf{R})), \quad \langle x\rangle^{-\frac{1}{2}-\delta} T_\beta T_q U \in L^2(0,T;H^{\frac{1}{4}}(\mathbf{R})),$$

and hence, since $\psi = U + T_{\mathfrak{B}}\eta$,
(7.3)

$$\langle x\rangle^{-\frac{1}{2}-\delta} T_\beta T_p \eta \in L^2(0,T;H^{\frac{1}{4}}(\mathbf{R})), \quad \langle x\rangle^{-\frac{1}{2}-\delta} T_\beta T_q \psi \in L^2(0,T;H^{\frac{1}{4}}(\mathbf{R})),$$

Since $\langle x\rangle^{-\frac{1}{2}-\delta} \in \Gamma^0_\rho(\mathbf{R}^d)$ for any $\rho \geq 0$, Theorem 3.7 implies that the commutators

$$\left[\langle x\rangle^{-\frac{1}{2}-\delta}, T_\beta T_p\right], \quad \left[\langle x\rangle^{-\frac{1}{2}-\delta}, T_\beta T_q\right]$$

are of order $s-1/2$ and $s-1$, respectively. Therefore, directly from (7.3) and the assumption

$$\eta \in C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R})), \quad \psi \in C^0([0,T];H^s(\mathbf{R})),$$

we obtain

$$T_\beta T_p \langle x\rangle^{-\frac{1}{2}-\delta} \eta \in L^2(0,T;H^{\frac{1}{4}}(\mathbf{R})), \quad T_\beta T_q \langle x\rangle^{-\frac{1}{2}-\delta} \psi \in L^2(0,T;H^{\frac{1}{4}}(\mathbf{R})).$$

Now since β, p, q are elliptic symbols of order s, $1/2$, 0, respectively, we conclude (cf Remark 3.9 or Proposition 4.6)

$$\langle x\rangle^{-\frac{1}{2}-\delta} \eta \in L^2(0,T;H^{s+\frac{3}{4}}(\mathbf{R})), \quad \langle x\rangle^{-\frac{1}{2}-\delta} \psi \in L^2(0,T;H^{s+\frac{1}{4}}(\mathbf{R})).$$

This proves Theorem 1.5. □

7.2. Proof of Proposition 7.1. To complete the proof of Theorem 1.5, it remains to prove Proposition 7.1. To do so, following the Doi approach, we begin with the following lemma which follows from the observation that $\partial_\xi(\frac{\xi}{|\xi|}) = 0$ for $\xi \in \mathbf{R}\setminus\{0\}$ (and the fact that c is uniformly bounded from below).

Lemma 7.2. *Let $\delta > 0$ and consider*

$$a(x,\xi) = \frac{\xi}{|\xi|}\int_0^x \frac{1}{\langle y\rangle^{1+\delta}}\,dy$$

Then

$$a \in \dot{\Gamma}^0_\infty(\mathbf{R}) := \bigcap_{\rho\geq 0} \dot{\Gamma}^0_\rho(\mathbf{R}),$$

and there exists $K > 0$ such that

$$\left\{c\,|\xi|^{\frac{3}{2}}, a\right\}(t,x,\xi) \geq K\langle x\rangle^{-1-\delta}\,|\xi|^{\frac{1}{2}},$$

for all $t \in [0,T]$, $x \in \mathbf{R}$, $\xi \in \mathbf{R}\setminus\{0\}$.

We are now in position to prove Proposition 7.1.

Proof of Proposition 7.1. We begin by remarking that we can assume without loss of generality that $\varphi \in C^1(I;L^2(\mathbf{R}))$ (A word of caution: to do so, instead of using the usual Friedrichs mollifiers, we need to use the operators J_ε introduced in §5.2). This allows us to write

$$\begin{aligned}\frac{d}{dt}\langle T_a\varphi,\varphi\rangle &= \langle T_{\partial_t a}\varphi,\varphi\rangle+\langle T_a\partial_t\varphi,\varphi\rangle+\langle T_a\varphi,\partial_t\varphi\rangle\\ &=\langle T_{\partial_t a}\varphi,\varphi\rangle\\ &\quad-\langle T_aT_V\partial_x\varphi+T_a iT_\gamma\varphi-T_af,\varphi\rangle\\ &\quad-\langle T_a\varphi,+T_V\partial_x\varphi+iT_\gamma\varphi-f\rangle\,,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 scalar product. Introduce the commutator

$$C:=[iT_\gamma,T_a]\,.$$

Since $\partial_t a=0$, the previous identity yields

$$\begin{aligned}\frac{d}{dt}\langle T_a\varphi,\varphi\rangle &= \langle C\varphi,\varphi\rangle+\left\langle i(T_\gamma^*-T_\gamma)T_a\varphi,\varphi\right\rangle\\ &\quad+\langle\partial_x(T_VT_a\varphi)-T_aT_V\partial_x\varphi,\varphi\rangle\\ &\quad+\langle T_af,\varphi\rangle+\langle T_a\varphi,f\rangle\end{aligned}\tag{7.4}$$

Since $a\in\dot\Gamma^0_0$, it follows from the usual estimates for paradifferential operators that

$$\left|\langle T_a\varphi,\varphi\rangle\right|\lesssim\left\|\varphi\right\|_{L^2}^2\,,$$

and

$$\left|\langle T_a\varphi,f\rangle\right|+\left|\langle T_af,\varphi\rangle\right|\le K\left\|\varphi\right\|_{L^2}^2+K\left\|f\right\|_{L^2}^2\,,$$

for some positive constant K. One easily obtains similar bounds for the second and third terms in the right hand-side of (7.4). Indeed, by definition of γ we know that $T_\gamma^*-T_\gamma$ is of order 0. On the other hand, as already seen, it follows from Theorems 3.7 that $\partial_x(T_VT_a\cdot)-T_aT_V\partial_x$ is of order 0. Therefore, integrating (7.4) in time, we end up with

$$\int_0^T\langle C\varphi,\varphi\rangle\,dt\le M\left\{\left\|\varphi(0)\right\|_{L^2}^2+\left\|\varphi(T)\right\|_{L^2}^2+\int_0^T\left(\left\|\varphi\right\|_{L^2}^2+\left\|f\right\|_{L^2}^2\right)\,dt\right\},$$

where M depends only on the $L^\infty(0,T;H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R}))$-norm of (η,ψ).

Hence to complete the proof it remains only to obtain a lower bound for the left hand-side. To do so, write

$$iT_\gamma=iT_c\left|D_x\right|^{\frac32}+\frac34T_{\frac{\xi}{|\xi|}\partial_xc}\left|D_x\right|^{\frac12}\,,$$

and recall that, by definition of a (see Lemma 7.2) there exists a constant K such that

$$\left\{c(t,x)\left|\xi\right|^{\frac34},a(x,\xi)\right\}\ge K\langle x\rangle^{-1-2\delta}\left|\xi\right|^{\frac12}\,,$$

for some positive constant $K>0$. Since

$$\left[T_a,T_{\frac{\xi}{|\xi|}\partial_xc}\left|D_x\right|^{\frac12}\right]\text{ is of order 0,}$$

Proposition 7.3 below then implies that

$$\langle C\varphi,\varphi\rangle\ge a\left\|\langle x\rangle^{-\frac12-\delta}\varphi\right\|_{H^{\frac14}}^2-A\left\|\varphi\right\|_{L^2}^2\,,$$

for some positive constants a,A. This completes the proof of Proposition 7.1 and hence of Theorem 1.5. □

Proposition 7.3. *Let $d \geq 1$ and $\delta > 0$. Assume that $d \in \Gamma^{1/2}_{1/2}(\mathbf{R}^d)$ is such that, for some positive constant K, we have*

$$d(x,\xi) \geq K\langle x\rangle^{-1-2\delta}\left|\xi\right|^{\frac{1}{2}},$$

for all $(x,\xi) \in \mathbf{R}^d \times \mathbf{R}^d \setminus \{0\}$. Then there exist two positive constants $0 < a < A$ such that

$$\langle T_d u, u\rangle \geq a\left\|\langle x\rangle^{-\frac{1}{2}-\delta}u\right\|^2_{H^{\frac{1}{4}}} - A\left\|u\right\|^2_{L^2}.$$

Remark 7.4. This proposition has been used for $d = 1$. However, it might be useful for $d \geq 1$.

Remark 7.5. Related results were previously proved by Bony [10] (see also [31]) in the much more general setting of sharp Gårding or Fefferman-Phong inequalities. Notice however that these results require much more regularity than Proposition 7.3 (where the symbol is only assumed to be $C^{\frac{1}{2}}$).

Proof. Again, the difficulty comes from our low regularity assumption. Indeed, with more regularity (say $d \in \Gamma^{1/2}_{\rho}(\mathbf{R}^d)$ with $\rho > 2$) this follows from the sharp Gårding inequality proved in [10].

Consider a partition of unity as a sum of squares, such that

$$1 = \theta_0^2(x) + \sum_{j=1}^{\infty}\theta^2(2^{-j}x) = \sum_{j=0}^{\infty}\theta_j^2(x),$$

where $\theta_0 \in C_0^\infty(\mathbf{R})$ and $\theta \in C^\infty(\mathbf{R})$ is supported in the annulus $\{x \in \mathbf{R} : 1 \leq |x| \leq 3\}$.

Then

$$I = \langle T_d u, u\rangle = \sum_{j=0}^{\infty}\left\langle \theta_j^2 T_d u, u\right\rangle.$$

The following result is an illustration of the pseudo-local property of paradifferential operators (see [11, p435] for similar results in this direction).

Lemma 7.6. *Let $\widetilde{\theta} \in C_0^\infty(]1/2,4[)$ equal to 1 on the support of θ, and set $\widetilde{\theta}_j(x) = \widetilde{\theta}(2^{-j}|x|)$ for $j \geq 1$. Also introduce $\widetilde{\theta}_0 \in C_0^\infty(\mathbf{R})$ equal to 1 on the support of θ_0. Then for all $\mu \in \mathbf{R}$, all $j \in \mathbf{N}$, and all $N \in \mathbf{N}$, the operator $R_j = \theta_j T_d(1-\widetilde{\theta}_j)$ is continuous from H^μ to $H^{\mu+N}$ with norm bounded by $C_N 2^{-jN}$.*

Proof. Writing (see (3.2))

$$\theta_j T_d(1-\widetilde{\theta}_j)u(x) = \frac{1}{(2\pi)^2}\int e^{i(x\cdot\xi - y\cdot\eta)}\theta_j(x)(1-\widetilde{\theta}_j(y)) \widehat{d}(\xi-\eta,\eta)\psi(\eta)\chi(\xi-\eta,\eta)u(y)\,dyd\eta d\xi,$$

we have

$$\theta_j T_d(1-\widetilde{\theta}_j)u(x) = \frac{1}{(2\pi)^2}\int e^{i(x-y)\cdot\eta}e^{ix\cdot\zeta}\theta_j(x)(1-\widetilde{\theta}_j(y)) \widehat{d}(\zeta,\eta)\psi(\eta)\chi(\zeta,\eta)u(y)\,dyd\eta d\zeta.$$

We then obtain the desired result from a non-stationary phase argument. Indeed, using that on the support of this integral we have $|x-y|>c2^j$, we can integrate by parts using the operator

$$L=\frac{(x-y)\cdot\partial_\eta}{|x-y|^2}.$$

Since $\chi(\zeta,\eta)$ is homogeneous of degree 0 in (ζ,η), we obtain that N such integration by parts gain N powers of 2^{-j} and of $|\eta|^{-1}$. $\square$

Now, write

$$\begin{aligned}
&\theta_j T_d u\\
&\quad=\theta_j T_d\widetilde{\theta}_j u+\theta_j T_d(1-\widetilde{\theta}_j)u\\
&\quad=T_d\theta_j\widetilde{\theta}_j+[\theta_j,T_d]\widetilde{\theta}_j+\theta_j T_d(1-\widetilde{\theta}_j)u\\
&\quad=T_dT_{\widetilde{\theta}_j}\theta_j+T_d(\widetilde{\theta}_j-T_{\widetilde{\theta}_j})\theta_j+[\theta_j,T_d]\widetilde{\theta}_j+\theta_j T_d(1-\widetilde{\theta}_j)u\\
&\quad=T_{\widetilde{\theta}_j d}\theta_j+(T_dT_{\widetilde{\theta}_j}-T_{\widetilde{\theta}_j d})\theta_j+T_d(\widetilde{\theta}_j-T_{\widetilde{\theta}_j})\theta_j+[\theta_j,T_d]\widetilde{\theta}_j+\theta_j T_d(1-\widetilde{\theta}_j)u.
\end{aligned}$$

The last term in the right hand side is estimated by means of Lemma 7.6. With regards to the second term in the right-hand side, we use (3.5) to obtain

$$\sup_{j\in\mathbf{N}}\left\|T_{\widetilde{\theta}_j}T_d-T_{\widetilde{\theta}_j d}\right\|_{L^2\to L^2}\lesssim\sup_{j\in\mathbf{N}}M^0_{1/2}(\widetilde{\theta}_j)M^{1/2}_{1/2}(d)\lesssim 1.$$

The third term is estimated by means of the following inequality (see [22])

$$\left\|\widetilde{\theta}_j-T_{\widetilde{\theta}_j}\right\|_{L^2\to H^1}\lesssim\|\theta_j\|_{W^{1,\infty}(\mathbf{R})}\lesssim 1.$$

Therefore, we conclude that

$$\langle(\theta_j)^2T_du,u\rangle=\left\langle T_{\widetilde{\theta}_j d}\theta_j u,\theta_j u\right\rangle+\langle U_j,\theta_j u\rangle$$

for some sequence (U_j) such that

$$\sum_{j=0}^{\infty}\|U_j\|^2_{L^2}\lesssim\sum_{j=0}^{\infty}\left(\left\|\widetilde{\theta}_j u\right\|_{L^2}\|\theta_j u\|_{L^2}+2^{-j}\|u\|_{L^2}\|\theta_j u\|_{L^2}\right)\lesssim\|u\|^2_{L^2}.$$

We want to prove

$$\sum_{j=0}^{\infty}\left\langle T_{\widetilde{\theta}_j d}\theta_j u,\theta_j u\right\rangle\geq a\left\|\langle x\rangle^{-\frac12-\delta}u\right\|^2_{H^{\frac14}}-A\|u\|^2_{L^2}.$$

To do this, it suffices to prove

$$\left\langle T_{\widetilde{\theta}_j d}\theta_j u,\theta_j u\right\rangle\geq a2^{-j(1+2\delta)}\|\theta_j u\|^2_{H^{\frac14}}-A\left\|U''_j\right\|^2_{L^2}$$

for some U''_j such that

$$\sum_{j=0}^{\infty}\left\|U''_j\right\|^2_{L^2}\leq A\|u\|^2_{L^2}.$$

Since $(\widetilde{\theta}_j d)^{\frac{1}{2}} \in \Gamma^{1/2}_{1/2}(\mathbf{R}^d)$, by applying Theorem 3.7 (with $m = m' = 1/2$ and $\rho = 1/2$), we have

$$\left\langle T_{\widetilde{\theta}_j d}\theta_j u, \theta_j u\right\rangle = \left\| T_{(\widetilde{\theta}_j d)^{\frac{1}{2}}}\theta_j u\right\|_{L^2}^2 + \langle R_j\theta_j u, \theta_j u\rangle ,$$

where R_j is uniformly bounded from L^2 to L^2. Now by assumption on d, we have

$$\left(\widetilde{\theta}_j(x)d(x,\xi)\right)^{\frac{1}{2}} \geq K\widetilde{\theta}_j(x)2^{-j\left(\frac{1}{2}+\delta\right)}\left|\xi\right|^{\frac{1}{4}} ,$$

where we used $0 \leq \widetilde{\theta}_j \leq 1$. Therefore the symbol e_j defined by

$$e_j(x,\xi) := \left(\widetilde{\theta}_j(x)d(x,\xi)\right)^{\frac{1}{2}} + K2^{-j\left(\frac{1}{2}+\delta\right)}(1-\widetilde{\theta}_j(x))\left|\xi\right|^{\frac{1}{4}} ,$$

satisfies the elliptic boundedness inequality

$$e_j(x,\xi) \geq K2^{-j\left(\frac{1}{2}+\delta\right)}\left|\xi\right|^{\frac{1}{4}} .$$

As a result

$$2^{-j\left(\frac{1}{2}+\delta\right)}\left\|\theta_j u\right\|_{H^{\frac{1}{4}}} \leq K\left\|T_{e_j}\theta_j u\right\|_{L^2} + K\left\|\theta_j u\right\|_{L^2} .$$

The desired result then follows from the fact that $(1-\widetilde{\theta}_j)\theta_j = 0$ which implies that

$$T_{(\widetilde{\theta}_j d)^{\frac{1}{2}}}\theta_j - T_{e_j}\theta_j = 2^{-j\left(\frac{1}{2}+\delta\right)}T_{(1-\widetilde{\theta}_j(x))\left|\xi\right|^{\frac{1}{4}}}\theta_j = R'_j\widetilde{\theta}_j,$$

for some operator R'_j uniformly bounded from L^2 to L^2.

This completes the proof of Proposition 7.3. □

References

[1] T. Alazard and G. Métivier, *Paralinearization of the Dirichlet to Neumann operator, and regularity of three dimensional water waves*, Comm. Partial Differential Equations 34 (2009), no. **10-12**, 1632–1704.

[2] T. Alazard, N. Burq and C. Zuily, *Strichartz estimates for the water waves system*,http://arxiv.org/abs/1002.0323 2010.

[3] T. Alazard, N. Burq and C. Zuily, *Local well posedness for the gravity water-waves system*, in preparation 2010.

[4] S. Alinhac, *Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels*, Comm. Partial Differential Equations **14** (1989), no. **2**, 173–230.

[5] B. Alvarez-Samaniego and D. Lannes, *Large time existence for 3D water-waves and asymptotics*, Invent. Math. **171** (2008), no. 3, 485–541.

[6] D. M. Ambrose and N. Masmoudi, *The zero surface tension limit of two-dimensional water waves*, Comm. Pure Appl. Math. **58** (2005), no. 10, 1287–1315.

[7] J.L. Bona, R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London Ser. A **278** (1975) 1287, 555601.

[8] K. Beyer, M. Günther, *On the Cauchy problem for a capillary drop. I. Irrotational motion*, Math. Methods Appl. Sci. **21** (1998), no. 12, 1149–1183.

[9] J. Bona, D. Lannes, and J.-C. Saut, *Asymptotic models for internal waves*, J. Math. Pures Appl. (9) **89** (2008), no. 6, 538–566.

[10] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 2, 209–246.

[11] J.-Y. Chemin, *Calcul paradifférentiel précisé et applications à des équations aux dérivées partielles non semilinéaires*, Duke Math. J. **56** (1988), no. 3, 431–469.

[12] H. Christianson, V. M. Hur, and G. Staffilani, *Strichartz estimates for the water-wave problem with surface tension*, preprint 2009 http://arxiv.org/abs/0908.3255.

[13] D. Coutand and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc. **20** (2007), no. 3, 829–930.

[14] S.-I. Doi, *On the Cauchy problem for Schrödinger type equations and the regularity of solutions*, J. Math. Kyoto Univ. **34** (1994), no. 2, 319–328.

[15] S.-I. Doi, *Remarks on the Cauchy problem for Schrödinger-type equations*, Comm. Partial Differential Equations **21** (1996), no. 1-2, 163–178.

[16] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1997.

[17] T. Iguchi, *A long wave approximation for capillary-gravity waves and an effect of the bottom*, Comm. Partial Differential Equations, **32** (2007), 37–85.

[18] T. Kano and T. Nishida Sur les ondes de surface de l'eau avec une justification mathmatique des quations des ondes en eau peu profonde, *J. Math. Kyoto Univ.* 19 (1979), no. 2, 335–370.

[19] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rational Mech. Anal. **58** (1975), no. 3, 181205

[20] G. Iooss and P. Plotnikov. Small divisor problem in the theory of three-dimensional water gravity waves. Memoirs of AMS, 200, 940, 2009. (128p.)

[21] D. Lannes, *Well-posedness of the water-waves equations*, J. Amer. Math. Soc. **18** (2005), no. 3, 605–654.

[22] G. Métivier, *Para-differential calculus and applications to the Cauchy problem for nonlinear systems*, Ennio de Giorgi Math. Res. Center Publ., Edizione della Normale, 2008.

[23] G. Métivier and J. Rauch, *Dispersive Stabilization*, Bull. Lond. Math. Soc. **42** (2010), no. 2, 250–262.

[24] G. Métivier and K. Zumbrun, *Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems*, Mem. Amer. Math. Soc. **175** (2005), no. 826, vi+107 pp.

[25] Y. Meyer, *Remarques sur un théorème de J.-M. Bony*, Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980), no. suppl. **1**, 1981, pp. 1–20.

[26] M. Ming and Z. Zhang, *Well-posedness of the water-wave problem with surface tension*, J. Math. Pures Appl. (9) **92** (2009), no. 5, 429–455.

[27] F. Rousset and N. Tzvetkov, *Transverse instability of the line solitary water-waves*, preprint http://fr.arxiv.org/abs/0906.2487.

[28] G. Schneider and E. Wayne, *The rigorous approximation of long-wavelength capillary-gravity waves*, Arch. Ration. Mech. Anal. **162** (2002), no. 3, 247–285.

[29] J. Shatah and C. Zeng, *Geometry and a priori estimates for free boundary problems of the Euler equation*, Comm. Pure Appl. Math. **61** (2008), no. 5, 698–744

[30] B. Schweizer, *On the three-dimensional Euler equations with a free boundary subject to surface tension*, Ann. Inst. H. Poincar Anal. Non Linaire **22** (2005), no. 6, 753–781.

[31] D. Tataru, *On the Fefferman-Phong inequality and related problems*, Comm. Partial Differential Equations **27** (2002), no. 11-12, 2101–2138.

[32] M. E. Taylor, *Pseudodifferential operators and nonlinear PDE*, Progress in Mathematics, vol. **100**, Birkhäuser Boston Inc., Boston, MA, 1991.

[33] Y. Trakhinin. Local existence for the free boundary problem for the non-relativistic and relativistic compressible euler equations with a vacuum boundary condition. *Comm. Pure Appl. Math.* 62 (2009), no. 11, 1551–1594.

[34] N. Tzvetkov, *Ill-posedness issues for nonlinear dispersive equations Lectures on nonlinear dispersive equations*, 63–103, GAKUTO Internat. Ser. Math. Sci. Appl., 27, Gakk?tosho, Tokyo, 2006.

[35] S. Wu *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc. **12** (1999), no. 2, 445–495

[36] S. Wu *Well-posedness in Sobolev spaces of the full water wave problem in* 2*-D*, Invent. Math. **130** (1997), no. 1, 39–72

[37] H. Yosihara, Capillary-gravity waves for an incompressible ideal fluid *J. Math. Kyoto Univ.*, 23 (1983), no. 4, 649–694.

[38] H. Yosihara, Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.*, 18 (1982), no. 1, 49–96.

[39] V.E. Zakharov, *Weakly nonlinear waves on the surface of an ideal finite depth fluid*, Amer. Math. Soc. Transl. **182** (1998), no. 2, 167-197.

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