LOSS OF REGULARITY FOR SUPERCRITICAL NONLINEAR SCHRÖDINGER EQUATIONS

THOMAS ALAZARD AND RÉMI CARLES

Abstract. We consider the nonlinear Schrödinger equation with defocusing, smooth, nonlinearity. Below the critical Sobolev regularity, it is known that the Cauchy problem is ill-posed. We show that this is even worse, namely that there is a loss of regularity, in the spirit of the result due to G. Lebeau in the case of the wave equation. As a consequence, the Cauchy problem for energy-supercritical equations is not well-posed in the sense of Hadamard. We reduce the problem to a supercritical WKB analysis. For super-cubic, smooth nonlinearity, this analysis is new, and relies on the introduction of a modulated energy functional *à la* Brenier.

1. Introduction

We consider the following defocusing nonlinear Schrödinger equation on $\mathbb{R}^n$:

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = |\psi|^{2\sigma}\psi \quad ; \quad \psi_{|t=0} = \varphi, \tag{1.1}$$

where $\sigma \geqslant 1$ is an integer, so that the nonlinearity is smooth. It is well-known that the critical Sobolev regularity corresponds to the value given by scaling arguments,

$$s_c := \frac{n}{2} - \frac{1}{\sigma}\cdot$$

Throughout this paper, we assume $s_c > 0$. If $\varphi \in H^s(\mathbb{R}^n)$ with $s \geqslant s_c$, then the Cauchy problem (1.1) is locally well-posed in $H^s(\mathbb{R}^n)$ [12]. On the other hand, if $s < s_c$, then the Cauchy problem (1.1) is ill-posed [14] (see also the appendices in [9, 10]). The worst phenomenon proved in [14] is the norm inflation. For $0 < s < s_c$, one can find a sequence $(\psi^h)_{0<h\leqslant 1}$ of solutions to (1.1) and $0 < t^h \to 0$, such that $\varphi^h \in \mathcal{S}(\mathbb{R}^n)$ and

$$\|\varphi^h\|_{H^s} \underset{h\to 0}{\longrightarrow} 0 \quad ; \quad \|\psi^h(t^h)\|_{H^s} \underset{h\to 0}{\longrightarrow} +\infty.$$

In this paper, we prove the stronger result:

Theorem 1.1. *Let $\sigma \geqslant 1$. Assume that $s_c = n/2 - 1/\sigma > 0$, and let $0 < s < s_c$. There exists a family $(\varphi^h)_{0<h\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^n)$ with*

$$\|\varphi^h\|_{H^s(\mathbb{R}^n)} \to 0 \text{ as } h \to 0,$$

a solution ψ^h to (1.1) and $0 < t^h \to 0$, such that:

$$\|\psi^h(t^h)\|_{H^k(\mathbb{R}^n)} \to +\infty \text{ as } h \to 0\,,\ \forall k > \frac{s}{1+\sigma(s_c - s)}\cdot$$

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Note that this result is not bound to the case $x \in \mathbb{R}^n$: for instance, it remains valid on a compact manifold, see §6.

In the case $\sigma = 1$ and $n \geqslant 3$, this result was established in [10]. It followed from a supercritical WKB analysis for the cubic nonlinear Schrödinger equation, which had been justified by E. Grenier [21]. For $\sigma \geqslant 2$, adapting the results of [21] seems to be a much more delicate issue, and a rigorous analysis in this setting for $n \leqslant 3$ has been given very recently [1]. An important remark, in the proof of Theorem 1.1 that we present here, is that it is not necessary to justify WKB analysis as precisely as in [21], or [1], to obtain this loss of regularity. From this point of view, our proof is very simple. On the other hand, it can be considered as highly nonlinear: it relies on a *quasilinear* analysis, as opposed to the *semilinear* analysis in [14] (see also Remark 5.4 at the end of this paper). In the opinion of the authors, the proof of Theorem 1.1 is at least as interesting as the result itself.

Remark 1.2. Shortly after this work was completed, an alternative proof was given by L. Thomann [32], based on the justification of WKB analysis in an analytic setting. This approach allows to consider focusing nonlinearities (the nonlinearity is treated as a semilinear perturbation in spaces based on analytic regularity), unlike the method followed in the present paper. On the other hand, the virial identity shows that for supercritical focusing nonlinearities, blow-up can happen for arbitrary small data in H^s ($s < s_c$) and arbitrary small times (see e.g. [30, Exercise 3.63]).

This result is to be compared with the main result in [25], which we recall with notations adapted to make the comparison with the Schrödinger case easier. For supercritical wave equations

$$\left(\partial_t^2 - \Delta\right) u + u^{2\sigma+1} = 0,$$

G. Lebeau shows that one can find a *fixed* initial datum in H^s, and a sequence of times $0 < t^h \to 0$, such that the H^k norms of the solution are unbounded along the sequence t^h, for $k \in]I(s), s]$. The expression for $I(s)$ is related to the critical Sobolev exponent

$$s_{\rm sob} = \frac{n}{2}\frac{\sigma}{\sigma+1},$$

which corresponds to the embedding $H^{s_{\rm sob}}(\mathbb{R}^n) \subset L^{2\sigma+2}(\mathbb{R}^n)$. In [25], we find:

$$I(s) = 1 \text{ if } 1 < s \leqslant s_{\rm sob} \quad ; \quad I(s) = \frac{s}{1+\sigma(s_c - s)} \text{ if } s_{\rm sob} \leqslant s < s_c. \tag{1.2}$$

Note that we have

$$\frac{s_{\rm sob}}{1+\sigma(s_c - s_{\rm sob})} = 1. \tag{1.3}$$

The approach in [25] consists in using an *anisotropic* scaling, as opposed to the isotropic scaling used in [24, 14]. Compare Theorem 1.1 with the approach of [25]. Recall that (1.1) has two important (formally) conserved quantities: mass and energy,

$$\begin{aligned} &M(t) = \int_{\mathbb{R}^n} |\psi(t,x)|^2 dx \equiv M(0),\\ &E(\psi(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\psi(t,x)|^2 dx + \frac{1}{\sigma+1}\int_{\mathbb{R}^n} |\psi(t,x)|^{2\sigma+2} dx \equiv E(\varphi). \end{aligned} \tag{1.4}$$

In view of (1.3), we obtain, for H^1-supercritical nonlinearities:

Corollary 1.3. *Let $n \geqslant 3$ and $\sigma > \frac{2}{n-2}$. There exists $(\varphi^h)_{0<h\leqslant 1}$ in $\mathcal{S}(\mathbb{R}^n)$ with*

$$\|\varphi^h\|_{H^1} + \|\varphi^h\|_{L^{2\sigma+2}} \to 0 \text{ as } h \to 0,$$

a solution ψ^h to (1.1) and $0 < t^h \to 0$, such that:

$$\|\psi^h(t^h)\|_{H^k(\mathbb{R}^n)} \to +\infty \text{ as } h \to 0\,,\ \forall k > 1.$$

We thus get the analogue of the result of G. Lebeau when $I(s) = 1$, with the drawback that we consider a *sequence* of initial data only. The information that we don't have for Schrödinger equations, and which is available for wave equations, is the finite speed of propagation, that is used in [25] to construct a fixed initial datum; see also the discussion in §6. On the other hand, our approach involves an *isotropic* scaling, which is recalled and generalized in Section 2. Moreover, our range for k is broader when $1 < s < s_{\rm sob}$, and also, we allow the range $0 < s \leqslant 1$, for which no analogous result is available for the wave equation. Note that unlike in [25], we perform no linearization in our analysis (the properties of the analogous linearized operator are not as interesting in the case of Schrödinger equations): despite the fact that for fixed ε, (1.1) is a semilinear equation, we consider a quasilinear system to prove our main result.

Before going further into details, let us focus on the notion of solution to (1.1). In view of Theorem 1.1, we assume that the initial data are in the Schwartz class: $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then (1.1) has a local smooth solution: for all $s > n/2$, there exists $T_s > 0$ such that (1.1) has a unique solution $\psi \in C([-T_s, T_s]; H^s)$. If $n \leqslant 2$, then (1.1) has a global smooth solution, $\psi \in C(\mathbb{R}; H^s)$ for all $s \geqslant 0$, and the identities (1.4) hold for all time. The same is true when $n = 3$ and $\sigma = 1$. These results are established in [19]. In the H^1-critical three dimensional case ($n = 3$ and $\sigma = 2$), it is proved in [15] that solutions with H^s regularity ($s > 1$) remain in H^s for all time; the same is true in the four dimensional case ($n = 4$ and $\sigma = 1$), from [29]. On the other hand, if the nonlinearity is H^1-supercritical ($\sigma > \frac{2}{n-2}$), then it is not known in general whether the solution remains smooth for all time or not. In Theorem 1.1, for $\sigma = 1$, the solution ψ^h is a smooth solution, that remains smooth up to time t^h, thanks to a result due to E. Grenier [21]. *A priori*, the solution we consider in Theorem 1.1 is a weak solution:

Definition 1.4. Let $\varphi \in H^1 \cap L^{2\sigma+2}(\mathbb{R}^n)$. A (global) weak solution to (1.1) is a function $\psi \in C(\mathbb{R}; L^2) \cap L^\infty(\mathbb{R}; H^1 \cap L^{2\sigma+2}) \cap C_w(\mathbb{R}; H^1 \cap L^{2\sigma+2})$ solving (1.1) in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^n)$, and such that:

- $\|\psi(t)\|_{L^2} = \|\varphi\|_{L^2}$, $\forall t \in \mathbb{R}$.
- $E(\psi(t)) \leqslant E(\varphi)$, $\forall t \in \mathbb{R}$.

From [20], for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, (1.1) has a global weak solution. The proof in [20] is based on Galerkin method. We use a different construction, as in [25], which is described in Section 4. Note that when the nonlinearity is H^1-subcritical, then the weak solution is unique, and coincides with the strong solution. Recall also that the existence of blowing-up solutions in the H^1-supercritical case is open so far. On the other hand, if the nonlinearity is focusing, many results are available (see [30] for an overview of the subject, and similar problems for other dispersive equations).

Note that in view of Definition 1.4, Corollary 1.3 is sharp.

As in [10], the idea for the proof of Theorem 1.1 consists in reducing the analysis to a supercritical WKB analysis, for an equation of the form:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2\sigma}u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0(x). \tag{1.5}$$

The parameter ε goes to zero. The above equation is supercritical as far as geometrical optics is concerned: if one plugs an approximate solution of the form

$$v^\varepsilon \sim e^{i\phi/\varepsilon}\left(\mathtt{a}_0 + \varepsilon \mathtt{a}_1 + \varepsilon^2 \mathtt{a}_2 + \dots\right)$$

into the equation, then closing the systems of equations for $\phi, \mathtt{a}_0, \mathtt{a}_1, \dots$ is a very delicate issue (see e.g. [11]). In the case $\sigma = 1$, this issue was resolved by E. Grenier [21]. However, the argument in [21] relies very strongly on the fact that the nonlinearity is defocusing, and cubic at the origin. In [1], we have proposed an approach that justifies WKB analysis in Sobolev spaces for (1.5) for any $\sigma \geqslant 2$, in space dimension $n \leqslant 3$ (higher dimensions could also be considered with the same proof, up to considering sufficiently large values of σ). In [18] and [32], WKB analysis was justified in spaces based on analytic regularity, in a periodical setting and for analytic manifolds, respectively. As noticed in [1], analytic regularity is necessary to justify WKB analysis with a focusing nonlinearity. The analytic regularity essentially allows to view the nonlinearity as a semilinear perturbation, and to construct an approximate solution that solves (1.5) up to a source term of order $e^{-\delta/\varepsilon}$ for some $\delta > 0$. Yet, such a justification is not needed to prove Theorem 1.1; see §2. In this paper, we use a functional that yields sufficiently many informations to infer Theorem 1.1. This functional may be viewed as a generalization of the one used in [26] in the cubic case, following an idea introduced by Y. Brenier [6]. The general form for this modulated energy functional was announced in [26]. However, we will see that making the corresponding analysis rigorous is not straightforward, since we consider weak solutions.

The main idea of the proof of Theorem 1.1 consists in noticing that for an ε-independent initial datum a_0 in (1.5), the solution u^ε becomes ε-oscillatory for times of order $\mathcal{O}(1)$ as $\varepsilon \to 0$. This phenomenon is typical of supercritical régimes, as far as geometrical optics is concerned (see also [13]). This crucial step is stated in Theorem 2.1, which in turn is proved thanks to the above mentioned modulated energy functional.

We end this introduction with a remark concerning the study of the Cauchy problem for (1.1). From [14], it is known that the Cauchy problem is not well posed in $H^s(\mathbb{R}^n)$ for $0 < s < s_c$. Yet, one can try to solve the Cauchy problem by searching the solutions in a larger space. Denote $H^\infty = \cap_{s>0} H^s(\mathbb{R}^n)$. Recall the notion of well-posedness in the sense of Hadamard:

Definition 1.5. Let $s \geqslant k \geqslant 0$. The Cauchy problem for (1.1) is well posed from $H^s(\mathbb{R}^n)$ to $H^k(\mathbb{R}^n)$ if, for all bounded subset $B \subset H^s(\mathbb{R}^n)$, there exist $T > 0$ and a Banach space $X_T \hookrightarrow C([0,T]; H^k(\mathbb{R}^n))$ such that:
(1) For all $\varphi \in B \cap H^\infty$, (1.1) has a unique solution $\psi \in C([0,T]; H^\infty)$.
(2) The mapping $\varphi \in (H^\infty, \|\cdot\|_B) \mapsto \psi \in X_T$ is continuous.

The following result is a direct consequence of our analysis (see Remark 2.2).

Corollary 1.6. *Let $n \geqslant 1$ and $\sigma \geqslant 1$ be such that $\sigma > 2/n$. The Cauchy problem for* (1.1) *is not well posed from $H^s(\mathbb{R}^n)$ to $H^k(\mathbb{R}^n)$ for all (s,k) such that*

$$0 < s < s_c = \frac{n}{2} - \frac{1}{\sigma}, \qquad k > \frac{s}{1+\sigma(s_c - s)}\cdot$$

2. Reduction of the problem

Let $0 < s < s_c$ and $a_0 \in \mathcal{S}(\mathbb{R}^n)$. For a sequence h aimed at going to zero, consider the family of initial data

$$\varphi^h(x) = h^{s-\frac{n}{2}} a_0\left(\frac{x}{h}\right). \tag{2.1}$$

Let $\varepsilon = h^{\sigma(s_c - s)}$. By assumption, ε and h go simultaneously to zero. Define the function u^ε by the relation:

$$u^\varepsilon(t,x) = h^{\frac{n}{2}-s}\psi^h\left(h^2\varepsilon t, hx\right). \tag{2.2}$$

Then (1.1) is equivalent to (1.5). Note that we have the relation:

$$\|\psi^h(t)\|_{\dot{H}^m} = h^{s-m}\left\| u^\varepsilon\left(\frac{t}{h^2\varepsilon}\right)\right\|_{\dot{H}^m}.$$

Our aim is to show that for some $\tau > 0$ independent of ε,

$$\liminf_{\varepsilon\to 0} \varepsilon^k \left\| u^\varepsilon\left(\tau\right)\right\|_{\dot{H}^k} > 0, \quad \forall k \geqslant 0. \tag{2.3}$$

Back to ψ, this will yield $t^h = \tau h^2 \varepsilon$ and

$$\|\psi^h(t^h)\|_{\dot{H}^k} \gtrsim h^{s-k}\varepsilon^{-k} = h^{s-k(1+\sigma(s_c-s))}.$$

To complete the above reduction, note that in view of Theorem 1.1, we only have to prove (2.3) for $k \in]0,1]$. Indeed, for $k > 1$, there exists $C_k > 0$ such that

$$\|f\|_{\dot{H}^1} \leqslant C_k \|f\|_{L^2}^{1-1/k}\|f\|_{\dot{H}^k}^{1/k}, \quad \forall f \in H^k(\mathbb{R}^n).$$

This inequality is straightforward thanks to Fourier analysis. Note also that thanks to the conservation of mass for u^ε, we have:

$$\|u^\varepsilon(t)\|_{\dot{H}^1} \leqslant C_k \|a_0\|_{L^2}^{1-1/k}\|u^\varepsilon(t)\|_{\dot{H}^k}^{1/k}.$$

Up to replacing a_0 with $|\log h|^{-1}a_0$ (for instance), the analysis of this section shows that Theorem 1.1 follows from:

Theorem 2.1. *Let $n \geqslant 1$, $a_0 \in \mathcal{S}(\mathbb{R}^n)$ be non-trivial, and $\sigma \geqslant 1$. There exists a solution u^ε to* (1.5) *and $\tau > 0$ such that for all $k \in]0,1]$,*

$$\liminf_{\varepsilon\to 0} \left\| |\varepsilon D_x|^k u^\varepsilon\left(\tau\right)\right\|_{L^2} > 0, \quad \textit{where } D_x = -i\nabla.$$

Remark 2.2. As we will see, the previous conclusion holds for all family of smooth solutions u^ε defined on a time interval independent of ε. In particular, Corollary 1.6 also follows from this analysis. To see this, suppose, by contradiction, that the Cauchy problem is well posed from $H^s(\mathbb{R}^n)$ to $H^k(\mathbb{R}^n)$. Since the family of initial data given by (2.1) is bounded in $H^s(\mathbb{R}^n)$, the first point in Definition 1.5 implies that the solutions ψ^h are defined for a time interval $[0,T]$ independent of h. As a result, the function u^ε, as given by (2.2), is defined for $t \in [0, T/(\varepsilon h^2)]$ with value in $H^\infty(\mathbb{R}^n)$, and hence on the fixed time interval $[0,T]$. Then, Theorem 4.1 implies that there exists $\tau > 0$ such that $\liminf \||\varepsilon D_x|^k u^\varepsilon(\tau)\|_{L^2} > 0$. Back to ψ^h this yields the existence of a sequence τ^h such that $\|\psi^h(\tau^h)\|_{H^k}$ tends to $+\infty$, which contradicts the continuity given by the second point of the definition.

Remark 2.3. If we could prove Theorem 2.1 for $k = 1$ only, then back to ψ^h, this would yield Theorem 1.1 for $I(s) < k \leqslant s$, where $I(s)$ is given by (1.2), like in [25].

Consider the case $k = 1$, and recall that the conservation of energy for u^ε reads, as long as u^ε is a strong solution of (1.5):

$$E^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{R}^n} |\varepsilon\nabla u^\varepsilon(t,x)|^2 dx + \frac{1}{\sigma+1}\int_{\mathbb{R}^n} |u^\varepsilon(t,x)|^{2\sigma+2}dx \equiv E^\varepsilon(0).$$

At time $t = 0$, the first term (kinetic energy) is of order $\mathcal{O}(\varepsilon^2)$, while the second (potential energy) is dominating, of order $\mathcal{O}(1)$. Therefore, the game consists in showing that there exists $\tau > 0$, time at which the kinetic energy is of the order of the total (initial) energy as $\varepsilon \to 0$.

Some important features of the proof of this result can be revealed by analyzing the linear case with variable coefficients:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = V(x)u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0(x).$$

Introduce the operator $H^\varepsilon := -(\varepsilon^2/2)\Delta + V(x)$, so that $u^\varepsilon(t) = e^{-itH^\varepsilon/\varepsilon}a_0$. Now, let $\operatorname{Op}_\varepsilon(q)$ be a semiclassical pseudo-differential operator with symbol $q(x,\xi) \in S^1_{1,0}$. Since $e^{itH^\varepsilon/\varepsilon}$ is unitary, by means of Egorov's Theorem (see [28]), we obtain

$$\| \operatorname{Op}_\varepsilon(q)u^\varepsilon\|_{L^2} = \|e^{itH^\varepsilon/\varepsilon}\operatorname{Op}_\varepsilon(q)e^{-itH^\varepsilon/\varepsilon}a_0\|_{L^2} = \|\operatorname{Op}_\varepsilon(q\circ\Phi_t)a_0\|_{L^2} + \mathcal{O}(\varepsilon),$$

where Φ_t is the Hamiltonian flow associated with H^ε. For small times, one can relate Φ_t to the solution $\phi(t,x)$ of the Hamilton–Jacobi equation:

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + V(x) = 0 \quad ; \quad \phi(0,x) = 0,$$

by the identity $\Phi_t(x,\xi) = (X(t,x) + t\xi\, ,\, \xi + (\nabla\phi)(t,X(t,x)))$, where X satisfies

$$\partial_t X(t,x) = (\nabla\phi)(t,X(t,x)) \quad ; \quad X(0,x) = x.$$

Hence, with $q(x,\xi) = i\xi$, we infer

$$\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2} = \|(\nabla\phi)(t,X(t,x))a_0\|_{L^2} + \mathcal{O}(\varepsilon),$$

so that the kinetic energy is of order $\mathcal{O}(1)$ provided that $(\nabla\phi)(t,X(t,\cdot))a_0 \neq 0$.

The previous argument can be made explicit for the harmonic oscillator

$$i\varepsilon\partial_t u^\varepsilon_\ell + \frac{\varepsilon^2}{2}\Delta u^\varepsilon_\ell = \frac{|x|^2}{2}u^\varepsilon_\ell \quad ; \quad u^\varepsilon_\ell(0,x) = a_0(x). \tag{2.4}$$

Lemma 2.4. *Let $n \geqslant 1$, and $a_0 \in \mathcal{S}(\mathbb{R}^n)$ (non-trivial). There exists $\tau > 0$ such that the solution u^ε_ℓ to* (2.4) *satisfies*

$$\liminf_{\varepsilon\to 0} \|\varepsilon\nabla u^\varepsilon_\ell(\tau)\|_{L^2} > 0.$$

Proof. The standard WKB approach yields, at leading order, the following approximate solution:

$$v^\varepsilon_\ell(t,x) = a_\ell(t,x)e^{i\phi_\ell(t,x)/\varepsilon},$$

where ϕ_ℓ and a_ℓ are given by an eikonal equation and a transport equation. Since we consider an harmonic oscillator, we can compute ϕ_ℓ and a_ℓ explicitly:

$$\partial_t \phi_\ell + \frac{1}{2}|\nabla \phi_\ell|^2 + \frac{|x|^2}{2} = 0;\ \phi_{\ell|t=0} = 0 : \quad \phi_\ell(t,x) = \frac{-|x|^2}{2}\tan t.$$
$$\partial_t a_\ell + \nabla\phi_\ell \cdot \nabla a_\ell + \frac{1}{2}a_\ell \Delta\phi_\ell = 0;\ a_{\ell|t=0} = a_0 :\ a_\ell(t,x) = \frac{1}{(\cos t)^{n/2}} a_0\left(\frac{x}{\cos t}\right).$$

Energy estimates then yield (see for instance [11, §3] for more details):

$$\|\varepsilon\nabla u_\ell^\varepsilon - \varepsilon\nabla v_\ell^\varepsilon\|_{L^\infty([0,T];L^2)} \leqslant C_T\varepsilon, \quad \forall T\in\left[0,\frac{\pi}{2}\right[.$$

Since

$$\liminf_{\varepsilon\to 0}\|\varepsilon\nabla v_\ell^\varepsilon(t)\|_{L^2} = \sin t\,\|xa_0\|_{L^2}, \quad \forall t\in\left[0,\frac{\pi}{2}\right[,$$

the lemma follows easily. □

The strategy for proving Theorem 2.1 is the same: we compare with the limit system. For nonlinear Schrödinger equation, the eikonal equation which gives the phase is coupled to the transport equation: the limiting system reads

$$\begin{cases} \partial_t\phi + \dfrac{1}{2}|\nabla\phi|^2 + |a|^{2\sigma} = 0 \quad ; \quad \phi_{|t=0} = 0, \\ \partial_t a + \nabla\phi\cdot\nabla a + \dfrac{1}{2}a\Delta\phi = 0 \quad ; \quad a_{|t=0} = a_0. \end{cases} \tag{2.5}$$

By introducing $v = \nabla\phi$, one can transform this system into a quasilinear system of nonlinear equations. An important feature of the system thus obtained is that it does not enter into the classical framework of symmetric hyperbolic systems for $\sigma \geqslant 2$. Nevertheless, one can solve the Cauchy problem (2.5) for all $\sigma\geqslant 1$ by a nonlinear change of variable. This is done in §3 following an idea due to T. Makino, S. Ukai and S. Kawashima [27].

For the general case $\sigma\geqslant 1$, we establish a modulated energy estimate, following the pioneering work of Y. Brenier [6]. The idea consists in obtaining an estimate for the L^2 norm of $\varepsilon\nabla a^\varepsilon$ where a^ε is the modulated unknown function $a^\varepsilon := u^\varepsilon e^{-i\phi/\varepsilon}$. It is found that a^ε satisfies

$$i\varepsilon\Big(\partial_t a^\varepsilon + \nabla\phi\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi\Big) + \frac{\varepsilon^2}{2}\Delta a^\varepsilon = \Big(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\Big)a^\varepsilon.$$

For $\sigma = 1$, one can obtain estimates uniform in ε, that is

$$\|\varepsilon\nabla a^\varepsilon\|_{L^\infty([0,T];L^2)} + \big\||a^\varepsilon|^2 - |a|^2\big\|_{L^\infty([0,T];L^2)} = \mathcal{O}(\varepsilon),$$

by an integration by parts argument. Again, this is based on the hyperbolicity in the case $\sigma = 1$ (see [26] for an application of this idea to the Gross–Pitaevskii equations). Using a modulated energy functional adapted to our problem, we prove the estimate (see Theorem 4.1 below):

$$\|\varepsilon\nabla a^\varepsilon\|_{L^\infty([0,T];L^2)} + \big\|\big(|a^\varepsilon|^2 - |a|^2\big)\big(|a^\varepsilon|^{\sigma-1} + |a|^{\sigma-1}\big)\big\|_{L^\infty([0,T];L^2)} = \mathcal{O}(\varepsilon).$$

This is enough to prove Theorem 2.1 for $k = 1$. Note that this suffices to infer Corollary 1.3. Finally, to cover the range $k\in]0,1]$, we microlocalize the previous estimate by means of wave packets operator.

3. The limiting system

Being optimistic, one would try to mimic the approach of E. Grenier [21], and write the solution u^ε to (1.5) as $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$, where

$$\begin{cases} \partial_t \phi^\varepsilon + \frac{1}{2}|\nabla \phi^\varepsilon|^2 + |a^\varepsilon|^{2\sigma} = 0 \quad ; \quad \phi^\varepsilon_{|t=0} = 0, \\ \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon_{|t=0} = a_0. \end{cases} \tag{3.1}$$

Considering the unknown $v^\varepsilon = \nabla \phi^\varepsilon$ instead of ϕ^ε, the first step in the analysis would be to solve

$$\begin{cases} \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + \nabla\left(|a^\varepsilon|^{2\sigma}\right) = 0 \quad ; \quad v^\varepsilon_{|t=0} = 0, \\ \partial_t a^\varepsilon + v^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \operatorname{div} v^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon_{|t=0} = a_0. \end{cases} \tag{3.2}$$

In [21], E. Grenier considers the unknown $\mathbf{u}^\varepsilon = (v^\varepsilon, \operatorname{Re} a^\varepsilon, \operatorname{Im} a^\varepsilon) \in \mathbb{R}^{n+2}$. It solves a partial differential equation of the form

$$\partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon = \frac{\varepsilon}{2} L\mathbf{u}^\varepsilon.$$

In the case $\sigma = 1$, the left-hand side of the above equation defines a symmetric quasi-linear hyperbolic system in the sense of Friedrichs, with a constant symmetrizer. The linear operator L corresponds to the term $i\Delta$ on the right hand side of (3.2): it is skew-symmetric, and does not appear in the energy estimates. Therefore, one can construct a smooth solution to (3.2) on some time interval $[0,T]$ with $T > 0$ independent of ε. In the case $\sigma \geqslant 2$, the symmetrizer of [21] would become

$$S = \begin{pmatrix} \frac{1}{4\sigma |a^\varepsilon|^{2\sigma-2}} I_n & 0 \\ 0 & I_2 \end{pmatrix}.$$

For $a^\varepsilon \in L^2(\mathbb{R}^n)$, this matrix is not uniformly bounded, and this is why the analysis in [21] is restricted to nonlinearities which are defocusing, and cubic at the origin.

This apparent lack of hyperbolicity is not a real problem for the homogeneous nonlinearity that we consider, provided that we analyze the limiting system only:

$$\begin{cases} \partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + |a|^{2\sigma} = 0 \quad ; \quad \phi_{|t=0} = 0, \\ \partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2} a \Delta \phi = 0 \quad ; \quad a_{|t=0} = a_0. \end{cases} \tag{3.3}$$

The above restriction remains apparently valid for this system: in the presence of vacuum (zeroes of a), the symmetrizer S is singular. This may lead to a loss of regularity in the energy estimates. However, we shall see that thanks to the special structure of (3.3), we can construct solutions to (3.3) in Sobolev spaces of sufficiently large order. Following an idea due to T. Makino, S. Ukai and S. Kawashima [27], we prove:

Lemma 3.1. *Let $a_0 \in \mathcal{S}(\mathbb{R}^n)$. There exists $T > 0$ such that (3.3) has a unique solution $(\phi, a) \in C^\infty([0,T]\times\mathbb{R}^n)^2$, with $(\phi,a) \in C([0,T],H^s)^2$ for all $s \geqslant 0$. Moreover, $\langle x\rangle^s \nabla\phi \in C([0,T],L^2)$ for all $s \geqslant 0$, where $\langle x \rangle = (1+|x|^2)^{1/2}$.*

Proof. Differentiating the first equation in (3.3), we first consider:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla\left(|a|^{2\sigma}\right) = 0 \quad ; \quad v_{|t=0} = 0, \\ \partial_t a + v\cdot\nabla a + \dfrac{1}{2} a \operatorname{div} v = 0 \quad ; \quad a_{|t=0} = a_0. \end{cases} \tag{3.4}$$

Adapting the idea of [27], consider the unknown $(v,u) = (v,a^\sigma)$. Even though the map $a \mapsto a^\sigma$ is not bijective, this will suffice to prove the lemma. The pair (v,u) solves:

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla\left(|u|^{2}\right) = 0 \quad ; \quad v_{|t=0} = 0, \\ \partial_t u + v\cdot\nabla u + \dfrac{\sigma}{2} u \operatorname{div} v = 0 \quad ; \quad u_{|t=0} = a_0^\sigma \in \mathcal{S}(\mathbb{R}^n). \end{cases} \tag{3.5}$$

This system is hyperbolic symmetric, with a constant symmetrizer. To see this, denote $\mathbf{u} = (v, \operatorname{Re} u, \operatorname{Im} u)^T \in \mathbb{R}^{n+2}$. Equation (3.5) is of the form

$$\partial_t \mathbf{u} + \sum_{j=1}^n A_j(\mathbf{u})\partial_j \mathbf{u} = 0,$$

where the matrices $A_j \in \mathcal{M}_{n+2}(\mathbb{R})$ are such that SA_j are symmetric, for

$$S = \begin{pmatrix} \sigma \mathrm{I}_n & 0 \\ 0 & 4\mathrm{I}_2 \end{pmatrix} \in \mathcal{S}_{n+2}(\mathbb{R}).$$

From classical theory on hyperbolic symmetric quasilinear systems (see e.g. [2, 31]), there exist $T>0$ and a unique solution $(v,u) \in C^\infty([0,T]\times\mathbb{R}^n)^2$, which is in $C([0,T],H^s)^2$ for all $s \geqslant 0$. The fact that $\langle x \rangle^s v \in C([0,T],L^2)$ follows easily by considering the momenta of u and v. Now that v is known, we define a as the solution of the transport equation

$$\partial_t a + v\cdot\nabla a + \frac{1}{2} a \operatorname{div} v = 0 \quad ; \quad a_{|t=0} = a_0.$$

The function a has the regularity announced in Lemma 3.1. We check that a^σ solves the second equation in (3.5). Since v is a smooth coefficient, by uniqueness for this linear equation, we have $u = a^\sigma$. Therefore, (v,a) solves (3.4). To conclude, we notice that v is irrotational, so there exists $\widetilde{\phi}$ such that $v = \nabla\widetilde{\phi}$. Setting $\phi = \widetilde{\phi} + F$, where $F = F(t)$ is a function of time only, (ϕ, a) solves (3.3). Uniqueness follows from the uniqueness for (3.5). □

Remark 3.2. The proof shows that if we assume only $a_0 \in H^s(\mathbb{R}^n)$ with $s > n/2+1$, then $u, v \in C([0,T];H^s)$. We infer $a \in C([0,T];H^{s-1})$: the possible loss of regularity due to the lack of hyperbolicity for (3.3) remains limited.

Remark 3.3. The nonlinear change of unknown function, $u = a^\sigma$, suggests that the above approach cannot be adapted to study (3.2), since we have to deal with the term $i\Delta a^\varepsilon$, and prevent the loss of regularity that it may cause in the energy estimates.

4. Semi-classical limit

Introduce the hydrodynamic variables:

$$\rho = |a|^2 \quad ; \quad \rho^\varepsilon = |u^\varepsilon|^2 \quad ; \quad J^\varepsilon = \operatorname{Im}\left(\varepsilon \overline{u}^\varepsilon \nabla u^\varepsilon\right).$$

The main result of this section is:

Theorem 4.1. *Let $n \geqslant 1$, and $\sigma \geqslant 1$ be an integer. Let $(v,a) \in C([0,T];H^\infty)^2$ given by Lemma 3.1, where $v = \nabla\phi$. Then we have the following estimate:*

$$\|(\varepsilon\nabla - iv)u^\varepsilon\|^2_{L^\infty([0,T];L^2)} + \left\|(\rho^\varepsilon - \rho)^2 \left((\rho^\varepsilon)^{\sigma-1} + \rho^{\sigma-1}\right)\right\|_{L^\infty([0,T];L^1)} = \mathcal{O}(\varepsilon^2).$$

Note that the above quantities are well-defined for weak solutions. We outline the argument in a formal proof, which is then made rigorous.

Formal proof. For $y \geqslant 0$, denote

$$f(y) = y^\sigma \quad ; \quad F(y) = \int_0^y f(z)dz = \frac{1}{\sigma+1}y^{\sigma+1} \quad ;$$
$$G(y) = \int_0^y zf'(z)dz = yf(y) - F(y) = \frac{\sigma}{\sigma+1}y^{\sigma+1}.$$

We check that $(\rho^\varepsilon, J^\varepsilon)$ satisfies, for $\sigma \geqslant 1$:

$$\begin{cases} \partial_t\rho^\varepsilon + \operatorname{div} J^\varepsilon = 0. \\ \partial_t J^\varepsilon_j + \dfrac{\varepsilon^2}{4}\displaystyle\sum_k \partial_k\left(4\operatorname{Re}\partial_j\overline{u}^\varepsilon\partial_k u^\varepsilon - \partial^2_{jk}\rho^\varepsilon\right) + \partial_j G(\rho^\varepsilon) = 0. \end{cases} \tag{4.1}$$

As suggested in [26, Remark 1, (2)], introduce the modulated energy functional:

$$H^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{R}^n} |(\varepsilon\nabla - iv)u^\varepsilon|^2\,dx + \int_{\mathbb{R}^n} \left(F(\rho^\varepsilon) - F(\rho) - (\rho^\varepsilon - \rho)f(\rho)\right)dx.$$

Denote

$$K^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{R}^n} |(\varepsilon\nabla - iv)u^\varepsilon|^2\,dx.$$

Integrations by parts, which are studied in more detail below, yield:

$$\frac{d}{dt}H^\varepsilon(t) = \mathcal{O}\left(K^\varepsilon + \varepsilon^2\right) - \int_{\mathbb{R}^n}\left(G(\rho^\varepsilon) - G(\rho) - (\rho^\varepsilon - \rho)G'(\rho)\right)\operatorname{div} v\,dx.$$

We check that there exists $K > 0$ such that

$$|G(\rho^\varepsilon) - G(\rho) - (\rho^\varepsilon - \rho)G'(\rho)| \leqslant K\,|F(\rho^\varepsilon) - F(\rho) - (\rho^\varepsilon - \rho)F'(\rho)|\,.$$

Therefore,

$$H^\varepsilon(t) \leqslant H^\varepsilon(0) + C\int_0^t \left(H^\varepsilon(s) + \varepsilon^2\right)ds.$$

We infer by Gronwall lemma that $H^\varepsilon(t) = \mathcal{O}(\varepsilon^2)$ so long as it is defined. Finally, Taylor's formula yields, since $F''(y) = f'(y) = \sigma y^{\sigma-1}$:

$$F(\rho^\varepsilon) - F(\rho) - (\rho^\varepsilon - \rho)F'(\rho) = \sigma(\rho^\varepsilon - \rho)^2\int_0^1 (1-\theta)\left(\rho + \theta(\rho^\varepsilon - \rho)\right)^{\sigma-1}d\theta.$$

The estimate, for all $\theta \in [0,1]$,

$$\left(\rho + \theta(\rho^\varepsilon - \rho)\right)^{\sigma-1} = \left((1-\theta)\rho + \theta\rho^\varepsilon\right)^{\sigma-1} \geqslant (1-\theta)^{\sigma-1}\rho^{\sigma-1} + \theta^{\sigma-1}(\rho^\varepsilon)^{\sigma-1}$$

shows that there exists $c > 0$ such that

$$H^\varepsilon(t) \geqslant K^\varepsilon(t) + c\int_{\mathbb{R}^n}(\rho^\varepsilon - \rho)^2\left((\rho^\varepsilon)^{\sigma-1} + \rho^{\sigma-1}\right)dx. \tag{4.2}$$

The result of Theorem 4.1 follows. ∎

Rigorous proof. In general, the above integrations by parts do not make sense for all $t \in [0,T]$, since we consider weak solutions only. Note however that for $\sigma \geqslant 2$ and $n \leqslant 3$, the analysis in [1] shows that we can work with strong solutions, so the following analysis is not needed in this case (for $\sigma = 1$ and $n \geqslant 1$, the same holds true, from [21]). Also, if one is just interested in proving Corollary 1.6 by contradiction, no further justification is needed for integrations by parts, and one can skip the end of this section.

We work on a sequence of global strong solutions, converging to a weak solution. For $(\delta_m)_m$ a sequence of positive numbers going to zero, introduce the saturated nonlinearity, defined for $y \geqslant 0$:

$$f_m(y) = \frac{y^\sigma}{1+(\delta_m y)^\sigma}.$$

Note that f_m is a symbol of degree 0. For fixed m, we have a global strong solution $u_m^\varepsilon \in C(\mathbb{R};H^1)$ to:

$$i\varepsilon\partial_t u_m^\varepsilon + \frac{\varepsilon^2}{2}\Delta u_m^\varepsilon = f_m\left(|u_m^\varepsilon|^2\right)u_m^\varepsilon \quad ; \quad u_m^\varepsilon(0,x) = a_0(x). \tag{4.3}$$

As $m \to \infty$, the sequence $(u_m^\varepsilon)_m$ converges to a weak solution of (1.5) (see [20, 25]). For $y \geqslant 0$, introduce also

$$F_m(y) = \int_0^y f_m(z)dz \quad ; \quad G_m(y) = \int_0^y z f_m'(z)dz = y f_m(y) - F_m(y).$$

The mass and energy associated to u_m^ε are conserved:

$$M_m^\varepsilon(t) = \int |u_m^\varepsilon(t,x)|^2 dx \equiv \|a_0\|_{L^2}^2.$$
$$E_m^\varepsilon(t) = \frac{1}{2}\|\varepsilon\nabla u_m^\varepsilon(t)\|_{L^2}^2 + \int_{\mathbb{R}^n} F_m\left(|u_m^\varepsilon(t,x)|^2\right)dx \equiv E_m^\varepsilon(0).$$

Moreover, the solution is in $H^2(\mathbb{R}^n)$ for all time: $u_m^\varepsilon \in C(\mathbb{R};H^2)$. To see this, we use an idea due to T. Kato [22, 23], and consider $\partial_t u_m^\varepsilon$. Energy estimates show that $\partial_t u_m^\varepsilon \in C(\mathbb{R};L^2)$, since f_m is a symbol of degree 0. Using (4.3) and the boundedness of f_m, we infer $\Delta u_m^\varepsilon \in C(\mathbb{R};L^2)$.

We consider the hydrodynamic variables:

$$\rho_m^\varepsilon = |u_m^\varepsilon|^2 \quad ; \quad J_m^\varepsilon = \operatorname{Im}\left(\varepsilon \overline{u}_m^\varepsilon \nabla u_m^\varepsilon\right).$$

From the above discussion, we have:

$$\rho_m^\varepsilon(t) \in W^{2,1}(\mathbb{R}^n) \text{ and } J_m^\varepsilon(t) \in W^{1,1}(\mathbb{R}^n), \quad \forall t \in \mathbb{R}. \tag{4.4}$$

The analogue of (4.1) is:

$$\begin{cases} \partial_t \rho_m^\varepsilon + \operatorname{div} J_m^\varepsilon = 0. \\ \partial_t (J_m^\varepsilon)_j + \dfrac{\varepsilon^2}{4}\displaystyle\sum_k \partial_k\left(4\operatorname{Re}\partial_j \overline{u}_m^\varepsilon \partial_k u_m^\varepsilon - \partial_{jk}^2\rho_m^\varepsilon\right) + \partial_j G_m(\rho_m^\varepsilon) = 0. \end{cases} \tag{4.5}$$

Introduce the modulated energy functional "adapted to (4.3)":

$$H_m^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{R}^n} |(\varepsilon\nabla - iv)u_m^\varepsilon|^2 dx + \int_{\mathbb{R}^n}\left(F_m(\rho_m^\varepsilon) - F_m(\rho) - (\rho_m^\varepsilon - \rho)f_m(\rho)\right)dx.$$

Notice that this functional is not exactly adapted to (4.3), since the limiting quantities (as $\varepsilon \to 0$) ρ and v are constructed with the nonlinearity f and not the nonlinearity f_m. We also distinguish the kinetic part:

$$K_m^\varepsilon(t) = \frac{1}{2}\int_{\mathbb{R}^n} |(\varepsilon\nabla - iv)u_m^\varepsilon|^2\, dx.$$

Thanks to the conservation of energy for u_m^ε, we have:

$$\begin{aligned}\frac{d}{dt}K_m^\varepsilon = &-\frac{d}{dt}\int F_m(\rho_m^\varepsilon)dx + \frac{1}{2}\int |v|^2\partial_t\rho_m^\varepsilon + \int \rho_m^\varepsilon v\cdot\partial_t v\\ &-\int J_m^\varepsilon\cdot\partial_t v - \int v\cdot\partial_t J_m^\varepsilon.\end{aligned}$$

Using Lemma 3.1, (4.4) and (4.5), (licit) integrations by parts yield:

$$\begin{aligned}\frac{d}{dt}K_m^\varepsilon = &-\frac{d}{dt}\int F_m(\rho_m^\varepsilon)dx - \frac{1}{2}\int |v|^2 \operatorname{div} J_m^\varepsilon - \sum_{j,k}\int \rho_m^\varepsilon v_j v_k \partial_j v_k\\ &-\int \rho_m^\varepsilon \nabla f(\rho)\cdot v + \int (v\cdot\nabla v)\cdot J_m^\varepsilon + \int \nabla f(\rho)\cdot J_m^\varepsilon\\ -\sum_{j,k}&\int \partial_k v_j \operatorname{Re}\left(\varepsilon\partial_j\overline{u}_m^\varepsilon \varepsilon\partial_k u_m^\varepsilon\right) - \frac{\varepsilon^2}{4}\int \nabla\left(\operatorname{div} v\right)\cdot\nabla\rho_m^\varepsilon + \int \rho_m^\varepsilon v\cdot\nabla f_m(\rho_m^\varepsilon).\end{aligned}$$

Proceeding as in [26], we have:

$$\begin{aligned}\varepsilon^2\int \nabla\left(\operatorname{div} v\right)\cdot\nabla\rho_m^\varepsilon &= \varepsilon\int \nabla\left(\operatorname{div} v\right)\cdot\left(\overline{u}_m^\varepsilon\varepsilon\nabla u_m^\varepsilon + u_m^\varepsilon\varepsilon\nabla\overline{u}_m^\varepsilon\right)\\ &= \varepsilon\int \nabla\left(\operatorname{div} v\right)\cdot\left(\overline{u}_m^\varepsilon(\varepsilon\nabla - iv)u_m^\varepsilon + u_m^\varepsilon\overline{(\varepsilon\nabla - iv)u^\varepsilon}_m\right)\\ &= \mathcal{O}\left(K_m^\varepsilon + \varepsilon^2\right),\end{aligned}$$

where we have used the conservation of mass and Young's inequality. From now on, we use the convention that the constant associated to the notation $\mathcal{O}$ is independent of m and ε. We have written the time derivative of K_m^ε as the sum of nine terms. The first one corresponds to the conservation of the energy, and will be canceled by the first term of the time derivative of $H_m^\varepsilon - K_m^\varepsilon$. We have just bounded the eighth one. We consider next the sum of four of the seven remaining terms: the second, third, fifth and seventh,

$$\begin{aligned}&\int \Big(-\frac{1}{2}|v|^2\operatorname{div} J_m^\varepsilon - \sum_{j,k}\rho_m^\varepsilon v_j v_k\partial_j v_k + (v\cdot\nabla v)\cdot J_m^\varepsilon - \sum_{j,k}\partial_k v_j \operatorname{Re}\left(\varepsilon\partial_j\overline{u}_m^\varepsilon\varepsilon\partial_k u_m^\varepsilon\right)\Big)\\ &= \sum_{j,k}\int \Big(v_k\partial_j v_k\left(J_m^\varepsilon\right)_j - |u_m^\varepsilon|^2 v_j v_k\partial_j v_k + v_j\partial_j v_k\left(J_m^\varepsilon\right)_k - \partial_j v_k \operatorname{Re}\left(\varepsilon\partial_k\overline{u}_m^\varepsilon\varepsilon\partial_j u_m^\varepsilon\right).\end{aligned}$$

Factoring out the term $\partial_j v_k$, and recalling that

$$J_m^\varepsilon = \operatorname{Im}\left(\overline{u}_m^\varepsilon\varepsilon\nabla u_m^\varepsilon\right),$$

the above sum can be simplified to:

$$-\sum_{j,k}\int \partial_j v_k \operatorname{Re}\left(\varepsilon\partial_j u_m^\varepsilon - iv_j u_m^\varepsilon\right)\left(\overline{\varepsilon\partial_k u_m^\varepsilon - iv_k u_m^\varepsilon}\right) = \mathcal{O}\left(K_m^\varepsilon\right),$$

from Cauchy–Schwarz inequality, since $\nabla v \in L^\infty([0,T]\times\mathbb{R}^n)$. We are now left with:

$$\frac{d}{dt}K_m^\varepsilon = \mathcal{O}\left(K_m^\varepsilon+\varepsilon^2\right) - \frac{d}{dt}\int F_m(\rho_m^\varepsilon) + \int \nabla f_m(\rho_m^\varepsilon)\cdot(\rho_m^\varepsilon v) - \int \nabla f(\rho)\cdot(\rho_m^\varepsilon v - J_m^\varepsilon)\,.$$

Since $G_m'(y) = y f_m'(y)$, we infer:

$$\frac{d}{dt}K_m^\varepsilon = \mathcal{O}\left(K_m^\varepsilon+\varepsilon^2\right) - \frac{d}{dt}\int F_m(\rho_m^\varepsilon) - \int G_m(\rho_m^\varepsilon)\operatorname{div} v - \int \nabla f(\rho)\cdot(\rho_m^\varepsilon v - J_m^\varepsilon)\,.$$

Direct computations yield

$$\begin{aligned}\frac{d}{dt}\left(H_m^\varepsilon - K_m^\varepsilon\right) =& \frac{d}{dt}\int F_m\left(\rho_m^\varepsilon\right) - \int \nabla f_m(\rho)\cdot J_m^\varepsilon + \int \left(\rho_m^\varepsilon-\rho\right) v\cdot\nabla f_m\left(\rho\right)\\ &+\int\left(\rho_m^\varepsilon-\rho\right)G_m'\left(\rho\right)\operatorname{div} v.\end{aligned}$$

We therefore come up with:

$$\begin{aligned}\frac{d}{dt}H_m^\varepsilon =&\mathcal{O}\left(K_m^\varepsilon+\varepsilon^2\right) - \int\left(G_m(\rho_m^\varepsilon) - G_m(\rho) - (\rho_m^\varepsilon-\rho)G_m'(\rho)\right)\operatorname{div} v\\ &+\int\nabla\left(f(\rho)-f_m(\rho)\right)\cdot\left(J_m^\varepsilon - \rho_m^\varepsilon v\right).\end{aligned}$$

Note that $f(\rho)-f_m(\rho)\to 0$ in $L^\infty([0,T];W^{1,\infty})$ as $m\to\infty$. We can thus write:

$$\begin{aligned}\frac{d}{dt}H_m^\varepsilon =&\mathcal{O}\left(K_m^\varepsilon+\varepsilon^2\right) + o_{m\to\infty}(1)\\ &-\int\left(G_m(\rho_m^\varepsilon) - G_m(\rho) - (\rho_m^\varepsilon-\rho)G_m'(\rho)\right)\operatorname{div} v.\end{aligned}\tag{4.6}$$

We conclude thanks to the following lemma, whose proof is postponed to the end of this section:

Lemma 4.2. *There exists $K>0$ independent of m such that $\forall\rho',\rho\geqslant 0$,*

$$\left|G_m(\rho') - G_m(\rho) - (\rho'-\rho)G_m'(\rho)\right| \leqslant K\left|F_m(\rho') - F_m(\rho) - (\rho'-\rho)F_m'(\rho)\right|.$$

Using this lemma and (4.6), we infer:

$$\frac{d}{dt}H_m^\varepsilon \leqslant C\left(H_m^\varepsilon+\varepsilon^2\right) + o_{m\to\infty}(1),$$

for some C independent of m. Gronwall lemma yields

$$\sup_{t\in[0,T]} H_m^\varepsilon(t)\leqslant C'\varepsilon^2 + o_{m\to\infty}(1),$$

for some C' independent of m. Letting $m\to\infty$, Fatou's lemma yields

$$\sup_{t\in[0,T]} H^\varepsilon(t)\leqslant C'\varepsilon^2.$$

Theorem 4.1 then follows from (4.2). □

Proof of Lemma 4.2. Taylor's formula yields

$$G_m(\rho') - G_m(\rho) - (\rho'-\rho)G_m'(\rho) = \left(\rho'-\rho\right)^2\int_0^1(1-\theta)G_m''\left(\rho+\theta(\rho'-\rho)\right)d\theta.$$

$$F_m(\rho') - F_m(\rho) - (\rho'-\rho)F_m'(\rho) = \left(\rho'-\rho\right)^2\int_0^1(1-\theta)F_m''\left(\rho+\theta(\rho'-\rho)\right)d\theta.$$

By definition,

$$F_m''(y) = f_m'(y) \quad ; \quad G_m''(y) = f_m'(y) + y f_m''(y).$$

Set, for $y \geqslant 0$, $h(y) = y^\sigma/(1+y^\sigma)$:

$$f_m'(y) = \delta_m^{1-\sigma} h'(\delta_m y) \quad ; \quad f_m''(y) = \delta_m^{2-\sigma} h''(\delta_m y).$$

Moreover,

$$h'(y) = \frac{\sigma y^{\sigma-1}}{(1+y^\sigma)^2} \geqslant 0.$$

Therefore, to prove the lemma, it suffices to show that for all $y \geqslant 0$,

$$|yh''(y)| \leqslant Ch'(y). \tag{4.7}$$

We check the identity

$$yh''(y) = h'(y) \times \frac{\sigma - 1 - (\sigma+1)y^\sigma}{1+y^\sigma}.$$

The estimate (4.7) is then straightforward, and the lemma follows. □

5. End of the proof of Theorem 1.1

To conclude, the heuristic argument is as follows. From Theorem 4.1, we expect

$$\|\varepsilon \nabla u^\varepsilon(t)\|_{L^2} \approx \|v(t) u^\varepsilon(t)\|_{L^2} \approx \|v(t)a(t)\|_{L^2}.$$

This follows easily from Hölder's inequality. For the values $k \in]0,1[$ in Theorem 4.1, we morally use an estimate of the form

$$\left\| |v(t)|^k u^\varepsilon(t)\right\|_{L^2} \lesssim \left\| |\varepsilon D_x|^k u^\varepsilon(t)\right\|_{L^2} + \left\| |\varepsilon D_x - v(t)|^k u^\varepsilon(t)\right\|_{L^2},$$

where the first term of the right-hand side goes to zero by interpolation between $k=0$ and $k=1$. The aim of the following lemma is to justify such a statement.

Lemma 5.1. *There exists a constant K such that, for all $\varepsilon \in]0,1]$, for all $s \in [0,1]$, for all $u \in H^1(\mathbb{R}^n)$ and for all $v \in W^{1,\infty}(\mathbb{R}^n)$,*

$$\||v|^s u\|_{L^2} \leqslant \||\varepsilon D_x|^s u\|_{L^2} + \|(\varepsilon \nabla - iv)u\|_{L^2}^s \|u\|_{L^2}^{1-s} + \varepsilon^{s/2} K \left(1 + \|\nabla v\|_{L^\infty}\right) \|u\|_{L^2}.$$

Proof. We begin with the following elementary inequality: For all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and all $s \in [0,1]$, there holds

$$|x|^s \leqslant |y|^s + |x-y|^s. \tag{5.1}$$

To see this, note that the result is obvious if $|x| \leqslant |y|$. Else, write $|y| = \lambda |x|$ with $\lambda \in [0,1]$ and use the inequalities $\lambda \leqslant \lambda^s$ and $(1-\lambda) \leqslant (1-\lambda)^s$.

With this preliminary established, introduce the wave-packets operator (see e.g. [16, 17, 28])

$$W^\varepsilon v(x,\xi) = c_n \varepsilon^{-3n/4} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi/\varepsilon - (x-y)^2/2\varepsilon} v(y)\, dy,$$

with $c_n = 2^{-n/2}\pi^{-3n/4}$. The mapping $v \mapsto W^\varepsilon v$ is continuous from the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$, and W^ε extends as an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$:

$$\|W^\varepsilon v\|_{L^2(\mathbb{R}^{2n})} = \|v\|_{L^2(\mathbb{R}^n)}.$$

By applying (5.1), we have

$$\left\| |v(x)|^s W^\varepsilon u\right\|_{L^2(\mathbb{R}^{2n})} \leqslant \left\| |\xi|^s W^\varepsilon u\right\|_{L^2(\mathbb{R}^{2n})} + \left\| |\xi - v(x)|^s W^\varepsilon u\right\|_{L^2(\mathbb{R}^{2n})}.$$

Therefore, since

$$\||\xi - v(x)|^s W^\varepsilon u\|_{L^2(\mathbb{R}^{2n})} \leqslant \|W^\varepsilon u\|_{L^2(\mathbb{R}^{2n})}^{1-s} \big\||\xi - v(x)|W^\varepsilon u\big\|_{L^2(\mathbb{R}^{2n})}^s$$
$$\leqslant \|u\|_{L^2(\mathbb{R}^n)}^{1-s} \big\|(\xi - v(x))W^\varepsilon u\big\|_{L^2(\mathbb{R}^{2n})}^s,$$

to obtain the desired estimate, we need only prove:

$$\big\||v(x)|^s W^\varepsilon u - W^\varepsilon(|v|^s u)\big\|_{L^2(\mathbb{R}^{2n})} \leqslant K\varepsilon^{s/2}\|\nabla v\|_{L^\infty}^s \|u\|_{L^2}, \tag{5.2}$$

$$\big\||\xi|^s W^\varepsilon u - W^\varepsilon(|\varepsilon D_x|^s u)\big\|_{L^2(\mathbb{R}^{2n})} \leqslant K\varepsilon^{s/2}\|u\|_{L^2}, \tag{5.3}$$

$$\big\|(i\xi - iv)W^\varepsilon u - W^\varepsilon\big((\varepsilon\nabla - iv)u\big)\big\|_{L^2(\mathbb{R}^{2n})} \leqslant K\varepsilon^{1/2}(1 + \|\nabla v\|_{L^\infty})\|u\|_{L^2}. \tag{5.4}$$

These properties follows from the fact that the wave packets operator conjugates the action of pseudo-differential operators, approximately, to multiplication by symbols. For smooth symbols, one has sharp results (see [16, 17, 28]). For the rough symbols $|v(x)|^s$ and $|\varepsilon\xi|^s$, one can proceed as follows.

To prove (5.2), directly from the definition, we compute

$$\||v|^s W^\varepsilon u - W^\varepsilon(|v|^s u)\|_{L^2(\mathbb{R}^{2n})}^2$$
$$= c_n^2(2\pi)^n\varepsilon^{-n/2} \iint e^{-(x-y)^2/\varepsilon} \left||v(x)|^s - |v(y)|^s\right|^2 |u(y)|^2 \, dydx.$$

Consequently, since $v \in W^{1,\infty}(\mathbb{R}^n)$, the inequality (5.1) implies

$$\||v|^s W^\varepsilon u - W^\varepsilon(|v|^s u)\|_{L^2(\mathbb{R}^{2n})}^2$$
$$\leqslant K\|\nabla v\|_{L^\infty}^{2s} \iint \varepsilon^{-n/2} e^{-(x-y)^2/\varepsilon} |x-y|^{2s} |u(y)|^2 \, dydx$$
$$\leqslant K\|\nabla v\|_{L^\infty}^{2s} \iint e^{-z^2} \left|\sqrt{\varepsilon}z\right|^{2s} \left|u(x - \sqrt{\varepsilon}z)\right|^2 \, dzdx,$$

which proves (5.2). We next compute $W^\varepsilon(|\varepsilon D_x|^s u)(x,\xi)$: it is given by

$$c_n(2\pi)^{-n/2}\varepsilon^{-7n/4} \iint e^{i(x-y)\cdot(\xi-\theta)/\varepsilon - (x-y)^2/2\varepsilon} e^{ix\cdot\theta/\varepsilon}|\theta|^s \widehat{u}\left(\frac{\theta}{\varepsilon}\right) \, d\theta dy,$$

where $\widehat{u}$ is the Fourier transform of u. Hence, by using

$$(2\pi)^{-n/2} \int e^{i(x-y)\cdot(\xi-\theta)/\varepsilon - (x-y)^2/2\varepsilon} \, dy = \varepsilon^{n/2} e^{-(\xi-\theta)^2/2\varepsilon},$$

we find

$$W^\varepsilon\big(|\varepsilon D_x|^s u\big)(x,\xi) := e^{ix\cdot\xi/\varepsilon} W^\varepsilon w^\varepsilon(\xi, -x),$$

with $w^\varepsilon(\tau) := |\tau|^s \varepsilon^{-n/2}\widehat{u}(\tau/\varepsilon)$. This leads us back to the situation of the previous step (with $|v(x)|^s$ replaced with $|x|^s$), and hence (5.3) is proved.

Finally, the arguments establishing (5.2) and (5.3) also yield the usual estimates

$$\big\|vW^\varepsilon u - W^\varepsilon(vu)\big\|_{L^2(\mathbb{R}^{2n})} \leqslant K\varepsilon^{1/2}\|\nabla v\|_{L^\infty}\|u\|_{L^2},$$
$$\big\|i\xi W^\varepsilon u - W^\varepsilon(\varepsilon\nabla u)\big\|_{L^2(\mathbb{R}^{2n})} \leqslant K\varepsilon^{1/2}\|u\|_{L^2},$$

which proves (5.4). This completes the proof of the lemma. □

We infer that the heuristic argument of the beginning of this section is justified:

Corollary 5.2. *For all $t \in [0,T]$ and all $k \in]0,1]$, we have:*

$$\liminf_{\varepsilon\to 0} \left\| |\varepsilon D_x|^k u^\varepsilon(t)\right\|_{L^2} \geqslant \left\| |v(t)|^k a(t)\right\|_{L^2}. \tag{5.5}$$

Proof. Let $t \in [0,T]$. It follows from the previous lemma that

$$\left\| |\varepsilon D_x|^k u^\varepsilon(t)\right\|_{L^2} = \| |v(t)|^k u^\varepsilon(t)\|_{L^2} + o(1).$$

Write

$$\left\| |v(t)|^k a(t)\right\|_{L^2} \leqslant \left\| |v(t)|^k u^\varepsilon(t)\right\|_{L^2} + \left\| |v(t)|^{2k}\left(|u^\varepsilon(t)|^2 - |a(t)|^2\right)\right\|_{L^1}.$$

From Hölder's inequality, the last term is bounded by

$$\left\| |v(t)|^{2k}\right\|_{L^{1+1/\sigma}} \left\| |u^\varepsilon(t)|^2 - |a(t)|^2\right\|_{L^{\sigma+1}}. \tag{5.6}$$

When $k \geqslant \sigma/(\sigma+1)$, Lemma 3.1 and Sobolev embedding show that the first term is bounded on $[0,T]$. When $0 < k < \sigma/(\sigma+1)$, Hölder's inequality yields:

$$\left\| |v(t)|^{2k}\right\|_{L^{1+1/\sigma}} \leqslant C_N \left\| \langle x\rangle^N v(t)\right\|_{L^2}^{2k\sigma/(\sigma+1)} \quad \text{for } N > \frac{n}{2k}\left(\frac{\sigma}{\sigma+1} - k\right).$$

Lemma 3.1 and Theorem 4.1 show that (5.6) goes to zero as ε tends to 0. □

To complete the proof of Theorem 2.1, it remains only to prove that the right-hand side in (5.5) is non trivial. To see this, we note that, from (3.4),

$$a_{|t=0} = a_0 \quad ; \quad v_{|t=0} = 0 \quad ; \quad \partial_t v_{|t=0} = -\nabla\left(|a_0|^{2\sigma}\right).$$

Therefore, by continuity (see Lemma 3.1), we obtain the following result.

Lemma 5.3. *There exists $\tau > 0$ such that*

$$\int |v(\tau,x)|^{2k} |a(\tau,x)|^2 dx > 0, \quad \forall k \in [0,1]. \tag{5.7}$$

This implies Theorem 2.1, hence Theorem 1.1.

Remark 5.4. We can compare the results of this paper with the analysis in [14]. The approximate solution used in [14] consists in neglecting the Laplacian in (1.5):

$$i\varepsilon\partial_t w^\varepsilon = |w^\varepsilon|^{2\sigma} w^\varepsilon \quad ; \quad w^\varepsilon_{|t=0} = a_0, \quad \text{hence} \quad w^\varepsilon(t,x) = a_0(x) e^{-it|a_0(x)|^{2\sigma}/\varepsilon}.$$

A direct application of Gronwall lemma shows that w^ε is a suitable approximation of u^ε up to time of order $c\varepsilon|\log\varepsilon|^\theta$, for some $c,\theta > 0$. The Taylor expansion in time for v shows that

$$v(t,x) = -t\nabla\left(|a_0(x)|^{2\sigma}\right) + \mathcal{O}\left(t^3\right).$$

The formal analysis of [10, §3.1] is thus justified also in this case: $w^\varepsilon(t)$ is a good approximation of $u^\varepsilon(t)$ for $t \ll \varepsilon^{1/3}$:

$$\| |\varepsilon D_x|^s u^\varepsilon(t)\|_{L^2} \approx \| |v(t)|^s a(t)\|_{L^2} \approx \| |\varepsilon D_x|^s w^\varepsilon(t)\|_{L^2} \quad \text{for } t \ll \varepsilon^{1/3}.$$

To prove this point, it seems necessary to perform a quasilinear analysis (see §3), and the semilinear approach based on Gronwall lemma is not enough.

6. Final remarks

To conclude this paper, we note that the approach presented here remains efficient in the case of non-trivial geometries. Indeed, the scaling argument that we have used in §2 is merely helpful for the intuition, to guess a suitable approximate solution. This argument meets the strategy adopted in the appendix of [9]. Introduce ω^h and A^h given by

$$v(t,x) = \omega^h\left(h^2\varepsilon t, hx\right) \quad ; \quad a(t,x) = h^{\frac{n}{2}-s} A^h\left(h^2\varepsilon t, hx\right).$$

The key approximation that we have used,

$$\left|\varepsilon \nabla u^\varepsilon(t,x)\right|^2 \approx \left|v(t,x)a(t,x)\right|^2, \quad 0 \leqslant t \leqslant T,$$

then reads

$$\left|\varepsilon h \nabla \psi^h(t,x)\right|^2 \approx \left|\omega^h(t,x)A^h(t,x)\right|^2, \quad 0 \leqslant t \leqslant h^2\varepsilon T.$$

More precisely, in terms of the initial problem (1.1), Theorem 4.1 reads:

$$\begin{aligned}
&h^{-s}\left\|\left(\varepsilon h\nabla - i\omega^h\right)\psi^h\right\|^2_{L^\infty([0,h^2\varepsilon T];L^2)} \\
&+h^{\left(\frac{n}{2}-s\right)2(\sigma+1)-\frac{n}{2}}\left\|\left(|\psi^h|^2-|A^h|^2\right)^2\left(|\psi^h|^{2\sigma-2}+|A^h|^{2\sigma-2}\right)\right\|_{L^\infty([0,h^2\varepsilon T];L^1)} \\
&= \mathcal{O}\left(\varepsilon^2\right).
\end{aligned}$$

It is essentially this estimate that we have used to prove Theorem 1.1 (and Corollary 1.3 stems exactly from this estimate).

Suppose for instance that x belongs to a bounded domain M, and not to all of $\mathbb{R}^n$, and that we consider (1.1) on M, with Dirichlet or Neumann boundary condition. If a_0 is compactly supported in a ball, contained in the interior of M,

$$\operatorname{supp} a_0 \subset B \Subset M,$$

then we can still consider (3.3), viewed as a system on $\mathbb{R}^n$. The key remark is that for a_0 compactly supported, the smooth solutions to (3.3) have a finite speed of propagation, which is *zero*. This is an important step in proving that smooth solutions develop singularities in finite time; see [27, 34]. Therefore, (ϕ, a) remains supported in B for $t \in [0,T]$; up to changing the origin to the center of B, so does (ω^h, A^h) on the time interval $[0, h^2\varepsilon T]$. In particular, (ω^h, A^h) is supported away from the boundary of M for $t \in [0, h^2\varepsilon T]$.

In the proof of Theorem 4.1, the integrations by parts affect v or a, that is, ω^h or A^h. Only two terms in the differentiation of the modulated energy functional do not contain ω^h or A^h: these two terms correspond to the global energy of ψ^h, which is a non-increasing function of time for strong solutions in the case of $\mathbb{R}^n$ (since it is in fact constant). As a matter of fact, this property is needed for strong solutions only, since it remains for weak solutions, by Fatou's lemma. Therefore, Theorem 1.1 remains valid on M, provided that we can construct strong solutions with a non-increasing energy. This is the case of compact surfaces when $\sigma \geqslant 1$, and of compact three dimensional manifolds when $\sigma = 1$, see [8]. This is also the case of bounded domains in $\mathbb{R}^2$ for $\sigma \geqslant 1$, see [5], of the ball in $\mathbb{R}^3$ for $\sigma = 1$ and radial data [3], and of exterior domains in $\mathbb{R}^3$, for $\sigma = 1$ [4].

Note however that the notion of criticality may differ on a curved space (see e.g. [7, 33]): the curvature of a manifold may create more ill-posedness phenomena, but since in the proof of ill-posedness in [14] (see also [10]), the Laplacian is neglected, the critical Sobolev exponent for local well-posedness cannot be less than in the case of $\mathbb{R}^n$.

This approach suggests that we can consider a more general manifold, up to working on a local chart, and provided that the energy associated to strong solutions of (1.1) is a non-increasing function of time, a question which we leave out at this stage.

Finally, we go back to the whole space case, $x \in \mathbb{R}^n$. As recalled above, if a_0 is compactly supported, then the ansatz that we consider remains supported in the same compact so long as the solution to (2.5) remains smooth; see [27], and also [34]. The justification of WKB analysis for short time shows that at least when $\sigma = 1$ ([21]), or $\sigma \in \mathbb{N}$ and $n \leqslant 3$ ([1]), we have, thanks to Borel lemma,

$$u^\varepsilon(t,x) = u^\varepsilon_{\rm app}(t,x) + \mathcal{O}\left(\varepsilon^\infty\right), \quad \text{in } C\left([0,T]; L^2 \cap L^\infty\right),$$

where $u^\varepsilon_{\rm app}$ is supported in the same compact as a_0. This seems to be an encouraging remark, in view of considering *fixed* initial data as in [25], instead of a sequence of initial data like here. However, the information that we do not have for the Schrödinger equation, and which is available for the wave equation, is a notion of finite speed of propagation for *weak* solutions to the nonlinear equation. This seems to be the only obstacle to consider fixed initial data in the Schrödinger case.

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CNRS & Université Paris-Sud, Mathématiques, Bât. 425, 91405 Orsay cedex, France
E-mail address: `Thomas.Alazard@math.cnrs.fr`

CNRS & Université Montpellier 2, Mathématiques, CC 051, Place Eugène Bataillon, 34095 Montpellier cedex 5, France
E-mail address: `Remi.Carles@math.cnrs.fr`