

A minicourse on the low Mach number limit

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1. Introduction

These lectures are devoted to the study of the so-called low Mach number limit for classical solutions of the compressible Navier-Stokes or Euler equations for non-isentropic fluids. The Mach number, hereafter denoted by ε , is a fundamental dimensionless number. By definition, it is the ratio of a characteristic velocity in the flow to the sound speed in the fluid. Hence, the target of the mathematical analysis of the low Mach number limit $\varepsilon \rightarrow 0$ is to justify some usual simplifications that are made when discussing the fluid dynamics of highly subsonic flows (which are very common).

For highly subsonic flows, a basic assumption that is usually made is that the compression due to pressure variations can be neglected. In particular, provided the sound propagation is adiabatic, it is the same as saying that the flow is incompressible. We can simplify the description of the governing equations by assuming that the fluid velocity is divergence-free. (The fact that the incompressible limit is a special case of the low Mach number limit explains why the limit $\varepsilon \rightarrow 0$ is a fundamental asymptotic limit in fluid mechanics.) On the other hand, if we include heat transfer in the problem, we cannot ignore the entropy variations. In particular we have to take into account the compression due to the combined effects of large temperature variations and thermal conduction. In this case, we find that the fluid velocity satisfies an inhomogeneous divergence constraint.

The main difficulty to study the low Mach number limit presents itself: the limit $\varepsilon \rightarrow 0$ is a singular limit which involves two time scales. We are thus led to study a nonlinear system of equations with a penalization operator. Our goal is precisely to study this problem for the full Navier–Stokes equations. More precisely, we will study the low Mach number limit for classical solutions of the full Navier-Stokes equations, in the general case where the combined effects of large temperature variations and thermal conduction are taken into account.

The mathematical analysis of the low Mach number limit begins with works of Ebin [30, 31], Klainerman and Majda [56, 57], Kreiss [60], Schochet [77, 78, 79] and many others. Concerning the Euler equations, after some rescalings and changes of variables (which are

explained below), we are thus led to analyze a quasi-linear symmetric hyperbolic system depending on the Mach number $\varepsilon \in (0, 1]$,

$$(1) \quad \begin{cases} g_1(\partial_t p + v \cdot \nabla p) + \varepsilon^{-1} \operatorname{div} v = 0, \\ g_2(\partial_t v + v \cdot \nabla v) + \varepsilon^{-1} \nabla p = 0, \\ \partial_t \sigma + v \cdot \nabla \sigma = 0. \end{cases}$$

The unknowns are the pressure p , the velocity v the entropy σ . The coefficients g_1 and g_2 are C^∞ positive functions of εp and σ .

It is clear on this system that the low Mach number limit is a singular limit which involves two time scales. The fast components are propagated by a wave equation and hence there are several geometrical factors that dictate the nature of the low Mach number limit. The domain may be the torus, the whole space or a domain $\Omega \subset \mathbb{R}^d$. The flow may be *isentropic* ($\sigma = 0$) or *non-isentropic* ($\sigma = O(1)$). The initial data may be *prepared* (namely $(\operatorname{div} v(0), \nabla p(0)) = O(\varepsilon)$), or *general*, which means here that $(p(0), v(0), \sigma(0))$ belongs to a given bounded subset of the Sobolev space H^s with $s > d/2 + 1$.

The analysis is in two steps. First, to study this singular limit one has to prove an existence and uniform boundedness result for a time independent of ε . Then, the next task is to analyze the limit of solutions as the Mach number ε tends to 0. The aim is to prove that the limits of (p, v, σ) are given by the incompressible Euler equation:

$$(2) \quad \begin{cases} \operatorname{div} v = 0, \\ g_2(\partial_t v + v \cdot \nabla v) + \nabla \pi = 0, \\ \partial_t \sigma + v \cdot \nabla \sigma = 0. \end{cases}$$

As first proved by Ukai [89], even if the initial data are compressible, the limit of the solutions for small ε is incompressible. For the isentropic equations with general initial data, the result is that the velocity is the sum of the limit flow, which is a solution of the incompressible equations whose initial data is the incompressible part of the original initial data, and a highly oscillatory term created by the sound waves (see [45]). In the whole space case, the solutions are known to converge, although this convergence is not uniform for time close to zero (see also the results of Asano [8], Iguchi [47] and Isozaki [48, 49, 50]). Furthermore, one can study the convergence with periodic boundary conditions by means of the filtering method (see Danchin [23], Gallagher [39, 40, 41, 42], Grenier [45], Joly–Métivier–Rauch [51], Schochet [80]). In bounded domain, for the barotropic Navier-Stokes system, Desjardins, Grenier, Lions and Masmoudi [27] have proved that the energy of the acoustic waves is dissipated in a boundary layer.

For the isentropic equations, the analysis is well-developed, even for solutions which are not regular. Indeed, the incompressible limit of the isentropic Navier-Stokes equations has been rigorously justified for weak solutions by Desjardins and Grenier [26], Lions and Masmoudi [66, 67], Bresch, Desjardins and Gérard-Varet [13] (see also [16, 27, 64]).

For viscous gases, global well-posedness in critical spaces was established by Danchin in [22], and the limit $\varepsilon \rightarrow 0$ was justified in the periodic case in [23], with the whole space case earlier achieved in [24]. We should also mention, among many others, the results of Hoff [46] and Dutrifoy and Hmidi [29]. Feireisl and Novotný initiated a program to extend the previous analysis to the incompressible limit of the weak solutions of the Navier–Stokes–Fourier system [36, 37, 38], which, in addition to the previous difficulties, require subtle energy estimates.

For the non-isentropic Euler equations with general initial data, Métivier and Schochet have proved some theorems [72, 73, 74] that supersede a number of earlier results. In particular, they have proved the existence of classical solutions on a time interval independent of ε (a part of their study is extended in [1] to the boundary case). The key point is to prove uniform estimates in Sobolev norms for the acoustic components. This is where the difference between almost isentropic and almost adiabatic enters. The reason is the following: the acoustics components are propagated by a wave equation whose coefficients are functions of the density, hence of the entropy. In the isentropic case, these coefficients are almost constant (the spatial derivatives are of order of $O(\varepsilon)$). By contrast, in the non-isentropic case, these coefficients are variable. This changes the nature of the linearized equations. The main obstacle is precisely that the linearized equations are not uniformly well-posed in Sobolev spaces. Hence, it is notable that one can prove that the solutions exist and are uniformly bounded for a time independent of ε .

In [2, 3] we start a rigorous analysis of the corresponding problems for the general case in which the combined effects of large temperature variations and thermal conduction are taken into account. We prove in [2, 3] that solutions exist and they are uniformly bounded for a time interval which is independent of the Mach number, the Reynolds number and the Péclet number (thereby including the Euler equation as well). Based on uniform estimates in Sobolev spaces, and using the decay to zero of the local energy of the acoustic waves in the whole space established by Métivier and Schochet in [72], we next prove that the penalized terms converge strongly to zero. In the end, this allows us to rigorously justify, at least in the whole space case, some well-known computations introduced by Majda [69] concerning the low Mach number limit.

The study of the incompressible limit is a vast subject of which we barely scratched the surface here. To fill in this gap we recommend the well written survey papers of Danchin [25], Desjardins and Lin [28], Feireisl [35], Gallagher [42], Masmoudi [70], Schochet [81, 82] and Villani [90] (see also [4]). Let us also point out that the research of numerical algorithms valid for all flow speeds is a very active field (see [34, 58, 59]).

Our goals in the next part of the introduction is to derive the non dimensionalized Navier-Stokes equations, explain the physical background and state the main theorems.

1.1. The equations. The general equations of fluid mechanics are the law of mass conservation, the conservation of momentum, the law of energy conservation and the laws

of thermodynamics. For a fluid with density ρ , velocity v , pressure P , temperature T , internal energy e , Lamé coefficients ζ, η and coefficient of thermal conductivity k , the full Navier-Stokes equations are

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P = \operatorname{div} \tau, \\ \partial_t(\rho e) + \operatorname{div}(\rho v e) + P \operatorname{div} v = \operatorname{div}(k \nabla T) + \tau \cdot Dv, \end{cases}$$

where τ denotes the viscous strain tensor given by (Newtonian gases):

$$\tau := 2\zeta Dv + \eta \operatorname{div} v I_d,$$

where $2Dv = \nabla v + (\nabla v)^t$ and I_d is the $d \times d$ identity matrix.

In order to be closed, the system is supplemented with a thermodynamic closure law, so that ρ, P, e, T are completely determined by only two of these variables. Also, it is assumed that ζ, η and k are smooth functions of the temperature.

The governing equations of fluid mechanics are merely written in this generality. Instead, one often prefers simplified forms. To obtain reduced systems, the easiest route is to introduce dimensionless numbers which quantify the importance of various physical processes. We distinguish three dimensionless parameters:

$$\varepsilon \in (0, 1], \quad \mu \in [0, 1], \quad \kappa \in [0, 1].$$

The first parameter ε is the Mach number (recall that it is the ratio of a characteristic velocity in the flow to the sound speed in the fluid). The parameters μ and κ are, up to multiplicative constants, the inverses of the Reynolds and Péclet numbers; they measure the importance of viscosity and heat-conduction.

To rescale the equations, one can cast equations in dimensionless form by scaling every variable by its characteristic value [69, 75]. Alternatively, one can consider one of the three changes of variables:

$$\begin{aligned} t \rightarrow \varepsilon^2 t, \quad x \rightarrow \varepsilon x, \quad v \rightarrow \varepsilon v, \quad \zeta \rightarrow \mu \zeta, \quad \eta \rightarrow \mu \eta, \quad k \rightarrow \kappa k, \\ t \rightarrow \varepsilon t, \quad x \rightarrow x, \quad v \rightarrow \varepsilon v, \quad \zeta \rightarrow \varepsilon \mu \zeta, \quad \eta \rightarrow \varepsilon \mu \eta, \quad k \rightarrow \varepsilon \kappa k, \\ t \rightarrow t, \quad x \rightarrow x/\varepsilon, \quad v \rightarrow \varepsilon v, \quad \zeta \rightarrow \varepsilon^2 \mu \zeta, \quad \eta \rightarrow \varepsilon^2 \mu \eta, \quad k \rightarrow \varepsilon^2 \kappa k. \end{aligned}$$

See [65, 91] for comments on the first two changes of variables. The third one is related to large-amplitude high-frequency solutions (see [20]).

These two approaches yield the same result. The full Navier-Stokes equations, written in a non-dimensional way, are:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \frac{\nabla P}{\varepsilon^2} = \mu \operatorname{div} \tau, \\ \partial_t(\rho e) + \operatorname{div}(\rho v e) + P \operatorname{div} v = \kappa \operatorname{div}(k \nabla T) + \varepsilon^2 \mu \tau \cdot Dv. \end{cases}$$

1.2. The limit constraint on the divergence of the velocity. We next give the usual formal computations which give the zero Mach number system. We also refer to [76] for formal computations including the Froude number, a case which will be studied in a forthcoming paper.

Before we proceed, let us pause to explain further why the nature of the low Mach number limit strongly depends on the size of the entropy variations. One can distinguish three cases: the almost isentropic regime where the entropy is constant except for perturbations of order of the Mach number; the almost adiabatic case where the entropy of each fluid particule is almost constant; the general non-adiabatic case. More precisely,

$$\begin{aligned} \text{Almost isentropic:} & \quad \nabla S = O(\varepsilon), \\ \text{Almost adiabatic:} & \quad \nabla S = O(1) \quad \text{and} \quad \partial_t S + v \cdot \nabla S = O(\varepsilon), \\ \text{Non-adiabatic:} & \quad \nabla S = O(1) \quad \text{and} \quad \partial_t S + v \cdot \nabla S = O(1). \end{aligned}$$

We claim that the fluid is asymptotically incompressible only in the almost isentropic or adiabatic cases. The heuristic argument is the following. When ε goes to 0, the pressure fluctuations converge to 0. Consequently, the limit entropy and density are functions of the temperature alone. Therefore, the evolution equation for the entropy and the continuity equation provide us with two values for the convective derivative of the temperature. By equating both values, it is found that the limit divergence constraint is of the form $\operatorname{div} v = \kappa C_p \operatorname{div}(k \nabla T)$. Hence, $\operatorname{div} v = 0$ implies that $\kappa \nabla T = 0$, which means that the limit flow is adiabatic (adiabatic means that each fluid particule has a constant entropy, which is obvious if the flow is isentropic).

Note that many problems are non-adiabatic (see [32, 33, 58, 59, 65, 69, 75, 76]). Such problems naturally arise, among others, in combustion which requires modeling of multicomponents flows which allow for large heat release and large deviations of the concentration of the chemical species. This is one motivation to study the general case where the combined effects of large temperature variations and thermal conduction are taken into account.

We now compute the limit system. We first make the following important observation: for the low Mach number limit problem, the point is not so much to use the conservative form of the equations, but instead to balance the acoustics components. This is one reason why it is interesting to work with the unknowns P, v, T (see [69]). We thus begin by forming evolution equations for the pressure and the temperature. To simplify the presentation, consider perfect gases. That is, assume that there exist two positive constants R and C_V such that

$$P = R\rho T \quad \text{and} \quad e = C_V T.$$

Performing linear algebra, we compute that,

$$\begin{cases} \partial_t P + v \cdot \nabla P + \gamma P \operatorname{div} v = (\gamma - 1) \{ \kappa \operatorname{div}(k \nabla T) + \varepsilon^2 \mu \tau \cdot Dv \}, \\ \rho(\partial_t v + v \cdot \nabla v) + \frac{\nabla P}{\varepsilon^2} = \mu \operatorname{div} \tau, \\ \rho C_V(\partial_t T + v \cdot \nabla T) + P \operatorname{div} v = \kappa \operatorname{div}(k \nabla T) + \varepsilon^2 \mu \tau \cdot Dv, \end{cases}$$

with $\gamma = 1 + R/C_V$.

Observe that, as the Mach number ε goes to zero, the pressure gradient becomes singular and hence the variations in pressure converge to zero. Under assumptions that will be made in the following, it implies that the pressure tends to a constant \underline{P} . With this notation, we find that the limit system reads

$$(3) \quad \begin{cases} \gamma \underline{P} \operatorname{div} v = (\gamma - 1) \kappa \operatorname{div}(k \nabla T), \\ \rho(\partial_t v + v \cdot \nabla v) + \nabla \pi = \mu \operatorname{div} \tau, \\ \rho C_P(\partial_t T + v \cdot \nabla T) = \kappa \operatorname{div}(k \nabla T), \end{cases}$$

where $\rho = \underline{P}/(RT)$ and $C_P = \gamma C_V$.

It is important to note that the incompressible limit is a special case of the low Mach number limit. Namely, the limit velocity is incompressible ($\operatorname{div} v = 0$) if and only if $\kappa \operatorname{div}(k \nabla T) = 0$, which in turn is equivalent to the fact that the limit entropy satisfies $\partial_t S + v \cdot \nabla S = 0$ (adiabatic regime).

Note that, one has a similar equation for general gases. For instance, if the gas obeys Mariotte's law ($P = R\rho T$ and $e = e(T)$ is a function of T alone), then we find that limit constraint on the divergence of the velocity reads:

$$\underline{P} \operatorname{div} v = \kappa \frac{R}{C_V(T) + R} \operatorname{div}(k \nabla T), \quad C_V = \frac{\partial e}{\partial T}.$$

This equation contains the main difference between perfect gases and general gases. For perfect gases, the limit constraint is linear in the sense that it reads $\operatorname{div} v_e = 0$ with $v_e = v - K \nabla T$ for some constant K . By contrast, for general equations of state, the limit constraint is nonlinear.

1.3. Main results. We consider classical solutions, that is, solutions valued in the Sobolev spaces $H^s(\mathbb{D})$ with s large enough, where the domain \mathbb{D} is either the whole space \mathbb{R}^d or the torus \mathbb{T}^d . Recall that, when $\mathbb{D} = \mathbb{R}^d$, the Sobolev spaces are endowed with the norms

$$\|u\|_{H^\sigma}^2 := (2\pi)^{-d} \int_{\mathbb{R}^d} \langle \xi \rangle^{2\sigma} |\widehat{u}(\xi)|^2 d\xi,$$

where \widehat{u} is the Fourier transform of u and $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. With regards to the case $\mathbb{D} = \mathbb{T}^d$, we replace the integrals in $\xi \in \mathbb{R}^d$ by sums on $k \in \mathbb{Z}^d$.

As above, to simplify the presentation of the introduction, we restrict ourselves to perfect gases. Recall that the system reads,

$$(4) \quad \begin{cases} \partial_t P + v \cdot \nabla P + \gamma P \operatorname{div} v = (\gamma - 1) \kappa \operatorname{div}(k \nabla T) + (\gamma - 1) \varepsilon \mathcal{Q}, \\ \rho(\partial_t v + v \cdot \nabla v) + \frac{\nabla P}{\varepsilon^2} = \mu \operatorname{div} \tau, \\ \rho C_V(\partial_t T + v \cdot \nabla T) + P \operatorname{div} v = \kappa \operatorname{div}(k \nabla T) + \varepsilon \mathcal{Q}, \end{cases}$$

where $\rho = P/(RT)$ and $\mathcal{Q} := \varepsilon \mu \tau \cdot Dv$. Equations (4) are supplemented with initial data:

$$(5) \quad P|_{t=0} = P_0, \quad v|_{t=0} = v_0 \quad \text{and} \quad T|_{t=0} = T_0.$$

We consider general initial data, that is we suppose that, initially,

$$\nabla v_0 = O(1), \quad \nabla P_0 = O(\varepsilon), \quad \nabla T_0 = O(1),$$

where $O(1)$ means uniformly bounded with respect to ε . In particular, we allow large temperature, density and entropy variations. Also, we allow two times scales (since $\partial_t v$ is of order of $\varepsilon^{-2} \nabla P$, the assumption $\nabla P_0 = O(\varepsilon)$ allow very large acceleration of order of ε^{-1}). The hypothesis $\nabla P_0 = O(\varepsilon)$ does not mean that we prepare the initial data. On the contrary, it is the natural scaling to balance the acoustic components (see (7) and [24, 56, 58, 69, 72]).

We assume that ζ , η and the coefficient of thermal conductivity k are C^∞ functions of the temperature T , satisfying

$$k(T) > 0, \quad \zeta(T) > 0 \quad \text{and} \quad \eta(T) + 2\zeta(T) > 0.$$

Also, we consider general equation: $\mu \in [0, 1]$ and $\kappa \in [0, 1]$. In particular we consider the full Navier-Stokes equations ($\mu = 1 = \kappa$) as well as the Euler equations ($\mu = 0 = \kappa$). The main result studied in these lectures asserts that the classical solutions exist and are uniformly bounded for a time interval independent of ε , μ and κ .

NOTATION. Hereafter, A denotes the set of non dimensionalized parameters:

$$A := \{ a = (\varepsilon, \mu, \kappa) \mid \varepsilon \in (0, 1], \mu \in [0, 1], \kappa \in [0, 1] \}.$$

THEOREM 1.1 ([2]). *Let $d \geq 1$ and \mathbb{D} denote either the whole space \mathbb{R}^d or the torus \mathbb{T}^d . Consider an integer $s > 1 + d/2$. For all positive \underline{P} , \underline{T} and M_0 , there is a positive time T such that for all $a = (\varepsilon, \mu, \kappa) \in A$ and all initial data (P_0^a, v_0^a, T_0^a) such that P_0^a and T_0^a take positive values and such that*

$$\varepsilon^{-1} \|P_0^a - \underline{P}\|_{H^{s+1}(\mathbb{D})} + \|v_0^a\|_{H^{s+1}(\mathbb{D})} + \|T_0^a - \underline{T}\|_{H^{s+1}(\mathbb{D})} \leq M_0,$$

the Cauchy problem for (4)–(5) has a unique classical solution (P^a, v^a, T^a) such that $(P^a - \underline{P}, v^a, T^a - \underline{T}) \in C^0([0, T]; H^{s+1}(\mathbb{D}))$ and such that P^a and T^a take positive values. In addition there exists a positive M , depending only on M_0 , \underline{P} and \underline{T} , such that

$$\sup_{a \in A} \sup_{t \in [0, T]} \left\{ \varepsilon^{-1} \|P^a(t) - \underline{P}\|_{H^s(\mathbb{D})} + \|v^a(t)\|_{H^s(\mathbb{D})} + \|T^a(t) - \underline{T}\|_{H^s(\mathbb{D})} \right\} \leq M.$$

REMARK 1.2. Note that we control the initial data in H^{s+1} and the solution in H^s . We shall give below our main result a refined form where the solutions satisfy the same estimates as the initial data do (see Theorem 4.10).

When ε tends to 0, the solutions of the full equations (4) converge to the unique solution of (3) whose initial velocity is the pseudo-incompressible part of the original velocity.

THEOREM 1.3 ([2]). *Fix $\mu \in [0, 1]$ and $\kappa \in [0, 1]$. Assume that $(P^\varepsilon, v^\varepsilon, T^\varepsilon)$ satisfy (4) and*

$$\sup_{\varepsilon \in (0,1]} \sup_{t \in [0,T]} \|\varepsilon^{-1}(P^\varepsilon(t) - \underline{P})\|_{H^s} + \|v^\varepsilon(t)\|_{H^s} + \|T^\varepsilon(t) - \underline{T}\|_{H^s} < +\infty,$$

for some fixed time $T > 0$, reference states $\underline{P}, \underline{T}$ and index s large enough. Suppose in addition that the initial data $T^\varepsilon|_{t=0} - \underline{T}$ are compactly supported. Then, for all $s' < s$, the pressure variations $\varepsilon^{-1}(P^\varepsilon - \underline{P})$ converges strongly to 0 in $L^2(0, T; H_{loc}^{s'}(\mathbb{R}^d))$. Moreover, for all $s' < s$, $(v^\varepsilon, T^\varepsilon)$ converges strongly in $L^2(0, T; H_{loc}^{s'}(\mathbb{R}^d))$ to a limit (v, T) satisfying the limit system (3).

Let us recall that the previous results concern perfect gases. The case of general gas is studied below following [3]. Yet, to simplify the presentation, we do not include separate statements in this introduction.

For the incompressible limit in the isentropic case, we will see that one can give a rate of convergence of $(\operatorname{div} v, \nabla p)$ to 0 by means of Strichartz' estimates. Here, we only state that the penalized terms converge to zero: it is an open problem to give a rate of convergence for the non-isentropic systems. One reason is that it is much more difficult to prove dispersive estimates for variable coefficients wave equations (however one can quote the notable results of Alinhac [6] and Burq [18]).

Also, in their pionnering work [72, 73, 74], Métivier and Schochet have shown that the study of the non-isentropic equations involves many other very interesting additional phenomena. In particular, under periodic boundary conditions, there are several difficult open problems concerning the justification of the low Mach number limit for the non-isentropic equations with large density variations (see also [14]).

With regards to Theorem 1.1, one technical reason why we are uniquely interested in the whole space \mathbb{R}^d or the Torus \mathbb{T}^d is that we will make use of the Fourier analysis. A more serious obstacle is that, in the boundary case, there should be boundary layers to analyze [14]. For the Euler equations (that is, $\mu = \kappa = 0$), however, Theorem 1.1 remains valid in the boundary case [1].

Another interesting problem is to study the previous systems in the multiple spatial scales case (see [81] and [58, 59] for formal computations).

2. Incompressible limit of the isentropic Euler equations

We start from scratch and consider first the isentropic Euler equations. Some important features of the analysis can be explained on this simplified system.

Consider the isentropic Euler equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \frac{\nabla P}{\varepsilon^2} = 0, \end{cases}$$

with $P = A\rho^\gamma$. As already explained, it is interesting to work with the unknowns P, v . It is easily found that one can rewrite the previous system into the form

$$\begin{cases} \partial_t P + v \cdot \nabla P + \gamma P \operatorname{div} v = 0, \\ P^{1/\gamma}(\partial_t v + v \cdot \nabla v) + \frac{\nabla P}{\varepsilon^2} = 0. \end{cases}$$

We next seek P in the form $P = \text{Const.} + O(\varepsilon)$. As in [72], since P is a positive function, it is reasonable to set $P = \underline{P}e^{\varepsilon p}$, where \underline{P} is a given positive constants, say the reference state at spatial infinity. Then, by writing $\partial_{t,x}P = \varepsilon P \partial_{t,x}p$ it is found that (p, v) satisfies a system of the form:

$$(6) \quad \begin{cases} g_1(\varepsilon p)(\partial_t p + v \cdot \nabla p) + \varepsilon^{-1} \operatorname{div} v = 0, \\ g_2(\varepsilon p)(\partial_t v + v \cdot \nabla v) + \varepsilon^{-1} \nabla p = 0. \end{cases}$$

We begin with the classical result that the solutions exist for a time independent of the Mach number ε (note that Sideris [85] proved that there is blowup in finite time for some initial data).

THEOREM 2.1 (from Klainerman & Majda [56, 57]). *Let $d \in \mathbb{N}^*$ and $s > d/2 + 1$. For all bounded subset \mathbb{B}_0 of $H^s(\mathbb{R}^d)$, there exist a time $T > 0$ and a bounded subset \mathbb{B} of $C^0([0, T]; H^s(\mathbb{R}^d))$ such that, for all $\varepsilon \in (0, 1]$ and all initial data $(p_0, v_0) \in \mathbb{B}_0$, the Cauchy problem for (6) has a unique classical solution $(p, v) \in \mathbb{B}$.*

PROOF. The system is symmetric hyperbolic; therefore the Cauchy problem is well-posed for fixed ε . We let $T_\varepsilon = T(\varepsilon, \mathbb{B}_0) > 0$ denote the lifespan, that is the supremum of all the positive times T such that the Cauchy problem for (6) has a unique solution in $C^0([0, T], H^s(\mathbb{R}^d))$. Furthermore, either $T_\varepsilon = +\infty$ or

$$\limsup_{t \rightarrow T_\varepsilon} \|(p, v)(t)\|_{H^s} = +\infty.$$

On account of this alternative the problem reduces to establishing uniform H^s estimates.

To obtain uniform estimates, it is convenient to rewrite system (6) in the form

$$\partial_t u + \sum_{1 \leq j \leq d} A_j(u, \varepsilon u) \partial_j u + \varepsilon^{-1} \sum_{1 \leq j \leq d} S_j \partial_j u = 0,$$

where the matrices A_j and S_j are symmetric. The result then follows from the usual estimates. Indeed, introduce

$$\dot{u} := (I - \Delta)^{s/2}u,$$

where $(I - \Delta)^{s/2}$ is the Fourier multiplier with symbol $(1 + |\xi|^2)^{s/2}$. Then \dot{u} satisfies

$$\partial_t \dot{u} + \sum_{1 \leq j \leq d} A_j(u, \varepsilon u) \partial_j \dot{u} + \varepsilon^{-1} \sum_{1 \leq j \leq d} S_j \partial_j \dot{u} = f := \sum_{1 \leq j \leq d} [A_j(u, \varepsilon u), \Lambda^s] \partial_j u,$$

where $[A, B]$ denotes the commutator $AB - BA$.

Denote by $\langle \cdot, \cdot \rangle$ the scalar product in L^2 . We obtain L^2 estimates for \dot{u} uniform in ε by a simple integration by parts in which the large terms in $1/\varepsilon$ cancel out. Indeed, since $S(\partial_x) = -S(\partial_x)^*$, one has

$$\frac{d}{dt} \|\dot{u}\|_{L^2}^2 = \sum_{1 \leq j \leq d} \langle (\partial_j A_j(u, \varepsilon u)) \dot{u}, \dot{u} \rangle + 2 \langle f, \dot{u} \rangle,$$

where we use the symmetry of the matrix A_j (a word of caution: the previous identity is clear if $\dot{u} \in C_t^1 L_x^2$. To prove the previous identity for $\dot{u} \in C_t^0 L_x^2$, one has to approximate \dot{u} by a sequence u^n in $C_t^0 H_x^1$ such that u^n is the unique solution of an approximating system. This can be accomplished by the usual Friedrichs Lemma).

Since

$$\|\partial_j A_j(u, \varepsilon u)\|_{L^\infty} \leq C(\|(u, \partial_j u)\|_{L^\infty}) \leq C(\|u\|_{H^s}),$$

it remains only to estimate the commutator f . To do that, we recall the following nonlinear estimates in Sobolev spaces.

a) If $s > d/2$, $F \in C^\infty$ and $F(0) = 0$ and $u \in H^s(\mathbb{R}^d)$, then

$$\|F(u)\|_{H^s} \leq C(\|u\|_{L^\infty}) \|u\|_{H^s}.$$

b) If $\sigma \geq 0$, $s > 0$ and $P \in \text{Op } S^m$ is a pseudo-differential operator of order m (say a Fourier multiplier or a differential operator), then

$$\|P(fu) - fPu\|_{H^\sigma} \leq K \|\nabla f\|_{L^\infty} \|u\|_{H^{\sigma+m-1}} + K \|f\|_{H^{\sigma+m}} \|u\|_{L^\infty}.$$

This implies that

$$\begin{aligned} \|\partial_j A_j(u, \varepsilon u)\|_{L^\infty} &\leq C(\|(u, \partial_j u)\|_{L^\infty}) \leq C(\|u\|_{H^s}), \\ \|[A_j(u, \varepsilon u), \Lambda^s] \partial_j u\|_{L^2} &\leq K \|(A_j - A_j(0))(u, \varepsilon u)\|_{H^s} \|\partial_j u\|_{H^{s-1}} \leq C(\|u\|_{H^s}). \end{aligned}$$

The Gronwall lemma completes the proof. \square

Our goal is to explain how to extend Theorem 2.1 to the full Navier-Stokes equations. Yet, we can extend this result in many other directions. Let us mention three.

- Schochet [77] has proved that the result remains true for the case with boundary, for the non-isentropic Euler equations with small entropy variations (the case with large entropy fluctuations is studied in [1] following [72]). An interesting open problem is to extend this result to the compressible Navier-Stokes equations.
- By a standard re-scaling, Theorem 2.1 just says that the classical solutions with small initial data of size δ exist for a time of order of $1/\delta$. See [5, 44, 86] for further much more refined results for the lifespan of the rotationally invariant or spherically symmetric solutions of the Euler equations whose initial data are obtained by adding a small smooth perturbation to a constant state.
- Alvarez-Samaniego and Lannes [7] have studied the well-posedness of the initial value problem for a wide class of singular evolution equations which allow losses of derivatives in energy estimates (for fixed ε , the solutions are constructed by a Nash-Moser iterative scheme).

Granted the uniform estimates established in the proof of the existence theorem, to prove that the limits satisfy the limit equations, we need only to establish compactness in time. We first consider well prepared initial data ($\operatorname{div} v_0^\varepsilon = O(\varepsilon)$ and $\nabla p_0^\varepsilon = O(\varepsilon)$) so that the first order time derivatives are bounded initially. For instance one can consider $p_0^\varepsilon = 0$ and a fixed divergence free vector field (say the data of the limit system).

For well prepared initial data, one can obtain compactness in time in the strong sense of Ascoli. This follows from the Arzela–Ascoli’s theorem and the following result.

PROPOSITION 2.2. *Let $T > 0$ and $\{(p^\varepsilon, v^\varepsilon)\}$ be a family of solutions uniformly bounded in $C^0([0, T]; H^s(\mathbb{R}^d))$ for some $s > 1 + d/2$. If $\varepsilon^{-1}(\nabla p^\varepsilon(0), \operatorname{div} v^\varepsilon(0))$ is uniformly bounded in $L^2(\mathbb{R}^d)$, then $\partial_t(p^\varepsilon, v^\varepsilon)$ is uniformly bounded in $C^0([0, T]; L^2(\mathbb{R}^d))$.*

PROOF. Note that $\dot{u}^\varepsilon := \partial_t(p^\varepsilon, v^\varepsilon)$ satisfies a system of the form

$$\partial_t \dot{u}^\varepsilon + \sum A_j^\varepsilon(t, x) \partial_j \dot{u}^\varepsilon + F^\varepsilon(t, x) \dot{u}^\varepsilon + \varepsilon^{-1} S \dot{u}^\varepsilon = 0,$$

where we can arrange that S is skew-symmetric and the matrices A_j^ε (resp. F^ε) are uniformly bounded in $W^{1,\infty}([0, T] \times \mathbb{R}^d)$ (resp. $L^\infty([0, T] \times \mathbb{R}^d)$). The result then follows by the usual L^2 estimate (multiply by \dot{u}^ε and integrate by parts on \mathbb{R}^d). \square

Note that the same argument applies if the space variable belongs to the d -dimensional Torus \mathbb{T}^d . In this case, by combining the previous uniform estimates for the time derivative with the uniform estimates in space established in the proof of the existence theorem, one immediately obtains convergence in $C^0(0, T; H^{s-\delta}(\mathbb{T}^d))$ and in $L^\infty(0, T; H^k(\mathbb{T}^d))$ weak-*. In [10, 11], Beirão da Veiga established convergence in the strong $C^0([0, T]; H^k(\mathbb{T}^d))$ norm (see also [83]).

Let us now consider the problem of convergence for general initial data. In the whole space case, the solutions converge, although this convergence is not uniform for time

close to zero for the oscillations on the acoustic time-scale prevent the convergence of the solutions on a small initial layer in time (see [8, 47, 48, 50, 89]).

THEOREM 2.3 (from Ukai [89]). *Let $T > 0$ and $\{(p^\varepsilon, v^\varepsilon)\}$ be a family of solutions uniformly bounded in $C^0([0, T]; H^s(\mathbb{R}^d))$ for some $s > 1 + d/2$. Suppose also that $(p^\varepsilon, v^\varepsilon)(0)$ is bounded in $L^1(\mathbb{R}^d)$ and that $Pv_\varepsilon(0)$ converge to v_0 in $H^s(\mathbb{R}^d)$, where P is the Leray projector onto divergence free vector fields. Define v as the unique solution of the Cauchy problem*

$$\partial_t v + P(v \cdot \nabla v) = 0, \quad v(0, x) = v_0(x).$$

Then, for all $s' < s$ and all $1 \leq p < +\infty$, $p^\varepsilon(t) \rightarrow 0$ and $v^\varepsilon(t) \rightarrow v(t)$ in $L^p(0, T; H_{loc}^{s'}(\mathbb{R}^d))$.

PROOF. Rewrite the equations in the form

$$\partial_t u^\varepsilon + \frac{i}{\varepsilon} L u^\varepsilon = f^\varepsilon,$$

with

$$u^\varepsilon = \begin{pmatrix} p^\varepsilon \\ v^\varepsilon \end{pmatrix}, \quad L = -i \begin{pmatrix} 0 & \nabla \\ \operatorname{div} & 0 \end{pmatrix},$$

so that

$$u^\varepsilon = e^{-itL/\varepsilon} u^\varepsilon(0) - \int_0^t e^{-i(t-s)L/\varepsilon} f^\varepsilon(s) ds.$$

Set $\Gamma = I - \Gamma_0$ where Γ_0 is the projection onto the null space of L . To prove Theorem 2.3, the key point is to establish the pointwise convergence

$$\forall t > 0, \quad \Gamma u^\varepsilon(t) \rightarrow 0 \text{ in } H_{loc}^{s'}(\mathbb{R}^d) \text{ as } \varepsilon \rightarrow 0.$$

Since $u^\varepsilon(t)$ is bounded in H^s , by Rellich's theorem, it is enough to prove that

$$\forall t > 0, \quad \Gamma u^\varepsilon(t) \rightarrow 0 \text{ in } \mathcal{D}'(\mathbb{R}^d) \text{ as } \varepsilon \rightarrow 0.$$

Write

$$(\Gamma u^\varepsilon(t), g) = (u^\varepsilon(0), e^{itL/\varepsilon} \Gamma g) - \int_0^t (f^\varepsilon, e^{i(t-s)L/\varepsilon} \Gamma g) ds.$$

Let us prove that the second term goes to 0. Since f^ε is bounded in $L_t^\infty L_x^2$, it is enough to prove that

$$\int_0^t (f^\varepsilon, e^{i(t-s)L/\varepsilon} \Gamma h) ds \rightarrow 0.$$

By density we can assume that $\widehat{h} \in C_0^\infty(\mathbb{R}^d \setminus 0)$. Also, since f^ε is bounded in $L_t^\infty L_x^1$, it is enough to prove that

$$e^{i(t-s)L/\varepsilon} \Gamma h \rightarrow 0 \text{ in } L^\infty(\mathbb{R}^d).$$

This in turn follows from the stationary phase on the sphere, which implies that one has convergence to zero in $L^\infty(\mathbb{R}^d)$:

$$\|e^{i\tau L/\varepsilon} \Gamma h\|_{L^\infty} \leq C \left(\frac{\varepsilon}{\tau}\right)^{(d-1)/2} \|h\|_{L^1}.$$

□

REMARK 2.4. As proved by Isozaki [49], the result holds if $x \in \Omega$ where Ω is the exterior of a bounded domain, with the solid wall boundary condition $v \cdot \nu = 0$ on the boundary $\partial\Omega$. One can use a scattering argument to reduce the analysis to establishing the result in the free space. Indeed, let S be the linearized operator of acoustics in $L^2(\Omega)$, and denote by Π the projection on the orthogonal complement of its kernel. By the completeness of the wave operators, for all $g \in L^2(\Omega)$ there exist $h_{\pm} \in L^2(\mathbb{R}^d)$ such that

$$\|e^{-i\tau S}\Pi g - e^{-i\tau L}\Gamma h_{\pm}\|_{L^2(\Omega)} \rightarrow 0 \text{ as } \tau \rightarrow \pm\infty.$$

REMARK 2.5. The fact that we use the stationary phase on the sphere is related to the fact that we study the wave equation. See Stein [87] for decay results for the Fourier transform of measures supported on smooth curved hypersurfaces. For further results about general constant coefficients symmetric hyperbolic systems, see [51]. In this paper, Joly, Métivier and Rauch proved that the contributions of the singular terms of the characteristic variety can be treated as error terms (see also [61]).

Strichartz' estimates for the linear wave equation can be used to show that the gradient part of the velocity converges strongly to zero; recall, in the Helmholtz decomposition of the velocity field, it is the gradient part that satisfies the linear wave equation. For instance one has the following estimate in the 3D case.

PROPOSITION 2.6 (from [52]). *There exists a constant C such that, for all $\delta \in (0, 1]$, all $\tau \geq 1$ and all $v \in C^0([0, +\infty); H^2(\mathbb{R}^3))$,*

$$\|j(\delta D_x)v\|_{L^2(0,\tau;L^\infty(\mathbb{R}^3))} \leq C\sqrt{\log(\tau/\delta)}\left(\|(\partial_{t,x}v)(0)\|_{L^2(\mathbb{R}^3)} + \|\partial_t^2 v - \Delta v\|_{L^1(0,\tau;L^2(\mathbb{R}^3))}\right),$$

where j is a smooth bump function satisfying $j(\xi) = 1$ if $|\xi| \leq 1$.

By applying this result with $\tau = T/\varepsilon$ and $v(t, x) = u(\varepsilon t, x)$, we find that, if

$$\varepsilon^2 \partial_t^2 u^\varepsilon - \Delta u^\varepsilon = O(\varepsilon) \quad \text{in } L^1(0, T; L^2(\mathbb{R}^3)),$$

and if $\varepsilon \partial_t u, \partial_x u = O(1)$ initially, then one has strong convergence

$$u^\varepsilon = O(\sqrt{\varepsilon} \log(1/\varepsilon)) \quad \text{in } L^2(0, T; L^\infty(\mathbb{R}^3)).$$

A very interesting point is that one has a decay rate in terms of a power of the Mach number. Such estimates have been proved in various contexts for the isentropic equations. In particular, this allows to study the convergence on $[0, T]$ for all $T < T_0$ where T_0 is the lifespan of the *limit* system (see [25, 42]). We refer the reader to the papers of Desjardins and Grenier [26] (for the incompressible limit of weak solutions of the compressible Navier-Stokes on the whole space $d = 2, 3$; note that the strong convergence of the divergence free part is shown in this paper by an alternative argument based on time-regularity); Danchin [24] (incompressible limit of the solutions of the compressible Navier-Stokes equations with critical regularity); Dutrifoy and Hmidi [29] (justification of the incompressible limit to solutions of Yudovich type and to vortex patches).

3. Low Mach number flows

3.1. The equations. We begin by rewriting the equations in the form

$$L(u, \partial_t, \partial_x)u + \frac{1}{\varepsilon}S(u, \partial_x)u = 0,$$

which is the classical framework of a singular limit problem.

For a fluid with density ρ , velocity v , pressure P , temperature T , internal energy e , Lamé coefficients ζ, η and coefficient of thermal conductivity k , the full Navier-Stokes equations, written in a non-dimensional way, are

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \frac{\nabla P}{\varepsilon^2} = \mu(2 \operatorname{div}(\zeta Dv) + \nabla(\eta \operatorname{div} v)), \\ \partial_t(\rho e) + \operatorname{div}(\rho v e) + P \operatorname{div} v = \kappa \operatorname{div}(k \nabla T) + Q, \end{cases}$$

where $\varepsilon \in (0, 1]$, $(\mu, \kappa) \in [0, 1]^2$ and Q is an additional given source term (see [32, 58, 69]), this notations includes the harmless term $\varepsilon \mu \tau \cdot Dv$. In order to be closed, the system is supplemented with a thermodynamic closure law, so that ρ, P, e, T are completely determined by only two of these variables. Also, it is assumed that ζ, η and k are smooth functions of the temperature.

By assuming that ρ and e are given smooth functions of (P, T) , it is found that, for smooth solutions, (P, v, T) satisfies a system of the form:

$$\begin{cases} \alpha(\partial_t P + v \cdot \nabla P) + \operatorname{div} v = \kappa \beta \operatorname{div}(k \nabla T) + \beta Q, \\ \rho(\partial_t v + v \cdot \nabla v) + \frac{\nabla P}{\varepsilon^2} = \mu(2 \operatorname{div}(\zeta Dv) + \nabla(\eta \operatorname{div} v)), \\ \gamma(\partial_t T + v \cdot \nabla T) + \operatorname{div} v = \kappa \delta \operatorname{div}(k \nabla T) + \delta Q, \end{cases}$$

where the coefficients α, β, γ and δ are smooth functions of (P, T) .

Since $\partial_t v$ is of order of $\varepsilon^{-2} \nabla P$, this suggests that we seek P in the form $P = \text{Const.} + O(\varepsilon)$. As in [72], since P and T are positive functions, it is reasonable to set

$$P = \underline{P}e^{\varepsilon p}, \quad T = \underline{T}e^{\theta},$$

where \underline{P} and \underline{T} are given positive constants, say the reference states at spatial infinity.

From now on, the unknown is (p, v, θ) with values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$. We are interested in the general case where p and θ are uniformly bounded in ε (so that $\nabla T = O(1)$ and $\partial_t v = O(\varepsilon^{-1})$). By writing $\partial_{t,x} P = \varepsilon P \partial_{t,x} p$, $\partial_{t,x} T = T \partial_{t,x} \theta$ and redefining the functions k, ζ and η , it is found that (p, v, θ) satisfies a system of the form:

$$(7) \quad \begin{cases} g_1(\phi)(\partial_t p + v \cdot \nabla p) + \frac{1}{\varepsilon} \operatorname{div} v = \frac{\kappa}{\varepsilon} \chi_1(\phi) \operatorname{div}(k(\theta) \nabla \theta) + \frac{1}{\varepsilon} \chi_1(\phi) Q, \\ g_2(\phi)(\partial_t v + v \cdot \nabla v) + \frac{1}{\varepsilon} \nabla p = \mu B_2(\phi, \partial_x) v, \\ g_3(\phi)(\partial_t \theta + v \cdot \nabla \theta) + \operatorname{div} v = \kappa \chi_3(\phi) \operatorname{div}(k(\theta) \nabla \theta) + \chi_3(\phi) Q, \end{cases}$$

where $\phi := (\theta, \varepsilon p)$ and $B_2(\phi, \partial_x) = \chi_2(\phi) \operatorname{div}(\zeta(\theta) D \cdot) + \chi_2(\phi) \nabla(\eta(\theta) \operatorname{div} \cdot)$.

The limit system reads:

$$\begin{cases} \operatorname{div} v = \kappa \chi_1 \operatorname{div}(k \nabla \theta) + \chi_1 Q, \\ g_2(\partial_t v + v \cdot \nabla v) + \nabla \Pi = \mu B_2(\partial_x), \\ g_3(\partial_t \theta + v \cdot \nabla \theta) = \kappa(\chi_3 - \chi_1) \operatorname{div}(k \nabla \theta) + (\chi_3 - \chi_1) Q, \end{cases}$$

where the coefficients g_i, χ_i are evaluated at $(\theta, 0)$.

In the following we make several structural assumptions.

ASSUMPTION 3.1. To avoid confusion, we denote by $(\vartheta, \wp) \in \mathbb{R}^2$ the place holder of the unknown $(\theta, \varepsilon p)$. Hereafter, it is assumed that:

(H1) The functions ζ, η and k are C^∞ functions of $\vartheta \in \mathbb{R}$, satisfying $k > 0$, $\zeta > 0$ and $\eta + 2\zeta > 0$.

(H2) The functions g_i and χ_i ($i = 1, 2, 3$) are C^∞ positive functions of $(\vartheta, \wp) \in \mathbb{R}^2$. Moreover,

$$\chi_1 < \chi_3.$$

The main hypothesis is the inequality $\chi_1 < \chi_3$. It plays a crucial role in proving L^2 estimates. Moreover, given the assumption $k(\vartheta) > 0$, it ensures that the operator $-(\chi_3 - \chi_1) \operatorname{div}(k \nabla \theta)$ [which appears in the last equation of the limit system] is positive. This means nothing but the fact that the limit temperature evolves according to the standard equation of heat diffusion! One can prove (see the appendix in [3]) that this assumption is satisfied by general gases.

PROPOSITION 3.2. (H2) is satisfied provided that the density ρ and the energy e are C^∞ functions of the pressure P and the temperature T , such that $\rho > 0$ and

$$P \frac{\partial \rho}{\partial P} + T \frac{\partial \rho}{\partial T} = \rho^2 \frac{\partial e}{\partial P}, \quad \frac{\partial \rho}{\partial P} > 0, \quad \frac{\partial \rho}{\partial T} < 0, \quad \frac{\partial e}{\partial T} \frac{\partial \rho}{\partial P} > \frac{\partial e}{\partial P} \frac{\partial \rho}{\partial T}.$$

REMARK 3.3. The first identity is the second principle of thermodynamics. The last three identities means that the coefficient of isothermal compressibility, the coefficient of thermal expansion and the specific heat at constant volume are positive.

In addition to Assumption 3.1, we need two compatibility conditions between the penalization operator and the viscous perturbation.

ASSUMPTION 3.4. We assume that there exist two functions S and ϱ such that $(\vartheta, \wp) \mapsto (S(\vartheta, \wp), \wp)$ and $(\vartheta, \wp) \mapsto (\vartheta, \varrho(\vartheta, \wp))$ are C^∞ diffeomorphisms from \mathbb{R}^2 onto \mathbb{R}^2 , $S(0, 0) = \varrho(0, 0) = 0$ and

$$(8) \quad g_1 \frac{\partial S}{\partial \vartheta} = -g_3 \frac{\partial S}{\partial \wp} > 0, \quad g_1 \chi_3 \frac{\partial \varrho}{\partial \vartheta} = -g_3 \chi_1 \frac{\partial \varrho}{\partial \wp} < 0.$$

REMARK 3.5. We claim several times that it is more convenient to work with the unknown P, v, T . Yet, an important feature of the proof of the uniform stability result is that we shall use all the thermodynamical variables. Indeed, $\sigma = S(\theta, \varepsilon p)$ is the entropy and $\rho = \varrho(\theta, \varepsilon p)$ is the density. The following computations explain why they play such a role in the analysis. Suppose (p, v, θ) is a smooth solution of (7) and let $\Psi = \Psi(\vartheta, \varphi) \in C^\infty(\mathbb{R}^2)$. Then $\psi := \Psi(\theta, \varepsilon p)$ satisfies

$$g_1 g_3 (\partial_t \psi + v \cdot \nabla \psi) + \underbrace{\left(g_1 \frac{\partial \Psi}{\partial \vartheta} + g_3 \frac{\partial \Psi}{\partial \varphi} \right)}{=: \Gamma_1(\Psi)} \operatorname{div} v = \kappa \underbrace{\left(g_1 \chi_3 \frac{\partial \Psi}{\partial \vartheta} + g_3 \chi_1 \frac{\partial \Psi}{\partial \varphi} \right)}{=: \Gamma_2(\Psi)} (\operatorname{div}(k(\theta) \nabla \theta) + Q),$$

where the coefficients $g_i, \chi_i, \partial \Psi / \partial \vartheta$ and $\partial \Psi / \partial \varphi$ are evaluated at $(\theta, \varepsilon p)$. Hence, with S and ϱ as the previous assumption, we have

$$\begin{aligned} \Gamma_1(S) &= 0 \text{ and } \Gamma_2(S) > 0; \\ \Gamma_1(\varrho) &> 0 \text{ and } \Gamma_2(\varrho) = 0. \end{aligned}$$

REMARK 3.6. A fundamental point observed by Feireisl and Novotný in [37] is that the Helmholtz' free energy plays a key role to prove uniform estimates. Yet the estimates proved in [37] and the estimates which we proved in [2, 3] are strongly different. Indeed, in [37], Feireisl and Novotný consider the framework of the incompressible limit of weak solutions to the Navier-Stokes-Fourier system in the case of small temperature variations. As a result, they need a very subtle set of energy estimates without “remainder” terms. Here, by contrast, we consider strong classical solutions and large temperature variations. The penalized operator is much more complicated¹, but we do not need an exact energy estimates for the solutions; this allows us to use additional transformations of the equations.

As alluded to in the introduction, there is a dichotomy between the case where χ_1 is independent of θ and the case where χ_1 depends on θ . An important remark is that χ_1 is independent of θ for perfect gases. In this direction, another important remark is that χ_1 depends on θ for common equations of states satisfying our assumptions and which are very close to the perfect gases laws. Indeed, one can easily prove the following result.

PROPOSITION 3.7. *Assume that the gas obeys Mariotte's law: $P = R\rho T$, for some positive constant R , and $e = e(T)$ satisfies $C_V := \partial e / \partial T > 0$. Then, (H2) is satisfied. Moreover, $\chi_1(\vartheta, \varphi)$ is independent of ϑ if and only if C_V is constant (that is for perfect gases).*

¹Indeed, as shown in §5.1, it is easy to prove a uniform stability result for classical solutions in the case of small temperature variations.

3.2. Uniform stability results. We state here various uniform stability results, which assert that the classical solutions of (7) exist and they are uniformly bounded for a time independent of ε , μ and κ . We concentrate below on the whole space problem ($x \in \mathbb{R}^d$) or the periodic case ($x \in \mathbb{T}^d$) and we work in the Sobolev spaces H^σ endowed with the norms $\|u\|_{H^\sigma} := \|(I - \Delta)^{\sigma/2}u\|_{L^2}$. To clarify the presentation, we separate the statements in four parts: 1) Euler equations, 2) Full Navier-Stokes system for perfect gases; 3) Full Navier-Stokes system for general gases without source terms; 4) Full Navier-Stokes system for general gases with source terms. The proofs are discussed in Sections 4 and 5.

The non-isentropic Euler equations. We begin with the non-isentropic Euler equations: consider the case $\mu = 0 = \kappa$ and introduce the entropy $\sigma = S(\theta, \varepsilon p)$. The coefficients g_1 and g_2 are now viewed as C^∞ positive functions of σ and εp . In this case, one can rewrite System (7) under the form:

$$(9) \quad \begin{cases} g_1(\sigma, \varepsilon p)(\partial_t p + v \cdot \nabla p) + \frac{1}{\varepsilon} \operatorname{div} v = 0, \\ g_2(\sigma, \varepsilon p)(\partial_t v + v \cdot \nabla v) + \frac{1}{\varepsilon} \nabla p = 0, \\ \partial_t \sigma + v \cdot \nabla \sigma = 0. \end{cases}$$

THEOREM 3.8 (from Métivier & Schochet [72]). *Let $\mathbb{N} \ni s > 1 + d/2$ and $\mathbb{D} = \mathbb{R}^d$ or \mathbb{T}^d . For all bounded subset $\mathbb{B}_0 \subset H^s(\mathbb{D})$, there exist $T > 0$ and a bounded subset $\mathbb{B} \subset C^0([0, T]; H^s(\mathbb{D}))$ such that, for all $\varepsilon \in (0, 1]$ and all initial data $(p_0, v_0, \sigma_0) \in \mathbb{B}_0$, the Cauchy problem for (9) has a unique solution $(p, v, \sigma) \in \mathbb{B}$.*

The following example, due to Métivier and Schochet, gives a typical example of instability for this system. Consider

$$\begin{cases} g(\sigma)\partial_t u + \varepsilon^{-1}\partial_x u = 0, \\ \partial_t \sigma + b(u)\partial_x \sigma = 0. \end{cases}$$

If $\sigma(0, x) = \underline{\sigma}$ is constant, then

$$u(t, x) = u(0, x - t/(\varepsilon g(\underline{\sigma}))).$$

Hence, a small perturbation of the initial entropy induces in small time a large perturbation of the velocity. This is why it is remarkable that one can prove uniform estimates in Sobolev spaces. In [72], the proof relies upon the fact that one can establish uniform bounds by applying some spatial operators with appropriate weights to the equations. The point is that the matrix multiplying the time derivatives depends on the unknown only through σ and εu . This special structure of the equations implies that the derivatives of the matrix multiplying the time derivatives are uniformly bounded with respect to ε . Thus, one can think of the problem as an evolution equation with a penalization operator with variable coefficients. Consider the problem

$$A_0(t, x)\partial_t u + \varepsilon^{-1}L(\partial_x)u = 0.$$

We cannot obtain Sobolev estimates by differentiating the equations since

$$A_0(t, x)\partial_t(\partial_j u) + \varepsilon^{-1}L(\partial_x)\partial_j u = -(\partial_j A_0(t, x))\partial_t u = O(\varepsilon^{-1}).$$

Simplify further the system and assume that A_0 does not depend on time. Then one has the following identity:

$$Ae^{tA} = e^{tA}A \quad \text{with} \quad A := A_0(x)^{-1}L(\partial_x),$$

which allows us to prove a uniform L^2 estimate for the penalised terms $L(\partial_x)u$. Indeed, one has

$$\|L(\partial_x)u(t)\|_{L^2} \lesssim \|Au(t)\|_{L^2} = \|Ae^{tA}u_0\|_{L^2} = \|e^{tA}(Au_0)\|_{L^2} \lesssim \|Au_0\|_{L^2} \lesssim \|L(\partial_x)u_0\|_{L^2},$$

where, as in the proof of Theorem 2.1, we used the fact that, since $L(\partial_x)^* = -L(\partial_x)$, we have a uniform L^2 estimate for the solution $u(t) = e^{tA}u_0$.

Alternatively, one can estimate first the time derivatives and next use the structure of the equations to estimate the spatial derivatives. The point is that time derivatives have the advantage of commuting with the boundary condition. By doing so, one can prove similar estimates in domains with boundary. In [1], it is proved that Theorem 3.8 is true with \mathbb{D} replaced by a bounded or unbounded domain.

Uniform stability result, perfect gases. We first state a uniform stability result for the case where $Q = 0$ and the coefficient χ_1 does not depend on θ . Consider the system:

$$(10) \quad \begin{cases} g_1(\phi)(\partial_t p + v \cdot \nabla p) + \frac{1}{\varepsilon} \operatorname{div} v = \frac{\kappa}{\varepsilon} \chi_1(\varepsilon p) \operatorname{div}(k(\theta) \nabla \theta), \\ g_2(\phi)(\partial_t v + v \cdot \nabla v) + \frac{1}{\varepsilon} \nabla p = \mu B_2(\phi, \partial_x) v, \\ g_3(\phi)(\partial_t \theta + v \cdot \nabla \theta) + \operatorname{div} v = \kappa \chi_3(\phi) \operatorname{div}(k(\theta) \nabla \theta), \end{cases}$$

where $\phi = (\theta, \varepsilon p)$.

THEOREM 3.9 (from [2]). *Suppose that χ_1 is independent of θ . Let $\mathbb{N} \ni s > 1 + d/2$ and $\mathbb{D} = \mathbb{R}^d$ or \mathbb{T}^d . For all bounded subset $\mathbb{B}_0 \subset H^{s+1}(\mathbb{D})$, there exist $T > 0$ and a bounded subset $\mathbb{B} \subset C^0([0, T]; H^s(\mathbb{D}))$, such that for all $(\varepsilon, \mu, \kappa) \in (0, 1] \times [0, 1]^2$ and all initial data $(p_0, v_0, \theta_0) \in \mathbb{B}_0$, the Cauchy problem for (10) has a unique solution $(p, v, \theta) \in \mathbb{B}$.*

REMARK 3.10. Recall from Proposition 3.7 that χ_1 is independent of θ for perfect gases. Hence, Theorem 1.1 as stated in the introduction is now a consequence of Theorem 3.9. Note that, for the study of classical solutions, we deliberately omitted the terms in $\varepsilon \mu \tau \cdot Dv$, nothing is changed in the statements of the results, nor in their proofs (this term is very important though for the study of weak solutions [12, 37]).

REMARK 3.11. See Theorem 4.10 below for a refined statement where the solutions satisfy the same estimates as the initial data.

Uniform stability result, general gases. We now consider the case where the coefficient χ_1 depends on θ :

$$(11) \quad \begin{cases} g_1(\phi)(\partial_t p + v \cdot \nabla p) + \frac{1}{\varepsilon} \operatorname{div} v = \frac{\kappa}{\varepsilon} \chi_1(\phi) \operatorname{div}(k(\theta) \nabla \theta), \\ g_2(\phi)(\partial_t v + v \cdot \nabla v) + \frac{1}{\varepsilon} \nabla p = \mu B_2(\phi, \partial_x) v, \\ g_3(\phi)(\partial_t \theta + v \cdot \nabla \theta) + \operatorname{div} v = \kappa \chi_3(\phi) \operatorname{div}(k(\theta) \nabla \theta), \end{cases}$$

where $\phi = (\theta, \varepsilon p)$.

In the free space \mathbb{R}^3 , Theorem 3.9 remains valid without the assumption on χ_1 .

THEOREM 3.12 (from [3]). *Suppose that $d \geq 3$. Let $\mathbb{N} \ni s > 1 + d/2$. For all bounded subset $\mathbb{B}_0 \subset H^{s+1}(\mathbb{R}^d)$, there exist $T > 0$ and a bounded subset $\mathbb{B} \subset C^0([0, T]; H^s(\mathbb{R}^d))$ such that, for all $(\varepsilon, \mu, \kappa) \in (0, 1] \times [0, 1]^2$ and all initial data $(p_0, v_0, \theta_0) \in \mathbb{B}_0$, the Cauchy problem for (11) has a unique solution $(p, v, \theta) \in \mathbb{B}$.*

REMARK 3.13. We state the result for $x \in \mathbb{R}^d$ only. The reason is that the analysis is easier in the periodic case. Moreover, in the periodic case, we have a uniform stability result without restriction on the dimension (see Theorem 3.18 below).

Uniform stability result, combustion equations. We now consider the full System 7, which we recall:

$$(12) \quad \begin{cases} g_1(\phi)(\partial_t p + v \cdot \nabla p) + \frac{1}{\varepsilon} \operatorname{div} v = \frac{\kappa}{\varepsilon} \chi_1(\phi) \operatorname{div}(k(\theta) \nabla \theta) + \frac{1}{\varepsilon} \chi_1(\phi) Q, \\ g_2(\phi)(\partial_t v + v \cdot \nabla v) + \frac{1}{\varepsilon} \nabla p = \mu B_2(\phi, \partial_x) v, \\ g_3(\phi)(\partial_t \theta + v \cdot \nabla \theta) + \operatorname{div} v = \kappa \chi_3(\phi) \operatorname{div}(k(\theta) \nabla \theta) + \chi_3(\phi) Q. \end{cases}$$

THEOREM 3.14 (from [3]). *Let $d = 1$ or $d \geq 3$ and $\mathbb{N} \ni s > 1 + d/2$. For all source term $Q = Q(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ and all $M_0 > 0$, there exist $T > 0$ and $M > 0$ such that, for all $(\varepsilon, \mu, \kappa) \in (0, 1] \times [0, 1] \times [0, 1]$ and all initial data $(p_0, v_0, \theta_0) \in H^{s+1}(\mathbb{R}^d)$ satisfying*

$$\|(\nabla p_0, \nabla v_0)\|_{H^{s-1}} + \|(\theta_0, \varepsilon p_0, \varepsilon v_0)\|_{H^{s+1}} \leq M_0,$$

the Cauchy problem for (12) has a unique solution (p, v, θ) in $C^0([0, T]; H^{s+1}(\mathbb{R}^d))$ such that

$$\sup_{t \in [0, T]} \|(\nabla p(t), \nabla v(t))\|_{H^{s-1}} + \|(\theta(t), \varepsilon p(t), \varepsilon v(t))\|_{H^s} \leq M.$$

3.3. L^2 estimates for the solutions. With regards to the low Mach number limit problem, we will see below that one can rigorously justify the low Mach number limit for general initial data provided that one can prove that the solutions are uniformly bounded in Sobolev spaces (see Proposition 3.20). The problem presents itself: Theorem 3.14 only gives uniform estimates for the derivatives of p and v . In this section, we give uniform bounds in Sobolev norms. Again, let us insist on the fact that, for perfect gases, we have uniform estimates in Sobolev spaces, and not only for the derivatives (see Theorem 3.9).

PROPOSITION 3.15. *Let $d \geq 3$. Consider a family of solutions (p^a, v^a, θ^a) ($a = (\varepsilon, \mu, \kappa)$) of (12) (for some source terms Q^a) uniformly bounded in the sense that*

$$\sup_{a \in A} \sup_{t \in [0, T]} \|(\nabla p^a(t), \nabla v^a(t))\|_{H^s} + \|\theta^a(t)\|_{H^{s+1}} < +\infty,$$

for some $s > 2 + d/2$ and fixed $T > 0$. Assume further that the source terms Q^a are uniformly bounded in $C^1([0, T]; L^1 \cap L^2(\mathbb{R}^d))$ and that the initial data $(p^a(0), v^a(0))$ are uniformly bounded in $L^2(\mathbb{R}^d)$. Then the solutions (p^a, v^a, θ^a) are uniformly bounded in $C^0([0, T]; L^2(\mathbb{R}^d))$.

REMARK 3.16. Theorem 3.12 is a consequence of Theorem 3.14 and Proposition 3.15.

PROOF. The strategy of the proof consists in transforming the system (7) so as to obtain L^2 estimates uniform in ε by a simple integration by parts argument.

The main argument is that, given $d \geq 3$ and $\sigma \in \mathbb{R}$, the Fourier multiplier $\nabla \Delta^{-1}$ is well defined on $L^1(\mathbb{R}^d) \cap H^\sigma(\mathbb{R}^d)$ with values in $H^{\sigma+1}(\mathbb{R}^d)$. Thus, we can introduce $U^a := (p^a, v^a - V^a)^T$ where

$$V^a := \kappa \chi_1(\phi^a) k(\theta^a) \nabla \theta^a + \nabla \Delta^{-1}(-\kappa \nabla \chi_1(\phi^a) \cdot k(\theta^a) \nabla \theta^a + \chi_1(\phi^a) Q^a).$$

U^a solves a system having the form

$$E^a(\partial_t U^a + v^a \cdot \nabla U^a) + \varepsilon^{-1} S(\partial_x) U^a = F^a,$$

where $S(\partial_x)$ is skew-symmetric, the symmetric matrices $E^a = E^a(t, x)$ are positive definite and one has the uniform bounds

$$\sup_{a \in A} \|\partial_t E^a\|_{L^\infty([0, T] \times \mathbb{R}^d)} + \|(E^a)^{-1}\|_{L^\infty([0, T] \times \mathbb{R}^d)}^{-1} + \|F^a\|_{L_T^1(L^2)} \leq C(R),$$

with

$$R := \sup_{a \in A} \sup_{t \in [0, T]} \left\{ \|(\nabla p^a(t), \nabla v^a(t))\|_{H^s} + \|\theta^a(t)\|_{H^{s+1}} \right\} + \sup_{a \in A} \|(p^a(0), v^a(0))\|_{L^2}.$$

This easily yields uniform L^2 estimates for U^a and hence for (p^a, v^a) since

$$\sup_{a \in A} \|(p^a, v^a)\|_{L_T^\infty(L^2)} \leq \sup_{a \in A} \|U^a\|_{L_T^\infty(L^2)} + C(R).$$

□

REMARK 3.17. For our purposes, one of the main differences between \mathbb{R}^3 and \mathbb{R} is the following. For all $f \in C_0^\infty(\mathbb{R}^3)$, there exists a vector field $u \in H^\infty(\mathbb{R}^3)$ such that $\operatorname{div} u = f$. In sharp contrast, the mean value of the divergence of a smooth vector field $u \in H^\infty(\mathbb{R})$ is zero. This implies that Proposition 3.15 is false with $d = 1$.

As noted by Klainerman & Majda [56, 57], the convergence for well prepared initial data allows us to prove that the limit system is well posed. The previous analysis thus implies that the limit system is well posed for $x \in \mathbb{R}^3$; which improves previous result by Embid on the Cauchy problem for zero Mach number equations [32, 33] when $x \in \mathbb{T}^d$.

We end this part by discussing further the difference between the cases $x \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$. In this direction, the main result is that, if $x \in \mathbb{T}^d$, then one has a uniform stability result without any restriction on the dimension (compare with Theorem 3.14).

THEOREM 3.18. *Let $d \geq 1$ and $\mathbb{N} \ni s > 1 + d/2$. For all source term $Q \in C^\infty(\mathbb{R} \times \mathbb{T}^d)$ and for all $M_0 > 0$, there exist $T > 0$ and $M > 0$ such that, for all $(\varepsilon, \mu, \kappa) \in (0, 1] \times [0, 1]^2$ and all initial data $(p_0, v_0, \theta_0) \in H^{s+1}(\mathbb{T}^d)$ satisfying*

$$\|(p_0, v_0)\|_{H^s} + \|(\theta_0, \varepsilon p_0, \varepsilon v_0)\|_{H^{s+1}} \leq M_0,$$

the Cauchy problem for (12) has a unique solution (p, v, θ) in $C^0([0, T]; H^{s+1}(\mathbb{T}^d))$ such that

$$\sup_{t \in [0, T]} \|\nabla p(t)\|_{H^{s-1}} + \|v(t)\|_{H^s} + \|(\theta(t), \varepsilon p(t))\|_{H^s} \leq M.$$

The proof follows from two observations: first, the analysis used to prove Theorems 3.9–3.14 applies *mutatis mutandis* in the periodic case; and second, the periodic case is easier in that one can prove uniform L^2 estimates for the velocity. This in turn implies that (as in [2, 72]) one can directly prove a closed set of estimates by means of the estimate:

$$\|v\|_{H^s(\mathbb{T}^d)} \leq C \|\operatorname{div} v\|_{H^{s-1}(\mathbb{T}^d)} + C \|\operatorname{curl}(\gamma v)\|_{H^{s-1}(\mathbb{T}^d)} + C \|v\|_{L^2(\mathbb{T}^d)},$$

for some constant C depending only on $\|\log \gamma\|_{H^s(\mathbb{T}^d)}$.

Let us concentrate on the main new qualitative property:

LEMMA 3.19. *Let $d \geq 1$, $s > 1 + d/2$ and fixed $T > 0$. Consider a family of solutions $(p^a, v^a, \theta^a) \in C^1([0, T] \times \mathbb{T}^d)$ of (7) (for some source terms Q^a) such that*

$$\sup_{a \in A} \sup_{t \in [0, T]} \|(\nabla p^a(t), \nabla v^a(t))\|_{H^s} + \|\theta^a(t)\|_{H^{s+1}} < +\infty.$$

If Q^a is uniformly bounded in $C^1([0, T]; L^2(\mathbb{T}^d))$ and $(p^a(0), v^a(0))$ is uniformly bounded in $L^2(\mathbb{T}^d)$, then v^a is uniformly bounded in $C^0([0, T]; L^2(\mathbb{T}^d))$.

PROOF. The argument is due to Schochet [78]. Set

$$f^a := \kappa \chi_1(\phi^a) \operatorname{div}(k(\phi^a) \nabla \theta^a) + \chi_1(\phi^a) Q^a,$$

and introduce the functions $V^a = V^a(t, x)$ and $P^a = P^a(t)$ by

$$P^a := \frac{\langle f^a \rangle}{\langle g_1(\phi^a) \rangle} \quad \text{and} \quad V^a := \nabla \Delta^{-1}(f^a - g_1(\phi^a) P^a).$$

Then $U^a := (q^a, v^a - V^a)^T$ with $q^a(t, x) = p^a(t, x) - P^a(t)$, satisfies

$$E^a(\partial_t U^a + v^a \cdot \nabla U^a) + \varepsilon^{-1} S(\partial_x) U^a = F^a,$$

where E^a, F^a satisfy similar to those obtained in the proof of Proposition 3.15. We obtain L^2 estimates uniform in ε by the same integration by parts argument. \square

3.4. The low Mach number limit. We now consider the behavior of the solutions of the full Navier-Stokes system in \mathbb{R}^d as the Mach number ε tends to zero. Fix μ and κ and consider a family of solutions $(p^\varepsilon, v^\varepsilon, \theta^\varepsilon)$ of system:

$$(13) \quad \begin{cases} g_1(\phi^\varepsilon)(\partial_t p^\varepsilon + v^\varepsilon \cdot \nabla p^\varepsilon) + \frac{1}{\varepsilon} \operatorname{div} v^\varepsilon = \frac{\kappa}{\varepsilon} \chi_1(\phi^\varepsilon) \operatorname{div}(k(\theta^\varepsilon) \nabla \theta^\varepsilon) + \frac{1}{\varepsilon} \chi_1(\phi^\varepsilon) Q, \\ g_2(\phi^\varepsilon)(\partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon) + \frac{1}{\varepsilon} \nabla p^\varepsilon = \mu B_2(\phi^\varepsilon, \partial_x) v^\varepsilon, \\ g_3(\phi^\varepsilon)(\partial_t \theta^\varepsilon + v^\varepsilon \cdot \nabla \theta^\varepsilon) + \operatorname{div} v^\varepsilon = \kappa \chi_3(\phi^\varepsilon) \operatorname{div}(k(\theta^\varepsilon) \nabla \theta^\varepsilon) + \chi_3(\phi^\varepsilon) Q, \end{cases}$$

where recall $\phi^\varepsilon := (\theta^\varepsilon, \varepsilon p^\varepsilon)$ and $B_2(\phi^\varepsilon, \partial_x) = \chi_2(\phi^\varepsilon) \operatorname{div}(\zeta(\theta^\varepsilon) D \cdot) + \chi_2(\phi^\varepsilon) \nabla(\eta(\theta^\varepsilon) \operatorname{div} \cdot)$.

We want to prove that the solutions converge to the unique solution of the limit system

$$\begin{cases} \operatorname{div} v = \kappa \chi_1 \operatorname{div}(k \nabla \theta) + \chi_1 Q, \\ g_2(\partial_t v + v \cdot \nabla v) + \nabla \Pi = \mu B_2(\partial_x), \\ g_3(\partial_t \theta + v \cdot \nabla \theta) = \kappa(\chi_3 - \chi_1) \operatorname{div}(k \nabla \theta) + (\chi_3 - \chi_1) Q, \end{cases}$$

whose initial velocity is the incompressible part of the original velocity.

It is assumed that the family $(p^\varepsilon, v^\varepsilon, \theta^\varepsilon)$ is bounded in $C^0([0, T]; H^s(\mathbb{R}^d))$ with s large enough and $T > 0$. Strong compactness of θ^ε is clear from uniform bounds for $\partial_t \theta^\varepsilon$. For the sequence $(p^\varepsilon, v^\varepsilon)$, however, the uniform bounds imply only weak compactness, insufficient to prove that the limits satisfy the limit equations. We remedy this by proving that the penalized terms converge strongly to zero. Namely, the key for proving the convergence result is to prove the decay to zero of the local energy of the acoustic waves.

PROPOSITION 3.20. *Fix $\mu \in [0, 1]$ and $\kappa \in [0, 1]$, and let $d \geq 1$. Assume that $(p^\varepsilon, v^\varepsilon, \theta^\varepsilon)$ satisfy (13) and are uniformly bounded in $C^0([0, T]; H^s(\mathbb{R}^d))$ for some fixed $T > 0$ and s large enough. Suppose that the initial data $\theta^\varepsilon(0)$ converge in $H^s(\mathbb{R}^d)$ to a function θ_0 decaying sufficiently rapidly at infinity in the sense that $\langle x \rangle^\delta \theta_0 \in H^s(\mathbb{R}^d)$ for some given $\delta > 2$. Then, for all indices $s' < s$,*

$$\begin{aligned} p^\varepsilon &\rightarrow 0 \quad \text{strongly in } L^2(0, T; H_{loc}^{s'}(\mathbb{R}^d)), \\ \operatorname{div} v^\varepsilon - \chi_1(\phi^\varepsilon) \operatorname{div}(\beta(\theta^\varepsilon) \nabla \theta^\varepsilon) - \chi_1(\phi^\varepsilon) Q &\rightarrow 0 \quad \text{strongly in } L^2(0, T; H_{loc}^{s'-1}(\mathbb{R}^d)). \end{aligned}$$

The proof of Proposition 3.20 is based on the following theorem of Métivier and Schochet about the decay to zero of the local energy for a class of wave operators with time dependent coefficients.

THEOREM 3.21 (from Métivier & Schochet [72]). *Let $T > 0$ and let u^ε be a bounded sequence in $C^0([0, T]; H^2(\mathbb{R}^d))$ such that*

$$\varepsilon^2 \partial_t(a^\varepsilon \partial_t u^\varepsilon) - \operatorname{div}(b^\varepsilon \nabla u^\varepsilon) = \varepsilon f^\varepsilon,$$

where the source term f^ε is bounded in $L^2(0, T; H^1(\mathbb{R}^d))$. Assume further that the coefficients $a^\varepsilon, b^\varepsilon$ are uniformly bounded in $C^1([0, T]; H^\sigma(\mathbb{R}^d))$, for some $\sigma > 1 + d/2$, and

converge in $C^0([0, T]; H_{loc}^\sigma(\mathbb{R}^d))$ to limits a, b satisfying the decay estimates

$$\begin{aligned} |a(t, x) - \underline{a}| &\leq K |x|^{-1-\gamma}, & |\nabla a(t, x)| &\leq K |x|^{-2-\gamma}, \\ |b(t, x) - \underline{b}| &\leq K |x|^{-1-\gamma}, & |\nabla b(t, x)| &\leq K |x|^{-2-\gamma}, \end{aligned}$$

for some given positive constants $\underline{a}, \underline{b}, K$ and γ .

Then, the sequence u^ε converges to 0 in $L^2(0, T; L_{loc}^2(\mathbb{R}^d))$.

PROOF OF PROPOSITION 3.20 GIVEN THEOREM 3.21. We can directly apply Theorem 3.21 to prove the first half of Proposition 3.20, that is, the convergence of p^ε to 0 in $L^2(0, T; L_{loc}^2(\mathbb{R}^d))$. Indeed, applying $\varepsilon^2 \partial_t$ to the equation for p^ε , we verify that

$$(14) \quad \varepsilon^2 \partial_t (a^\varepsilon \partial_t p^\varepsilon) - \operatorname{div}(b^\varepsilon \nabla p^\varepsilon) = \varepsilon f^\varepsilon,$$

with $a^\varepsilon := g_1(\phi^\varepsilon)$, $b^\varepsilon := 1/g_2(\phi^\varepsilon)$ and f^ε is bounded in $C^0([0, T]; H^1(\mathbb{R}^d))$. Hence Theorem 3.21 applies.

To prove the second half of Proposition 3.20, we begin by proving that $\dot{p}^\varepsilon := (\varepsilon \partial_t) p^\varepsilon$ converges to 0 in $L^2(0, T; L_{loc}^2(\mathbb{R}^d))$. To do so we apply $(\varepsilon \partial_t)$ on equation (14), to obtain

$$\varepsilon^2 \partial_t (a^\varepsilon \partial_t \dot{p}^\varepsilon) - \operatorname{div}(b^\varepsilon \nabla \dot{p}^\varepsilon) = \varepsilon \tilde{f}^\varepsilon,$$

with

$$\tilde{f}^\varepsilon := \varepsilon \partial_t f^\varepsilon - \varepsilon \partial_t (\partial_t a^\varepsilon (\varepsilon \partial_t) p^\varepsilon) + \operatorname{div}(\partial_t b^\varepsilon \nabla p^\varepsilon).$$

Again one can verify that \tilde{f}^ε is a bounded sequence in $C^0([0, T]; H^1(\mathbb{R}^d))$, which proves the desired result.

To complete the proof, observe that

$$\operatorname{div} v^\varepsilon - \chi_1(\phi^\varepsilon) \operatorname{div}(\beta(\theta^\varepsilon) \nabla \theta^\varepsilon) - \chi_1(\phi^\varepsilon) Q = -g_1(\psi^\varepsilon) (\varepsilon \partial_t) p^\varepsilon + O(\varepsilon).$$

Hence, the fact that $\operatorname{div} v^\varepsilon - \chi_1(\theta^\varepsilon, p^\varepsilon) \operatorname{div}(\beta(\theta^\varepsilon) \nabla \theta^\varepsilon)$ converges to 0 in $L^2(0, T; L_{loc}^2(\mathbb{R}^d))$ follows from the previous step and the fact that $g_1(\psi^\varepsilon) - g_1(0)$ is uniformly bounded in $C^0([0, T]; H^s(\mathbb{R}^d))$. \square

4. Linear estimates for the linearized system

A key step in the analysis is to estimate the solution $(\tilde{p}, \tilde{v}, \tilde{\theta})$ of linearized equations. We consider the system:

$$(15) \quad \begin{cases} g_1(\phi) (\partial_t \tilde{p} + v \cdot \nabla \tilde{p}) + \frac{1}{\varepsilon} \operatorname{div} \tilde{v} - \frac{\kappa}{\varepsilon} \operatorname{div}(k_1(\phi) \nabla \tilde{\theta}) = F_1, \\ g_2(\phi) (\partial_t \tilde{v} + v \cdot \nabla \tilde{v}) + \frac{1}{\varepsilon} \nabla \tilde{p} - \mu B_2(\phi, \partial_x) \tilde{v} = F_2, \\ g_3(\phi) (\partial_t \tilde{\theta} + v \cdot \nabla \tilde{\theta}) + \operatorname{div} \tilde{v} - \kappa \chi_3(\phi) \operatorname{div}(k(\phi) \nabla \tilde{\theta}) = F_3, \end{cases}$$

where

$$k_1 = \chi_1 k,$$

the unknown is $\tilde{U} := (\tilde{p}, \tilde{v}, \tilde{\theta})$, which is a function of $(t, x) \in \mathbb{R} \times \mathbb{D}$ with values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$; v and ϕ are coefficients: $v = v(t, x) \in \mathbb{R}^d$ and $\phi = \phi(t, x) \in \mathbb{R}^2$; F_1, F_2, F_3 are given source terms; g_i, χ_i and are our given coefficients (recall that they are smooth positive functions of $\phi \in \mathbb{R}^2$ and that $\chi_1 < \chi_3$).

REMARK 4.1. One can wonder why we consider this system and not the system obtained by replacing the operator $\operatorname{div}(k_1 \nabla \cdot)$ with the apparently more natural operator $\chi_1 \operatorname{div}(k \nabla \cdot)$. A first observation is that we cannot prove uniform estimates for the system thus obtained! A key feature of the system (15) is that we can write the equation for \tilde{p} under the form

$$g_1(\psi)(\partial_t \tilde{p} + v \cdot \nabla \tilde{p}) + \frac{1}{\varepsilon} \operatorname{div} v_e = F_1,$$

for some velocity v_e . This will allow us to symmetrize the equations.

REMARK 4.2. One can also wonder why we can chose to work with System (15). A notable fact in the forthcoming analysis of nonlinear estimates is that we can see unsigned large terms $\varepsilon^{-1} f^\varepsilon(t, x)$ in the equations for p and v as source terms provided that: 1) they do not convey fast oscillations in time: $\partial_t f^\varepsilon = O(1)$; 2) it does not implies a loss of derivatives. To be more precise: in the nonlinear estimates, we can see terms of the form $\varepsilon^{-1} F(\varepsilon p, \theta, \sqrt{\kappa} \nabla \theta)$ as source terms. In particular we will be able to handle $\sqrt{\kappa} \varepsilon^{-1} k \nabla \chi_1 \cdot \nabla \theta$ as a source term (note that, for perfect gases, $\sqrt{\kappa} \varepsilon^{-1} k \nabla \chi_1 \cdot \nabla \theta = O(1)$ since $\chi_1 = \chi_1(\varepsilon p)$ is a function of εp for perfect gases). However, this may cause a loss of derivatives in the estimates! We avoided this technical point by assuming that k does not depend on εp .

Many results have been obtained concerning the symmetrization of the Navier Stokes equations (see, e.g., [12, 15, 22, 37, 54, 55, 71]). Yet, the previous works do not include the dimensionless numbers. Here we prove estimates valid for all $a = (\varepsilon, \mu, \kappa)$ in $A := (0, 1] \times [0, 1] \times [0, 1]$. Our result improves earlier works [1, 56, 72] on allowing $\kappa \neq 0$. Indeed, when $\kappa = 0$, the penalization operator is skew-symmetric and hence the perturbation terms do not appear in the L^2 estimate, so that the classical proof for solutions to the unperturbed equations holds. In sharp contrast (as observed in [69]), when $\kappa \neq 0$ and the initial temperature variations are large, the problem is more involved. Several difficulties also specifically arise for the purpose of proving estimates that are independent of μ and κ . In this regard we prove some additional damping effects for $\operatorname{div} v$ and ∇p (similar additional damping effects have been previously used by Danchin [22, 25] to study the Cauchy problem in critical spaces).

Recall the notation:

$$\forall \sigma \in \mathbb{R}, \forall \varrho \geq 0, \quad \|f\|_{H^\sigma} := \|f\|_{H^{\sigma-1}} + \varrho \|f\|_{H^\sigma},$$

and introduce the energy:

$$\begin{aligned} \|(\tilde{p}, \tilde{v}, \tilde{\theta})\|_{a,T} &:= \sup_{t \in [0, T]} \{ \|(\tilde{p}, \tilde{v})(t)\|_{H_{\varepsilon\nu}^1} + \|\tilde{\theta}(t)\|_{H_{\nu}^1} \} \\ &+ \left(\int_0^T \kappa \|\nabla \tilde{\theta}\|_{H_{\nu}^1}^2 + \mu \|\nabla \tilde{v}\|_{H_{\varepsilon\nu}^1}^2 + \kappa \|\operatorname{div} \tilde{v}\|_{L^2}^2 + (\mu + \kappa) \|\nabla \tilde{p}\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

We estimate $\|(\tilde{p}, \tilde{v}, \tilde{\theta})\|_{a,T}$ in terms of the norm $\|(\tilde{p}, \tilde{v}, \tilde{\theta})\|_{a,0} := \|(\tilde{p}, \tilde{v})(0)\|_{H_{\varepsilon\nu}^1} + \|\tilde{\theta}(0)\|_{H_{\nu}^1}$ of the data.

THEOREM 4.3. *There exists a function $\mathcal{C}(\cdot)$ such that*

$$\|(\tilde{p}, \tilde{v}, \tilde{\theta})\|_{a,T} \leq C_0 e^{TC} (\|(\tilde{p}, \tilde{v})(0)\|_{H_{\varepsilon\nu}^1} + \|\tilde{\theta}(0)\|_{H_{\nu}^1}) + C \int_0^T \|(F_1, F_2)\|_{H_{\varepsilon\nu}^1} + \|F_3\|_{H_{\nu}^1} dt,$$

where

$$C_0 = \mathcal{C}(\|\psi(0)\|_{L^\infty(\mathbb{D})}), \quad C := \mathcal{C}\left(\sup_{t \in [0, T]} \|(\psi, \partial_t \psi + v \cdot \nabla \psi, \nabla \psi, \nabla v, \nu \nabla^2 \psi)\|_{L^\infty(\mathbb{D})}\right).$$

REMARK 4.4. The above mentioned damping effects correspond to the fact that:

→ we control $\operatorname{div} v$ in $L_{t,x}^2$ even if $\mu = 0$ provided that $\kappa > 0$.

→ we control ∇p in $L_{t,x}^2$ whenever $\mu + \kappa > 0$.

These damping effects are not smoothing effect since we assume that (p, v) belongs to $H_{\varepsilon\nu}^1(\mathbb{R}^d)$. Yet, as ε tend to zero, note that we control $\nu(\operatorname{div} v, \nabla p)$ in $L_{t,x}^2$ while we only assume that $\varepsilon(\operatorname{div} v, \nabla p)$ is uniformly bounded in L_x^2 initially.

To prove this estimate, first observe that the energy contains three main components:

$$\begin{aligned} \|(\tilde{p}, \tilde{v}, \tilde{\theta})\|_{a,T} &\approx \sup_{t \in [0, T]} \{ \|(\tilde{p}, \tilde{v}, \tilde{\theta})(t)\|_{L^2} \} + \left(\int_0^T \kappa \|\nabla \tilde{\theta}\|_{L^2}^2 + \mu \|\nabla \tilde{v}\|_{L^2}^2 dt \right)^{\frac{1}{2}} \\ &+ \nu \sup_{t \in [0, T]} \{ \|(\tilde{\theta}, \varepsilon \tilde{p}, \varepsilon \tilde{v})\|_{H^1} \} + \nu \left(\int_0^T \kappa \|\nabla \tilde{\theta}\|_{H^1}^2 + \mu \|\varepsilon \nabla \tilde{v}\|_{H^1}^2 dt \right)^{\frac{1}{2}} \\ &+ \left(\int_0^T \kappa \|\operatorname{div} \tilde{v}\|_{L^2}^2 + (\mu + \kappa) \|\nabla \tilde{p}\|_{L^2}^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

I briefly present the analysis in a simplified case. Consider the system

$$(16) \quad \begin{cases} \partial_t p + \frac{1}{\varepsilon} \operatorname{div} v - \frac{1}{\varepsilon} \Delta \theta = 0, \\ \partial_t v + \frac{1}{\varepsilon} \nabla p = 0, \\ \partial_t \theta + \operatorname{div} v - \beta \Delta \theta = 0. \end{cases}$$

Parallel to the assumption $\chi_1 < \chi_3$, suppose

$$(17) \quad \beta > 1.$$

LEMMA 4.5. *We have*

$$\|(p, v, \nabla\theta)(t)\|_{L^2}^2 + \int_0^t \|\operatorname{div} v - \beta\Delta\theta\|_{L^2}^2 d\tau \leq K_\beta \|(p, v, \nabla\theta)(0)\|_{L^2}^2.$$

PROOF. To symmetrize the large terms in ε^{-1} , we introduce $v_e := v - \nabla\theta$. This change of variables transforms (16) into

$$\begin{cases} \partial_t p + \varepsilon^{-1} \operatorname{div} v_e = 0, \\ \partial_t v_e + \varepsilon^{-1} \nabla p - \nabla \operatorname{div} v_e + (\beta - 1) \nabla \Delta\theta = 0, \\ \partial_t \theta + \operatorname{div} v_e - (\beta - 1) \Delta\theta = 0. \end{cases}$$

Multiply by $(p, v_e, -\eta\Delta\theta)$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|(p, v_e, \sqrt{\eta} \nabla\theta)\|_{L^2}^2 + \|\operatorname{div} v_e\|_{L^2}^2 - (\beta - 1 + \eta) \langle \Delta\theta, \operatorname{div} v_e \rangle + \eta(\beta - 1) \|\Delta\theta\|_{L^2}^2 = 0.$$

With $\eta := \beta - 1$ (> 0 by assumption), this yields

$$\frac{1}{2} \frac{d}{dt} \|(p, v_e, \sqrt{\beta - 1} \nabla\theta)\|_{L^2}^2 + \|\operatorname{div} v_e - (\beta - 1) \Delta\theta\|_{L^2}^2 = 0.$$

and hence the desired estimate

$$\|(p, v, \nabla\theta)(t)\|_{L^2}^2 + \int_0^t \|\operatorname{div} v - \beta\Delta\theta\|_{L^2}^2 d\tau \leq K_\beta \|(p, v, \nabla\theta)(0)\|_{L^2}^2.$$

□

We thus have proved an L^2 estimate independent of ε . To go beyond and obtain smoothing effect on $\operatorname{div} v$ it is sufficient to estimate $\Delta\theta$ independently.

LEMMA 4.6. *We have*

$$\|(\varepsilon p, \varepsilon v, \theta)(t)\|_{H^1}^2 + \int_0^t \|\nabla\theta\|_{H^1}^2 d\tau \leq K_\beta \|(\varepsilon p, \varepsilon v, \theta)(0)\|_{H^1}^2.$$

PROOF. The strategy is to incorporate the troublesome term $\operatorname{div} v$ [in the equation for θ] into a skew-symmetric operator. To do so introduce

$$\zeta := \varepsilon\beta p - \theta \quad \text{and} \quad v_\varepsilon := \varepsilon v.$$

We compute

$$\begin{cases} \partial_t \zeta + \frac{\beta - 1}{\varepsilon} \operatorname{div} v_\varepsilon = 0, \\ \partial_t v_\varepsilon + \frac{1}{\beta\varepsilon} \nabla \zeta + \frac{1}{\beta\varepsilon} \nabla \theta = 0, \\ \partial_t \theta + \frac{1}{\varepsilon} \operatorname{div} v_\varepsilon - \beta \Delta\theta = 0. \end{cases}$$

This yields

$$\frac{1}{2} \frac{d}{dt} \|(\sqrt{1/(\beta - 1)} \zeta, \sqrt{\beta} v_\varepsilon, \theta)\|_{L^2}^2 + \beta \|\nabla\theta\|_{L^2}^2 = 0.$$

Note that we implicitly used the assumption $\beta > 1$.

Integrate the previous inequality, to obtain

$$\|(\zeta, v_\varepsilon, \theta)(t)\|_{L^2}^2 + \int_0^t \|\nabla\theta\|_{L^2}^2 d\tau \leq K_\beta \|(\zeta, v_\varepsilon, \theta)(0)\|_{L^2}^2.$$

Since the coefficients are constants, we obtain

$$\|\nabla(\zeta, v_\varepsilon, \theta)(t)\|_{L^2}^2 + \int_0^t \|\nabla^2\theta\|_{L^2}^2 d\tau \leq K_\beta \|\nabla(\zeta, v_\varepsilon, \theta)(0)\|_{L^2}^2.$$

On applying the triangle inequality, one can replace ζ with εp in the previous estimates. \square

REMARK 4.7. How to prove Lemma 2 for the full linearized system (15)?

For the full system, set

$$\tilde{\zeta} := \varepsilon g_1(\phi)\chi_3(\phi)k(\phi)\tilde{p} - g_3(\phi)\chi_1(\phi)k(\phi)\tilde{\theta},$$

then $\mathcal{U} := (\tilde{\zeta}, \varepsilon\tilde{v}, \tilde{\theta})$ satisfies a mixed hyperbolic/parabolic system of the form

$$\mathcal{L}_1(v, \phi)\mathcal{U} - \mathcal{L}_2(\mu, \kappa, \phi)\mathcal{U} + \frac{1}{\varepsilon}\mathcal{S}(\phi)\mathcal{U} = F,$$

with

$$\mathcal{S} = \begin{pmatrix} 0 & \gamma_1 \operatorname{div} & 0 \\ \nabla(\gamma_1 \cdot) & 0 & \nabla(\gamma_2 \cdot) \\ 0 & \gamma_2 \operatorname{div} & 0 \end{pmatrix} \quad \text{with} \quad \gamma_1 := \frac{1}{g_1\chi_3k} \quad \text{and} \quad \gamma_2 := \frac{g_3\chi_1}{g_1\chi_3}.$$

This operator (with variable coefficients!) is skew-symmetric.

By combining the previous estimates, we get

$$(18) \quad \|(p, v, \theta)(t)\|_{L^2} + \|\nabla(\theta, \varepsilon p, \varepsilon v)(t)\|_{L^2} + \left(\int_0^t \|\operatorname{div} v\|_{L^2}^2 + \|\nabla\theta\|_{H^1}^2 d\tau \right)^{1/2} \leq K_\beta \|(p, v, \theta)(0)\|_{L^2} + K_\beta \|\nabla(\theta, \varepsilon p, \varepsilon v)(0)\|_{L^2}.$$

To conclude, it remains to estimate $\int_0^t \|\nabla p\|_{L^2}^2 d\tau$.

LEMMA 4.8. *There holds*

$$(19) \quad \int_0^t \|\nabla p\|_{L^2}^2 d\tau = \int_0^t \|\operatorname{div} v\|_{L^2}^2 - \langle \operatorname{div} v, \Delta\theta \rangle d\tau + \varepsilon [\langle v(\tau), \nabla p(\tau) \rangle]_{\tau=0}^{\tau=t}.$$

PROOF. Multiply the second equation in (16) by $\varepsilon\nabla p$ and integrate over the strip $[0, t] \times \mathbb{D}$, to obtain

$$\int_0^t \|\nabla p\|_{L^2}^2 d\tau = - \int_0^t \langle \partial_t v, \varepsilon\nabla p \rangle d\tau.$$

Integrating by parts both in space and time yields

$$\begin{aligned} \int_0^t \langle \varepsilon\partial_t v, \nabla p \rangle d\tau &= - \int_0^t \langle v, \varepsilon\partial_t \nabla p \rangle d\tau + \varepsilon [\langle v(\tau), \nabla p(\tau) \rangle]_{\tau=0}^{\tau=t} \\ &= \int_0^t \langle \operatorname{div} v, \varepsilon\partial_t p \rangle d\tau + \varepsilon [\langle v(\tau), \nabla p(\tau) \rangle]_{\tau=0}^{\tau=t}, \end{aligned}$$

which implies the desired identity.

This strategy has a long history and we refer the reader to Matsumura and Nishida [71] for references and an application to the global in time Cauchy problem for the compressible Navier-Stokes equation. \square

All the terms that appear in the right hand side of (19) have been estimated previously. As a consequence the estimate (18) holds true if we include $\int_0^t \|\nabla p\|_{L^2}^2 d\tau$ in its left hand side. By doing so, we obtain the exact analogue of the estimate given in Theorem 4.3 (for $\mu = 0$ and $\kappa = 1$).

4.1. Back to the uniform stability result. We are now in position to give Theorem 3.9 a refined form where the solutions satisfy the same estimates as the initial data do.

DEFINITION 4.9. Recall the notation: $\|u\|_{H_a^s} := \|u\|_{H^{\sigma-1}} + \varrho \|u\|_{H^\sigma}$.

Let $T \in [0, +\infty)$, $s \in \mathbb{R}$ and $a = (\varepsilon, \mu, \kappa) \in A$. Define

$$\begin{aligned} & \| (p, v, \theta) \|_{\mathcal{H}_a^s(T)} \\ & := \| (I - \Delta)^{s/2} (p, v, \theta) \|_{a, T} \\ & = \sup_{t \in [0, T]} \{ \| (p(t), v(t)) \|_{H_{\varepsilon\nu}^{s+1}} + \| \theta(t) \|_{H_\nu^{s+1}} \} \\ & \quad + \left(\int_0^T \mu \| \nabla v \|_{H_{\varepsilon\nu}^{s+1}}^2 + \kappa \| \nabla \theta \|_{H_\nu^{s+1}}^2 + \kappa \| \operatorname{div} v \|_{H^s}^2 + (\mu + \kappa) \| \nabla p \|_{H^s}^2 dt \right)^{1/2}, \end{aligned}$$

with $\nu := \sqrt{\mu + \kappa}$. Similarly, define

$$\| (p, v, \theta) \|_{\mathcal{H}_{a,0}^s} := \| (I - \Delta)^{s/2} (p, v, \theta) \|_{a,0} = \| (p, v) \|_{H_{\varepsilon\nu}^{s+1}} + \| \theta \|_{H_\nu^{s+1}}.$$

(The hybrid norm $\|\cdot\|_{H_{\varepsilon\nu}^{s+1}}$ was already used by Danchin in [23].)

THEOREM 4.10. *Let $d \neq 2$, for all integer $s > 1 + d/2$ and for all positive M_0 , there exists a positive T and a positive M such that for all $a \in A$ and all initial data in $B(\mathcal{H}_{a,0}^s; M_0)$, the Cauchy problem has a unique classical solution in $B(\mathcal{H}_a^s(T); M)$.*

One can also give refined statement for Theorem 3.12 and Theorem 3.14 (see [3]).

5. Nonlinear estimates

To prove the uniform stability results, the first task is to establish the local well posedness of the Cauchy problem for fixed $a = (\varepsilon, \mu, \kappa) \in A$. Secondly, one has to establish uniform estimates, which is very long and technical. We merely give the scheme of the proof, indicate the main tools and refer the reader to the original papers [2, 3] for details.

To establish the desired nonlinear estimates, the analysis is divided into four steps. This happens for two reasons. Firstly, on the technical side, most of the work concerns the separation of the estimates into high and low frequency components, where the division

occurs at frequencies of order of $1/\varepsilon$ (since the second-derivative terms with $O(1)$ coefficients and the first-derivative terms with $O(\varepsilon^{-1})$ coefficients balance there). Secondly, there is a division into terms whose evolution is estimated directly by eliminating large terms of size $O(\varepsilon^{-1})$, and terms whose size is estimated by means of Theorem 4.3 and the special structure of the equations.

We thus divide the estimates into:

- A. High frequencies: $(I - J_{\varepsilon\nu})(p, v, \theta)$;
- B. Low frequencies of the fast components: $J_{\varepsilon\nu}(\operatorname{div} v, \nabla p)$;
- C. Slow components: $\operatorname{curl}(e^\varphi v)$ (sor some appropriate weight φ) and the low frequency component of the temperature $J_{\varepsilon\nu}\theta$.

This scheme of estimates has two useful properties. Firstly, it avoids estimating the L^2 norm of p and v . Secondly, it allows us to overcome the factor $1/\varepsilon$ in front of the source term Q . Indeed, the linear estimate in Theorem 4.3 is applied only to high-frequencies and weighted time derivatives $(\varepsilon\partial_t)^m$. Hence, the fact that the source term is assumed to be neither of high frequency nor have rapid time oscillations allows us to recover the lost factor of ε in the nonlinear estimates.

Notable technical aspects include the use of new tools to localize in the frequency space as well as a proof of a variable coefficients Friedrichs' estimate for $\operatorname{div}/\operatorname{curl}$ system.

5.1. Local existence for fixed $(\varepsilon, \mu, \kappa)$. Case $\nabla P = O(\varepsilon)$, $\nabla T = O(\varepsilon)$, $\operatorname{div} v = O(1)$. The following result contains an analysis of the easy case where initially $\theta_0 = O(\varepsilon)$. This regime is interesting for the incompressible limit (see [9]).

PROPOSITION 5.1. *Let $d \geq 1$ and $\mathbb{R} \ni s > 1 + d/2$. For all $M_0 > 0$, there exists $T > 0$ and $M > 0$ such that for all $a \in A$ and all initial data $(p_0, v_0, \theta_0) \in H^s(\mathbb{R}^d)$ satisfying*

$$(20) \quad \|(p_0, v_0)\|_{H^s} + \varepsilon^{-1} \|\theta_0\|_{H^s} \leq M_0,$$

the Cauchy problem for (7) has a unique classical solution (p, v, θ) in $C^0([0, T]; H^s(\mathbb{R}^d))$ such that

$$(21) \quad \sup_{t \in [0, T]} \|(p(t), v(t))\|_{H^s} + \varepsilon^{-1} \|\theta(t)\|_{H^s} \leq M.$$

PROOF. The proof is based on the change of unknown $(p, v, \theta) \mapsto (\varrho(\theta, \varepsilon p), v, \theta)$ where ϱ is as given by Assumption 3.4. By setting $\rho = \varrho(\theta, \varepsilon p)$ it is found that (p, v, θ) satisfies (7) if and only if

$$\begin{cases} \chi_3(\partial_t \rho + v \cdot \nabla \rho) + (\chi_3 - \chi_1) \operatorname{div} v = 0, \\ g_2(\partial_t v + v \cdot \nabla v) + \varepsilon^{-2} \gamma_1 \nabla \theta + \varepsilon^{-2} \gamma_2 \nabla \rho - \mu B_2 v = 0, \\ g_3(\partial_t \theta + v \cdot \nabla \theta) + \operatorname{div} v - \kappa \chi_3 \operatorname{div}(k \nabla \theta) = 0, \end{cases}$$

where $\gamma_1 = (\chi_1 g_3)/(\chi_3 g_1)$ and $\gamma_2 = 1/g_1$. Notice that Assumption (H2) implies that the coefficients g_i , γ_i , χ_3 and $\chi_3 - \chi_1$ are positive.

We note that the unknown $u := (\tilde{\rho}, v, \tilde{\theta})$, where $\tilde{\rho} := \varepsilon^{-1}\rho(\theta, \varepsilon p)$ and $\tilde{\theta} := \varepsilon^{-1}\theta$, solves a coupled hyperbolic/parabolic system of the form

$$(22) \quad A_0(\varepsilon u)\partial_t u + \sum_{1 \leq j \leq d} A_j(u, \varepsilon u)\partial_j u + \varepsilon^{-1}S(\varepsilon u, \partial_x)u = B(\mu, \kappa, u, \partial_x)u.$$

Thanks to assumption (20), the fact that the change of unknowns is singular in ε causes no difficulty: the H^s -norm of the initial data $u(0) = (\tilde{\rho}(0), v(0), \tilde{\theta}(0))$ is estimated by a constant which depends only on M_0 . Since A_0 depends only on the unknown through εu , one easily verify that the proof of Theorem 2.1 applies. By that proof, we conclude that the solutions of (22) exist and are uniformly bounded for a time T independent of ε .

Once this is granted, it remains to verify that the solutions (p, v, θ) of System (7) exist and are uniformly bounded in the sense of (21). Because $(\vartheta, \wp) \mapsto (\vartheta, \varrho(\vartheta, \wp))$ is a C^∞ diffeomorphism with $\varrho(0, 0) = 0$, one can write $\varepsilon p = P(\theta, \varrho(\theta, \varepsilon p)) = P(\theta, \rho)$, for some C^∞ function P vanishing at the origin. Therefore

$$\|p\|_{H^s} = \varepsilon^{-1} \|P(\theta, \rho)\|_{H^s} \leq \varepsilon^{-1} C_P(\|(\theta, \rho)\|_{L^\infty}) \|(\theta, \rho)\|_{H^s} \leq C(\|(\tilde{\theta}, \tilde{\rho})\|_{H^s}),$$

so that $\|(p, v)\|_{H^s} + \varepsilon^{-1} \|\theta\|_{H^s} \leq C(\|u\|_{H^s})$. □

5.2. Div/Curl estimates. Recall the estimate

$$(23) \quad \|\nabla v\|_{H^s} \leq \|\operatorname{div} v\|_{H^s} + \|\operatorname{curl} v\|_{H^s},$$

which is immediate using Fourier transform. We prove a variant where $\operatorname{curl} v$ is replaced by $\operatorname{curl}(\rho v)$ where ρ is a positive weight.

Since the commutator $[\operatorname{curl}, \rho]$ is a zero-order operator, it follows from the product rule in Sobolev spaces that, if $s > d/2$, then (see [72])

$$\|\nabla v\|_{H^s(\mathbb{R}^d)} \leq C \|\operatorname{div} v\|_{H^s(\mathbb{R}^d)} + C \|\operatorname{curl}(e^\varphi v)\|_{H^s(\mathbb{R}^d)} + C \|v\|_{L^2(\mathbb{R}^d)},$$

for some constant C depending only on $\|\varphi\|_{H^{s+1}}$.

If $d \neq 2$, then we can prove that this estimate remains valid without control of the low frequencies.

PROPOSITION 5.2. *Let $d \geq 3$ and $\mathbb{N} \ni s > d/2$. There exists a function C such that,*

$$(24) \quad \|\nabla v\|_{H^s(\mathbb{R}^d)} \leq C(\|\varphi\|_{H^{s+1}(\mathbb{R}^d)}) \left(\|\operatorname{div} v\|_{H^s(\mathbb{R}^d)} + \|\operatorname{curl}(e^\varphi v)\|_{H^s(\mathbb{R}^d)} \right).$$

This estimate is obvious if $d = 1$. The fact that Theorem 3.14 precludes the case $d = 2$ is a consequence of the fact that I do not know if (24) holds for $d = 2$.

5.3. Localization in the frequency space. We separate the estimates into high and low frequency components, where the division occurs at frequencies of order of the inverse of ε (since the second-derivative terms with $O(1)$ coefficients and the first-derivative terms with $O(\varepsilon^{-1})$ coefficients balance there). We now develop the analysis needed to localize in the frequency space.

We shall consider two families of smoothing operators. Firstly, consider the family

$$\{ J_h := j(hD_x) \mid h \in (0, 1] \},$$

where

$$0 \leq j \leq 1, \quad j(\xi) = 1 \text{ for } |\xi| \leq 1, \quad j(\xi) = 0 \text{ for } |\xi| \geq 2, \quad j(\xi) = j(-\xi).$$

The Friedrichs mollifiers J_h are interesting because they are essentially projection operators $J_h = J_h J_{ch}$ (for all $0 \leq c \leq 2^{-1}$). Alternatively, it is also interesting to use a family of invertible smoothing operators. A good candidate is the family

$$\{ \Lambda_h^m := (I - h^2 \Delta)^{m/2} \mid h \in (0, 1] \}.$$

The fact that these operators are invertible allows us to derive a product estimate, which, in words, says that the smoothing effect of the operators Λ_h^{-m} is distributive.

PROPOSITION 5.3. *Let $\sigma_0 > d/2$, $(\sigma_1, \sigma_2) \in \mathbb{R}_+^2$ and $(m_1, m_2) \in \mathbb{R}_+^2$ be such that*

$$\sigma_1 + \sigma_2 + m_1 + m_2 \leq 2\sigma_0.$$

There exists $K = K(d, \sigma, \sigma_i, m_i)$, such that for all $h \in (0, 1]$ and $u_i \in H^{\sigma_0 - \sigma_i - m_i}$,

$$\| \Lambda_h^{-m_1 - m_2} (u_1 u_2) \|_{H^{\sigma_0 - \sigma_1 - \sigma_2}} \leq K \| \Lambda_h^{-m_1} u_1 \|_{H^{\sigma_0 - \sigma_1}} \| \Lambda_h^{-m_2} u_2 \|_{H^{\sigma_0 - \sigma_2}}.$$

The main result we use to localize in the frequency space is the following, which complements the usual Friedrichs' Lemma. The thing of interest is that we give a precise rate of convergence which does not require much on the high wave number part of u .

PROPOSITION 5.4. *Let $s > d/2 + 1$ and $m \in [0, 1]$. For all $\sigma \in (-s + m, s - 1]$, there exists a constant K , such that for all $h \in (0, 1]$, all $f \in H^s(\mathbb{R}^d)$ and all $u \in H^{-s}(\mathbb{R}^d)$,*

$$\| J_h(fu) - f J_h u \|_{H^{\sigma - m + 1}} \leq h^m K \| f \|_{H^s} \| \Lambda_h^{-(s+\sigma)} u \|_{H^\sigma}.$$

PROOF. Let us prove a slightly weaker estimate

$$\| [J_h, f] \Lambda_h^{s+\sigma-1} \|_{H^\sigma \rightarrow H^\sigma} \lesssim h \| f \|_{H^s}.$$

Split

$$[J_h, f] = [J_h, f] J_{h/5} + [J_h, f] (I - J_{h/5}).$$

To estimate the first term, we use the following commutator estimate: let $s > d/2 + 1$, $m \in [0, +\infty)$, $\sigma \in [-s + m, s - 1]$ and consider a Fourier multiplier P of order m , then

$$\| P(fu) - f P u \|_{H^{\sigma - m + 1}} \leq K \| f \|_{H^s} \| u \|_{H^\sigma}.$$

This implies that

$$\begin{aligned} \|[J_h, f] \Lambda_h^{s+\sigma-1} J_{h/5}\|_{H^\sigma \rightarrow H^\sigma} &\lesssim \|[J_h, f]\|_{H^\sigma \rightarrow H^\sigma} = h \|[h^{-1}(I - J_h), f]\|_{H^\sigma \rightarrow H^\sigma} \\ &\lesssim h \|f\|_{H^s}, \end{aligned}$$

since the symbols of the operators $h^{-1}(I - J_h)$, viewed as Fourier multipliers of order 1, are uniformly bounded.

For the second term: set $U := \Lambda_h^{s+\sigma-1}(I - J_{h/5})u$. Use $J_h(I - J_{h/5}) = 0$ to obtain

$$[J_h, f]U = J_h(fU) = J_h(((I - J_h)f)U),$$

and conclude via the easily proved estimates:

$$\begin{aligned} \|J_h\{((I - J_h)f)U\}\|_{H^\sigma} &\lesssim h^{-(s+\sigma-1)} \|((I - J_h)f)U\|_{H^{-(s-1)}}. \\ \|((I - J_h)f)U\|_{H^{-(s-1)}} &\lesssim \|(I - J_h)f\|_{H^{s-1}} \|U\|_{H^{-s+1}} \lesssim h \|f\|_{H^s} \|U\|_{H^{-s+1}}, \\ \|U\|_{H^{-(s-1)}} &\lesssim \|h^{s+\sigma-1} |D_x|^{s+\sigma-1} u\|_{H^{-(s-1)}} \lesssim h^{s+\sigma-1} \|u\|_{H^\sigma}. \end{aligned}$$

□

5.4. High frequency regime. To obtain estimates in Sobolev norms, the classical approach consists in differentiating the equations so as to apply the energy estimates proved for the linearized system (see Theorem 4.3). This certainly fails here since it reveals terms in ε^{-1} . Yet, one can follow this strategy in the high frequency regime where the parabolic behavior prevails.

Set $U = (p, v, \theta)$. We estimate the size of $(I - J_h)\Lambda^s U$ by means of Theorem 4.3, where recall $J_h := j(hD_x)$ where j is a bump function, and $\Lambda^s := (I - \Delta)^{s/2}$.

The parameter h has the form $\varepsilon\nu$ where ν measures the smoothing effect. We take $\nu = \sqrt{\mu + \kappa}$ since the main smoothing effect concerns the penalized terms:

$$\left(\int_0^T (\mu + \kappa) \|\operatorname{div} v\|_{H^s}^2 + (\mu + \kappa) \|\nabla p\|_{H^s}^2 + \kappa^2 \|\nabla \theta\|_{H^{s+1}}^2 dt \right)^{1/2} \leq \|(p, v, \theta)\|_{\mathcal{H}_a^s(T)},$$

where $\|(p, v, \theta)\|_{\mathcal{H}_a^s(T)}$ is as defined in Definition 4.9.

To apply Theorem 4.3, we have to estimate the commutators:

$$\begin{aligned} f_{1,\text{HF}}^{a,\nu}(U) &:= [g_1(\phi), \mathcal{Q}_{\varepsilon\nu}] D_t p + g_1(\phi) [v, \mathcal{Q}_{\varepsilon\nu}] \cdot \nabla p + \frac{\kappa}{\varepsilon} [B_1(\phi), \mathcal{Q}_{\varepsilon\nu}] \theta, \\ f_{2,\text{HF}}^{a,\nu}(U) &:= [g_2(\phi), \mathcal{Q}_{\varepsilon\nu}] D_t v + g_2(\phi) [v, \mathcal{Q}_{\varepsilon\nu}] \cdot \nabla v + \mu [B_2(\phi), \mathcal{Q}_{\varepsilon\nu}] v, \end{aligned}$$

where $\phi := (\theta, \varepsilon p)$, $D_t := \partial_t + v \cdot \nabla$ and

$$\mathcal{Q}_{\varepsilon\nu} := (I - J_{\varepsilon\nu})\Lambda^s.$$

We use the following estimates: there exists a constant $K = K(d, s)$ such that

$$\begin{aligned} \|[f, P]u\|_{H_{\varepsilon\nu}^1} &\leq \varepsilon\nu K \|\nabla f\|_{L^\infty} \|u\|_{H^s} + \varepsilon\nu K \|\nabla f\|_{H^s} \|u\|_{L^\infty}, \\ \|[f, P]u\|_{H_\nu^1} &\lesssim \nu K \|\nabla f\|_{L^\infty} \|u\|_{H^s} + \nu K \|\nabla f\|_{H^s} \|u\|_{L^\infty}. \end{aligned}$$

The fact that the right-hand side only involves ∇f follows from the most simple of all the sharp commutator estimates established in [62]: for all $s > 1 + d/2$ and all Fourier multiplier $A(D_x)$ of order s , there exists a constant K such that, for all $f \in H^s(\mathbb{R}^d)$ and all $u \in H^s(\mathbb{R}^d)$,

$$\|[f, A(D_x)]u\|_{L^2} \leq K \|\nabla f\|_{L^\infty} \|u\|_{H^{s-1}} + K \|\nabla f\|_{H^{s-1}} \|u\|_{L^\infty}.$$

With this preliminary established, we obtain tame estimates (which are linear with respect to the biggest norms):

$$\begin{aligned} \|f_{1,\text{HF}}^{a,\nu}(U)\|_{H_{\varepsilon\nu}^1} &\leq C(R) \{1 + \|\varepsilon D_t p\|_{H_{\varepsilon\nu}^s} + \kappa \|\theta\|_{H_{\varepsilon\nu}^{s+2}}\}, \\ \|f_{2,\text{HF}}^{a,\nu}(U)\|_{H_{\varepsilon\nu}^1} &\leq C(R) \{1 + \|\varepsilon D_t v\|_{H_{\varepsilon\nu}^s} + \mu \|\varepsilon v\|_{H_{\varepsilon\nu}^{s+2}}\}, \end{aligned}$$

with $R := \|(\theta, \varepsilon p, \varepsilon v)\|_{H_{\varepsilon\nu}^{s+1}}$.

Also, to apply Theorem 4.3, we have to estimate the commutator of $\mathcal{Q}_{\varepsilon\nu}$ and the equation for θ in $H_{\varepsilon\nu}^1$. Yet, this commutator is not uniformly bounded. To overcome this difficulty, we first note that one has uniform estimates for the $H_{\varepsilon\nu}^1$ norm of:

$$f_{3,\text{HF}}^{a,\nu}(U) := [g_3(\phi), \mathcal{Q}_{\varepsilon\nu}] D_t \theta + g_3(\phi) \{v \cdot \nabla \theta; \mathcal{Q}_{\varepsilon\nu}\} + \kappa [B_3(\phi), \mathcal{Q}_{\varepsilon\nu}] \theta,$$

where

$$\{v; P\} \cdot \nabla \theta := v \cdot \nabla P \theta + (Pv) \cdot \nabla \theta - P(v \cdot \nabla \theta).$$

Indeed, the fact that we linearized the product $v \cdot \nabla \theta$ allows us to obtain

$$\|f_{3,\text{HF}}^{a,\nu}(U)\|_{H_{\varepsilon\nu}^1} \leq C(R) \{1 + \|D_t \theta\|_{H_{\varepsilon\nu}^s} + \kappa \|\theta\|_{H_{\varepsilon\nu}^{s+2}}\}.$$

To conclude, we have to verify that Theorem 4.3 remains valid if the system (15) is replaced with

$$\begin{cases} g_1(\phi)(\partial_t \tilde{p} + v \cdot \nabla \tilde{p}) + \frac{1}{\varepsilon} \operatorname{div} \tilde{v} - \frac{\kappa}{\varepsilon} \operatorname{div}(k_1(\phi) \nabla \tilde{\theta}) = F_1, \\ g_2(\phi)(\partial_t \tilde{v} + v \cdot \nabla \tilde{v}) + \frac{1}{\varepsilon} \nabla \tilde{p} - \mu B_2(\phi, \partial_x) \tilde{v} = F_2, \\ g_3(\phi)(\partial_t \tilde{\theta} + v \cdot \nabla \tilde{\theta}) + G(\phi, \nabla \phi) \cdot \tilde{v} + \operatorname{div} \tilde{v} - \kappa \chi_3(\phi) \operatorname{div}(k(\phi) \nabla \tilde{\theta}) = F_3, \end{cases}$$

where G is smooth in its arguments with values in \mathbb{R}^d . The fact that Theorem 4.3 remains valid for this system is not obvious; we use the additional damping for the pressure.

5.5. Estimates in the low frequency region. It remains to estimate $J_{\varepsilon\nu} U$ where $\nu := \sqrt{\mu + \kappa}$. This is the most delicate part. Note that, for the Euler equations ($\mu = 0 = \kappa$), one has $J_{\varepsilon\nu} = I$, and hence it is clear that it is the main part!

As alluded to previously, the nonlinear energy estimates cannot be obtained from the L^2 estimates by an elementary argument using differentiation of the equations with respect to spatial derivatives. For such problems a general strategy can be used. First we apply to the equations some operators based on $(\varepsilon \partial_t)$. Next, one uses the special structure of the equations to estimate the spatial derivatives.

To clarify matters, consider the example of a 1D scalar equation:

$$a_0(t, x)\partial_t u + a(u)\partial_x u + \varepsilon^{-1}\partial_x u = 0.$$

The link between $(\varepsilon\partial_t)$ and ∂_x is clear:

$$(25) \quad \forall k \leq s \in]d/2 + 1, +\infty[, \quad \|(\varepsilon\partial_t)^k u\|_{H^{s-k}} \leq C(\|u\|_{H^s}).$$

The L^2 norm of $(\varepsilon\partial_t)^s u$ is uniformly bounded since $[a_0, (\varepsilon\partial_t)^s]\partial_t u = O(1)$. Since $(\varepsilon\partial_t)^s u = a_0^{-1}(\varepsilon\partial_t)^{s-1}\partial_x u$ plus a term of order of $O(\varepsilon)$, for ε small enough, this yields an L^2 estimate for $(\varepsilon\partial_t)^{s-1}\partial_x u = \partial_x(\varepsilon\partial_t)^{s-1}u$. Hence $(\varepsilon\partial_t)^{s-1}u$ is estimated in H^1 . By induction one estimates $(\varepsilon\partial_t)^k u$ in H^{s-k} , and hence u in H^s .

For the full Navier-Stokes equations, we begin by estimating $(\varepsilon\partial_t)^s J_{\varepsilon\nu} U$. We next use the structure of the equation to estimate $((\varepsilon\partial_t)^{s-1} \operatorname{div} J_{\varepsilon\nu} v, (\varepsilon\partial_t)^{s-1} \nabla J_{\varepsilon\nu} p)$.

As stated, this strategy works only for perfect gases [2]. For the case of greatest physical interest ($d = 3$), it applies (with only minor changes) to the general gases as well. Yet, if $d \leq 2$, because of the lack of L^2 estimates for the velocity, we cannot use the time derivatives. For this problem, we use an idea introduced by Secchi in [84]. Namely, we replace ∂_t by the convective derivative $D_t = \partial_t + v \cdot \nabla$. To simplify the presentation, we avoid such technical points below, and work with time derivatives instead of convective derivatives.

Note that this basic strategy has many roots, at least for hyperbolic problems (see, e.g., [1, 48, 83, 84]). For our purposes, the key point is that the hyperbolic behavior prevails in the low frequency regime. Yet, in sharp contrast with the Euler equations ($\mu = \kappa = 0$), the form of the equations (13) shows that the time derivative and the spatial derivatives do not have the same weight. In particular, our analysis requires some preparation.

Introduce the function Ψ defined by

$$\Psi := (\psi, \partial_t \psi, \nabla \psi) \quad \text{with} \quad \psi := (\theta, \varepsilon p, \varepsilon v).$$

One can verify that Ψ satisfies an equation of the form

$$\varepsilon\partial_t \Psi = \sum_{1 \leq j \leq d} B_{a,j}(\Psi)\partial_j \Psi + \varepsilon(\mu + \kappa) \sum_{1 \leq j, k \leq d} \partial_j (B_{a,jk}(\Psi)\partial_k \Psi).$$

for some $B_{a,\cdot}$ uniformly bounded in C^∞ .

We want to introduce an operator based on $(\varepsilon\partial_t)$ which has the weight of a spatial derivative. The previous result suggests introducing:

$$Z_{\varepsilon,\nu}^\ell := \Lambda_{\varepsilon\nu}^{-\ell} (\varepsilon\partial_t)^\ell.$$

The operators $Z_{\varepsilon,\nu}^1$ do have the weight of a spatial derivative. Indeed, one can prove that, for $m \in \mathbb{N}$, $Z_{\varepsilon,\nu}^m \Psi$ satisfies the same estimates as $\Lambda^m F(\Psi)$ does (where F is a given

function):

$$\begin{aligned} \sum_{\ell=0}^s \|Z_{\varepsilon,\nu}^\ell \Psi\|_{H^{s-\ell-1}} &\leq C(\|\Psi\|_{H^{s-1}}), \\ \sum_{\ell=0}^s \|Z_{\varepsilon,\nu}^\ell \Psi\|_{H_\nu^{s-\ell}} &\leq C(\|\Psi\|_{H^{s-1}}) \|\Psi\|_{H_\nu^s}. \end{aligned}$$

The proof of these estimates uses the fact that one can distribute the smoothing effect according to (see Proposition 5.3):

$$\|\Lambda_h^{-m_1-m_2}(u_1 u_2)\|_{H^{\sigma-\sigma_1-\sigma_2}} \lesssim \|\Lambda_h^{-m_1} u_1\|_{H^{\sigma-\sigma_1}} \|\Lambda_h^{-m_2} u_2\|_{H^{\sigma-\sigma_2}}.$$

To localize in the low frequency region, we next have to prove commutator estimates with gain of a factor ε . We establish some estimates which allows us to commute $J_{\varepsilon\nu}(\varepsilon\partial_t)^m$ with the equations. In this direction, the main result is that there exists a constant K such that, for all $\varepsilon > 0$ and all $\nu \geq 0$, all $T > 0$, all $\mathbb{N} \ni m \leq s$, and all smooth functions defined on $[0, T]$,

$$\begin{aligned} &\| [f, J_{\varepsilon\nu}(\varepsilon\partial_t)^m] u \|_{H_{\varepsilon\nu}^{s-m+1}} \\ &\lesssim \varepsilon\nu \|f\|_{H^s} \|Z_{\varepsilon,\nu}^m u\|_{H^{s-m}} + \varepsilon \sum_{\ell=0}^{m-1} \|Z_{\varepsilon,\nu}^\ell \partial_t f\|_{H^{s-1-\ell}} \|Z_{\varepsilon,\nu}^\ell u\|_{H^{s-\ell-1}}. \end{aligned}$$

Here, we used Proposition 5.4: if $s > d/2+1$ and $m \in [0, 1]$, then for all $\sigma \in (-s+m, s-1]$,

$$\|J_h(fu) - fJ_h u\|_{H^{\sigma-m+1}} \lesssim h^m \|f\|_{H^s} \|\Lambda_h^{-(s+\sigma)} u\|_{H^\sigma}.$$

With these preliminaries established, we can proceed to give an estimate for $J_{\varepsilon\nu}(\varepsilon\partial_t)^s U$ by means of Theorem 4.3. The fast components $J_{\varepsilon\nu}(\operatorname{div} v, \nabla p)$ are estimated next by using the following induction argument: Set

$$\|u\|_{\mathcal{K}_\nu^\sigma(T)} := \|u\|_{L^\infty(0,T;H^{\sigma-1})} + \nu \|u\|_{L^2(0,T;H^\sigma)},$$

and let $\tilde{U} := (\tilde{p}, \tilde{v}, \tilde{\theta})$ solve

$$\begin{cases} g_1(\phi)(\partial_t \tilde{p} + v \cdot \nabla \tilde{p}) + \varepsilon^{-1} \operatorname{div} \tilde{v} - \kappa \varepsilon^{-1} \chi_1(\phi) \operatorname{div}(k(\theta) \nabla \tilde{\theta}) = f_1, \\ g_2(\phi)(\partial_t \tilde{v} + v \cdot \nabla \tilde{v}) + \varepsilon^{-1} \nabla \tilde{p} - \mu B_2(\phi, \partial_x) \tilde{v} = f_2, \\ g_3(\phi)(\partial_t \tilde{\theta} + v \cdot \nabla \tilde{\theta}) + \operatorname{div} \tilde{v} - \kappa \chi_3(\phi) \operatorname{div}(k(\theta) \nabla \tilde{\theta}) = f_3. \end{cases}$$

If the support of the Fourier transform of \tilde{U} is included in the ball $\{|\xi| \leq 2/\varepsilon\nu\}$, then

$$\begin{aligned} &\|\nabla \tilde{p}\|_{\mathcal{K}_\nu^\sigma(T)} + \|\operatorname{div} \tilde{v}\|_{\mathcal{K}_\nu^\sigma(T)} \\ &\leq \tilde{C} \|(\varepsilon\partial_t) \tilde{p}\|_{\mathcal{K}_\nu^\sigma(T)} + \tilde{C} \|(\varepsilon\partial_t) \operatorname{div} \tilde{v}\|_{\mathcal{K}_\nu^{\sigma-1}(T)} \\ &\quad + \tilde{C} \|\nabla \tilde{p}\|_{L_T^\infty(L^2)} + \tilde{C} \|\tilde{\theta}(0)\|_{H_\nu^{\sigma+1}} + \varepsilon C \|\mu \tilde{v}\|_{\mathcal{K}_\nu^{\sigma+1}(T)} \\ &\quad + \varepsilon C \|(f_1, f_2)\|_{\mathcal{K}_\nu^\sigma(T)} + \nu \tilde{C} \|f_3\|_{L_T^2(H^\sigma)}, \end{aligned}$$

with $\tilde{C} := C_0 e^{(\sqrt{T}+\varepsilon)C}$, where C depends only on the norm $\|(p, v, \theta)\|_{\mathcal{H}_a^s(T)}$.

5.6. Slow components. To complete the proof, it remains to estimate $\text{curl } v$ and θ ; which is not straightforward.

A basic idea is to apply the curl operator to the equation for the velocity so as to cancel the large term $\varepsilon^{-1}\nabla p$. However, by doing so, we obtain

$$g_2\partial_t\omega + g_2v \cdot \nabla\omega - \mu B(\partial_x)\omega = F - \nabla g_2 \times \partial_tv,$$

with $F = O(1)$. This does not yield the desired result since $\nabla g_2 \times \partial_tv$ is not uniformly bounded. In particular, we cannot estimate $\text{curl } v$ directly. The idea in [72] is to estimate $\omega_2 := \text{curl}(g_2v)$ which satisfies an equation of the form:

$$\partial_t\omega_2 + v \cdot \nabla\omega_2 - \mu B(\phi, \partial_x)\omega_2 = F_2 - (\partial_t g_2)v,$$

where F_2 and $(\partial_t g_2)v$ are uniformly bounded in ε (since $g_2 = g_2(\theta, \varepsilon p)$ and $\partial_t\theta, (\varepsilon\partial_t)p$ are uniformly bounded in ε). Hence, $\text{curl}(g_2v)$ is a slow component: $\partial_t \text{curl}(g_2v) = O(1)$. However, this does not suffice to prove the desired estimates in Sobolev spaces, uniformly in μ , for the Navier-Stokes equations. Indeed, if we compute the term F_2 , it is found that F_2 contains second order derivatives of the velocity which causes a loss of one derivatives in the estimates. As shown by Métivier and Schochet, the good slow component is $\text{curl}(e^\varphi v)$ for a weight φ which is a function of the entropy σ alone ($\sigma = S(\theta, \varepsilon p)$ with S as given in the assumptions, see (8)).

One interesting feature of the entropy evolution equation is that it is coupled to the momentum equation only through the convective term. Indeed, for the purpose of proving estimates independent of κ , we cannot see the term $\text{div } v$ in the equation for θ as a source term. Starting from (see (8))

$$(26) \quad dS(\vartheta, \varphi) = g_3(\vartheta, \varphi) d\vartheta - g_1(\vartheta, \varphi) d\varphi,$$

we find:

$$(27) \quad \partial_t\sigma + v \cdot \nabla\sigma = \kappa G_1(\phi, \nabla\theta) + \kappa G_2(\phi)\Delta\theta$$

for some C^∞ functions G_1 and G_2 , with $G_1(0, 0) = 0$.

This yields uniform estimates for the $L_t^\infty H_x^s$ norm of σ by the usual H^s -estimates for hyperbolic equations. An interesting point is that the low frequency component $\Lambda_{\varepsilon\nu}^{-1}\sigma$ satisfies parabolic-type estimates, where recall $\Lambda_{\varepsilon\nu}^{-1} = (I - \varepsilon\nu\Delta)^{-1/2}$.

Let us form a parabolic evolution equation for $\dot{\sigma} := \nu\Lambda_{\varepsilon\nu}^{-1}\nabla\sigma$. Writing the identity (26) in the form $d\vartheta = c_1(\vartheta, \varphi) dS + c_2(\vartheta, \varphi) d\varphi$ with $c_1 := 1/g_3^{-1}$ and $c_2 := g_1/g_3$, yields

$$(28) \quad \nabla\theta = c_1(\phi)\nabla\sigma + \varepsilon c_2(\phi)\nabla p.$$

Inserting this expression for $\nabla\theta$ into the equation (27), yields

$$(29) \quad \partial_t\sigma + v \cdot \nabla\sigma - \kappa k(\phi)\Delta\sigma = \mathbf{G}_3 + \mathbf{G}_4 := \kappa G_3(\phi, \nabla\phi) + \kappa\varepsilon G_4(\phi, \nabla\phi)\Delta p,$$

where $k := (\chi_3 - \chi_1)kc_1$ is a smooth positive function; $G_3, G_4 \in C^\infty$ and $G_3(0, 0) = 0$. Consequently, we find that

$$(30) \quad \partial_t \dot{\sigma} + v \cdot \nabla \dot{\sigma} - \kappa k(\phi) \Delta \dot{\sigma} = \mathcal{G},$$

where the source term is given by

$$\begin{aligned} \mathcal{G} &:= -\nu \Lambda_{\varepsilon\nu}^{-1} (\nabla v \cdot \nabla \sigma) + \nu [v, \Lambda_{\varepsilon\nu}^{-1}] \cdot \nabla \nabla \sigma \\ &\quad + \kappa \Lambda_{\varepsilon\nu}^{-1} (\nabla k(\phi) \Delta \sigma) + \kappa \nu [k(\phi), \Lambda_{\varepsilon\nu}^{-1}] \Delta \nabla \sigma \\ &\quad + \nu \kappa \Lambda_{\varepsilon\nu}^{-1} \nabla G_3 + \nu \kappa \Lambda_{\varepsilon\nu}^{-1} \nabla G_4. \end{aligned}$$

In light of the standard parabolic estimates we need only to prove that the source term \mathcal{G} can be split as $\mathcal{G}_1 + \sqrt{\kappa} \mathcal{G}_2$ with \mathcal{G}_1 estimated in $L^1(0, T; H^s)$, and \mathcal{G}_2 estimated in $L^2(0, T; H^{s-1})$. The main point is to show why we can see the term with three derivatives acting on p as a source term. Write:

$$\begin{aligned} &\|\nu \sqrt{\kappa} \Lambda_{\varepsilon\nu}^{-1} \nabla G_4\|_{L^2(0, T; H^{s-1})} \\ &:= \|\nu \sqrt{\kappa} \Lambda_{\varepsilon\nu}^{-1} \nabla \{\varepsilon G_4(\phi, \nabla \phi) \Delta p\}\|_{L^2(0, T; H^{s-1})} \\ &\lesssim \|\sqrt{\kappa} G_4(\phi, \nabla \phi) \Delta p\|_{L^2(0, T; H^{s-1})} \quad (\varepsilon \nu \nabla \Lambda_{\varepsilon\nu}^{-1} \lesssim I) \\ &\lesssim (1 + \|\tilde{G}_4(\phi, \nabla \phi)\|_{L^\infty(0, T; H^{s-1})}) \|\sqrt{\kappa} p\|_{L^2(0, T; H^{s+1})} \quad (\text{straightforward}). \end{aligned}$$

We next use that we have already proved that p is well estimated, since we have already estimated its high and low frequency components.

6. Decay to zero of the local energy

We conclude this part by explaining the proof of Theorem 3.21. The strategy in [72] consists in introducing some semi-classical measures and prove that they vanish, which implies the strong compactness in time. Together with the strong compactness in space, this gives the desired result. The semi-classical measures will be defined as defect measures of wave-packets transforms. The definitions rely upon the works of Gérard [43], Lions and Paul [68] and Tartar [88]. In [43] microlocal defect measures are defined for bounded sequences in $L^2_{\text{loc}}(\mathcal{V}, H)$, where \mathcal{V} is an open set of \mathbb{R}^d and H is a separable Hilbert space. It leads to positive measure on the cosphere bundle of \mathcal{V} by use of Garding's inequality. And in [68] semi-classical measures are defined by means of Wigner transform. It leads to positive measures via the Husimi's transform. Garding's inequality has been known to be related to the wave-packets transforms since the work of Córdoba and Fefferman [21]. Moreover, in [68], the authors point out the connection between Husimi's transform and the wave-packets transform. Detailed accounts of the subject can be found in [17, 19].

Let us start by defining semi-classical defect measures for scalar valued-functions. For all bounded sequence in $L^2(\mathbb{R}^d)$, one can extract a subsequence which converges weakly in $L^2(\mathbb{R}^d)$. Defect measures are measures which help us understand what can prevent the strong convergence.

THEOREM 6.1 (from Gérard [43]). *Let (u_n) be a bounded sequence in $L^2_{\text{loc}}(\mathbb{R}^d)$. Assume that (u_n) converges weakly to 0. Then there exist a subsequence $(u_{\varphi(n)})$ and a positive Radon measure μ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ such that, for all classical pseudo-differential operator A of order $m \leq 0$ on \mathbb{R}^d , with principal symbol $\sigma(A)$ of order 0, and for all bump function χ such that $\sigma = \chi\sigma$,*

$$(A(\chi u_{\varphi(n)}), \chi u_{\varphi(n)}) \rightarrow \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \sigma(A)(x, \xi) d\mu(x, \xi).$$

The measure μ is called a micro-local defect measure. For further uses, an important remark is that this could be extended to functions with values in Hilbert spaces.

The proof of this result is strongly based on the Garding inequality, which implies that

$$\limsup_{n \rightarrow +\infty} \|A(\chi u_n)\|_{L^2} \leq C(\chi) \sup |\sigma(A)|.$$

Indeed, to define μ , consider a compact $K \subset \mathbb{R}^d$ and a subset $\{a_k \mid k \in \mathbb{N}\} \subset C_0^\infty(\mathbb{R}^{2n})$ which is dense in $\{u \in C_0(\mathbb{R}^{2n}) : \text{supp } u \subset K\}$. Let, for $k \in \mathbb{N}$, A_k be a pseudo-differential operator with principal symbol $\sigma_0(A_k) = a_k$. By a standard diagonal argument, construct a sequence ε_n converging to 0 such that, for all $k \in \mathbb{N}$, $(A_k(\chi u_{\varphi(n)}), u_{\varphi(n)})$ converges towards a limit denoted by $\Phi(a_k) \in \mathbb{C}$. By continuity and density, and a further diagonal argument, the map Φ can be extended to a bounded linear functional on $C_0(\mathbb{R}^{2n})$. The Riesz representation theorem then implies the existence of the measure μ . In addition, the Garding inequality also implies that μ is a (real) nonnegative measure.

An alternative to the Garding inequality is to use a positive quantization rule (see [63]) to define pseudo-differentials operators. We will not explicitly introduce the Wick quantization rule. Instead, we introduce the wave-packets transform associated to the scale ε^{-1} . Since we will work with vector valued function, we consider functions that depend on the space variable (viewed as an extra parameter). Define

$$W^\varepsilon v(t, \tau, x) = c\varepsilon^{-3/4} \int_{\mathbb{R}} e^{i(t-s)\tau - (t-s)^2/\varepsilon} v(s, x) ds.$$

Then, with $c = (2\pi^3)^{-1/4}$, W^ε extends as an isometry from $L^2(\mathbb{R} \times \mathbb{R}^d)$ to $L^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d)$.

To introduce the wave-packets transform of u^ε , we have to carefully extend the functions to $t \in \mathbb{R}$. Hereafter, a subscript zero indicates compact support. Let $\chi_\varepsilon \in C_0^\infty((0, T))$ be a family of functions such that $\chi_\varepsilon(t) = 1$ for $t \in [\varepsilon^{1/2}, T - \varepsilon^{1/2}]$ and $\|\varepsilon \partial_t \chi_\varepsilon\|_{L^\infty} \leq 2\varepsilon^{1/2}$. We set $\tilde{u}^\varepsilon := \chi_\varepsilon u^\varepsilon$. Next, we choose extensions $\tilde{a}^\varepsilon, \tilde{b}^\varepsilon$ of $a^\varepsilon, b^\varepsilon$, supported in $t \in [-1, T + 1]$, uniformly bounded in $C_0^1(\mathbb{R}; H^\sigma(\mathbb{R}^d))$, and converging to \tilde{a}, \tilde{b} in $C_0^1(\mathbb{R}; H'_{\text{loc}}(\mathbb{R}^d))$. Note that \tilde{u}^ε satisfies

$$\varepsilon^2 \partial_t (\tilde{a}^\varepsilon \partial_t \tilde{u}^\varepsilon) - \text{div}(\tilde{b}^\varepsilon \nabla \tilde{u}^\varepsilon) = \varepsilon \tilde{f}^\varepsilon,$$

where (\tilde{f}^ε) is a bounded family in $C_0^0(\mathbb{R}; H^1(\mathbb{R}^d))$.

The wave packets transform is a nice tool to measure in the phase plane how much of a function oscillates at frequencies $O(\varepsilon^{-1})$. In particular, one has

LEMMA 6.2. Let $U^\varepsilon = W^\varepsilon \tilde{u}^\varepsilon$. As ε tends to 0,

$$(31) \quad F^\varepsilon := \tau^2 \tilde{a}^\varepsilon U^\varepsilon + \operatorname{div}(\tilde{b}^\varepsilon \nabla U^\varepsilon) \rightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^2; H^1(\mathbb{R}^d)).$$

PROOF. This follows from the fact that the wave-packets operators conjugate the action of pseudo-differential operators, approximately, to multiplication by symbols (see [21, 63]). As an example, let $a \in C_0^1(\mathbb{R}; H^\sigma(\mathbb{R}^d))$ with $\sigma > 1 + d/2$. We have

$$aW^\varepsilon v - W^\varepsilon(av) = c\varepsilon^{-3/4} \int e^{i(t-s)\tau - (t-s)^2/\varepsilon} (a(t, x) - a(s, x)) v(s, x) ds.$$

Using the identity $\int e^{i(u-s)\tau/\varepsilon} d\tau = 2\pi\varepsilon\delta_0(u-s)$, we find

$$(32) \quad \|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^d)}^2 \\ = c_2\varepsilon^{-1/2} \iiint e^{-2(t-s)^2/\varepsilon} |a(t, x) - a(s, x)|^2 |v(s, x)|^2 ds dt dx.$$

Since $\sigma - 1 > d/2$, the Sobolev embedding implies that

$$\|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^d)}^2 \\ \leq c_2\varepsilon^{-1/2} \iint e^{-2(t-s)^2/\varepsilon} \|a(t) - a(s)\|_{H^{\sigma-1}(\mathbb{R}^d)}^2 \|v(s)\|_{L^2(\mathbb{R}^d)}^2 dt ds.$$

From the previous bound and $[W^\varepsilon, \partial_x] = 0$, we easily get

$$\|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2; H^1(\mathbb{R}^d))} \leq K\sqrt{\varepsilon} \|a\|_{C_0^1(\mathbb{R}; H^\sigma(\mathbb{R}^d))} \|v\|_{L^2(\mathbb{R}; H^1(\mathbb{R}^d))}.$$

Similarly, one has

$$(33) \quad \|W^\varepsilon(\varepsilon\partial_t v) - i\tau W^\varepsilon v\|_{L^2(\mathbb{R}^2; H^1(\mathbb{R}^d))} \leq K\sqrt{\varepsilon} \|v\|_{L^2(\mathbb{R}; H^1(\mathbb{R}^d))}.$$

□

We are now in position to define the semi-classical measures. For technical reasons, it appears convenient to state the result for the family

$$\Theta^\varepsilon := U^\varepsilon - \Delta U^\varepsilon.$$

We denote by \mathcal{L} , \mathcal{K} , and \mathcal{L}^1 the spaces of bounded, compact, and trace class operators in $L^2(\Omega)$, respectively, and we denote by \mathcal{K}_+ and \mathcal{L}_+^1 the subspaces of non-negative self-adjoint operators in \mathcal{K} and \mathcal{L}^1 , respectively (we refer to [53] for details and definitions). If $A \in \mathcal{L}^1$, we denote by $\operatorname{tr}(A)$ the trace of A . Recall that the dual space of $(\mathcal{K}, \|\cdot\|_{\mathcal{L}})$ is $(\mathcal{L}^1, \|\cdot\|_{\mathcal{L}^1})$ with the duality bracket $\operatorname{tr}(AB)$. If $A \in C_0(\mathbb{R}^2, \mathcal{K})$, then A acts on $\Theta \in L^2(\mathbb{R}^2 \times \mathbb{R}^d)$ following $(A\Theta)(t, \tau, x) = (A\Theta(t, \tau, \cdot))(x)$.

The semi-classical defect measures are then defined as defect measures for the wave packets transforms.

PROPOSITION 6.3. Let $\Theta^\varepsilon = U^\varepsilon - \Delta U^\varepsilon$. There are a subsequence Θ^{ε_n} , a finite nonnegative Borel measure μ on \mathbb{R}^2 , and $M \in L^1(\mathbb{R}^2, \mathcal{L}_+^1, \mu)$, such that for all $A \in C_0(\mathbb{R}^2, \mathcal{K})$,

$$(34) \quad \int_{\mathbb{R}^2} (A(t, \tau) \Theta^{\varepsilon_n}(t, \tau), \Theta^{\varepsilon_n}(t, \tau)) dt d\tau \rightarrow \int \text{tr}(A(t, \tau) M(t, \tau)) \mu(dt, d\tau),$$

and

$$(35) \quad \left(\tilde{a}(t) \tau^2 + \text{div}(\tilde{b}(t) \nabla \cdot) \right) (I - \Delta)^{-1} M(t, \tau) = 0 \quad \mu\text{-a.e.}$$

A difficult result proved in [72] is that the characteristic variety is trivial: that is, the kernel of the operator $(\tilde{a}(t) \tau^2 + \text{div}(\tilde{b}(t) \nabla \cdot))(I - \Delta)^{-1}$ in $L^2(\mathbb{R}^d)$ is reduced to $\{0\}$. Then, (34) and (35) imply that, for all $\varphi \in C_0(\mathbb{R}^2)$ and all $K \in \mathcal{K}_+$,

$$(36) \quad \int_{\mathbb{R}^2} \varphi(t, \tau) (K \Theta^{\varepsilon_n}(t, \tau), K \Theta^{\varepsilon_n}(t, \tau)) dt d\tau \xrightarrow{n \rightarrow \infty} 0.$$

We want to show that this convergence holds for $\varphi(t, \tau) = 1$. The idea is that, on the one hand, \tilde{u}^ε is compactly supported in time; so is Θ^ε in the sense given below by (37). And, on the other hand, \tilde{u}^ε oscillates in time at most at frequencies $O(\varepsilon^{-1})$.

First, let $\zeta \in C_0^\infty((-1, T+1))$ be such that $\zeta(t) = 1$ for $t \in [0, T]$. In view of (32), one infers that

$$\|\zeta \Theta^\varepsilon - W^\varepsilon((1 - \Delta) \zeta \tilde{u}^\varepsilon)\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^d)} \leq C \sqrt{\varepsilon} \|\partial_t \zeta\|_{L^\infty(\mathbb{R})} \|(1 - \Delta) \tilde{u}^\varepsilon\|_{L^2(\mathbb{R} \times \mathbb{R}^d)}.$$

Furthermore, from the definition of \tilde{u}^ε we have $\zeta \tilde{u}^\varepsilon = \tilde{u}^\varepsilon$. Therefore, the previous inequality implies that

$$(37) \quad \int_{\mathbb{R}^2} (1 - \zeta(t))^2 \|\Theta^\varepsilon(t, \tau)\|_{L^2(\mathbb{R}^d)}^2 dt d\tau \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Note that $(\varepsilon \partial_t \tilde{u}^\varepsilon)_\varepsilon$ is a bounded family in $C_0(\mathbb{R}, H^{s-1}(\mathbb{R}^d))$, it follows from (33) that $(\tau \Theta^\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^2 \times \mathbb{R}^d)$. Thus with (36) and (37) we conclude that for all $K \in \mathcal{K}_+$,

$$(38) \quad \int_{\mathbb{R}^2} \|K \Theta^{\varepsilon_n}(t, \tau)\|_{L^2(\mathbb{R}^d)}^2 dt d\tau \xrightarrow{n \rightarrow \infty} 0.$$

Recall that, by the definition of $\Theta^\varepsilon = (1 - \Delta) W^\varepsilon \tilde{u}^\varepsilon$. Since W^ε is an isometry from $L^2(\mathbb{R} \times \mathbb{R}^d)$ to $L^2(\mathbb{R}^2 \times \mathbb{R}^d)$, and since W^ε commutes with $K(1 - \Delta)$, the convergence (38) implies that

$$(39) \quad \forall K \in \mathcal{K}_+, \quad \|K(1 - \Delta) \tilde{u}^{\varepsilon_n}\|_{L^2(\mathbb{R} \times \mathbb{R}^d)} \xrightarrow{n \rightarrow \infty} 0.$$

Given that \tilde{u}^ε is bounded in $L^2(\mathbb{R}; H^s(\mathbb{R}^d))$, the convergence (39) implies the convergence of $\tilde{u}^{\varepsilon_n}$ to 0 in $L^2(\mathbb{R}, H_{\text{loc}}^{s'}(\mathbb{R}^d))$ for all $s' < s$. Since the limit is zero the convergence holds for the given family \tilde{u}^ε . It concludes the proof of Theorem 3.21 and hence concludes the proof of Proposition 3.20.

About exterior domains. As alluded to above, the analysis extends to exterior domains (see [1]). I conclude this part by explaining where the boundary condition enters. Consider the equations

$$\begin{aligned} a^\varepsilon(\varepsilon\partial_t)p^\varepsilon + \operatorname{div} v^\varepsilon &= \varepsilon f_1^\varepsilon, \\ r^\varepsilon(\varepsilon\partial_t)v^\varepsilon + \nabla p^\varepsilon &= \varepsilon f_2^\varepsilon, \end{aligned}$$

supplemented with the solid wall boundary condition $v^\varepsilon \cdot \nu = 0$ on $\partial\Omega$ where ν is the normal to the boundary. With $U^\varepsilon = (\Psi^\varepsilon, m^\varepsilon) = (W^\varepsilon \tilde{q}^\varepsilon, W^\varepsilon \tilde{v}^\varepsilon) \in L^2(\mathbb{R}^2; H^s(\Omega))$, one has

$$\begin{aligned} i\tau a^\varepsilon \Psi^\varepsilon + \operatorname{div}(m^\varepsilon) &= F_1^\varepsilon, \\ i\tau r^\varepsilon m^\varepsilon + \nabla \Psi^\varepsilon &= F_2^\varepsilon, \end{aligned}$$

where F^ε converge to 0 in $L^2(\mathbb{R}^2; H^1(\Omega))$. Using $v_{|\mathbb{R} \times \partial\Omega}^\varepsilon \cdot \nu = 0$, we obtain that $m^\varepsilon \cdot \nu = 0$ on the boundary $\mathbb{R} \times \mathbb{R} \times \partial\Omega$. Taking the inner product of the second equation with ν , we infer that

$$\partial_\nu \Psi^\varepsilon := \nabla \Psi^\varepsilon \cdot \nu = F_2^\varepsilon \cdot \nu \quad \text{on } \mathbb{R} \times \mathbb{R} \times \partial\Omega,$$

which is meaningful since $F^\varepsilon \in L^2(\mathbb{R}^2; H^1(\Omega))$. Conversely, we can recover Ψ^ε from $\Theta^\varepsilon := (I - \Delta)\Psi^\varepsilon$ and F_2^ε . One has

$$\Psi^\varepsilon = (I - \Delta_N)^{-1} \Theta^\varepsilon + N(F_2^\varepsilon \cdot \nu),$$

where we used the following definitions: given $g \in L^2(\Omega)$, $\varphi \in H^{1/2}(\partial\Omega)$,

$$\begin{aligned} f &= (I - \Delta_N)^{-1} g \quad \text{if and only if} \quad (I - \Delta)f = g \text{ in } \Omega \text{ and } \partial_\nu f = 0 \text{ on } \partial\Omega, \\ f &= N(\varphi) \quad \text{if and only if} \quad (I - \Delta)f = 0 \text{ in } \Omega \text{ and } \partial_\nu f = \varphi \text{ on } \partial\Omega. \end{aligned}$$

The important point is that $N(F_2^\varepsilon \cdot \nu)$ converges to 0 in $L^2(\mathbb{R}^2 \times \Omega)$.

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