A STATIONARY PHASE TYPE ESTIMATE

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ABSTRACT. The purpose of this note is to prove a stationary phase estimate well adapted to parameter dependent phases. In particular, no discussion is made on the positions (and behavior) of critical points, no lower or upper bound on the gradient of the phase is assumed, and the dependence of the constants with respect to derivatives of the phase and symbols is explicit.

1. INTRODUCTION

For a fixed phase, the stationary phase lemma (and its simplified version, the stationary phase estimate) is a very well understood tool which provides very good estimates for oscillatory integrals of the type

$$I_{\phi,b}(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(\xi)} b(\xi) d\xi \Rightarrow |I_{\Phi,b}(\lambda)| \leq C\lambda^{-\frac{d}{2}}. \tag{1}$$

The method of proof is quite standard and follows the classical path:

(1) Using the non degeneracy of the hessian of the phase, one knows that the critical points are isolated, hence for a compactly supported symbol there are finitely many such critical points.
(2) Away from the critical points, the non stationary estimates, obtained for example by integrating by parts N times with the operator
$$L = \frac{\nabla_\xi \Phi \cdot \nabla_\xi}{i\lambda |\nabla_\xi \Phi|^2},$$
gives an estimate bounded by $C_N \lambda^{-N}$.
(3) Near each critical point, performing first a change of variables (the Morse Lemma) to reduce to the case where the phase is quadratic, and then an exact calculation in Fourier variables gives the estimate (1).

When $d = 1$, Van der Corput Lemma provides a very robust estimate.

However, in higher dimensions, the situation is less simple, in particular when considering parameter dependent phases (with parameters living in a non-compact domain), where

(1) even away from the critical points, $\nabla_\xi \Phi$ can degenerate,
(2) the determinant of the hessian can degenerate,
(3) the number of critical points can blow up.

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In view of numerous applications (for example dispersion estimates for solutions to PDE's), a precise control of the behavior, with respect to the phase and symbol, of the constant C in (1) is necessary. Many robust methods to prove (1) have been developed (see for example Bahouri-Chemin-Danchin[2], Hörmander [4], Muscalu-Schlag [5], Farah-Rousset-Tzvetkov [3], Ruzhanski [6] and Stein [7]). However, it seems that none of these results gives an estimate directly applicable to general situations. This was the motivation for this note.

2. Preliminaries

- We consider
$$I(\lambda) = \int_{\mathbf{R}^d} e^{i\lambda\Phi(\xi)} b(\xi)\, d\xi,$$
where Φ is a complex phase, $\Phi = \Phi_1 + i\Phi_2 \in C^\infty$ with Φ_j real and $b \in C_0^\infty$ is a symbol.
- Let $K = \operatorname{supp} b$ and $\varepsilon_0 > 0$ be a small constant. We set
$$K_{\varepsilon_0} = \{\xi \in \mathbf{R}^d : \operatorname{dist}(\xi, K) \leq \varepsilon_0\}, \tag{2}$$
where "dist" is the sup distance on $\mathbf{R}^d$.
- We assume that the phase Φ is smooth on a neighborhood of K_{ε_0} and we set for $k \geq 2, l \geq 0$,
$$\mathcal{M}_k := \sum_{2\leq|\alpha|\leq k} \sup_{\xi\in K_{\varepsilon_0}} |D_\xi^\alpha \Phi(\xi)|, \quad \mathcal{N}_l := \sum_{|\alpha|\leq l} \sup_{\xi\in K} |D_\xi^\alpha b(\xi)|$$
- Let Hess Φ_1 be the Hessian matrix of Φ_1. We shall set
$$a_0 = \inf_{\xi\in K_{\varepsilon_0}} |\det \text{Hess } \Phi_1(\xi)|. \tag{3}$$
- We fix now several constants. First of all since Hess Φ_1 is a real symmetric matrix its eigenvalues $(\lambda_j)_{j=1,\ldots,d}$ are real and satisfy
$$|\lambda_j| \leq C_1 \mathcal{M}_2$$
where C_1 depends only on the dimension d. It follows from (3) that
$$|\lambda_j| \geq \frac{a_0}{(C_1\mathcal{M}_2)^{d-1}}. \tag{4}$$
This implies that
$$|\text{Hess } \Phi_1(\xi)X| \geq \frac{a_0}{(C_1\mathcal{M}_2)^{d-1}}|X|, \quad \forall X \in \mathbf{R}^d. \tag{5}$$
- Now if B is an open ball contained in the interior of K_{ε_0} there exists a positive constant C_2 depending only on the dimension d such that for all $\xi, \eta \in B$,
$$\begin{aligned} \nabla\Phi_1(\xi) - \nabla\Phi_1(\eta) &= \text{Hess } \Phi_1(\eta)(\xi-\eta) + R(\xi,\eta) \\ |R(\xi,\eta)| &\leq C_2\mathcal{M}_3|\xi-\eta|^2. \end{aligned} \tag{6}$$
- According to (5) and (6) we set, when $\mathcal{M}_2 > 0$,
$$\delta_{\varepsilon_0} = \frac{a_0}{4(C_1\mathcal{M}_2)^{d-1}C_2\mathcal{M}_3}, \quad \delta = \min(\delta_{\varepsilon_0}, \frac{\varepsilon_0}{4}). \tag{7}$$

3. Main results

Theorem 1. *Assume*

$$(8)\qquad \begin{aligned} &(i) \quad \mathcal{M}_{d+2} < +\infty, \quad \mathcal{N}_{d+1} < +\infty, \\ &(ii) \quad a_0 > 0, \\ &(iii) \quad \Phi_2 \geq 0 \ \textit{on } K. \end{aligned}$$

Then there exists $C > 0$ depending only on the dimension d such that, for all $\lambda \geq 1$,

$$(9)\qquad |I(\lambda)| \leq \frac{C|K_{\varepsilon_0}|}{a_0 \delta^d}(1 + \mathcal{M}_{d+2}^{\frac{d}{2}})\mathcal{N}_{d+1}\lambda^{-\frac{d}{2}},$$

where $|K_{\varepsilon_0}|$ denotes the Lebesgue measure of the set K_{ε_0}.

Remarks 2. 1. We notice that no upper bound (nor lower bound) on $\nabla\Phi$ is required. This is important in particular in the case where the phase Φ depends on parameters. For instance, in some cases the phase Φ is of the form $\Phi(x, y, \xi) = (x-y)\cdot\xi + \phi(x, y, \xi)$ where x, y are in $\mathbf{R}^d$. In these case $\nabla\Phi = x - y + \nabla\phi$ and there is no natural upper nor lower bound for it.

2. Here is another example (see [1]). Assume $\Phi(x, y, \xi) = (x-y)\cdot\xi + t\theta(x, y, \xi)$ where $t \in (0, T)$ and x, y are in $\mathbf{R}^d$. Assume that Φ and b satisfiy $(i), (ii)$ and that $|\det \text{Hess } \theta| \geq c > 0$ where c depends only on the dimension d. Then setting $X = \frac{x}{t}, Y = \frac{y}{t}$ we write $i\lambda\Phi = i\lambda t\big((X-Y)\cdot\xi + \theta(t, tX, tY, \xi)\big)$ and we may apply Theorem 1 with $a_0 = c$ and λ replaced by λt. We obtain an estimate of $I(\lambda)$ by $t^{-\frac{d}{2}}\lambda^{-\frac{d}{2}}$ as soon as $t \geq \lambda^{-1}$.

3. When a_0 is very small (meaning that $\delta = \frac{a_0}{4(C_1\mathcal{M}_2)^{d-1}C_2\mathcal{M}_3}$) then $\frac{1}{\delta^d}$ is proportional to $\frac{1}{a_0^d}$.The term $\frac{1}{a_0^d}$ in (9) comes from the possible occurence of a_0^{-d} critical points of the phase on the support of b. In the case where Φ has only one non degenerate critical point this term could be avoided. In this direction we have the following result.

Theorem 3. *Assume that Φ and b satisfy the assumptions (8) and that the map*

$$(10)\qquad K_{\varepsilon_0} \to \mathbf{R}^d, \quad \xi \mapsto \nabla\Phi_1(\xi) \quad \textit{is injective.}$$

Then one can find $C > 0$ depending only on the dimension d such that

$$(11)\qquad |I(\lambda)| \leq \frac{C}{a_0}(1 + \mathcal{M}_{d+2}^{\frac{d}{2}})\mathcal{N}_{d+1}\lambda^{-\frac{d}{2}}.$$

Here are two examples where Theorem 3 applies.

Examples 4. 1. Assume besides (8) that K_{ε_0} is convex and that

$$\langle \text{Hess } \Phi_1(\xi)X, X\rangle > 0, \quad \forall\xi \in K_{\varepsilon_0}, \quad \forall X \in \mathbf{R}^d.$$

Then (10) is satisfied.

First of all, since the symmetric matrix Hess Φ_1 is a non negative, its eigenvalues are non negative. It follows from (4) that

$$(12)\qquad \langle \text{Hess } \Phi_1(\xi)X, X\rangle \geq \frac{a_0}{(C_1\mathcal{M}_2)^{d-1}}|X|^2, \quad \forall\xi \in K_{\varepsilon_0}, \quad \forall X \in \mathbf{R}^d.$$

With $\xi, \eta \in K_{\varepsilon_0}$ we write

$$\begin{aligned}\nabla\Phi_1(\xi) - \nabla\Phi_1(\eta) &= \int_0^1 \frac{d}{ds}\big[\nabla\Phi_1(s\xi + (1-s)\eta)\big]\, ds \\ &= \int_0^1 \text{Hess } \Phi_1(s\xi + (1-s)\eta)\cdot(\xi - \eta)\, ds.\end{aligned}$$

It follows from (12) that

$$\langle \nabla\Phi_1(\xi) - \nabla\Phi_1(\eta), \xi - \eta\rangle \geq \frac{a_0}{(C_d\mathcal{M}_2)^{d-1}}|\xi - \eta|^2,$$

from which we deduce that

$$\frac{a_0}{(C_d\mathcal{M}_2)^{d-1}}|\xi - \eta| \leq |\nabla\Phi_1(\xi) - \nabla\Phi_1(\eta)|,$$

which completes the proof.

2. Let A be a real, symmetric, non singular $d \times d$ matrix and Ψ be a smooth phase such that $\mathcal{M}_{d+2}(\Psi) < +\infty$. Set $\Phi(\xi) = \frac{1}{2}\langle A\xi, \xi\rangle + \varepsilon\Psi(\xi)$. Then if ε is small enough the assumptions in Theorem 3 are satisfied.

Remark 5. Notice that the estimates (9), (11) do not seem to be optimal with respect to the power of a_0 (for small a_0) since according to the usual stationary phase method one could expect to have $a_0^{-\frac{1}{2}}$ in the right hand side.

Actually it is sufficient to prove the following weaker inequality.

Theorem 6. *1. Under the hypotheses of Theorem 1 there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ increasing such that for every $\lambda \geq 1$*

$$|I(\lambda)| \leq |K_{\varepsilon_0}|\mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\frac{1}{a_0\delta^d}\lambda^{-\frac{d}{2}}. \tag{13}$$

2. Under the hypotheses of Theorem 3 there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ increasing such that for every $\lambda \geq 1$

$$|I(\lambda)| \leq \mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\frac{1}{a_0}\lambda^{-\frac{d}{2}}. \tag{14}$$

Proof of Theorems 1 and 3 given Theorem 6. We assume that (13) is proved and our goal is to deduce that (9) holds with $C = \mathcal{F}(1)$. Set $t = 1 + \mathcal{M}_{d+2}$ and consider $\lambda \geq 1$. Since $t\lambda \geq 1$ we can apply (13) with (λ, Φ) replaced with $(t\lambda, \Phi(\xi)/t)$. Notice that a_0 is then changed to $t^{-d}a_0$ and $\mathcal{M}_k(\frac{\Phi}{t}) = \frac{1}{t}\mathcal{M}_k(\Phi)$. Therefore δ remains unchanged. It follows that we can write,

$$\begin{aligned}\left|\int_{\mathbf{R}^d} e^{i\lambda\Phi(\xi)}b(\xi)\, d\xi\right| &= \left|\int_{\mathbf{R}^d} e^{it\lambda\frac{\Phi(\xi)}{t}}b(\xi)\, d\xi\right| \\ &\leq |K_{\varepsilon_0}|\mathcal{F}\left(\mathcal{M}_{d+2}\left(\frac{\Phi}{t}\right)\right)\mathcal{N}_{d+1}\frac{1}{t^{-d}a_0\delta^d}(t\lambda)^{-\frac{d}{2}} \\ &\leq |K_{\varepsilon_0}|\mathcal{F}\left(\frac{\mathcal{M}_{d+2}(\Phi)}{t}\right)\mathcal{N}_{d+1}\frac{1}{a_0\delta^d}\lambda^{-\frac{d}{2}}t^{\frac{d}{2}},\end{aligned}$$

which yields the desired estimate. The case 2. is similar. □

We are left with the proof of Theorem 6.

4. Proof of Theorem 6.

4.1. **Case 1.** In what follows we shall denote by C_d a positive constant depending only on the dimension d and by $\mathcal{F}$ an increasing function from $\mathbf{R}^+$ to $\mathbf{R}^+$ which can change from line to line.

Point 1. First of all we may assume that

$$\lambda^{\frac{1}{2}}\delta \geq 1. \tag{15}$$

Indeed assume $\lambda^{\frac{1}{2}}\delta \leq 1$; then since by (8), (ii) we have $a_0 \leq C\mathcal{M}_2^d$ we have

$$|I(\lambda)| \leq \|b\|_{L^1(\mathbf{R}^d)} \leq \|b\|_{L^1(\mathbf{R}^d)} \frac{a_0}{a_0} \frac{1}{(\lambda^{\frac{1}{2}}\delta)^d} \leq C|K|\mathcal{N}_0\mathcal{M}_2^d \frac{1}{a_0\delta^d}\lambda^{-\frac{d}{2}}.$$

Point 2. We first localize the problem in small balls. Let δ be defined in (7). We can write

$$\mathbf{R}^d = \cup_{j\in\mathbf{N}} B(\xi_j^*, \delta)$$

where the $B(\xi_j^*, \delta)$ are cubes of size δ such that there exists k_0 depending only on the dimension d such that every point in $\mathbf{R}^d$ belongs to at most k_0 cubes. This implies in particular that for every finite subset $J \subset \mathbf{N}$ we have

$$\sum_{j\in J} |B(\xi_j^*, \delta)| \leq k_0 \Big| \bigcup_{j\in J} B(\xi_j^*, \delta)\Big|. \tag{16}$$

Let $(\chi_j)_{j\in\mathbf{N}}$ be a partition of unity associated to this covering. Notice that χ_j can be taken of the form $\chi_0(\frac{\xi-\xi_j^*}{\delta})$ so that

$$|\partial_\xi^\alpha \chi_j(\xi)| \leq C_\alpha \delta^{-|\alpha|}. \tag{17}$$

Let

$$\Lambda = \{j \in \mathbf{N} : B(\xi_j^*, \delta) \cap K \neq \emptyset\}. \tag{18}$$

This is a finite set whose cardinal will be denoted by $|\Lambda|$. It follows that we have $K \subset \cup_{j\in\Lambda} B(\xi_j^*, \delta)$.

Now if $\xi \in B(\xi_j^*, \delta)$, where $j \in \Lambda$, by definition one can find $\eta \in B(\xi_j^*, \delta) \cap K$. Then by (7) we have

$$\mathrm{dist}(\xi, K) \leq |\xi - \eta| \leq 2\delta \leq \frac{\varepsilon_0}{2}.$$

It follows that $\cup_{j\in\Lambda} B(\xi_j^*, \delta)$ is contained in the interior of K_{ε_0} defined in (2). Using (16) we obtain $\sum_{j\in\Lambda} |B(\xi_j^*, \delta)| \leq k_0|K_{\varepsilon_0}|$ which implies that

$$|\Lambda| \leq k_0|K_{\varepsilon_0}|\,\delta^{-d}. \tag{19}$$

Lemma 7.

(20) *On each the ball $B(\xi_j^*, \delta)$ the map $\xi \mapsto \nabla\Phi_1(\xi)$ is injective.*

Proof. Indeed as we saw above, these balls are contained in the interior of K_{ε_0} where the hypotheses (8) are satisfied. By (6) and (5) if $\xi, \eta \in B(\xi_j^*, \delta)$ we can write

$$|\nabla\Phi_1(\xi) - \nabla\Phi_1(\eta)| \geq \Big(\frac{a_0}{(C_1\mathcal{M}_2)^{d-1}} - 2\delta C_2\mathcal{M}_3\Big)|\xi - \eta| \geq \frac{a_0}{2(C_1\mathcal{M}_2)^{d-1}}|\xi - \eta|.$$

□

Setting $b_j = \chi_j b$ we can write

$$I(\lambda) = \sum_{j\in\Lambda} I_j(\lambda), \quad I_j(\lambda) = \int_{\mathbf{R}^d} e^{i\lambda\Phi(\xi)} b_j(\xi)\, d\xi. \tag{21}$$

Now we fix $j \in \Lambda$ and estimate the integral $I_j(\lambda)$.

Lemma 8. *Let $i \in \{1,\dots,d\}$ and $A_i = \frac{\partial_i\overline{\Phi}}{|\nabla\Phi|^2}$. One can find $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ increasing such that*

$$|D_\xi^\alpha A_i(\xi)| \leq \mathcal{F}(M_{1+|\alpha|}) \sum_{k=2}^{1+|\alpha|} \frac{1}{|\nabla\Phi(\xi)|^k}, \quad |\alpha| \geq 1. \tag{22}$$

Proof. We proceed by induction on $|\alpha|$. A simple computation shows that (22) is true for $|\alpha| = 1$. Assume it is true for $|\alpha| \leq l$ and let $|\gamma| = l+1 \geq 2$. Differentiating $|\gamma|$ times the equality $|\nabla\Phi|^2 A_i = \partial_i\overline{\Phi}$ we obtain

$$\begin{aligned} &|\nabla\Phi|^2 D_\xi^\gamma A_i = (1) - (2) - (3), \quad \text{with} \quad (1) = D_\xi^\gamma \partial_i\overline{\Phi} \\ &(2) = \sum_{|\beta|=1} \binom{\gamma}{\beta} (D_\xi^\beta |\nabla\Phi|^2) D^{\gamma-\beta} A_i, \quad (3) = \sum_{2\leq|\beta|\leq|\gamma|} \binom{\gamma}{\beta} (D_\xi^\beta |\nabla\Phi|^2) D^{\gamma-\beta} A_i. \end{aligned} \tag{23}$$

We have $|(1)| \leq \mathcal{M}_{l+2}$. By the induction, $|(2)| \leq C\mathcal{M}_2|\nabla\Phi|\mathcal{F}(M_{l+1}) \sum_{k=2}^{l+1} \frac{1}{|\nabla\Phi|^k}$. Now for $|\beta| \geq 2$ we have $|\gamma| - |\beta| \leq l-1$. Since $|D_\xi^\beta |\nabla\Phi|^2| \leq C_2\mathcal{M}_{|\beta|+1}|\nabla\Phi| + \mathcal{F}(\mathcal{M}_{|\beta|})$ the induction shows that $|(3)| \leq C\mathcal{F}(\mathcal{M}_{l+2})(1+|\nabla\Phi|)\sum_{k=2}^{l} \frac{1}{|\nabla\Phi|^k}$. Dividing both members of the first equation in (23) by $|\nabla\Phi|^2$ we obtain eventually

$$|D_\xi^\gamma A_i| \leq \mathcal{F}(\mathcal{M}_{l+2}) \sum_{k=2}^{l+2} \frac{1}{|\nabla\Phi|^k}, \quad |\gamma| = l+1.$$

This completes the proof of (22). ☐

Lemma 9. *Let $\mathcal{L} = A\cdot\nabla$ where $A_i = \frac{\partial_i\overline{\Phi}}{|\nabla\Phi|^2}$. For any $N \in \mathbf{N}$ we have*

$$\begin{aligned} &({}^t\mathcal{L})^N = \sum_{|\alpha|\leq N} c_{\alpha,N}\partial^\alpha, \quad \textit{with} \\ &|\partial_\xi^\beta c_{\alpha,N}| \leq \mathcal{F}(\mathcal{M}_{N-|\alpha|+|\beta|+1}) \sum_{k=N}^{2N-|\alpha|+|\beta|} \frac{1}{|\nabla\Phi_1|^k}. \end{aligned} \tag{24}$$

(Here we set $\mathcal{M}_1 = 1$. It occurs when $\beta = 0, |\alpha| = N$.)

Proof. Again we proceed by induction on N. We shall prove the estimate in (24) with Φ instead of Φ_1 in the right hand side and then we use the trivial estimate $|\nabla\Phi| \geq |\nabla\Phi_1|$. For $N = 1$ we have $c_{\alpha,N} = A_i$ if $|\alpha| = 1$ and $c_{\alpha,N} = \mathrm{div}A$ if $|\alpha| = 0$. Then (24) follows immediately from (22). Assume that (24) is true up to the order

N and let us prove it for $N+1$. We write

$$({}^t\mathcal{L})^{N+1} = {}^t\mathcal{L}({}^t\mathcal{L})^N = -(\nabla \cdot A)({}^t\mathcal{L})^N = -\sum_{|\alpha|\le N}\sum_{i=1}^{d} \partial_i (A_i c_{\alpha,N}\partial^\alpha),$$
$$= -\sum_{|\alpha|\le N} (\text{div } A) c_{\alpha,N}\partial^\alpha - \sum_{|\alpha|\le N} A\cdot\nabla c_{\alpha,N}\partial^\alpha - \sum_{|\alpha|\le N}\sum_{i=1}^{d} A_i c_{\alpha,N}\partial_i\partial^\alpha,$$
$$= \sum_{|\gamma|\le N+1} c_{\gamma,N+1}\partial^\gamma,$$

where

$$\begin{aligned}
&c_{0,N+1} = -(\text{div } A)c_{0,N} - A\cdot\nabla c_{0,N}, \quad \text{if} \quad |\gamma| = 0,\\
&c_{\gamma,N+1} = -(\text{div } A)c_{\gamma,N} - A\cdot\nabla c_{\gamma,N} - A_i c_{\alpha,N} \quad |\alpha| = |\gamma|-1, \text{ if } \quad 1\le|\gamma|\le N,\\
&c_{\gamma,N+1} = -A_i c_{\alpha,N}, \quad \text{if } \partial^\gamma = \partial_i\partial^\alpha, |\alpha| = N, \text{ if } \quad |\gamma| = N+1.
\end{aligned}$$

We estimate now each coefficient. First of all $\partial^\beta c_{0,N+1}$ is a finite sum of terms of the form $(\partial^{\beta_1}\partial_i A_i)(\partial^{\beta_2}c_{0,N})$ and $(\partial^{\beta_1}A_i)(\partial^{\beta_2}\partial_i c_{0,N})$ with $\beta = \beta_1+\beta_2$. Using (22) and the induction the first term is bounded by

$$\mathcal{F}(\mathcal{M}_{|\beta_1|+2}) \sum_{k=2}^{|\beta_1|+2} |\nabla\Phi|^{-k}\mathcal{F}(\mathcal{M}_{N+|\beta_2|+1}) \sum_{l=N}^{2N+|\beta_2|} |\nabla\Phi|^{-l}.$$

Concerning the second term, if $\beta_1 = 0, \beta_2 = \beta$ it is bounded by

$$\frac{1}{|\nabla\Phi|}\mathcal{F}(\mathcal{M}_{N+|\beta|+2}) \sum_{l=N}^{2N+|\beta|+1} |\nabla\Phi|^{-l} \le \mathcal{F}(\mathcal{M}_{N+1+|\beta|+1}) \sum_{l=N+1}^{2(N+1)+|\beta|} |\nabla\Phi|^{-l}.$$

If $\beta_1 \ne 0$ it is bounded by

$$\mathcal{F}(\mathcal{M}_{|\beta_1|+1}) \sum_{k=2}^{|\beta_1|+1} |\nabla\Phi|^{-k}\mathcal{F}(\mathcal{M}_{N+|\beta_2|+1}) \sum_{l=N}^{2N+|\beta_2|+1} |\nabla\Phi|^{-l}.$$

Since $N+2 \le k+l \le 2N+2+|\beta_1|+|\beta_2| = 2(N+1)+|\beta|$ we see that $\partial^\beta c_{0,N+1}$ satisfies the estimate in (24) with N replaced by $N+1$.

Let us look to the term $\partial^\beta c_{\gamma,N+1}$ with $|\gamma| = N+1$. This term is also a finite sum of terms of the form $(\partial^{\beta_1}A_i)(\partial^{\beta_2}c_{\alpha,N}), |\alpha| = |\gamma|-1$. As above, if $\beta_1 = 0$, using (22) and the induction it is bounded by

$$\frac{1}{|\nabla\Phi|}\mathcal{F}(\mathcal{M}_{N-|\gamma|+1+|\beta|+1}) \sum_{l=N}^{2N-|\gamma|+1+|\beta|} |\nabla\Phi|^{-l}$$
$$\le \mathcal{F}(\mathcal{M}_{N+1-|\gamma|+|\beta|+1}) \sum_{l=N+1}^{2(N+1)-|\gamma|+|\beta|} |\nabla\Phi|^{-l}.$$

If $\beta_1 \ne 0$ it is bounded by

$$\mathcal{F}(\mathcal{M}_{|\beta_1|+1}) \sum_{k=2}^{|\beta_1|+1} |\nabla\Phi|^{-k}\mathcal{F}(\mathcal{M}_{N-|\gamma|+1+|\beta_2|}) \sum_{l=N}^{2N-|\gamma|+1+|\beta_2} |\nabla\Phi|^{-l}.$$

Since $N+2 \leq k+l \leq 2N+2-|\gamma|+|\beta|$ we see that $\partial^\beta c_{\gamma,N+1}$ satisfies also the estimate in (24). The estimates of the other terms are similar and left to the reader. $\square$

Let $\psi \in C_0^\infty(\mathbf{R}^d)$ be such that $\psi(x) = 1$ if $|x| \leq 1$, $\psi(x) = 0$ if $|x| \geq 2$. With the notation in (21), j being fixed, we write

(25)
$$I_j(\lambda) = \int e^{i\lambda\Phi(\xi)}\psi\big(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\big)\chi_j(\xi)b(\xi)\,d\xi \\ + \int e^{i\lambda\Phi(\xi)}\big(1-\psi\big(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\big)\big)\chi_j(\xi)b(\xi)\,d\xi =: K_j(\lambda) + L_j(\lambda).$$

We shall use (see (20)) the fact that on the support of χ_j the map $\xi \mapsto \nabla\Phi_1(\xi)$ is injective. Let us estimate K_j. We write

$$|K_j(\lambda)| \leq \int |\psi\big(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\big)\chi_j(\xi)b(\xi)|\,d\xi,$$

and we set $\eta = \lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)$ then $d\eta = \lambda^{\frac{d}{2}}|\det \text{Hess }\Phi_1(\xi)|\,d\xi$. Then using (8) (ii) and the notations therein we obtain

$$(26) \qquad |K_j(\lambda)| \leq \frac{C_d}{a_0}\mathcal{N}_0\lambda^{-\frac{d}{2}}.$$

To estimate L_j we introduce the vector field $X = \frac{1}{i\lambda}\frac{\nabla\overline{\Phi}}{|\nabla\Phi|^2}\cdot\nabla$ which satisfies

$$Xe^{i\lambda\Phi} = e^{i\lambda\Phi}.$$

Now with $N \geq 1$ to be chosen we write

$$L_j(\lambda) = \int e^{i\lambda\Phi(\xi)}({}^tX)^N\Big\{(1-\psi\big(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\big))\chi_j(\xi)b(\xi)\Big\}\,d\xi.$$

Since $X = \frac{1}{i\lambda}\mathcal{L}$ we can use (24) and we obtain

$$|L_j(\lambda)| \leq C_N \sum_{\substack{\alpha=\alpha_1+\alpha_2+\alpha_3\\|\alpha|\leq N}} S_{\alpha,N}, \quad \text{where}$$

$$S_{\alpha,N} = \lambda^{-N}\int |c_{\alpha,N}|\,\Big|\partial_\xi^{\alpha_1}\big[1-\psi\big(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\big)\big]\Big|\,\Big|\partial_\xi^{\alpha_2}\chi_j(\xi)\Big|\,\Big|\partial_\xi^{\alpha_3}b(\xi)\Big|\,d\xi.$$

Our aim is to prove that with an appropriate choice of N we have

$$(27) \qquad |L_j(\lambda)| \leq \mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\frac{1}{a_0}\lambda^{-\frac{d}{2}}.$$

Step 1. $\alpha_1 = 0$. Here we integrate on the set $|\nabla\Phi_1(\xi)| \geq \lambda^{-\frac{1}{2}}$. We use (20), the bounds (17), (24), (8) (ii) and we make the change of variable $\eta = \nabla\Phi_1(\xi)$; then $d\eta = |\det\text{Hess }\Phi_1(\xi)|\,d\xi$ then

$$|S_{\alpha,N}| \leq \frac{\lambda^{-N}}{a_0}\delta^{-|\alpha_2|}\mathcal{N}_{|\alpha_3|}\mathcal{F}(\mathcal{M}_{N+1})\sum_{k=N}^{2N-|\alpha|}\int_{|\eta|\geq\lambda^{-\frac{1}{2}}}\frac{d\eta}{|\eta|^k}, \\ \leq \frac{C_d\lambda^{-N}}{a_0}\delta^{-|\alpha_2|}\mathcal{N}_{|\alpha_3|}\mathcal{F}(\mathcal{M}_{N+1})\sum_{k=N}^{2N-|\alpha|}\int_{\lambda^{-\frac{1}{2}}}^{+\infty} r^{d-1-k}\,dr.$$

We take $N = d+1$. Since $|\alpha_j| \leq |\alpha| \leq N$ we see that

$$|S_{\alpha,N}| \leq \frac{\lambda^{-N}}{a_0}\delta^{-|\alpha|}\mathcal{N}_{d+1}\mathcal{F}(\mathcal{M}_{d+2})\lambda^{N-\frac{1}{2}|\alpha|-\frac{d}{2}},$$
$$\leq \frac{1}{a_0}\mathcal{N}_{d+1}\mathcal{F}(\mathcal{M}_{d+2})\lambda^{-\frac{d}{2}}\frac{1}{(\lambda^{\frac{1}{2}}\delta)^{|\alpha|}}.$$

Since (see (15)) we assumed that $\lambda^{\frac{1}{2}}\delta \geq 1$ we obtain $|S_{\alpha,N}| \leq \frac{1}{a_0}\mathcal{N}_{d+1}\mathcal{F}(\mathcal{M}_{d+2})\lambda^{-\frac{d}{2}}$.

Step 2. $\alpha_1 \neq 0$. Here, since we differentiate ψ, we are integrating on the set $\lambda^{-\frac{1}{2}} \leq |\nabla\Phi_1(\xi)| \leq 2\lambda^{-\frac{1}{2}} \leq 1$.

We have to estimate

$$S_{\alpha,N} = \lambda^{-N}\int |c_{\alpha,N}| \left|\partial_\xi^{\alpha_1}\left[\psi\left(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\right)\right]\right| \left|\partial_\xi^{\alpha_2}\chi_j(\xi)\right| \left|\partial_\xi^{\alpha_3}b(\xi)\right| \, d\xi.$$

By the Faa-di-Bruno formula we have

$$\partial_\xi^{\alpha_1}\left[\psi\left(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\right)\right] = \sum_{1\leq|\beta|\leq|\alpha_1|} a_{\alpha_1,\beta}\psi^{(\beta)}\left(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\right)\prod_{i=1}^{s}\left(\lambda^{\frac{1}{2}}\partial_\xi^{l_i}\nabla\Phi_1\right)^{k_i},$$

where $a_{\alpha,\beta}$ are absolute constants, $\sum_{i=1}^s k_i = \beta, \sum_{i=1}^s k_i|l_i| = \alpha_1$.We deduce that

$$\left|\partial_\xi^{\alpha_1}\left[\psi\left(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\right)\right]\right| \leq \mathcal{F}(\mathcal{M}_{|\alpha_1|+1})\sum_{1\leq|\beta|\leq|\alpha_1|}\lambda^{\frac{|\beta|}{2}}\left|\psi^{(\beta)}\left(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\right)\right|,$$
$$\leq \mathcal{F}(\mathcal{M}_{|\alpha_1|+1})\lambda^{\frac{|\alpha_1|}{2}}\sum_{1\leq|\beta|\leq|\alpha_1|}\left|\psi^{(\beta)}\left(\lambda^{\frac{1}{2}}\nabla\Phi_1(\xi)\right)\right|.$$

Using (24) and the fact that $|\nabla\Phi_1(\xi)| \leq 1$ we have $|c_{\alpha,N}| \leq \mathcal{F}(\mathcal{M}_{N+1})|\nabla\Phi_1(\xi)|^{|\alpha|-2N}$. Eventually we have $\left|\partial_\xi^{\alpha_2}\chi_j(\xi)\right| \leq C_\alpha\delta^{-|\alpha_2|}$. Performing as above the change of variables $\eta = \nabla\Phi_1(\xi)$ we will have

$$|S_{\alpha,N}| \leq \frac{\lambda^{-N}\delta^{-|\alpha_2|}}{a_0}\mathcal{F}(\mathcal{M}_{N+1})\mathcal{N}_{|\alpha|}\lambda^{\frac{|\alpha_1|}{2}}\int_{\lambda^{-\frac{1}{2}}\leq|\eta|\leq 2\lambda^{-\frac{1}{2}}}|\eta|^{|\alpha|-2N}\,d\eta,$$
$$\leq \frac{1}{a_0}\mathcal{F}(\mathcal{M}_{N+1})\mathcal{N}_{|\alpha|}\lambda^{-\frac{d}{2}}(\lambda^{\frac{1}{2}}\delta)^{-|\alpha_2|}.$$

Since $\lambda^{\frac{1}{2}}\delta \geq 1$ we obtain $|S| \leq \frac{1}{a_0}\mathcal{F}(\mathcal{M}_{N+1})\mathcal{N}_N\lambda^{-\frac{d}{2}}$. Therefore (27) is proved. Using (25), (26), (27) since $N = d+1$ we obtain

$$|I_j| \leq \frac{1}{a_0}\mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\lambda^{-\frac{d}{2}}, \quad 1 \leq j \leq J.$$

Now since by (19) we have $|\Lambda| \leq C_d|K_{\varepsilon_0}|\delta^{-d}$ (where C_d depends only on the dimensions d), using (21) we obtain eventually

$$|I(\lambda)| \leq \frac{|K_{\varepsilon_0}|}{a_0\delta^d}\mathcal{F}(\mathcal{M}_{d+2})\mathcal{N}_{d+1}\lambda^{-\frac{d}{2}}$$

which completes the proof of the first case of the theorem.

4.2. **Case 2.** In that case it is not necessary to make a localization of $I(\lambda)$ in small balls of size δ as in the first case.

Then as before we write

$$I(\lambda) = \int e^{i\lambda\Phi(\xi)}\psi\big(\lambda^{\frac{1}{2}}|\nabla\Phi_1(\xi)|\big)b(\xi)\,d\xi \\ + \int e^{i\lambda\Phi(\xi)}\big(1-\psi\big(\lambda^{\frac{1}{2}}|\nabla\Phi_1(\xi)|\big)\big)b(\xi)\,d\xi =: K(\lambda)+L(\lambda)$$

and the final estimate follows from (26) and (27).

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