WKB ANALYSIS FOR THE GROSS–PITAEVSKII EQUATION WITH NON-TRIVIAL BOUNDARY CONDITIONS AT INFINITY

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Abstract. We consider the semi-classical limit for the Gross–Pitaevskii equation. In order to consider non-trivial boundary conditions at infinity, we work in Zhidkov spaces rather than in Sobolev spaces. For the usual cubic nonlinearity, we obtain a point-wise description of the wave function as the Planck constant goes to zero, so long as no singularity appears in the limit system. For a cubic-quintic nonlinearity, we show that working with analytic data may be necessary and sufficient to obtain a similar result.

1. Introduction

We study the semi-classical limit $\hbar \to 0$ for the Gross–Pitaevskii equation

$$i\hbar\partial_t u + \frac{\hbar^2}{2m}\Delta u = Vu + f\left(|u|^2\right)u,$$

where $x \in \mathbb{R}^n$. In the case of Bose–Einstein condensation (BEC), the external potential $V = V(t,x)$ models an external trap, and the nonlinearity f describes the nonlinear interactions of the particles (see e.g. [10, 24, 18]). We consider two types of nonlinearity f (after renormalization):

- Cubic nonlinearity: $f(|u|^2)u = \left(|u|^2 - 1\right)u$.
- Cubic-quintic nonlinearity: $f(|u|^2)u = \left(|u|^4 + \lambda|u|^2\right)u$, $\lambda \in \mathbb{R}$.

The cubic nonlinearity is certainly the most commonly used model in BEC. The defocusing nonlinearity corresponds to a positive scattering length, as in the case of ^{87}Rb, ^{23}Na and ^{1}H. Note that this model is also used in superfluid theory. See e.g. [10, 24, 18] and references therein. The cubic-quintic nonlinearity, which is mostly used as an envelope equation in optics, is also considered in BEC for alkalimetal gases (see e.g. [13, 1, 23]), in which case $\lambda < 0$. The cubic term corresponds to a negative scattering length, and the quintic term to a repulsive three-body elastic interaction. We also consider the case $\lambda > 0$ (positive scattering length).

1.1. **Cubic nonlinearity.** Up to rescaling the Planck constant, we consider the limit $\varepsilon \to 0$ for:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + \left(|u^\varepsilon|^2 - 1\right)u^\varepsilon, \quad x \in \mathbb{R}^n,\ n \geqslant 1, \tag{1.1}$$

$$u^\varepsilon(0,x) = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon}. \tag{1.2}$$

Our initial data do not necessarily decay to zero at infinity. Typically, we do not assume $a_0^\varepsilon \in L^2(\mathbb{R}^n)$ (see Theorem 1.3 below). Recently, the Cauchy problem [11, 16] and the semi-classical limit [20] for (1.1) with $V \equiv 0$ have been studied more systematically. When the external potential V is zero, $V \equiv 0$, the Hamiltonian structure yields, at least formally:

$$\frac{d}{dt}\left(\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}^2 + \left\||u^\varepsilon(t)|^2 - 1\right\|_{L^2}^2\right) = 0.$$

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In this case, a natural space to study the Cauchy problem associated to (1.1) is the energy space (see e.g. [5, 16] and references therein)

$$E = \{u \in H^1_{\rm loc}(\mathbb{R}^n) \ ; \ \nabla u \in L^2(\mathbb{R}^n), \ |u|^2 - 1 \in L^2(\mathbb{R}^n)\}.$$

For this quantity to be well defined, one cannot assume that u^ε is in $L^2(\mathbb{R}^n)$; morally, the modulus of u^ε goes to one at infinity. To study solutions which are bounded, but not in $L^2(\mathbb{R}^n)$, P. E. Zhidkov introduced in the one-dimensional case in [28] (see also [29]):

$$X^s(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n) \ ; \ \nabla u \in H^{s-1}(\mathbb{R}^n)\}, \quad s > n/2. \tag{1.3}$$

We also denote $X^\infty := \cap_{s>n/2} X^s$. The study of these spaces was generalized in the multidimensional case by C. Gallo [11]. They make it possible to consider solutions to (1.1) whose modulus has a non-zero limit as $|x| \to \infty$, but not necessarily satisfying $|u^\varepsilon(t,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$. We shall also use these spaces.

Recently, P. Gérard [16] has solved the Cauchy problem for the Gross–Pitaevskii equation in the more natural space E, in space dimensions two and three. The main novelty consists in working with distances instead of norms, in order to apply a fixed point argument in E. In particular, the constraint $|u^\varepsilon(t,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$ is satisfied.

To our knowledge, if the initial data do not vanish at infinity, the introduction of an (unbounded) external potential in Gross–Pitaevskii equation has no physical motivation. Note also that if V is an harmonic potential, then the formal Hamiltonian corresponding to (1.1) is necessarily infinite (see § 2.3). On the other hand, introducing a quadratic external potential or considering a quadratic initial phase ϕ_0 makes no difference in our analysis. The model (1.1)–(1.2) with $V \equiv 0$ and ϕ_0 quadratic is certainly more physically relevant, and does not seem to enter into the framework of the previous mathematical studies. Another motivation to introduce this external potential stems from the study of the semi-classical limit of the Schrödinger–Poisson system, where $|u^\varepsilon|^2 - 1$ is replaced with $V_{\rm p}^\varepsilon$ given by $\Delta V_{\rm p}^\varepsilon = q\left(|u^\varepsilon|^2 - c\right)$. This models appears in the semi-conductor theory where the real number q models a charge, which we may take equal to one here, and the function $c = c(x)$ models a doping profile, which we may take to be $c \equiv 1$. As in [2], we will prove that if V grows quadratically in space, then if $|u^\varepsilon(t=0,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$, one must not expect $|u^\varepsilon(t,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$ for $t > 0$.

Assumptions. *We assume that the potential and the initial phase are of the form:*

- *$V \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$, and $V = V_{\rm quad} + V_{\rm lin}$, where $V_{\rm quad}(t,x) = {}^t x M(t) x$ is a quadratic form, with $M(t) \in \mathcal{S}_n(\mathbb{R})$ a symmetric $n \times n$ matrix, depending smoothly on t, and $\nabla V_{\rm lin} \in C^\infty(\mathbb{R}_t; X^s)$ for all $s > n/2$.*
- *$\phi_0 \in C^\infty(\mathbb{R}^n)$, and $\phi_0 = \phi_{\rm quad} + \phi_{\rm lin}$, where $\phi_{\rm quad}(x) = {}^t x Q_0 x$ is a quadratic form, with Q_0 a symmetric matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$, and $\nabla \phi_{\rm lin} \in X^\infty$.*

Note that our assumptions include the case where $V_{\rm lin}$ and $\phi_{\rm lin}$ are linear in x. In general, these functions are sub-linear in x, since their gradient is bounded.

Lemma 1.1. *There exist $T > 0$ and a unique solution $\phi_{\rm eik} \in C^\infty([0,T] \times \mathbb{R}^n)$ to:*

$$\partial_t \phi_{\rm eik} + \frac{1}{2}|\nabla_x \phi_{\rm eik}|^2 + V_{\rm quad} = 0 \quad ; \quad \phi_{{\rm eik}|t=0} = \phi_{\rm quad}\,. \tag{1.4}$$

Moreover, $\phi_{\rm eik}$ is a quadratic form in x:

$$\phi_{\rm eik}(t,x) = {}^t x Q(t) x, \tag{1.5}$$

where $Q(t) \in \mathcal{S}_n(\mathbb{R})$ is a smooth function of t.

Proof. Existence and uniqueness follow from [8, Lemma 1]. To prove that $\phi_{\rm eik}$ is quadratic in x, seek $\phi_{\rm eik}$ of the form (1.5). Then (1.4) is equivalent to the system of ordinary differential equations

$$\dot{Q}(t) + 2Q(t)^2 + M(t) = 0 \quad ; \quad Q(0) = Q_0.$$

The lemma then follows from Cauchy–Lipschitz Theorem. □

Remark 1.2. As in [2], we shall use the following geometrical interpretation of the above lemma. The time T is such that for $t \in [0,T]$, the map given by

$$\partial_t x(t,y) = \nabla_x \phi_{\rm eik}\left(t, x(t,y)\right) = Q(t)x(t,y) \quad ; \quad x(0,y) = y,$$

defines a global diffeomorphism on $\mathbb{R}^n$. Therefore, the characteristics associated to the operator $\partial_t + \nabla\phi_{\rm eik}\cdot\nabla$ do not meet for $t \in [0,T]$, and this operator is a smooth transport operator:

$$\left(\partial_t + \nabla\phi_{\rm eik}\cdot\nabla\right)\left(f(t,y)\right) = \partial_t f\left(t, x(t,y)\right).$$

Note that if $Q(t)$ and its anti-derivative commute, then we have

$$x(t,y) = \exp\left(\int_0^t Q(\tau)d\tau\right) y.$$

Theorem 1.3. *Suppose that there exist* $a_0, a_1 \in X^\infty$ *such that:*

$$\|a_0^\varepsilon - a_0 - \varepsilon a_1\|_{X^s} = o(\varepsilon), \quad \forall s > n/2. \tag{1.6}$$

There exist $T_* \in]0,T]$ *independent of* $\varepsilon \in]0,1]$*, and a unique solution* $u^\varepsilon \in C^\infty \cap L^\infty([0,T_*]\times\mathbb{R}^n)$ *to* (1.1)–(1.2)*. Moreover, there exist* $a, \phi \in C^\infty([0,T_*]\times\mathbb{R}^n)$ *with* $a, \nabla\phi \in C([0,T_*];X^s)$ *for all* $s > n/2$*, such that:*

$$\limsup_{\varepsilon\to 0}\left\|u^\varepsilon(t,\cdot) - a(t,\cdot)e^{i(\phi(t,\cdot)+\phi_{\rm eik}(t,\cdot))/\varepsilon}\right\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(t) \quad \textit{as } t\to 0. \tag{1.7}$$

The functions a *and* ϕ *depend nonlinearly on* ϕ_0 *and* a_0 *(see* (3.1) *below). There exists* $\phi^{(1)} \in L^\infty([0,T_*]\times\mathbb{R}^n)$*, real-valued, with* $\nabla\phi^{(1)} \in C([0,T_*];X^s)$ *for all* $s > n/2$*, such that:*

$$\limsup_{\varepsilon\to 0}\left\|u^\varepsilon - ae^{i\phi^{(1)}}e^{i(\phi+\phi_{\rm eik})/\varepsilon}\right\|_{L^\infty([0,T_*]\times\mathbb{R}^n)} = 0. \tag{1.8}$$

The modulation $\phi^{(1)}$ *is a nonlinear function of* ϕ_0*,* a_0 *and* a_1 *(see* (3.2) *below).*

Remark 1.4. Several applications of this general results are given, in §3, §4 and §5.

Remark 1.5. If we assume moreover

$$\|a_0^\varepsilon - a_0 - \varepsilon a_1\|_{X^s} = \mathcal{O}(\varepsilon^2), \quad \forall s > n/2,$$

then the above error estimate can be improved:

$$\left\|u^\varepsilon - ae^{i\phi^{(1)}}e^{i(\phi+\phi_{\rm eik})/\varepsilon}\right\|_{L^\infty([0,T_*]\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon).$$

Remark 1.6. The above result and System (3.2) below show that in general, it is necessary to know the initial amplitude a_0^ε up to the order $o(\varepsilon)$ to describe the leading order behavior of the *wave function* u^ε. It is not necessary to know a_0^ε with such precision to study the convergence of quadratic observables. See §6. In particular, in Theorem 6.1, we extend the result of [20] to the three-dimensional case (on a bounded domain, or outside a bounded domain).

Remark 1.7. Most of the results that we present here remain valid in a space-periodic setting, that is if we assume $x \in \mathbb{T}^n$. In that case, compactness arguments show that the proof of Theorem 1.3 remains valid when $V \in C^\infty(\mathbb{R}_t \times \mathbb{T}^n_x)$ and $\phi_0 \in C^\infty(\mathbb{T}^n)$. On the other hand, the discussions in §2.3 and §5 become irrelevant on the torus. Finally, note that it is equivalent to work in Sobolev spaces, since $X^s(\mathbb{T}^n) = H^s(\mathbb{T}^n)$ for $s > n/2$.

The analysis detailed in §2 and §3 shows that the formal part of [7] can be justified in the present framework. We shall only state a typical consequence of this approach:

Corollary 1.8 (Instability). *Let $n \geqslant 1$, $a_0, a_1 \in C^\infty \cap X^\infty(\mathbb{R}^n)$, with $\mathrm{Re}(\overline{a}_0 a_1) \not\equiv 0$, and $\phi_0 \in C^\infty(\mathbb{R}^n)$, with $\nabla\phi_0 \in X^\infty$. Let u^ε and v^ε solve the initial value problems:*

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \left(|u^\varepsilon|^2 - 1\right) u^\varepsilon \ ; \ u^\varepsilon\big|_{t=0} = a_0 e^{i\phi_0/\varepsilon}.$$
$$i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2}\Delta v^\varepsilon = \left(|v^\varepsilon|^2 - 1\right) v^\varepsilon \ ; \ v^\varepsilon\big|_{t=0} = \left(a_0 + \delta^\varepsilon a_1\right) e^{i\phi_0/\varepsilon},$$

where $\delta^\varepsilon \to 0$. Assume that there exists $N \in \mathbb{N}$ such that $\delta^\varepsilon/\varepsilon^{1-\frac{1}{N}} \to +\infty$. Then we can find $t^\varepsilon \to 0$ such that $\liminf_{\varepsilon\to 0} \|u^\varepsilon(t^\varepsilon) - v^\varepsilon(t^\varepsilon)\|_{L^\infty} > 0$. In particular,

$$\liminf_{\varepsilon\to 0} \frac{\|u^\varepsilon - v^\varepsilon\|_{L^\infty([0,t^\varepsilon]\times\mathbb{R}^n)}}{\left\| u^\varepsilon_{|t=0} - v^\varepsilon_{|t=0}\right\|_{L^\infty(\mathbb{R}^n)}} = +\infty.$$

Remark 1.9. Note that if $\phi_0 \equiv 0$, then we also have:

$$\liminf_{\varepsilon\to 0} \frac{\|u^\varepsilon - v^\varepsilon\|_{L^\infty([0,t^\varepsilon]\times\mathbb{R}^n)}}{\left\| u^\varepsilon_{|t=0} - v^\varepsilon_{|t=0}\right\|_{X^s}} = +\infty, \quad \forall s > n/2.$$

This shows that the instability mechanism is not due to regularity issues. It is due to the fact that (1.1) is super-critical as far as WKB analysis is concerned: the small initial perturbation (of order δ^ε) yields a high-frequency perturbation of the evolution (a multiplicative factor of the form $e^{-2it\delta^\varepsilon \mathrm{Re}(\overline{a}_0 a_1)/\varepsilon}$).

1.2. **Cubic-quintic nonlinearity.** Denote $f_\lambda(y) = y^2 + \lambda y$. We now consider

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta u^\varepsilon = f_\lambda\left(|u^\varepsilon|^2\right) u^\varepsilon, \quad x \in \mathbb{R}^n, \ n \geqslant 1, \\ \qquad u^\varepsilon(0,x) = a_0^\varepsilon(x) e^{i\phi_0(x)/\varepsilon}. \end{cases} \tag{1.9}$$

Note that in (1.9), we assume that there is no external potential, $V = 0$. We also assume that there is no initial quadratic oscillation: $\phi_0 \in C^\infty(\mathbb{R}^n;\mathbb{R})$, with $\nabla\phi_0 \in X^\infty$. The case $\lambda > 0$, $V \neq 0$, with $a_0^\varepsilon \in H^\infty$, is contained in [8]. We assume $V_{\mathrm{quad}} = 0$ here in order to consider non-zero boundary conditions at infinity. We also assume $V_{\mathrm{lin}} = 0$ for simplicity only.

Plugging an approximate solution of the form $u^\varepsilon \approx a e^{i\phi/\varepsilon}$, with a and ϕ independent of ε, and passing to the limit $\varepsilon \to 0$ as in [14, 20], we find formally that $(\rho, v) := (|a|^2, \nabla\phi)$ solves:

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho v) = 0. \\ \partial_t v + v\cdot\nabla v + \nabla\left(f_\lambda(\rho)\right) = 0. \end{cases} \tag{1.10}$$

If $\lambda > 0$, then the problem is hyperbolic. Essentially, the result of Theorem 1.3 remains valid. When $\lambda < 0$, the above problem is hyperbolic for $\rho > |\lambda|/2$ and elliptic for $\rho < |\lambda|/2$. This feature is reminiscent of Euler equations of gas dynamics in Lagrangian coordinates:

$$\begin{cases} \partial_t u + \partial_x v = 0. \\ \partial_t v + \partial_x \left(p(u)\right) = 0. \end{cases} \tag{1.11}$$

As recalled in [22], a typical mathematical example for van der Waals state laws is given by $p(u) = (u^2-1)u$. The problem is hyperbolic if $u > 1/\sqrt{3}$, and elliptic if $u < 1/\sqrt{3}$. Hadamard's argument implies that the only reasonable framework to study (1.10) or (1.11) is that of analytic functions (see [22]). In this case, we refer to the approach of [15, 27]. More details are given in §7. When the elliptic region for (1.10) is avoided, then essentially, Theorem 1.3 remains valid:

Theorem 1.10. *Suppose that there exist $a_0, a_1 \in X^\infty$ such that:*

$$\|a_0^\varepsilon - a_0 - \varepsilon a_1\|_{X^s} = o(\varepsilon), \quad \forall s > n/2.$$

Assume moreover that $\phi_0 \in C^\infty(\mathbb{R}^n;\mathbb{R})$ with $\nabla\phi_0 \in X^\infty$, and:

- *Either $\lambda > 0$,*
- *Or $\lambda < 0$ and there exists $\delta > 0$ such that $|a_0(x)|^2 \geqslant \delta + \frac{|\lambda|}{2}$, $\forall x \in \mathbb{R}^n$.*

Then there exist $\varepsilon_*, T_* > 0$, and a unique solution $u^\varepsilon \in C^\infty \cap L^\infty([0,T_*]\times\mathbb{R}^n)$ to (1.9) for all $\varepsilon \in]0,\varepsilon_*]$. Moreover, there exist $a, \phi \in C^\infty([0,T_*]\times\mathbb{R}^n)$ with $a, \nabla\phi \in C([0,T_*];X^s)$ for all $s > n/2$, such that:

$$\limsup_{\varepsilon\to 0} \left\| u^\varepsilon(t,\cdot) - a(t,\cdot)e^{i\phi(t,\cdot)/\varepsilon} \right\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(t) \quad \text{as } t \to 0.$$

There exists $\phi^{(1)} \in L^\infty([0,T_*]\times\mathbb{R}^n)$, real-valued, with $\nabla\phi^{(1)} \in C([0,T_*];X^s)$ for all $s > n/2$, such that:

$$\limsup_{\varepsilon\to 0} \left\| u^\varepsilon - a e^{i\phi^{(1)}} e^{i\phi/\varepsilon} \right\|_{L^\infty([0,T_*]\times\mathbb{R}^n)} = 0.$$

1.3. **Structure of the paper.** In §2, we construct the solution u^ε as $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$, where a^ε is complex-valued and Φ^ε is real-valued. This yields the existence part of Theorems 1.3 and 1.10. The proof of these theorems is completed in §3, where the limit of $(a^\varepsilon, \Phi^\varepsilon)$ as ε goes to zero is studied. We give three examples of applications of Theorem 1.3 in §4, in the case $\phi_{\rm eik} = 0$. In §5, we study the time evolution of a non-trivial boundary condition at infinity when $\phi_{\rm eik} \neq 0$. In §6, we investigate the limit of the position and current densities. Finally, we explain why working in an analytic setting is often necessary (and always sufficient) in the case of the cubic-quintic nonlinearity.

2. Construction of the solution

2.1. **Phase-amplitude representation: the case $\phi_{\rm eik} = V = 0$.** When V and ϕ_0 are identically zero, the existence and uniqueness part of Theorem 1.3 was established by C. Gallo [11]. Note however that with our scaling, the fact that T_* is independent of $\varepsilon \in]0,1]$ does not follow from [11]. Since the approach in Zhidkov spaces is rather similar to the one in Sobolev spaces, we shall essentially explain the new aspects of the proof. To treat both cubic and cubic-quintic nonlinearities, consider the general equation

$$\begin{cases} i\varepsilon\partial_t u^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta u^\varepsilon = f\left(|u^\varepsilon|^2\right)u^\varepsilon, \quad x \in \mathbb{R}^n,\ n \geqslant 1, \\ u^\varepsilon(0,x) = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon}, \end{cases} \tag{2.1}$$

where $f \in C^\infty(\mathbb{R}_+;\mathbb{R})$. We keep the hierarchy introduced by E. Grenier [17]: seek $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon}$, where a^ε is complex-valued, and Φ^ε is real-valued. We impose

$$
\left\{
\begin{aligned}
&\partial_t \Phi^\varepsilon + \frac{1}{2}\left|\nabla \Phi^\varepsilon\right|^2 + f\left(|a^\varepsilon|^2\right) = 0 \quad ; \quad \Phi^\varepsilon\big|_{t=0} = \phi_0,\\
&\partial_t a^\varepsilon + \nabla\Phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \Phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,.
\end{aligned}
\right.
\tag{2.2}
$$

As an intermediary unknown function, introduce the "velocity" $v^\varepsilon = \nabla\Phi^\varepsilon$. Separate real and imaginary parts of a^ε, $a^\varepsilon = a_1^\varepsilon + i a_2^\varepsilon$, and introduce:

$$
\mathbf{u}^\varepsilon = \begin{pmatrix} a_1^\varepsilon \\ a_2^\varepsilon \\ v_1^\varepsilon \\ \vdots \\ v_n^\varepsilon \end{pmatrix}, \quad \mathbf{u}_0^\varepsilon = \begin{pmatrix} \mathrm{Re}(a_0^\varepsilon) \\ \mathrm{Im}(a_0^\varepsilon) \\ \partial_1\phi_0 \\ \vdots \\ \partial_n \phi_0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & -\Delta & 0 & \dots & 0 \\ \Delta & 0 & 0 & \dots & 0 \\ 0 & 0 & & 0_{n\times n} & \end{pmatrix},
$$

$$
\text{and} \quad A(\mathbf{u},\xi) = \sum_{j=1}^n A_j(\mathbf{u})\xi_j = \begin{pmatrix} v\cdot\xi & 0 & \frac{a_1}{2}\,{}^t\xi \\ 0 & v\cdot\xi & \frac{a_2}{2}\,{}^t\xi \\ 2f' a_1\,\xi & 2f' a_2\,\xi & v\cdot\xi I_n \end{pmatrix},
$$

where f' stands for $f'(|a_1|^2+|a_2|^2)$. We now have the system:

$$
\partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon = \frac{\varepsilon}{2} L\mathbf{u}^\varepsilon \quad ; \quad \mathbf{u}^\varepsilon_{|t=0} = \mathbf{u}_0^\varepsilon. \tag{2.3}
$$

The matrices A_j are symmetrized by the matrix

$$
S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{4f'} I_n \end{pmatrix},
$$

which is symmetric positive if and only if $f'\left(|a_1|^2+|a_2|^2\right) > 0$: this includes the case of the decofusing cubic nonlinearity (1.1), of the cubic-quintic nonlinearity (1.9) with $\lambda > 0$, and of the cubic-quintic nonlinearity (1.9) with $\lambda < 0$, provided that $|a_1|^2 + |a_2|^2 > |\lambda|/2$.

Proposition 2.1. *Assume that $\mathbf{u}_0^\varepsilon$ is bounded in X^s for all $s > n/2$, uniformly for $\varepsilon \in [0,1]$, and that there exists $\varepsilon_* > 0$ and $\delta > 0$ such that*

$$
f'\left(|a_0^\varepsilon|^2\right) \geqslant \delta > 0, \quad \forall x \in \mathbb{R}^n,\ \forall \varepsilon \in [0,\varepsilon_*].
$$

Then for $s > n/2+2$, there exist $T_* > 0$ and a unique solution $\mathbf{u}^\varepsilon \in C([0,T_*];X^s)$ to (2.3) for all $\varepsilon \in [0,\varepsilon_*]$. In addition, this solution is in $C([0,T_*];X^m)$ for all $m > n/2$, with bounds independent of $\varepsilon \in [0,\varepsilon_*]$.

Proof. Let $s > n/2+2$. As usual, the main point consists in obtaining *a priori* estimates for the system (2.3), so we shall focus our attention on this aspect. We have an *a priori* bound for $\mathbf{u}^\varepsilon$ in L^∞:

$$
\begin{aligned}
\|\mathbf{u}^\varepsilon(t)\|_{L^\infty} \leqslant & \|\mathbf{u}_0^\varepsilon\|_{L^\infty} + \int_0^t \sum_{j=1}^n \|A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon(\tau)\|_{L^\infty} d\tau + \int_0^t \|\Delta \mathbf{u}^\varepsilon(\tau)\|_{L^\infty} d\tau \\
\leqslant & \|\mathbf{u}_0^\varepsilon\|_{L^\infty} + \int_0^t F\left(\|\mathbf{u}^\varepsilon(\tau)\|_{L^\infty}\right) \|\nabla \mathbf{u}^\varepsilon(\tau)\|_{H^{s-1}} d\tau \\
& + C\int_0^t \|\Delta \mathbf{u}^\varepsilon(\tau)\|_{H^{s-2}} d\tau.
\end{aligned}
$$

We infer:

$$
\|\mathbf{u}^\varepsilon(t)\|_{L^\infty} \leqslant \|\mathbf{u}_0^\varepsilon\|_{L^\infty} + \int_0^t G\left(\|\mathbf{u}^\varepsilon(\tau)\|_{X^s}\right) \|\mathbf{u}^\varepsilon(\tau)\|_{X^s} d\tau. \tag{2.4}
$$

To have a closed system of estimates, introduce $P = (I - \Delta)^{(s-1)/2}\nabla$, so that $\|f\|_{X^s} \approx \|f\|_{L^\infty} + \|Pf\|_{L^2}$. Denote

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx,$$

the scalar product in L^2. Since S is symmetric, we have

$$\frac{d}{dt}\left\langle SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle = \left\langle \partial_t SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle + 2\,\mathrm{Re}\left\langle S\partial_t P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle,$$

So long as

$$f'\left(|a^\varepsilon|^2\right) \geqslant \frac{\delta}{2} > 0, \tag{2.5}$$

we have the following set of estimates. First,

$$\begin{aligned}\left\langle \partial_t SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle &\leqslant \|\partial_t S\|_{L^\infty}\|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2\\ &\leqslant C_\delta\left(\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}\right)\|\partial_t\mathbf{u}^\varepsilon(t)\|_{L^\infty}\|\mathbf{u}^\varepsilon(t)\|_{X^s}^2.\end{aligned}$$

Directly from (2.3), we have:

$$\begin{aligned}\|\partial_t\mathbf{u}^\varepsilon(t)\|_{L^\infty} &\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}\right)\|\nabla\mathbf{u}^\varepsilon(t)\|_{L^\infty} + \|\Delta\mathbf{u}^\varepsilon(t)\|_{L^\infty}\\ &\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{X^s}\right)\|\mathbf{u}^\varepsilon(t)\|_{X^s}.\end{aligned}$$

Since SL is skew-symmetric, we have

$$\mathrm{Re}\left\langle SLP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle = 0,$$

which prevents any loss of regularity in the estimates. For the quasi-linear term involving the matrices A_j, we note that since SA_j is symmetric, commutator estimates (see [19]) yield:

$$\begin{aligned}\sum_{j=1}^n \left\langle SP\left(A_j(\mathbf{u}^\varepsilon)\partial_j\mathbf{u}^\varepsilon\right), P\mathbf{u}^\varepsilon(t)\right\rangle &\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{L^\infty}\right)\|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2\|\nabla\mathbf{u}^\varepsilon(t)\|_{L^\infty}\\ &\leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{X^s}\right)\|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2.\end{aligned}$$

Finally, we have:

$$\frac{d}{dt}\left\langle SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle \leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{X^s}\right)\|\mathbf{u}^\varepsilon(t)\|_{X^s}^2.$$

This estimate, along with (2.4), shows that on a sufficiently small time interval $[0, T_*]$, with $T_* > 0$ independent of $\varepsilon \in [0, \varepsilon_*]$, (2.5) holds. This yields the first part of Proposition 2.1.

The fact that the local existence time does not depend on $s > n/2 + 2$ follows from the continuation principle based on Moser's calculus and tame estimates (see e.g. [21, Section 2.2] or [26, Section 16.1]). □

The existence part of Theorem 1.10 and of Theorem 1.3 when $\phi_{\rm eik} = 0$ follows. Indeed, define Φ^ε by

$$\Phi^\varepsilon(t) = \phi_0 - \int_0^t \left(\frac{1}{2}|v^\varepsilon(\tau)|^2 + f\left(|a^\varepsilon(\tau)|^2\right)\right)d\tau.$$

We check that $\partial_t(\nabla\Phi^\varepsilon - v^\varepsilon) = \nabla\partial_t\Phi^\varepsilon - \partial_t v^\varepsilon = 0$, so that $\nabla\Phi^\varepsilon = v^\varepsilon$, and $(\Phi^\varepsilon, a^\varepsilon)$ solves (2.2). Finally, uniqueness for (2.1) follows from energy estimates. If $u^\varepsilon, v^\varepsilon \in C^\infty \cap L^\infty([0, T_*]\times\mathbb{R}^n)$ solve (2.1), then $w^\varepsilon := u^\varepsilon - v^\varepsilon$ satisfies:

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = f\left(|u^\varepsilon|^2\right)u^\varepsilon - f\left(|v^\varepsilon|^2\right)v^\varepsilon\ ;\ w^\varepsilon_{|t=0} = 0.$$

We have, for $t \in [0, T_*]$,

$$\|w^\varepsilon\|_{L^\infty(0,t;L^2)} \leqslant C\left(\|u^\varepsilon\|_{L^\infty([0,T_*]\times\mathbb{R}^n)}, \|v^\varepsilon\|_{L^\infty([0,T_*]\times\mathbb{R}^n)}\right)\|w^\varepsilon\|_{L^1(0,t;L^2)},$$

and Gronwall lemma yields $w^\varepsilon \equiv 0$.

2.2. **Phase-amplitude representation: the case $\phi_{\rm eik} \neq 0$.** We know consider (1.1)–(1.2) only: the nonlinearity is exactly cubic. To take the presence of V and $\phi_{\rm quad}$ into account, we proceed as in [8]: we construct the solution as $u^\varepsilon = a^\varepsilon e^{i(\phi^\varepsilon+\phi_{\rm eik})/\varepsilon}$. The analogue of (2.2) is:

$$\begin{cases} \partial_t \Phi^\varepsilon + \dfrac{1}{2}\left|\nabla \Phi^\varepsilon\right|^2 + V + |a^\varepsilon|^2 - 1 = 0 \quad ; \quad \Phi^\varepsilon\big|_{t=0} = \phi_0,\\ \partial_t a^\varepsilon + \nabla\Phi^\varepsilon\cdot\nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\Phi^\varepsilon = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,. \end{cases}$$

Set $\Phi^\varepsilon = \phi^\varepsilon + \phi_{\rm eik}$. The introduction of $\phi_{\rm eik}$ allows us to get rid of the terms $V_{\rm quad}$ and $\phi_{\rm quad}$, and work in Zhidkov spaces. The above problem reads, in terms of $(\phi^\varepsilon, a^\varepsilon)$:

$$(2.6)\qquad \begin{cases} \partial_t \phi^\varepsilon + \dfrac{1}{2}\left|\nabla \phi^\varepsilon\right|^2 + \nabla\phi_{\rm eik}\cdot\nabla\phi^\varepsilon + V_{\rm lin} + |a^\varepsilon|^2 - 1 = 0,\\ \partial_t a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi^\varepsilon + \dfrac{1}{2}a^\varepsilon\Delta\phi_{\rm eik} = i\dfrac{\varepsilon}{2}\Delta a^\varepsilon,\\ \phi^\varepsilon\big|_{t=0} = \phi_{\rm lin} \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,. \end{cases}$$

Resume the notations of the previous paragraph, with now:

$$\mathbf{\Sigma} = \begin{pmatrix} 0\\ 0\\ \partial_1 V_{\rm lin}\\ \vdots\\ \partial_n V_{\rm lin}\end{pmatrix}, \quad \text{and} \quad A(\mathbf{u},\xi) = \sum_{j=1}^n A_j(\mathbf{u})\xi_j = \begin{pmatrix} v\cdot\xi & 0 & \frac{a_1}{2}\,{}^t\xi\\ 0 & v\cdot\xi & \frac{a_2}{2}\,{}^t\xi\\ 2a_1\,\xi & 2a_2\,\xi & v\cdot\xi I_n\end{pmatrix}.$$

The system (2.3) is replaced by:

$$(2.7)\quad \partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j\mathbf{u}^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla\mathbf{u}^\varepsilon + \tilde{M}\mathbf{u}^\varepsilon + \mathbf{\Sigma} = \frac{\varepsilon}{2}L\mathbf{u}^\varepsilon \quad ; \quad \mathbf{u}^\varepsilon_{|t=0} = \mathbf{u}^\varepsilon_0,$$

where $\tilde{M} = \tilde{M}(t)$ is a matrix depending on time only, since $\phi_{\rm eik}$ is exactly quadratic in x. This aspect seems necessary in the proof of Proposition 2.2 below. This explains why we make Assumptions 1.1, and do not content ourselves with general sub-quadratic potential and initial phase as in [8]. The important aspect to notice is that since the nonlinearity in (1.1) is *exactly* cubic, then the matrices A_j are symmetrized by a *constant* matrix, namely:

$$S = \begin{pmatrix} I_2 & 0\\ 0 & \frac{1}{4}I_n\end{pmatrix}.$$

In [8], nonlinearities which are cubic *at the origin* were considered (as in [17]), and the possibly quadratic phase $\phi_{\rm eik}$ made the assumption $xa_0^\varepsilon \in L^2(\mathbb{R}^n)$ apparently necessary, to control the time derivative of the symmetrizer. Of course, we want to avoid this decay assumption for the Gross–Pitaevskii equation, so working with a constant symmetrizer is important.

Proposition 2.2. *Assume that $\mathbf{u}_0^\varepsilon$ is bounded in X^s for all $s > n/2$, uniformly for $\varepsilon \in [0,1]$. Then for $s > n/2+2$, there exist $T_* \in]0,T]$, independent of $\varepsilon \in [0,1]$, and a unique solution $\mathbf{u}^\varepsilon \in C([0,T_*];X^s)$ to (2.7). In addition, this solution is in $C([0,T_*];X^m)$ for all $m > n/2$, with bounds independent of $\varepsilon \in [0,1]$.*

Sketch of the proof. The proof follows the same lines as the proof of Proposition 2.1, so we shall only point out the differences.

Let $s > n/2+2$. By construction, the operator $\partial_t + \nabla\phi_{\rm eik}\cdot\nabla$ is a transport operator along the characteristics associated to $\phi_{\rm eik}$, which do not intersect for $t\in[0,T]$. Therefore, we have an *a priori* bound for $\mathbf{u}^\varepsilon$ in L^∞:

$$\begin{aligned}\|\mathbf{u}^\varepsilon(t)\|_{L^\infty} \leqslant & \|\mathbf{u}_0^\varepsilon\|_{L^\infty} + \int_0^t \sum_{j=1}^n \|A_j(u)\partial_j u(\tau)\|_{L^\infty} d\tau \\ & + \int_0^t \Big(C\|\mathbf{u}^\varepsilon(\tau)\|_{L^\infty} + \|\mathbf{\Sigma}(\tau)\|_{L^\infty} + \|\Delta \mathbf{u}^\varepsilon(\tau)\|_{L^\infty}\Big) d\tau \\ \leqslant & \|\mathbf{u}_0^\varepsilon\|_{L^\infty} + C\int_0^t \left(1+\|\mathbf{u}^\varepsilon(\tau)\|_{X^s}\right)\|\mathbf{u}^\varepsilon(\tau)\|_{X^s} d\tau + \|\mathbf{\Sigma}\|_{L^\infty([0,T];X^s)}. \end{aligned} \tag{2.8}$$

To have a closed system of estimates, resume the operator $P = (I-\Delta)^{(s-1)/2}\nabla$, so that $\|f\|_{X^s} \approx \|f\|_{L^\infty} + \|Pf\|_{L^2}$. We have

$$\frac{d}{dt}\left\langle SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle = 2\,\mathrm{Re}\left\langle S\partial_t P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle,$$

since S is constant symmetric. Since SL is skew-symmetric, we have

$$\mathrm{Re}\left\langle SLP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle = 0.$$

For the quasi-linear term involving the matrices A_j, we note that since SA_j is symmetric, commutator estimates yield:

$$\sum_{j=1}^n \left\langle SP\left(A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon\right), P\mathbf{u}^\varepsilon(t)\right\rangle \leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{X^s}\right)\|P\mathbf{u}^\varepsilon(t)\|_{L^2}^2.$$

Next, write

$$\begin{aligned}\left\langle SP\left(\nabla\phi_{\rm eik}\cdot\nabla\mathbf{u}^\varepsilon(t)\right), P\mathbf{u}^\varepsilon(t)\right\rangle = & \left\langle S\nabla\phi_{\rm eik}\cdot\nabla P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle \\ & + \left\langle S[P,\nabla\phi_{\rm eik}\cdot\nabla]\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle.\end{aligned}$$

The first term of the right-hand side is estimated thanks to an integration by parts:

$$\begin{aligned}2\,\mathrm{Re}\left\langle S\nabla\phi_{\rm eik}\cdot\nabla P\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle &= \int S\nabla\phi_{\rm eik}(t,x)\cdot\nabla|P\mathbf{u}^\varepsilon(t,x)|^2 dx \\ &= -\int S\Delta\phi_{\rm eik}(t,x)|P\mathbf{u}^\varepsilon(t,x)|^2 dx.\end{aligned}$$

For the second term, we notice that $[P,\nabla\phi_{\rm eik}\cdot\nabla] = \psi\nabla$, where $\psi = \psi(t,D)$ is a pseudo-differential operator in x, of order $s-1$, depending smoothly of $t\in[0,T]$. Therefore,

$$2\,\mathrm{Re}\left\langle SP\left(\nabla\phi_{\rm eik}\cdot\nabla\mathbf{u}^\varepsilon(t)\right), P\mathbf{u}^\varepsilon(t)\right\rangle \lesssim \|\mathbf{u}^\varepsilon(t)\|_{X^s}^2.$$

The fact that $\tilde M$ is independent of x is crucial here, to ensure that $P(\tilde M\mathbf{u}^\varepsilon)\in L^2$ for $\mathbf{u}^\varepsilon\in X^s$. If $\tilde M$ depended on x, that is if $\phi_{\rm eik}$ was not a polynomial of order at most two, the low frequencies might be a problem at this step of the proof. Finally, we have:

$$\frac{d}{dt}\left\langle SP\mathbf{u}^\varepsilon(t), P\mathbf{u}^\varepsilon(t)\right\rangle \leqslant C\left(\|\mathbf{u}^\varepsilon(t)\|_{X^s}\right)\|\|\mathbf{u}^\varepsilon(t)\|_{X^s}^2.$$

This estimate, along with (2.8), yields the first part of Proposition 2.2. We conclude like in the proof of Proposition 2.1. □

The existence part of Theorem 1.3 follows from the above result, by setting

$$\phi^\varepsilon(t) = \phi_{\rm lin} - \int_0^t \left(\frac{1}{2}\left|v^\varepsilon(\tau)\right|^2 + \nabla\phi_{\rm eik}(\tau)\cdot v^\varepsilon(\tau) + V_{\rm lin}(\tau) + |a^\varepsilon(\tau)|^2 - 1\right) d\tau.$$

Finally, uniqueness for (1.1)–(1.2) follows from energy estimates. If $u^\varepsilon, v^\varepsilon \in C^\infty \cap L^\infty([0,T_*]\times\mathbb{R}^n)$ solve (1.1)–(1.2), then $w^\varepsilon := u^\varepsilon - v^\varepsilon$ satisfies:

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\Delta w^\varepsilon = (V-1)w^\varepsilon + |u^\varepsilon|^2 u^\varepsilon - |v^\varepsilon|^2 v^\varepsilon \ ; \ w^\varepsilon_{|t=0} = 0.$$

We have, for $t \in [0,T_*]$,

$$\|w^\varepsilon\|_{L^\infty(0,t;L^2)} \lesssim \left(\|u^\varepsilon\|^2_{L^\infty([0,T_*]\times\mathbb{R}^n)} + \|v^\varepsilon\|^2_{L^\infty([0,T_*]\times\mathbb{R}^n)}\right)\|w^\varepsilon\|_{L^1(0,t;L^2)},$$

and Gronwall lemma yields $w^\varepsilon \equiv 0$.

2.3. **On the Hamiltonian structure.** When $V = V(x)$ is time-independent, (1.1) formally has a Hamiltonian structure, with

$$H = \frac{1}{2}\|\varepsilon\nabla u^\varepsilon(t)\|^2_{L^2} + \int_{\mathbb{R}^n} V(x)|u^\varepsilon(t,x)|^2 dx + \frac{1}{2}\left\||u^\varepsilon(t)|^2 - 1\right\|^2_{L^2}.$$

When $V \equiv 0$, this structure is used in [16] to prove the global existence of solutions in the energy space. On the other hand, suppose that V is, say, harmonic:

$$V(x) = \sum_{j=1}^n \lambda_j x_j^2,$$

where the constants $\lambda_j \geqslant 0$ are not all equal to zero. Then necessarily, H is infinite: suppose for instance that $\lambda_1 > 0$. Then if $\partial_{x_1}u^\varepsilon(t,\cdot), x_1 u^\varepsilon(t,\cdot) \in L^2(\mathbb{R}^n)$, the uncertainty principle (a simple integration by parts, plus Cauchy–Schwarz inequality in this case) yields:

$$u^\varepsilon(t,\cdot) \in L^2(\mathbb{R}^n).$$

Therefore, the constraint $|u^\varepsilon(t,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$ cannot be satisfied, for otherwise, $1 = 1 - |u^\varepsilon(t,\cdot)|^2 + |u^\varepsilon(t,\cdot)|^2 \in L^2(\mathbb{R}^n) + L^1(\mathbb{R}^n)$.

Similarly, assume that $V \equiv 0$, but $\phi_{\rm quad} \neq 0$: rapid quadratic oscillations are present in the initial data. We have

$$\varepsilon\nabla u^\varepsilon_{|t=0} = \left(\varepsilon\nabla a_0^\varepsilon + i a_0^\varepsilon\nabla\phi_0\right)e^{i\phi_0/\varepsilon}.$$

Therefore, the above quantity is in L^2 provided that $\nabla a_0^\varepsilon, a_0^\varepsilon\nabla\phi_{\rm quad} \in L^2(\mathbb{R}^n)$. If for instance $\phi_{\rm quad}(x) = cx_1^2$ with $c \neq 0$, the last assumption means that $x_1 a_0^\varepsilon \in L^2(\mathbb{R}^n)$, which brings us back to the previous discussion.

We shall see in Section 5 that if $\phi_{\rm eik} \not\equiv 0$, and if $a_0^\varepsilon \in X^\infty$ is such that

$$|a_0^\varepsilon|^2 - 1 \in L^2(\mathbb{R}^n),$$

then the last constraint present in H is not propagated in general. In small time at least, one has generically

$$|u^\varepsilon(t,\cdot)|^2 - 1 \notin L^2(\mathbb{R}^n).$$

3. Semi-classical analysis

We now complete the proof of Theorem 1.3. The end of the proof of Theorem 1.10 follows essentially the same lines, so we omit it. The main adaptation is due to the fact that when the nonlinearity is not exactly cubic, the symmetrizer S is not constant. We refer to [17] or [8], to see that the proof below is easily adapted.

Introduce (ϕ, a), solution to (2.6) with $\varepsilon = 0$, that is

$$\left\{\begin{aligned} &\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \nabla\phi_{\rm eik}\cdot\nabla\phi + V_{\rm lin} + |a|^2 - 1 = 0 \ ; \ \phi\big|_{t=0} = \phi_{\rm lin}, \\ &\partial_t a + \nabla\phi\cdot\nabla a + \nabla\phi_{\rm eik}\cdot\nabla a + \frac{1}{2}a\Delta\phi + \frac{1}{2}a\Delta\phi_{\rm eik} = 0 \ ; \ a\big|_{t=0} = a_0. \end{aligned}\right. \tag{3.1}$$

It is a particular case of Proposition 2.2 that (3.1) has a unique solution, such that $a, \nabla\phi \in C([0,T_*];X^s)$ for all $s > n/2$.

Proposition 3.1. *Under the assumptions of Theorem 1.3, let $(\phi^\varepsilon, a^\varepsilon)$ and (ϕ, a) be given by (2.6) and (3.1) respectively. For all $s > n/2$, there exists C_s such that*

$$\|\nabla(\phi^\varepsilon - \phi)\|_{L^\infty([0,T_*];X^s)} + \|a^\varepsilon - a\|_{L^\infty([0,T_*];X^s)} \leqslant C_s\varepsilon.$$

Sketch of the proof. We shall give the outline of the proof, since it is very similar to the case of Sobolev spaces [8]. The differences are those pointed out in the proof of Proposition 2.2. Resuming the notations of §2, set

$$\mathbf{u} = \begin{pmatrix} \operatorname{Re} a \\ \operatorname{Im} a \\ \partial_1\phi \\ \vdots \\ \partial_n\phi \end{pmatrix}, \quad \mathbf{w}_0^\varepsilon = \begin{pmatrix} \operatorname{Re}(a_0^\varepsilon - a_0) \\ \operatorname{Im}(a_0^\varepsilon - a_0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Denoting $\mathbf{w}^\varepsilon = \mathbf{u}^\varepsilon - \mathbf{u}$, (2.7) yields:

$$\begin{cases} \partial_t \mathbf{w}^\varepsilon + \displaystyle\sum_{j=1}^n \left(A_j(\mathbf{u}^\varepsilon)\partial_j\mathbf{u}^\varepsilon - A_j(\mathbf{u})\partial_j\mathbf{u}\right) + \nabla\phi_{\rm eik}\cdot\nabla\mathbf{w}^\varepsilon + \tilde{M}\mathbf{w}^\varepsilon = \dfrac{\varepsilon}{2}L\mathbf{w}^\varepsilon + \dfrac{\varepsilon}{2}L\mathbf{u}\,, \\ \mathbf{w}^\varepsilon_{|t=0} = \mathbf{w}_0^\varepsilon. \end{cases}$$

We know by Proposition 2.2 that $\mathbf{u}^\varepsilon$ and $\mathbf{u}$ are bounded in $C([0,T_*];X^s)$ for all $s > n/2$. The source term $\mathbf{\Sigma}$ in (2.7) is now replaced by $\frac{\varepsilon}{2}L\mathbf{u}$, which is of order $\mathcal{O}(\varepsilon)$ in $C([0,T_*];X^s)$, and we have easily, for $s > n/2$ and $t \in [0,T_*]$:

$$\|\mathbf{w}^\varepsilon\|_{L^\infty([0,t];X^s)} \leqslant \|\mathbf{w}_0^\varepsilon\|_{X^s} + \mathcal{O}(\varepsilon) + \int_0^t \|\mathbf{w}^\varepsilon(\tau)\|_{X^s}d\tau.$$

The proposition follows from Gronwall lemma. □

Remark 3.2. Note that for the time T_* in Proposition 3.1 (as well as in Proposition 3.4 below), we can pick the life-span of (ϕ, a), the solution of (3.1). Indeed, the error estimate and the standard continuity argument show that $(\phi^\varepsilon, a^\varepsilon)$ cannot blow-up as long as (ϕ, a) remains smooth, provided that ε is chosen sufficiently small. In particular, if (ϕ, a) remains smooth globally in time, then for any $\tau > 0$, we can find $\varepsilon(\tau) > 0$ such that Proposition 3.1 and Proposition 3.4 below remain valid on $[0,\tau]$ for $\varepsilon \in]0, \varepsilon(\tau)]$. On the other hand, one must not expect $T_* = \infty$ in general: the solution to (6.1) may not remain smooth for all time. See [25].

Corollary 3.3. *There exists C such that for all $t \in [0,T_*]$,*

$$\|\phi^\varepsilon(t,\cdot) - \phi(t,\cdot)\|_{L^\infty} \leqslant C\varepsilon t.$$

Proof. Set $w_\phi^\varepsilon = \phi^\varepsilon - \phi$. It satisfies

$$(\partial_t + \nabla\phi_{\rm eik}\cdot\nabla)w_\phi^\varepsilon = \frac{1}{2}\left(|\nabla\phi|^2 - |\nabla\phi^\varepsilon|^2\right) + |a|^2 - |a^\varepsilon|^2 \ ; \ w^\varepsilon_{\phi|t=0} = 0.$$

By Proposition 3.1, the right hand side is $\mathcal{O}(\varepsilon)$ in L^∞. Integration along the characteristics associated to $\partial_t + \nabla\phi_{\rm eik}\cdot\nabla$ (see Remark 1.2) yields the result. □

The first estimate (1.7) of Theorem 1.3 follows easily:

$$\begin{aligned} u^\varepsilon - ae^{i\phi/\varepsilon} &= a^\varepsilon e^{i\phi^\varepsilon/\varepsilon} - ae^{i\phi/\varepsilon} = (a^\varepsilon - a)\,e^{i\phi^\varepsilon/\varepsilon} + a\left(e^{i\phi^\varepsilon/\varepsilon} - e^{i\phi/\varepsilon}\right) \\ &= \mathcal{O}(\varepsilon) + ae^{i(\phi^\varepsilon+\phi)/(2\varepsilon)}2i\sin\left(\frac{\phi^\varepsilon - \phi}{2\varepsilon}\right) = \mathcal{O}(\varepsilon) + \mathcal{O}(t), \end{aligned}$$

where the $\mathcal{O}(\cdot)$'s stand for estimates in $L^\infty([0,T_*]\times\mathbb{R}^n)$.

To improve (1.7) to (1.8), we need the next term in the asymptotic expansion of $(\phi^\varepsilon, a^\varepsilon)$ in terms of powers of ε. Introduce the system:

$$\tag{3.2}\begin{cases} \partial_t\phi^{(1)} + \nabla(\phi_{\rm eik}+\phi)\cdot\nabla\phi^{(1)} + 2\,\mathrm{Re}\left(\overline{a}a^{(1)}\right) = 0\ ;\ \phi^{(1)}_{|t=0} = 0. \\ \partial_t a^{(1)} + \nabla(\phi_{\rm eik}+\phi)\cdot\nabla a^{(1)} + \nabla\phi^{(1)}\cdot\nabla a + \dfrac{1}{2}a^{(1)}\Delta(\phi_{\rm eik}+\phi) \\ \qquad\qquad + \dfrac{1}{2}a\Delta\phi^{(1)} = \dfrac{i}{2}\Delta a\ ;\ a^{(1)}_{|t=0} = a_1. \end{cases}$$

It is easy to see that this linear system has a unique classical solution such that $a^{(1)}, \nabla\phi^{(1)} \in C([0,T_*];X^s)$ for all $s > n/2$. Reasoning as in the proof of Corollary 3.3, we see that we have also $\phi^{(1)} \in C([0,T_*];X^s)$. Moreover, mimicking the proofs of Proposition 3.1 and Corollary 3.3, we have the following result, whose proof is left out:

Proposition 3.4. *Let $(\phi^\varepsilon, a^\varepsilon)$, (ϕ, a) and $(\phi^{(1)}, a^{(1)})$ be given by* (2.6), (3.1) *and* (3.2) *respectively. Denote $r_0^\varepsilon = a_0^\varepsilon - a_0 - \varepsilon a_1$. For all $s > n/2+2$,*

$$\|\nabla(\phi^\varepsilon - \phi - \varepsilon\phi^{(1)})\|_{L^\infty([0,T_*];X^s)} + \|a^\varepsilon - a - \varepsilon a^{(1)}\|_{L^\infty([0,T_*];X^s)} \leqslant \widetilde{C}_s\left(\varepsilon^2 + \|r_0^\varepsilon\|_{X^s}\right).$$

In addition, there exists $\widetilde{C}$ such that if $s > n/2+2$,

$$\|\phi^\varepsilon - \phi - \varepsilon\phi^{(1)}\|_{L^\infty([0,T_*]\times\mathbb{R}^n)} \leqslant \widetilde{C}\left(\varepsilon^2 + \|r_0^\varepsilon\|_{X^s}\right).$$

We can now complete the proof of Theorem 1.3:

$$\begin{aligned} u^\varepsilon - ae^{i\phi^{(1)}}e^{i\phi/\varepsilon} &= a^\varepsilon e^{i\phi^\varepsilon/\varepsilon} - ae^{i(\phi+\varepsilon\phi^{(1)})/\varepsilon} \\ &= (a^\varepsilon - a)\,e^{i\phi^\varepsilon/\varepsilon} + a\left(e^{i\phi^\varepsilon/\varepsilon} - e^{i(\phi+\varepsilon\phi^{(1)})/\varepsilon}\right) \\ &= \mathcal{O}(\varepsilon) + ae^{i(\phi^\varepsilon+\phi+\varepsilon\phi^{(1)})/(2\varepsilon)}2i\sin\left(\frac{\phi^\varepsilon - \phi - \varepsilon\phi^{(1)}}{2\varepsilon}\right) \\ &= \mathcal{O}(\varepsilon) + \mathcal{O}\left(\frac{\|r_0^\varepsilon\|_{X^s}}{\varepsilon}\right). \end{aligned}$$

This yields (1.8), along with Remark 1.5.

Remark 3.5. Following the same lines, we see that if a_0^ε is known up to order $\mathcal{O}(\varepsilon^{N+1})$ in X^s for some $s > n/2+2$, $N \in \mathbb{N}$, then we can construct an approximate solution v_N^ε such that

$$\|u^\varepsilon - v_N^\varepsilon\|_{L^\infty([0,T_*];X^s)} = \mathcal{O}\left(\varepsilon^N\right).$$

To conclude this paragraph, we note that if we know that the initial corrector a_1 is not only in X^∞, but in H^∞, then Theorem 1.3 becomes more precise.

Corollary 3.6. *Under the same assumptions as in Theorem 1.3, suppose moreover that $a_1 \in H^\infty$, and*

$$\|a_0^\varepsilon - a_0 - \varepsilon a_1\|_{H^s} = \mathcal{O}\left(\delta^\varepsilon\right), \quad \forall s > 0, \quad \textit{with } \delta^\varepsilon = o(\varepsilon).$$

Then (1.8) *can be improved to:*

$$\tag{3.3} \sup_{t\in[0,T_*]}\left\|u^\varepsilon(t,\cdot) - a(t,\cdot)e^{i\phi^{(1)}(t,\cdot)}e^{i(\phi(t,\cdot)+\phi_{\rm eik}(t,\cdot))/\varepsilon}\right\|_{L^\infty\cap L^2} = \mathcal{O}\left(\varepsilon + \frac{\delta^\varepsilon}{\varepsilon}\right).$$

Essentially, one just has to notice that the error estimates in Propositions 3.1 and 3.4 can then be measured in H^s instead of X^s. Note also that in (3.3), it may happen that none of the two functions is in L^2.

4. Examples when $\phi_{\rm eik} \equiv 0$

In this paragraph, we consider (1.1)–(1.2), and we assume $\phi_{\rm eik} \equiv 0$.

4.1. **An example from** [9]**.** As an application, we can recover and improve the result of [9], in the case of the whole space (the space variable x lies in a bounded domain in [9]). Assume that

$$a_0^\varepsilon(x) = a_0(x) = e^{i\theta_0(x)}, \quad \theta_0 \in H^\infty(\mathbb{R}^n;\mathbb{R}) \quad ; \quad \phi_0 = V = 0.$$

That is, we consider:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = \left(|u^\varepsilon|^2 - 1\right) u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = e^{i\theta_0(x)}.$$

Then $a_0^\varepsilon = a_0 \in X^\infty$, and we see that:

- $\phi \equiv 0$ and a is independent of time: $a(t,x) = a_0(x) = e^{i\theta_0(x)}$.
- $\phi^{(1)}$ solves
$$\partial_t^2 \phi^{(1)} = \operatorname{Im}\left(\overline{a}\Delta a\right),$$
so that $\theta(t,x) := \phi^{(1)}(t,x) + \theta_0(x)$ solves:
$$\left(\partial_t^2 - \Delta\right)\theta = 0 \quad ; \quad \theta(0,x) = \theta_0(x) \quad ; \quad \partial_t\theta(0,x) = 0.$$

Note that (ϕ, a) remains smooth for all time, so we can take T_* arbitrarily large (see Remark 3.2). Since from Theorem 1.3 and the above corollary,

$$\sup_{t\in[0,T_*]} \|u^\varepsilon(t,\cdot) - a(t,\cdot)e^{i\phi^{(1)}(t,\cdot)}\|_{L^\infty\cap L^2} = \sup_{t\in[0,T_*]} \|u^\varepsilon(t,\cdot) - e^{i\theta(t,\cdot)}\|_{L^\infty\cap L^2} = \mathcal{O}(\varepsilon),$$

where $T_* > 0$ is arbitrary. We recover [9, Theorem 2] in the case of the whole space, with no restriction on the space dimension, and a precised error estimate. Note also that in view of Remark 3.5, we can justify [9, Proposition 5] (giving the ε-order corrector for u^ε), and get a complete asymptotic expansion for u^ε.

4.2. **When** $|a_0^\varepsilon|^2 - 1 \in L^2$**.** As in Corollary 3.6, assume that (1.6) is precised to

$$\|a_0^\varepsilon - a_0 - \varepsilon a_1\|_{H^s} = o(\varepsilon), \quad \forall s > 0.$$

where $a_0 \in X^\infty$ and $a_1 \in H^\infty$. Assume moreover that

$$|a_0|^2 - 1 \in L^2(\mathbb{R}^n).$$

Then (2.2) yields:

$$\begin{aligned}
\frac{d}{dt}\left\||a^\varepsilon(t)|^2 - 1\right\|_{L^2}^2 &= 4\int_{\mathbb{R}^n} \left||a^\varepsilon(t,x)|^2 - 1\right| \operatorname{Re}\left(\overline{a}^\varepsilon(t,x)\partial_t a^\varepsilon(t,x)\right)dx \\
&\lesssim \left\||a^\varepsilon(t)|^2 - 1\right\|_{L^2} \|a^\varepsilon\|_{L^\infty}\left(\|\nabla\Phi^\varepsilon\cdot\nabla a^\varepsilon\|_{L^2} + \|a^\varepsilon\Delta\Phi^\varepsilon\|_{L^2} + \|\Delta a^\varepsilon\|_{L^2}\right) \\
&\lesssim \left\||a^\varepsilon(t)|^2 - 1\right\|_{L^2} \|a^\varepsilon\|_{X^s}^2\left(\|\nabla\Phi^\varepsilon\|_{L^\infty} + \|\Delta\Phi^\varepsilon\|_{L^2} + 1\right),
\end{aligned}$$

where we consider $s > n/2 + 2$. Therefore, Proposition 2.1 shows that

$$|u^\varepsilon|^2 - 1 \in C([0,T_*];L^2(\mathbb{R}^n)).$$

Note that this property holds even if $V = V_{\mathrm{lin}} \neq 0$.

4.3. **When** $a_0^\varepsilon(x) \sim 1$ **as** $|x| \to \infty$**.** In a spirit similar to [20] (where the authors choose $\theta_0 \equiv 0$), assume that $V = 0$, $\phi_0(x) = v^\infty\cdot x$ for some $v^\infty \in \mathbb{R}^n$, and

$$\left\|a_0^\varepsilon - e^{i\theta_0(x)} - \varepsilon a_1\right\|_{H^s} = \mathcal{O}\left(\delta^\varepsilon\right), \quad \forall s > 0, \quad \text{where } \theta_0 \in H^\infty \text{ is real-valued.}$$

Then as in §4.1, we compute:

$$\phi(t,x) = v^\infty\cdot x - \frac{|v^\infty|^2}{2}t \quad ; \quad a(t,x) = a_0\left(x - v^\infty t\right) = e^{i\theta_0(x - v^\infty t)}.$$

We also note that $T_* > 0$ can be taken arbitrarily large. In addition, we check that $\phi^{(1)}$ is such that $\widetilde{\phi}^{(1)}(t,y) = \phi^{(1)}(t,x+v^\infty t)$ solves:

$$\left(\partial_t^2 - \Delta\right)\widetilde{\phi}^{(1)} = \operatorname{Im}\left(\overline{a}_0\Delta a_0\right) = \Delta\theta_0 \quad ; \quad \widetilde{\phi}^{(1)}(0,x) = 0 \; ; \; \partial_t\widetilde{\phi}^{(1)}(0,x) = -2\operatorname{Re}\left(\overline{a}_0 a_1\right).$$

Therefore, Corollary 3.6 yields

$$\sup_{t\in[0,T]} \left\| u^\varepsilon(t,\cdot) - e^{i\theta(t,\cdot)} e^{i\phi(t,\cdot)/\varepsilon} \right\|_{L^\infty\cap L^2} = \mathcal{O}\left(\varepsilon + \frac{\delta^\varepsilon}{\varepsilon}\right),$$

where θ is given by $\theta(t,x) = \widetilde{\theta}(t,y)\big|_{y=x-v^\infty t}$, with:

$$\left(\partial_t^2 - \Delta\right)\widetilde{\theta} = 0 \quad ; \quad \widetilde{\theta}_{|t=0} = \theta_0 \ ; \ \partial_t \widetilde{\theta}_{|t=0} = -2\operatorname{Re}\left(\overline{a}_0 a_1\right).$$

5. Time propagation of the condition at infinity: $\phi_{\rm eik} \neq 0$

In this section, we assume that $|a_0^\varepsilon|^2 - 1 \in L^2(\mathbb{R}^n)$, and aim at understanding how this condition is propagated on the time interval $[0,T_*]$ when $\phi_{\rm eik} \neq 0$. Essentially, we have $|u^\varepsilon(t,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$ for $t\in[0,T_*]$ if and only if $\phi_{\rm eik} \equiv 0$. The function $\phi_{\rm eik}$ is identically zero if and only if $V_{\rm quad} = \phi_{\rm quad} = 0$: that case was developed in §4.2. We compute

$$\frac{d}{dt}\left\| |u^\varepsilon(t)|^2 - 1\right\|_{L^2}^2 \leqslant 4 \left\| |u^\varepsilon(t)|^2 - 1\right\|_{L^2} \|a^\varepsilon(t)\|_{L^\infty} \|\partial_t a^\varepsilon(t)\|_{L^2}.$$

In the above estimate, we assumed that $\partial_t a^\varepsilon(t,\cdot) \in L^2$. Let us now examine this condition. In view of Proposition 2.2, we know that all the terms in the second equation of (2.6) are in $L^2(\mathbb{R}^n)$, except possibly $\partial_t a^\varepsilon$, $\nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon$ and $a^\varepsilon\Delta\phi_{\rm eik}$. Therefore if $\phi_{\rm eik}\equiv 0$, we infer that $|u^\varepsilon(t,\cdot)|^2 - 1 \in L^2(\mathbb{R}^n)$ for all $t \in [0,T_*]$.

Assume now that $\phi_{\rm eik}$ is not zero. To gather the terms $\partial_t a^\varepsilon$ and $\nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon$ together, consider the change of variable of Remark 1.2, and set

$$\widetilde{a}^\varepsilon(t,y) = a^\varepsilon(t,x(t,y)).$$

Since the Jacobi determinant $\det\nabla_y x(t,y) > 0$ is bounded from above, and from below away from zero for $t\in[0,T_*]\subset[0,T]$, $\partial_t a^\varepsilon(t,\cdot)$ and $\partial_t\widetilde{a}^\varepsilon(t,\cdot)$ are simultaneously in $L^2(\mathbb{R}^n)$. Given $\Delta\phi_{\rm eik}$ is a function of time only, we have

$$\partial_t\widetilde{a}^\varepsilon = -\frac{1}{2}\widetilde{a}^\varepsilon\Delta\phi_{\rm eik} + C([0,T_*];L^2).$$

We are in a case where $\widetilde{a}^\varepsilon\Delta\phi_{\rm eik}\not\in L^2$. To overcome this issue, consider

$$\left\|\left|u^\varepsilon(t)e^{\frac12\int_0^t\Delta\phi_{\rm eik}(\tau)d\tau}\right|^2 - 1\right\|_{L^2}^2 = \left\|\left|a^\varepsilon(t)e^{\int_0^t \operatorname{Tr}Q(\tau)d\tau}\right|^2 - 1\right\|_{L^2}^2,$$

where Q is given by Lemma 1.1. For $t\in[0,T_*]$, this quantity is equivalent to:

$$\left\|\left|\widetilde{a}^\varepsilon(t)e^{\int_0^t \operatorname{Tr}Q(\tau)d\tau}\right|^2 - 1\right\|_{L^2}^2.$$

We have:

$$\begin{aligned}\frac{d}{dt}\left\|\left|\widetilde{a}^\varepsilon(t)e^{\int_0^t \operatorname{Tr}Q(\tau)d\tau}\right|^2 - 1\right\|_{L^2}^2 \leqslant C\left\|\left|\widetilde{a}^\varepsilon(t)e^{\int_0^t \operatorname{Tr}Q(\tau)d\tau}\right|^2 - 1\right\|_{L^2} \|\widetilde{a}^\varepsilon(t)\|_{L^\infty}\times \\ \times\left\|\partial_t\widetilde{a}^\varepsilon(t) + \frac12\widetilde{a}^\varepsilon(t)\Delta\phi_{\rm eik}(t)\right\|_{L^2}.\end{aligned}$$

We infer that $\left|\widetilde{a}^\varepsilon(t)e^{\int_0^t \operatorname{Tr}Q(\tau)d\tau}\right|^2 - 1 \in C([0,T_*];L^2)$, hence

$$\left|u^\varepsilon(t)e^{\int_0^t \operatorname{Tr}Q(\tau)d\tau}\right|^2 - 1 \in C([0,T_*];L^2).$$

Morally, for $t\in[0,T_*]$, the modulus of u^ε goes to $\exp(-\int_0^t \operatorname{Tr}Q(\tau)d\tau)$ as $|x|\to\infty$. We conclude by some examples that illustrate this analysis.

Example 1. Consider the case where $\phi_{\rm quad} = 0$, and $V_{\rm quad}(x) = \omega^2 \frac{|x|^2}{2}$ is an isotropic harmonic potential ($\omega > 0$). Then we compute

$$\phi_{\rm eik}(t,x) = -\omega \frac{|x|^2}{2} \tan(\omega t), \quad t \in [0,T] \subset \left[0, \frac{\pi}{2\omega}\right[,$$

$$\text{and} \qquad \exp\left(-\int_0^t \operatorname{Tr} Q(\tau) d\tau\right) = \exp\left(\frac{n\omega}{2}\int_0^t \tan(\omega\tau)d\tau\right) = (\cos(\omega t))^{-n/2}.$$

Therefore, the "limit of the modulus of u^ε at infinity" grows at time evolves. If in Proposition 2.2, we can take T_* arbitrarily close to $\pi/(2\omega)$, this suggests that there is some sort of "blow-up at infinity" at t approaches $\pi/(2\omega)$.

Example 2. Consider the case where $\phi_{\rm quad} = 0$, and $V_{\rm quad}(x) = -\omega^2 \frac{|x|^2}{2}$ is an isotropic *repulsive* harmonic potential ($\omega > 0$). We have

$$\phi_{\rm eik}(t,x) = \omega \frac{|x|^2}{2} \tanh(\omega t), \quad t \in [0,+\infty[,$$

$$\text{and} \qquad \exp\left(-\int_0^t \operatorname{Tr} Q(\tau) d\tau\right) = (\cosh(\omega t))^{-n/2}.$$

Therefore, the "limit of the modulus of u^ε at infinity" decays at time evolves.

Example 3. Consider the case $\phi_{\rm quad} = -|x|^2/2$, and $V_{\rm quad}(x) = 0$. We compute

$$\phi_{\rm eik}(t,x) = \frac{|x|^2}{2(t-1)}, \quad t \in [0,1[, \quad \text{and } \exp\left(-\int_0^t \operatorname{Tr} Q(\tau) d\tau\right) = (1-t)^{-n/2}.$$

This case is similar to the first example.

Example 4. Consider the case $\phi_{\rm quad} = |x|^2/2$, and $V_{\rm quad}(x) = 0$. We have

$$\phi_{\rm eik}(t,x) = \frac{|x|^2}{2(t+1)}, \quad t \in [0,+\infty[, \quad \text{and } \exp\left(-\int_0^t \operatorname{Tr} Q(\tau) d\tau\right) = (1+t)^{-n/2}.$$

This case is similar to the second example, provided that we consider positive times.

6. On the hydrodynamic limit

In this paragraph, we consider the setting of either Theorem 1.3 or Theorem 1.10. That is, the semi-classical limit is justified for small time in Zhidkov spaces. Let $\Phi = \phi_{\rm eik} + \phi$, $\mathbf{v} = \nabla\Phi$ and $\rho = |a|^2$. As is easily checked, $(\rho, \mathbf{v})$ solves the following compressible Euler equation:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \quad ; \quad \rho_{t=0} = |a_0|^2. \\ \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla V + \nabla f(\rho) = 0 \quad ; \quad \mathbf{v}_{|t=0} = \nabla\phi_0, \end{cases} \tag{6.1}$$

where $f(\rho) = \rho - 1$ in the cubic case, and $f(\rho) = \rho^2 + \lambda\rho$ in the cubic-quintic case. To simplify the discussion, assume in this paragraph that $V_{\rm quad} = \phi_{\rm quad} = 0$, hence $\phi_{\rm eik} = 0$. Proposition 3.1 implies in particular the convergence of the main two quadratic quantities, as $\varepsilon \to 0$:

- Density: $|u^\varepsilon|^2 \to \rho$ in $L^\infty([0,T_*]\times\mathbb{R}^n)$.
- Momentum: $\operatorname{Im}(\varepsilon \overline{u}^\varepsilon \nabla u^\varepsilon) \to \rho\mathbf{v}$ in $L^\infty([0,T_*]\times\mathbb{R}^n)$.

It should be noted that if we assume only that for some $s > n/2+2$,

$$\|a_0^\varepsilon - a_0\|_{X^s} = \delta_0^\varepsilon = o(1) \quad \text{as } \varepsilon \to 0,$$

the proof of Proposition 3.1 shows that we have:

$$\|\nabla(\phi^\varepsilon - \phi)\|_{L^\infty([0,T_*];X^s)} + \|a^\varepsilon - a\|_{L^\infty([0,T_*];X^s)} = \mathcal{O}\left(\varepsilon + \delta_0^\varepsilon\right).$$

Therefore,

$$|u^\varepsilon|^2 = \rho + \mathcal{O}\left(\varepsilon+\delta_0^\varepsilon\right) \quad ; \quad \operatorname{Im}(\varepsilon\overline{u}^\varepsilon\nabla u^\varepsilon) = \rho\mathbf{v} + \mathcal{O}\left(\varepsilon+\delta_0^\varepsilon\right).$$

To have a more precise asymptotics, it is necessary to work with the assumption of Theorem 1.3. If for some $s > n/2+2$,

$$\|a_0^\varepsilon - a_0 - \varepsilon a_1\|_{X^s} = \delta_1^\varepsilon = o(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

we get:

$$|u^\varepsilon|^2 = \rho + 2\varepsilon \operatorname{Re}\left(\overline{a} a^{(1)}\right) + \mathcal{O}\left(\varepsilon^2 + \delta_1^\varepsilon\right).$$

$$\operatorname{Im}(\varepsilon \overline{u}^\varepsilon \nabla u^\varepsilon) = \rho \mathbf{v} + \varepsilon\left(2\operatorname{Re}\left(\overline{a} a^{(1)}\right)\mathbf{v} + \rho \nabla \phi^{(1)}\right) + \mathcal{O}\left(\varepsilon^2 + \delta_1^\varepsilon\right).$$

Finally, note that in general, even if $a_1 = 0$, the modulation $\phi^{(1)}$ is not trivial. Suppose that $a_1 = 0$: (3.2) shows that $\partial_t a^{(1)}_{|t=0} \neq 0$, because of the source term $\frac{i}{2}\Delta a$. Therefore, even if $\phi^{(1)}_{|t=0} = \partial_t \phi^{(1)}_{|t=0} = 0$, we have $\partial_t^2 \phi^{(1)}_{|t=0} \neq 0$ in general, and the correctors of order ε in the above asymptotics are not trivial.

However, if a_0 is *real-valued* and $a_1 = 0$, then a is real-valued, $a^{(1)}$ is purely imaginary, so $\phi^{(1)} \equiv 0$. The same holds if a_0 is real-valued and a_1 is purely imaginary.

We end this section by studying the hydrodynamic limit in the case when $\Omega \subset \mathbb{R}^n$ is a regular domain with bounded boundary $\partial\Omega$ and $n \in \{2,3\}$ (either a bounded domain or an exterior domain). To simplify the presentation, we consider the case without external potential and without linear or quadratic initial phase. The Gross–Pitaevskii equation is then supplemented with the Neumann boundary condition:

$$\begin{cases} i\varepsilon \partial_t u^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta u^\varepsilon = \left(|u^\varepsilon|^2 - 1\right) u^\varepsilon & \text{in } \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

where n is the unit outward normal to $\partial\Omega$. Consider the corresponding limit system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \nabla \phi) = 0 & \text{in } \Omega, \\ \partial_t \phi + \dfrac{1}{2}|\nabla \phi|^2 + \rho - 1 = 0 & \text{in } \Omega, \\ \nabla \phi \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

In [20], Lin and Zhang proved that if $n = 2$, then the quadratic observables $|u^\varepsilon|^2$ and $\varepsilon \operatorname{Im}(\overline{u}^\varepsilon \nabla u^\varepsilon)$ converge towards the density ρ and the momentum $\rho\nabla\phi$. In the spirit of the pionnering work of Brenier [6], in [20] the strategy of the proof is to estimate the modulated energy functional

$$E^\varepsilon := \frac{1}{\varepsilon^2}\int_\Omega |\varepsilon \nabla u^\varepsilon - i u^\varepsilon \nabla \phi|^2 + \left(|u^\varepsilon|^2 - \rho\right)^2 \, dx.$$

The assumption $n = 2$ does not enter into the analysis of E^ε and only corresponds to the fact that they used the Brezis–Gallouët inequality (see also [9]) to define sufficiently smooth solutions to the Gross–Pitaevskii equation. They are now several 3D results (see [16, 4, 11, 12]), and hence one can justify the hydronamic limit for $n \in \{2,3\}$. In particular, Theorem 6.1 below is not new, but rather an update. Yet, our main purpose here is to establish a local version of the modulated energy functional. This is done in the proof of Theorem 6.1 (see (6.4)), by following the approach introduced in [3].

Theorem 6.1. *Let u^ε and (ρ, ϕ) be classical solutions of* (6.2) *and* (6.3) *satisfying, for some fixed $T > 0$,*

$$\begin{aligned}
&u^\varepsilon \in C([0,T]; X^2(\Omega)), \quad |u^\varepsilon|^2 - 1 \in C([0,T]; L^2(\Omega)),\\
&\rho \in C([0,T]; X^1(\Omega)), \quad \rho - 1 \in C([0,T]; L^2(\Omega)),\\
&\nabla\phi, \nabla^2\phi, \nabla^3\phi \in C([0,T]; L^2(\Omega) \cap L^\infty(\Omega)).
\end{aligned}$$

Assume that initially

$$\|\varepsilon\nabla u_0^\varepsilon - iu_0^\varepsilon \nabla\phi_0\|_{L^2(\Omega)} + \left\||u_0^\varepsilon|^2 - \rho_0\right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon),$$

then

$$\begin{aligned}
&|u^\varepsilon|^2 - \rho = \mathcal{O}(\varepsilon) \quad in \quad L^\infty([0,T]; L^2(\Omega)),\\
&\varepsilon \operatorname{Im}(\overline{u}^\varepsilon \nabla u^\varepsilon) - \rho\nabla\phi = \mathcal{O}(\varepsilon) \quad in \quad L^\infty([0,T]; L^1_{\mathrm{loc}}(\Omega)).
\end{aligned}$$

Proof. The idea consists in filtering out the oscillations by the change of unknown

$$a^\varepsilon(t,x) := u^\varepsilon(t,x)e^{-i\phi(t,x)/\varepsilon}.$$

The amplitude a^ε solves

$$\partial_t a^\varepsilon + \nabla\phi \cdot \nabla a^\varepsilon + \frac{1}{2}a^\varepsilon \Delta\phi - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{i}{\varepsilon}\left(|a^\varepsilon|^2 - \rho\right)a^\varepsilon.$$

Next set

$$q^\varepsilon := \frac{|a^\varepsilon|^2 - \rho}{\varepsilon}\cdot$$

We easily find that

$$\partial_t q^\varepsilon + \operatorname{div}\left(\operatorname{Im}\left(\overline{a}^\varepsilon \nabla a^\varepsilon\right)\right) + \operatorname{div}(q^\varepsilon \nabla\phi) = 0.$$

Furthermore, with this notation, the equations for $\psi^\varepsilon := \nabla a^\varepsilon$ read

$$\partial_t \psi^\varepsilon + \nabla\phi \cdot \nabla\psi^\varepsilon + \frac{1}{2}\psi^\varepsilon\Delta\phi + \psi^\varepsilon \cdot \nabla\nabla\phi + \frac{1}{2}a^\varepsilon \nabla\Delta\phi + iq^\varepsilon\psi^\varepsilon + ia^\varepsilon\nabla q^\varepsilon = i\frac{\varepsilon}{2}\Delta\psi^\varepsilon.$$

Also, note that $\psi^\varepsilon \cdot n = e^{-i\phi/\varepsilon}(\nabla u^\varepsilon - i\varepsilon^{-1}u^\varepsilon\nabla\phi) \cdot n = 0$ on $\partial\Omega$.

We now introduce the modulated energy

$$e^\varepsilon := |\psi^\varepsilon|^2 + (q^\varepsilon)^2.$$

The key point is that

$$q^\varepsilon \operatorname{div}\left(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)\right) + \operatorname{Re}\left(ia^\varepsilon(\nabla q^\varepsilon) \cdot \overline{\psi}^\varepsilon\right) = \operatorname{div}\left(\operatorname{Im}(q^\varepsilon \overline{a}^\varepsilon \psi^\varepsilon)\right).$$

Hence, directly from the previous equations, we have

$$\begin{aligned}
(6.4) \quad \partial_t e^\varepsilon + \operatorname{div}(e^\varepsilon\nabla\phi) + \operatorname{div}\big(2\operatorname{Im}(q^\varepsilon\overline{a}^\varepsilon\psi^\varepsilon)\big) + \operatorname{div}\left(\varepsilon \operatorname{Im}\left(\overline{\psi}^\varepsilon \cdot \nabla\psi^\varepsilon\right)\right)&\\
= -(q^\varepsilon)^2\Delta\phi - \operatorname{Re}\left((2\psi^\varepsilon \cdot \nabla\nabla\phi + a^\varepsilon\nabla\Delta\phi)\cdot\overline{\psi}^\varepsilon\right)&.
\end{aligned}$$

We claim that

$$(6.5) \qquad E^\varepsilon(t) = \|e^\varepsilon(t)\|_{L^1(\Omega)} \leqslant \|e^\varepsilon(0)\|_{L^1(\Omega)} \exp(Ct) + C,$$

for some constant C independent of ε. Since $v \cdot n = 0$ and $\psi^\varepsilon \cdot n = 0$ on $\partial\Omega$, by integrating in space and using the Gronwall's lemma, to prove (6.6), the only delicate point is to prove that,

$$(6.6) \qquad \int \left|a^\varepsilon \nabla\Delta\phi \cdot \overline{\psi}^\varepsilon\right| dx \leqslant C\|e^\varepsilon\|_{L^1(\Omega)} + C.$$

To do so, as in Lemma 1 in [16], let $\chi \in C_0(\mathbb{C})$ be such that $0 \leqslant \chi \leqslant 1$, $\chi(z) = 1$ for $|z| \leqslant 2$, and $\chi(z) = 0$ for $|z| \geqslant 3$. Then write $a^\varepsilon = b^\varepsilon + c^\varepsilon$ where $b^\varepsilon = \chi(a^\varepsilon)a^\varepsilon$ and $c^\varepsilon = (1 - \chi(a^\varepsilon))a^\varepsilon$. We have $|b^\varepsilon| \leqslant 3$, $|c^\varepsilon| \leqslant \left||a^\varepsilon|^2 - 1\right|$ and hence

$$\|b^\varepsilon\|_{L^\infty(\Omega)} \leqslant 3, \qquad \|c^\varepsilon\|_{L^2(\Omega)} \leqslant \left\||a^\varepsilon|^2 - 1\right\|_{L^2(\Omega)} \leqslant \varepsilon\,\|q^\varepsilon\|_{L^2(\Omega)} + \|\rho - 1\|_{L^2(\Omega)}\,.$$

The desired estimate (6.6) then follows from

$$\begin{aligned}
\left\| b^\varepsilon \nabla\Delta\phi \cdot \overline{\psi}^\varepsilon \right\|_{L^1(\Omega)} &\leqslant \left\| b^\varepsilon \right\|_{L^\infty(\Omega)} \left\| \nabla\Delta\phi \right\|_{L^2(\Omega)} \left\| \psi^\varepsilon \right\|_{L^2(\Omega)} , \\
\left\| c^\varepsilon \nabla\Delta\phi \cdot \overline{\psi}^\varepsilon \right\|_{L^1(\Omega)} &\leqslant \left\| c^\varepsilon \right\|_{L^2(\Omega)} \left\| \nabla\Delta\phi \right\|_{L^\infty(\Omega)} \left\| \psi^\varepsilon \right\|_{L^2(\Omega)} ,
\end{aligned}$$

and the elementary inequality $\sqrt{x} \leqslant 1 + x$.

Since

$$e^\varepsilon = \frac{1}{\varepsilon^2} \left| \varepsilon\nabla u^\varepsilon - iu^\varepsilon \nabla\phi \right|^2 + \frac{1}{\varepsilon^2} \left(|u^\varepsilon|^2 - \rho \right)^2 ,$$

the family $(e^\varepsilon(0))_{\varepsilon\in]0,1]}$ is bounded in $L^1(\Omega)$ by assumption. Consequently, it follows from (6.5) that $(e^\varepsilon)_{\varepsilon\in]0,1]}$ is bounded in $L^\infty([0,T];L^1(\Omega))$.

By definition, this implies that $|u^\varepsilon|^2 - \rho = \mathcal{O}(\varepsilon)$ in $L^\infty([0,T];L^2(\Omega))$. It remains to prove that

$$\varepsilon \operatorname{Im}(\overline{u}^\varepsilon \nabla u^\varepsilon) - \rho\nabla\phi = \mathcal{O}(\varepsilon) \text{ in } L^\infty([0,T];L^1_{\mathrm{loc}}(\Omega)).$$

Write

$$\varepsilon \operatorname{Im}(\overline{u}^\varepsilon \nabla u^\varepsilon) - \rho\nabla\phi = \varepsilon \operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon) + \left(|a^\varepsilon|^2 - \rho \right) \nabla\phi.$$

Since $\nabla\phi \in L^\infty([0,T]\times\Omega)$, the previous result implies that the second term is $\mathcal{O}(\varepsilon)$ in $L^\infty([0,T];L^2(\Omega))$. With regards to the first one, again write $a^\varepsilon = b^\varepsilon + c^\varepsilon$ and use the obvious estimates

$$\begin{aligned}
&\| \varepsilon \operatorname{Im}(\overline{b}^\varepsilon \nabla a^\varepsilon) \|_{L^2(\Omega)} \leqslant \varepsilon \| b^\varepsilon \|_{L^\infty(\Omega)} \| \nabla a^\varepsilon \|_{L^2(\Omega)} \leqslant 3\varepsilon \left\| e^\varepsilon \right\|^{1/2}_{L^1(\Omega)} , \\
&\| \varepsilon \operatorname{Im}(\overline{c}^\varepsilon \nabla a^\varepsilon) \|_{L^1(\Omega)} \leqslant \varepsilon \| c^\varepsilon \|_{L^2(\Omega)} \| \nabla a^\varepsilon \|_{L^2(\Omega)} \leqslant C\varepsilon \left\| e^\varepsilon \right\|^{1/2}_{L^1(\Omega)} + C\varepsilon^2 \left\| e^\varepsilon \right\|_{L^1(\Omega)} .
\end{aligned}$$

This completes the proof. □

7. Cubic-quintic nonlinearity

In view of Theorem 1.10, we now consider (1.9) in the case where the elliptic region becomes relevant: $\lambda < 0$, and assume for instance that there exists $\underline{x} \in \mathbb{R}^n$ such that $|a_0(\underline{x})|^2 < |\lambda|/2$. If we write $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$, where $(a^\varepsilon, \Phi^\varepsilon)$ is given by (2.2), then we naturally have to consider the limit system:

$$\tag{7.1} \left\{ \begin{aligned} &\partial_t \phi + \frac{1}{2} \left| \nabla\phi \right|^2 + f_\lambda \left(|a|^2 \right) = 0 \quad ; \quad \phi\big|_{t=0} = \phi_0 , \\ &\partial_t a + \nabla\phi \cdot \nabla a + \frac{1}{2} a \Delta\phi = 0 \quad ; \quad a\big|_{t=0} = a_0 . \end{aligned} \right.$$

Setting $v = \nabla\phi$, we find:

$$\tag{7.2} \left\{ \begin{aligned} &\partial_t v + v \cdot \nabla v + \nabla f_\lambda \left(|a|^2 \right) = 0 \quad ; \quad v\big|_{t=0} = \nabla\phi_0 , \\ &\partial_t a + v \cdot \nabla a + \frac{1}{2} a \operatorname{div} v = 0 \quad ; \quad a\big|_{t=0} = a_0 . \end{aligned} \right.$$

Then [22, Theorem 3.2] shows that (7.2) is strongly ill-posed in Sobolev spaces. The problem remains in Zhidkov spaces, since analyticity is essentially necessary. Indeed, Hadamard's argument (see [22] and references therein) shows for instance that if ϕ_0 is analytic near $\underline{x}$, then (7.2) has a C^1-solution only if a_0 is also analytic near $\underline{x}$. So it may happen that (7.2) has no solution in X^s, even for s large.

On the other hand, if one is ready to work with analytic regularity, then it becomes possible to justify the semi-classical limit for (1.9); see [15, 27].

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