# Introduction to microlocal analysis

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**Abstract**

These lecture notes provide a self-contained introduction to microlocal analysis, a branch of modern analysis used today in many fields. The main goal is to give complete proofs of the continuity of pseudodifferential operators on Sobolev spaces, of the symbolic calculus for pseudodifferential operators, and of Hörmander's theorem on the propagation of singularities. This book also contains a self-contained introduction to the study of the Fourier transform as well as exercises for those who wish to test their understanding of the theory by practice. Students will find many additional problems in the book [1] that Claude Zuily and I have recently written.

# Contents

# Introduction

How to act on a function to study its regularity? How to act on the solutions of a partial differential equation to conjugate it to a simpler equation? There exist many possible answers to these general questions. The ones we will study in this course come from a theory developed since the sixties, known as microlocal analysis, and used today in many fields. It has its origin in the discovery made two centuries ago by Fourier that the heat equation

$$\partial_t u - \Delta u = 0 \quad \text{where} \quad \Delta = \sum_{j=1}^{d} \partial_{x_j}^2,$$

can be reduced to an ordinary differential equation

$$\partial_t v + |\xi|^2 v = 0 \quad \text{where} \quad |\xi|^2 = \sum_{j=1}^{d} \xi_j^2.$$

Fourier observed already in 1812 that any function is a sum of oscillatory exponentials, which are the functions $\mathbb{R}^d \ni x \mapsto e^{ix\cdot\xi} \in \mathbb{C}$ where $x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d$. It turns out that this idea of decomposing a function into oscillatory exponentials is extremely fruitful. It is now used in all sciences. In mathematical analysis, this idea allows to solve many linear partial differential equations, but also to study subtle qualitative properties of the solutions of nonlinear partial differential equations.

The study of microlocal analysis is a very vast subject. We refer the reader to the books of Hörmander for a thorough presentation of this field. One of the main objects studied by microlocal analysis are the so-called pseudo-differential operators, and we limit the scope of these lectures to the study of the main properties of these operators.

There exists many definitions with many variants of a pseudo-differential operator. We follow here the classical definition introduced in the work of Kohn-Nirenberg and Hördmander. We say that $T$ is a pseudo-differential operator if we can define it from a function $a = a(x, \xi)$ by the relation

$$T\left(e^{ix\cdot\xi}\right) = a(x,\xi)e^{ix\cdot\xi}. \tag{0.0.1}$$

We then say that $a$ is the symbol for $T$ and we denote $T = \mathrm{Op}(a)$. For instance, the operator associated with the symbol $a = \sum_\alpha a_\alpha(x)(i\xi)^\alpha$ is simply the differential operator $\mathrm{Op}(a) = \sum_\alpha a_\alpha(x)\partial_x^\alpha$ (with classical notations).

One of the main goal of these lectures is to show that the pseudo-differential calculus is a process that associates to a symbol $a = a(x, \xi)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ an operator $\mathrm{Op}(a)$ such that one can understand the properties of these operators (product, adjoint, boundedness on the usual spaces of functions...) simply by looking at the properties of the symbols. Then, by using symbolic calculus we will be able to study the microlocal regularity of a function, i.e. study its wavefront set.

The main goal of this course is to give complete proofs of the continuity of pseudo-differential operators on Sobolev spaces, of the symbolic calculus for pseudo-differential operators, and of Hörmander's theorem on the propagation of singularities.

The application that associates an operator $\mathrm{Op}(a)$ to the symbol $a$ is called a quantization. There are very many quantizations that are known to be useful, which are variants of (0.0.1). We will also discuss Bony's quantization, which is perfectly suited for non-linear problems.

This book also contains a self-contained introduction to the study of the Fourier transform as well as exercises for those who wish to test their understanding of the theory by practice. For additional applications and problems, the readers are referred to the book [1] that Claude Zuily and I have recently written.

The study of pseudo-differential operators is a very vast subject. I refer to Hörmander [17] for the general theory as well as Alinhac and Gérard [2], Grigis and Sjöstrand [14], Lerner [18], Métivier [19], Saint-Raymond [23], Taylor [25] or Zworski [29] for other introductions to this theory.

# Part I

# The Fourier transform

# Chapter 1

# Functional analysis

We begin by studying some classical results in functional analysis that explain why trigonometric polynomials are dense in the space of continuous or square integrable periodic functions.

## 1.1 Stone–Weierstrass Theorem

The space of continuous functions has the property of being an algebra. It is natural to try to determine whether some remarkable sub-algebras are dense. The main example concerns the study of the approximation of continuous functions by polynomial functions. There are two fundamental results. The first one, due to Weierstrass, states that any continuous function can be approximated on a compact interval by means of algebraic polynomials $P(x) = \sum_{n=0}^{N} a_n x^n$. The second shows that a continuous periodic function can be approximated by trigometric polynomials of the form $P(x) = c_0 + \sum_{n=0}^{N} (a_n \cos(nx) + b_n \sin(nx))$. Both results are in fact consequences of the following abstract result.

**Theorem 1.1.1** (Stone–Weierstrass)**.** *Let $X$ be a compact metric space and equip the space $C(X;\mathbb{R})$ of continuous functions with real values of the uniform norm, $\|f\| = \sup_{x\in X} |f(x)|$. Consider a unitary sub-algebra $A$ of $C(X;\mathbb{R})$ (this means a subset of $C(X;\mathbb{R})$ that contains the constant functions and which is stable by addition and multiplication: if $f, g$ are in $A$ then $f + g$ and $fg$ are in $A$). It is further assumed that $A$ separates the points of $X$ in the sense that, for all $x, y$ in $X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$. Then $A$ is dense in $C(X;\mathbb{R})$.*

*Proof.* The proof is based on three different ideas. The first one concerns the solution of the problem of the approximation of the absolute value function by polynomials.

**Lemma 1.1.2.** *For all $a > 0$, there exists a sequence of polynomials $\{p_n\}_{n\in\mathbb{N}}$ which converges uniformly to the absolute value function on $[-a, a]$.*

*Proof.* By the elementary change of variables $u(x) \mapsto u(x/a)$, we return to the case $a = 1$. We will start by constructing an auxiliary sequence of polynomials that converges uniformly to the square root function over the interval $[0, 1]$. Consider the sequence of functions $P_n \colon [0, 1] \to \mathbb{R}$ defined by induction:

$$P_0 = 0 \quad \text{and} \quad P_{n+1}(x) = P_n(x) + \frac{1}{2}\big(x - P_n(x)^2\big). \tag{1.1.1}$$

Then $P_n$ is a polynomial function such that $0 \leq P_n(x)$. Let us show by recurrence that $P_n(x) \leq \sqrt{x}$ for all $x$ in $[0, 1]$. For this we write

$$P_{n+1}(x) - P_n(x) = \frac{1}{2}(\sqrt{x} - P_n(x))(\sqrt{x} + P_n(x))$$

then the induction hypothesis is used to study the right-hand side. We obtain that the first factor $\sqrt{x} - P_n(x)$ is non negative and that the second factor $\sqrt{x} + P_n(x)$ is bounded by 2, hence the desired result:

$$P_{n+1}(x) \leq P_n(x) + (\sqrt{x} - P_n(x)) = \sqrt{x}.$$

Now, going back to the definition of the sequence (1.1.1), we see that the property $0 \leq P_n(x) \leq \sqrt{x}$ implies that, for all $x$ in $[0, 1]$, the sequence $\{P_n(x)\}_{n\geq 0}$ is increasing and bounded; and therefore convergent. The limit $P(x)$ of this sequence satisfies the equation $P(x) = P(x) + \frac{1}{2}(x - P(x)^2)$, hence $P(x) = \sqrt{x}$. Then, we use the classical Dini's lemma (see Lemma 1.1.5) to show that the sequence $\{P_n\}_{n\geq 0}$ converges uniformly to its limit $\sqrt{x}$ on $[0, 1]$.

Now, define $p_n(x) = P_n(x^2)$ to obtain a sequence of polynomials that converges uniformly to the absolute value function over the symmetric interval $[-1, 1]$. This completes the proof. □

The second idea of the proof is that $\overline{A}$ satisfies the following stability property.

**Lemma 1.1.3.** *If $f$ and $g$ are two elements of $\overline{A}$, then $\max\{f, g\}$ and $\min\{f, g\}$ belong to $\overline{A}$.*

*Proof.* Let $f, g$ in $\overline{A}$. Then $f+g$ and $f-g$ belong to $\overline{A}$. Thus, to prove this lemma, it is sufficient to prove that $\overline{A}$ is stable by taking the absolute value (that is, to prove that if $f$ belongs to $A$, then $|f|$ too). Indeed, we will deduce the desired result from the elementary identities

$$\max\{x, y\} = \frac{x+y+|x-y|}{2} \quad , \quad \min\{x, y\} = \frac{x+y-|x-y|}{2}.$$

Consider a function $f$ in $\overline{A}$, and let us show that $|f|$ is in $\overline{A}$. There exists a sequence $\{f_p\}_{p\in\mathbb{N}}$ with $f_p \in A$ and which converges to $f$ uniformly on $X$. Since $X$ is compact, $f(X)$ is bounded in $\mathbb{R}$, from which we deduce that there exists $a > 0$ such that $f_p(X) \subset [-a, a]$ for all $p \in \mathbb{N}$.

Lemma 1.1.2 implies that there exists a sequence of polynomials $\{p_n(t)\}$ that converges to the absolute value $|t|$ uniformly on $[-a, a]$. Then, for all $\varepsilon > 0$, we can find two indices $n$ and $p$ such that $\left\|f_p - f\right\| \leq \varepsilon/2$ and $\sup_{t\in[-a,a]} |p_n(t) - |t|| \leq \varepsilon/2$. It follows directly that $\left\|p_n(f_p) - |f|\right\| \leq \varepsilon$. Since $A$ is an algebra, $p_n(f_p)$ belongs to $A$, which implies that $|f|$ belongs to $\overline{A}$. □

It then remains to explain how this property of being stable by switching to absolute value comes into play. This is the object of the following lemma.

**Lemma 1.1.4.** *Let $X$ be a compact topological space that contains at least two elements. Assume that $H \subset C(X;\mathbb{R})$ satisfies the two following conditions:*

*a. For all $u, v$ in $H$, the functions* $\max\{u, v\}$ *and* $\min\{u, v\}$ *are in $H$;*

*b. for any pair of distinct points of $X$, if $\alpha_1$ and $\alpha_2$ are two real numbers, there exists $u \in H$ such that $u(x_1) = \alpha_1$ and $u(x_2) = \alpha_2$.*

*Then $H$ is dense in $C(X;\mathbb{R})$.*

*Proof.* Let $f \in C(X;\mathbb{R})$ and $\varepsilon > 0$. Let us fix a point $x \in X$. For all $y \neq x$, there exists $v_y \in H$ such that $v_y(x) = f(x)$ and $v_y(y) = f(y)$. Set

$$O_y = \{z \in X \ : \ v_y(z) > f(z) - \varepsilon\}.$$

For all $y \in X$, $O_y$ is an open set which contains both $y$ and $x$, so $X = \cup_{y\neq x} O_y$. By compactness we can extract a finite subcover, which means that $X = \cup_{j=1}^{r} O_{y_j}$, with

$y_j \neq x$ for all $j$. Let us then set $u_x = \max\{v_{y_1}, \ldots, v_{y_r}\}$. Then this function satisfies $u_x \in H$ and furthermore

$$u_x(x) = f(x), \quad \text{and} \quad \forall x' \in X,\ u_x(x') > f(x') - \varepsilon.$$

We now vary $x$ and define for each $x \in X$,

$$\Omega_x = \{x' \in X \ :\ u_x(x') < f(x') + \varepsilon\}.$$

Thus $\Omega_x$ is an open set by continuity of $v$. Moreover $\Omega_x$ contains $x$. The compactness of $X$ can be used again to get a finite number of points such that $X = \cup_{i=1}^{p} \Omega_{x_i}$. Finally, let us set $u = \min\{u_{x_1}, \ldots, u_{x_p}\}$. Then $u \in H$ and, for all $x \in X$, we have

$$f(x) - \varepsilon < u(x) < f(x) + \varepsilon.$$

This proves that $\|f - u\| \leq \varepsilon$, which ends the proof. □

The end of the proof of the Stone-Weierstrass theorem is easy. Firstly, if $X$ is reduced to a single element then the result is trivial because $C(X;\mathbb{R})$ consists of constant functions, which belong to $A$ by assumption. Otherwise, if $X$ contains at least two elements, then Lemma 1.1.3 shows that $H = \overline{A}$ satisfies the first hypothesis of Lemma 1.1.4. It only remains to check that $H = \overline{A}$ satisfies the second hypothesis. For that, consider $x_1, x_2$ in $X$ and two real numbers $\alpha_1$ and $\alpha_2$. As $A$ separates the points, there exists $f$ in $A$ such that $f(x_1) \neq f(x_2)$. We then define $u$ by

$$u(x) = \alpha_1 + (\alpha_2 - \alpha_1)\frac{f(x) - f(x_1)}{f(x_2) - f(x_1)}.$$

This function belongs to $A$ (thus to $\overline{A}$) and satisfies the desired property. This completes the proof of the theorem of Stone-Weierstrass. □

For the sake of completeness, we recall Dini's lemma that we used in the proof of the Stone-Weierstrass theorem.

**Lemma 1.1.5** (Dini)**.** *Let $I$ be a compact interval of $\mathbb{R}$. Consider a sequence of continuous functions $f_n \colon I \to \mathbb{R}$ with $n \geq 0$, which is increasing in the sense of $f_n \leq f_{n+1}$. If the sequence converges pointwise to a continuous function $f \in C^0(I)$, then it converges uniformly.*

*Proof.* Let $\varepsilon > 0$. For all integer $n$ we put $\Omega_n = \{x \in I \ :\ f_n(x) > f(x) - \varepsilon\}$. The sets $\Omega_n$ are open because the functions $f_n$ and $f$ are continuous by assumption. Moreover these sets satisfy $\Omega_n \subset \Omega_{n+1}$ and $I = \cup_{n\in\mathbb{N}}\Omega_n$ (because the sequence $(f_n)_{n\in\mathbb{N}}$ is increasing and converges pointwise to $f$). By the Borel-Lebesgue property, there exists an integer $m$ such that $I = \Omega_m$, and thus such that for all $x \in I$, $f_m(x) > f(x)-\varepsilon$. By increasing convergence one has the other inequality: $f_m(x) \leq f(x)$. Thus, $\|f - f_m\|_\infty \leq \varepsilon$, which proves that the convergence is uniform. □

The next result is an easy extension of the Stone–Weierstrass result to the case of complex valued functions.

**Corollary 1.1.6** (Stone–Weierstrass, complex version)**.** *Consider a compact metric space $X$. Let us equip the space $C(X;\mathbb{C})$ of those continuous functions with complex values with the uniform norm, $\|f\| = \sup_{x\in X}|f(x)|$. Consider a sub-algebra $A$ of $C(X;\mathbb{C})$, unitary, stable by complex conjugation (if $f \in A$ then $\overline{f}$ belongs to $A$) and separating the points of $X$. Then $A$ is dense in $C(X;\mathbb{C})$.*

*Proof.* Note that if $f \in A$ then $\operatorname{Re} f$ and $\operatorname{Im} f$ belong to $A$. Set $H = A \cap C(X;\mathbb{R})$. Then $H$ is a sub-algebra of $C^0(X;\mathbb{R})$ which is unitary and which separates the points (if $x \neq y$ and if $f \in A$ is such that $f(x) \neq f(y)$, then either $\operatorname{Re} f$ is suitable, or $\operatorname{Im} f$ is suitable). So $H$ is dense in $C(X;\mathbb{R})$. This implies the desired result by decomposition into real and imaginary parts. □

We are going to apply the previous result to the study of periodic functions $f\colon \mathbb{R}^d \to \mathbb{C}$. To simplify the notation, rather than considering arbitrary periods, we will assume that the functions are $2\pi$-periodic with respect to each variable (otherwise use change of variables of the form $f(x_1,\dots,x_d) \mapsto f(T_1x_1/(2\pi),\dots,T_dx_d/(2\pi)))$. By definition, a function $f\colon \mathbb{R}^d \to \mathbb{C}$ is $2\pi$-periodic with respect to each variable if

$$f(x + 2\pi e_j) = f(x).$$

We will simply say below that $f$ is periodic.

**Definition 1.1.7.** *A trigonometric polynomial is a function $P\colon \mathbb{R}^d \to \mathbb{C}$ of the form*

$$P(x) = \sum_{|n|\leq N} c_n e^{in\cdot x},$$

*with $N \in \mathbb{N}$, $n = (n_1,\dots,n_d)$, $|n| = n_1+\cdots+n_d$, $c_n \in \mathbb{C}$ and $n\cdot x = n_1x_1+\cdots+n_dx_d$.*

**Corollary 1.1.8** (Density of trigonometric polynomials)**.** *Consider a continuous and $2\pi$-periodic function $f\colon \mathbb{R}^d \to \mathbb{C}$. For all $\varepsilon > 0$, there exists a trigonometric polynomial $P$ such that $\sup_{x\in\mathbb{R}^d}|f(x) - P(x)| < \varepsilon$.*

*Proof.* Let us note $\mathbb{S}^1$ the circle of complex numbers of modulus 1 and introduce $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, the product of $d$ copies of $\mathbb{S}^1$. Consider the algebra of functions $f: \mathbb{T}^d \to \mathbb{C}$ which are continuous. Denote by $A$ the sub-algebra formed of functions of the form $P(z) = \sum_{|n|\le N} c_n z^n$ where $z^n = z_1^{n_1} \cdots z_d^{n_d}$. Then $A$ is a sub-algebra of $C(\mathbb{T}^d;\mathbb{C})$, unitary, stable by conjugation and separating the points (trivially). The Stone-Weierstrass theorem (in the complex version) implies that $A$ is dense. To conclude the proof, let us now consider $f: \mathbb{R}^d \to \mathbb{C}$ continuous and $2\pi$-periodic with respect to each variable. Then we can define a function $F: \mathbb{T}^d \to \mathbb{C}$ by $F(e^{ix_1}, \dots, e^{ix_d}) = f(x_1, \dots, x_d)$ and apply the previous result. □

## 1.2 Hilbertian bases

Consider a complex Hilbert space $H$ equipped with a scalar product $(\cdot,\cdot)$.

A sequence of elements $(e_n)_{n\in\mathbb{N}}$ in a Hilbert $H$ space is called an orthonormal system if and only if

$$(e_n, e_m) = \delta_n^m \qquad \forall n, m \in \mathbb{N},$$

where $\delta_n^m = 1$ if $n = m$ and 0 otherwise.

The following result states that one can always obtain such orthonormal systems starting from a family of linearly independent vectors.

**Proposition 1.2.1** (Gram-Schmidt orthonormalization)**.** *Let $H$ be a vector space with a scalar product. Consider a family $(u_n)_{n\in\mathbb{N}}$ of linearly independent vectors. Then there exists an orthonormal system $(e_n)_{n\in I}$ such that, for all $N \in \mathbb{N}$,*

$$\operatorname{vect}\{e_0, \dots, e_N\} = \operatorname{vect}\{u_0, \dots, u_N\}.$$

*Proof.* We set $e_0 = u_0/\|u_0\|$ and define the following elements by recurrence, so that

$$e_n = v_n/\|v_n\| \quad \text{where} \quad v_n = u_n - (u_n, e_0)e_0 - \cdots - (u_n, e_{n-1})e_{n-1}.$$

We check that $e_n$ is orthogonal to $\operatorname{vect}\{e_0, \dots, e_{n-1}\}$. □

The main inequality concerning orthonormal system is given by the following result.

**Lemma 1.2.2** (Bessel Inequality)**.** *Consider an orthonormal system $(e_n)_{n\in\mathbb{N}}$ and $f$ an element of $H$. Then*

$$\sum_{n=0}^{\infty} |(f, e_n)|^2 \leq \|f\|^2 .$$

*Proof.* Set $S_N f = \sum_{n=0}^{N} (f, e_n) e_n$. We have

$$\|S_N f\|^2 = \sum_{0\leq n_1, n_2 \leq N} (f, e_{n_1}) \overline{(f, e_{n_2})} (e_{n_1}, e_{n_2}) = \sum_{n=0}^{N} |(f, e_n)|^2 .$$

We deduce that

$$(f, S_N f) = \sum_{n=0}^{N} (f, (f, e_n) e_n) = \sum_{n=0}^{N} |(f, e_n)|^2 = \|S_N f\|^2 .$$

Consequently, the Cauchy-Schwarz inequality implies that

$$\|S_N f\|^2 = (f, S_N f) \leq \|f\| \, \|S_N f\| ,$$

hence $\|S_N f\| \leq \|f\|$ and the wanted result follows by letting $N$ goes to $+\infty$. □

**Theorem 1.2.3.** *Consider a Hilbert space $H$. The following properties are equivalent:*

$i$) *The vector space generated by $\{e_n\}$ is dense in $H$.*

$ii$) *For all $f \in H$, $\|f\|^2 = \sum_{n=0}^{+\infty} |(f, e_n)|^2$.*

$iii$) *For all $f \in H$, the series $\sum (f, e_n) e_n$ converges to $f$.*

$iv$) *If $f \in H$ satisfies $(f, e_n) = 0$ for all $n \in \mathbb{N}$ then $f = 0$.*

*Proof.* The implications $iii) \Rightarrow i)$ and $iii) \Rightarrow iv)$ are trivial. Let us prove that $ii) \Rightarrow iii)$. To do so, we use the identity $(f, S_N f) = \|S_N f\|^2$ (see the proof of the above lemma) to deduce that

$$\|f - S_N f\|^2 = \|f\|^2 + \|S_N f\|^2 - 2\,\mathrm{Re}(f, S_N f) = \|f\|^2 - \|S_N f\|^2 ,$$

which implies

$$\left\| f - \sum_{n=0}^{N} (f, e_n) e_n \right\|^2 = \|f\|^2 - \sum_{n=0}^{N} |(f, e_n)|^2 . \tag{1.2.1}$$

This proves that $f - \sum_{n=0}^{N} (f, e_n) e_n$ converges to 0 if $\sum_{n=0}^{N} |(f, e_n)|^2$ converges to $\|f\|^2$.

Consider the implication $i) \Rightarrow ii)$. Let us recall that $\|S_N f\| \leq \|f\|$ for all $f \in H$. Let $E$ be the vector space generated by $\{e_n\}_{n\in\mathbb{N}}$. Let $\varepsilon > 0$ and let $f' \in E$ such that $\|f - f'\| < \varepsilon$. For $N$ large enough we have $S_N f' = f'$. In addition

$$\|S_N f - S_N f'\| = \|S_N (f - f')\| \leq \|f - f'\| \leq \varepsilon,$$

so

$$\|S_N f - f\| \leq \|S_N f - S_N f'\| + \|S_N f' - f'\| + \|f' - f\| \leq \varepsilon + 0 + \varepsilon.$$

Therefore, $(f - S_N f)$ converges to 0. Now we can pass to the limit in (1.2.1). and we get $\|f\|^2 = \sum_{n=0}^{+\infty} |(f, e_n)|^2$, which concludes the proof of $i) \Rightarrow ii)$.

We now move to the implication $iv) \Rightarrow iii)$ (this is where we use the fact that $H$ is complete). Set $a_n = (f, e_n)$ and $f_p = \sum_{n=1}^{p} a_n e_n$. Bessel's inequality results in $(a_n) \in \ell^2$. Now, for $m > p$ we have $\left\| f_m - f_p \right\|^2 = \sum_{n=p+1}^{m} |a_n|^2$ and thus the sequence $(f_p)$ is a Cauchy sequence, hence converges to an element denoted by $f'$. But then (considering the partial sums and passing to the limit) we find that $(f', e_n) = a_n$ for all $n$, which means that $(f - f', e_n) = 0$ for all $n$. We deduce that $f = \sum_{n=1}^{\infty} a_n e_n$, which concludes the proof. □

## 1.3 Fourier series

For $p$ in $[1, \infty]$, we will note $L^p_{\text{per}}(\mathbb{R}^d)$ the space of measurable functions $f \colon \mathbb{R}^d \to \mathbb{C}$, which are periodic and such that $|f|^p$ is integrable on the cube $[0, 2\pi]^d$ (then $|f|^p$ is integrable on any compact of $\mathbb{R}^d$ by periodicity). We quotient the spaces $L^p_{\text{per}}(\mathbb{R}^d)$ by the equivalence relation of equality almost everywhere.

A trigonometric polynomial is a function of the form

$$P(x) = \sum_{|k| \leq N} a_k e^{ik\cdot x}, \qquad a_k \in \mathbb{C},\ k \in \mathbb{Z}^d, \tag{1.3.1}$$

where we used the notations

$$k \cdot x = k_1 x_1 + \cdots + k_d x_d, \quad |k| = \sqrt{k_1^2 + \cdots + k_d^2}.$$

Let $e_k$ be the function (called the oscillatory exponential) defined by

$$e_k(x) = e^{ik\cdot x} = \exp(ik \cdot x),$$

and introduce the scalar product $(\cdot, \cdot)$ defined on $L^2_{\text{per}}(\mathbb{R}^d)$ by

$$(f, g) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} f(x)\overline{g(x)}\, \mathrm{d}x.$$

The key point is the orthogonality relation

$$(e_k, e_{k'}) = \delta_k^{k'} \qquad \text{(equal to 1 if } k = k' \text{ and 0 otherwise)},$$

which is satisfied by a direct calculation. We deduce that the coefficients $a_k$ in (1.3.1) satisfy

$$a_k = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} P(x)e^{-ik\cdot x}\, \mathrm{d}x, \quad \forall k \in \mathbb{Z}^d.$$

This motivates the following definition.

**Definition 1.3.1.** *Given a function $f \in L^1_{\text{per}}(\mathbb{R}^d)$, we define the $k^{th}$-Fourier coefficient of $f$ by*

$$\hat{f}(k) = (f, e_k) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} f(x)e^{-ik\cdot x}\, \mathrm{d}x,$$

*and we call Fourier series of $f$ the sequence $(S_N(f))_{N\in\mathbb{N}}$ defined by*

$$S_N f(x) = \sum_{|k|\leq N} \hat{f}(k)e^{ik\cdot x}.$$

There exists $g \in L^1_{\text{per}}(\mathbb{R}^d)$ such that $\|S_N g - g\|_{L^1}$ does not tend to 0 as $N$ goes to $+\infty$. This explains that one has to seek convergence in other spaces. The simplest and perhaps most important result to know about Fourier series is the following theorem.

**Theorem 1.3.2.** *For all $f \in L^2_{\text{per}}(\mathbb{R}^d)$, we have*

$$f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)e^{ik\cdot x}$$

*with convergence in* $L^2_{\text{per}}(\mathbb{R}^d)$*, which means that*

$$\lim_{N\to+\infty} \|f - S_N f\|_{L^2} = 0 \quad \textit{where} \quad S_N f(x) = \sum_{|k|\le N} \hat{f}(k) e^{ik\cdot x}.$$

*In addition*

$$\|f\|^2_{L^2} = \sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2.$$

*Conversely, if* $c = (c_k) \in \ell^2(\mathbb{Z}^d)$*, then the series* $\displaystyle\sum_{k\in\mathbb{Z}^d} c_k e^{ik\cdot x}$ *converges in* $L^2_{\text{per}}(\mathbb{R}^d)$ *to a function* $f$ *satisfying* $\hat{f}(k) = c_k$*.*

*Proof.* We have already noticed that $(e_k)_{k\in\mathbb{Z}^d}$ is an orthonormal family : $(e_k, e_{k'}) = 0$ if $k \neq k'$ and $\|e_k\|_{L^2} = 1$. Thus, according to the Theorem 1.2.3 on Hilbertian bases, to prove this result, it is sufficient to show that the vector space spanned by $F = \text{vect}\{e_k \,:\, k \in \mathbb{Z}^d\}$ is dense in $L^2_{\text{per}}(\mathbb{R}^d)$. Let $f \in L^2_{\text{per}}(\mathbb{R}^d)$ and $\varepsilon > 0$. Since the set $C^0_{\text{per}}(\mathbb{R}^d)$ of continuous and $2\pi$-periodic functions is dense in $L^2_{\text{per}}(\mathbb{R}^d)$, there exists $g \in C^0_{\text{per}}(\mathbb{R}^d)$ such that $\|f - g\|_{L^2} \le \varepsilon$. Moreover, we have seen that the Stone-Weierstrass theorem implies that trigonometric polynomials are dense in $C^0_{\text{per}}(\mathbb{R}^d)$. Precisely, the Proposition 1.1.8 implies that there exists $h \in \text{vect}\{e_k \,:\, k \in \mathbb{Z}^d\}$ such that $\|g - h\|_{L^\infty} \le \varepsilon$. We deduce that

$$\|f - h\|_{L^2} \le \|f - g\|_{L^2} + \|g - h\|_{L^2} \le \varepsilon + (2\pi)^d \|g - h\|_{L^\infty} \le (1 + (2\pi)^d)\varepsilon.$$

This shows that $F$ is dense in $L^2_{\text{per}}(\mathbb{R}^d)$, which ends the proof. □

# Chapter 2

# The Fourier transform

## 2.1 From sums to integrals

We will study a decomposition analogous to the Fourier series decomposition, but without making any periodicity hypothesis. Here also the goal is to write a function as a sum of oscillatory exponentials. Recall that an oscillatory exponential is by definition a function of the form $x \mapsto \exp(ix \cdot \xi)$ with $\xi \in \mathbb{R}^d$. The difference with Fourier series is that this sum will be an integral on $\mathbb{R}^d$ instead of being a sum indexed by $k \in \mathbb{Z}^d$.

The Fourier series decomposition of a periodic function is well understood: it is the decomposition of an element of a Hilbert space on a Hilbertian basis. On the other hand, the decomposition of a non-periodic function as a sum (in the sense of integrals) of oscillatory exponentials is less intuitive. To understand how this decomposition is obtained, we will start from the Fourier series decomposition for functions which are $2T$-periodic with respect to each variable, and then make $T$ go to $+\infty$. The heuristic idea is to see a function defined on $\mathbb{R}^d$ as a periodic function of period $+\infty$ with respect to each variable.

Consider a function $f$ which is $C^\infty$ and has compact support. For $T$ large enough, the support of $f$ is included in $Q_T = ]-T, T[^d$. We will compute the Fourier decomposition of $f$ in $L^2(Q_T)$ and make $T$ tend to $+\infty$. To do so, let us let us introduce the scalar product $(f, g) = (2T)^{-d} \int_{Q_T} f(x)\overline{g(x)}\,\mathrm{d}x$ on $L^2(Q_T)$ and set $e_k(x) = \exp(i\pi k \cdot x/T)$ where $k \in \mathbb{Z}^d$. These functions are $2T$-periodic with respect to each variable and we have $(e_k, e_l) = \delta_k^l$. The Fourier coefficients of $f$ are given by

$$\hat{f}_k = (f, e_k) = \frac{1}{(2T)^d} \int_{Q_T} f(x) \exp\Big( - \frac{i\pi k \cdot x}{T}\Big) \,\mathrm{d}x.$$

Let us fix $x \in \mathbb{R}^d$. Since $f \in C_0^\infty(Q_T)$, we have

$$f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}_k e_k(x)$$

(with normal convergence and thus pointwise). We can therefore write that

$$f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}_k e_k(x) = \sum_{k \in \mathbb{Z}^d} \frac{1}{(2T)^d} \Big( \int_{Q_T} f(y) \exp\Big( - \frac{i\pi k \cdot y}{T}\Big) \,\mathrm{d}y\Big) \exp\Big(\frac{i\pi k \cdot x}{T}\Big).$$

As the support of $f$ is included in $Q_T$, we observe that

$$\frac{1}{2^d} \Big( \int_{Q_T} f(y) \exp\Big( - \frac{i\pi k \cdot y}{T}\Big) \,\mathrm{d}y\Big) \exp\Big(\frac{i\pi k \cdot x}{T}\Big) = F\Big(\frac{k}{T}\Big)$$

with

$$F(\xi) := \frac{1}{2^d} \exp\big(i\pi\xi \cdot x\big) \int_{\mathbb{R}^d} f(y) \exp\big( - i\pi\xi \cdot y\big) \,\mathrm{d}y.$$

If we put $h = 1/T$, then $f(x)$ is equal to $\sum_{k\in\mathbb{Z}^d} h^d F(kh)$. When $T$ tends to $+\infty$, the step $h$ tends to 0 and this sum is a Riemann sum which converges, formally [1], to $\int_{\mathbb{R}^d} F(\xi) \,\mathrm{d}\xi$. We find that

$$f(x) = \frac{1}{2^d} \int_{\mathbb{R}^d} e^{i\pi x\cdot\xi} \Big( \int_{\mathbb{R}^d} e^{-i\pi y\cdot\xi} f(y) \,\mathrm{d}y\Big) \,\mathrm{d}\xi.$$

We prefer to write the previous relation in the form

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \Big( \int_{\mathbb{R}^d} e^{-iy\cdot\xi} f(y) \,\mathrm{d}y\Big) \,\mathrm{d}\xi. \tag{2.1.1}$$

This formula corresponds to a frequency description of the function $f$ (in the physical literature, $\xi$ is called the wave vector and $|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}$ is said to be the frequency).

[1] As we will see later, it is easy to show that if $f$ is $C^\infty$ with compact support on $\mathbb{R}^d$, then the function $F$ is integrable on $\mathbb{R}^d$. This would allow to justify the passage to the limit when $T$ tends to $+\infty$.

**Definition 2.1.1.** *Let $f \in L^1(\mathbb{R}^d)$. We call Fourier transform of $f$ the function, denoted $\widehat{f}$ or $\mathcal{F}(f)$, defined for all $\xi \in \mathbb{R}^d$ by*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x. \tag{2.1.2}$$

The assumption that $f$ is an integrable function is the minimal assumption for the formula (2.1.2) to make sense for the Lebesgue integral. This is why we start by defining the Fourier transform on $L^1(\mathbb{R}^d)$. But we will see that it is natural to work with other function spaces. We will see two results in this direction. As for the Fourier series, an essential result is that the Fourier transform preserves the $L^2$-norm (up to a constant depending on $\pi$). This allows us to extend the definition of the Fourier transform from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. We will also see how to define the Fourier transform on a much larger space, the space of tempered distributions, which contains all the Lebesgue spaces $L^p(\mathbb{R}^d)$ as well as the Lebesgue spaces $L^p_{\text{per}}(\mathbb{R}^d)$ of periodic functions, and this whatever $1 \le p \le \infty$. In particular, this Fourier transform extended to the space of tempered distributions also contains the theory of Fourier series. Let us add that the space of tempered distributions contains many other spaces useful in the theory of partial differential equations. We will study the case of the Hölder spaces, and introduce the Littlewood-Paley decomposition to give a characterization of these spaces which is very useful.

## 2.2 Schwartz class

To construct a Fourier transform on a space which is as big as possible, we will use a duality principle. The process is the following: if $T$ is a continuous linear application from $E$ into $E$ then $T^*$ is continuous from $E^*$ into $E^*$. Moreover if $E \subset L^1(\mathbb{R}^d)^*$, then $L^1(\mathbb{R}^d) \subset E^*$. So to extend the definition of the Fourier transform to a space larger than $L^1(\mathbb{R}^d)$, we will try to define it as the adjoint of an isomorphism of a space $E$ included in $L^1(\mathbb{R}^d)$. (A word of caution: this corresponds very roughly to what we are going to do. Indeed, we will not be able to work in the framework of Banach spaces. We will have to work in the framework of Fréchet spaces.)

We have to look for a space, the smallest possible, such that the Fourier transform is an isomorphism of this space in itself. This principle is very simple but we will see that its implementation is subtle. Indeed, the following proposition shows that we cannot use the space we spontaneously think of (the space of functions $C^\infty$ with compact support).

**Proposition 2.2.1.** *There exists no non-zero $f \in L^1(\mathbb{R})$ function with compact support and whose whose Fourier transform is also compactly supported.*

*Proof.* Let $f \in L^1(\mathbb{R})$ have compact support. Then we can define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \int_{\mathbb{R}} e^{-ixz} f(x)\,\mathrm{d}x.$$

Note that $F(\xi) = \widehat{f}(\xi)$ for all $\xi$ in $\mathbb{R}$. Hence, $F$ vanishes on an interval. As $F$ is an entire function (holomorphic on $\mathbb{C}$), we get that $F = 0$ because a non zero entire function can only vanish on a discrete set. □

Instead of working with $C^\infty$ functions with compact support, we will work with work with functions $C^\infty$ which are rapidly decayingat infinity, in the sense of the definition below.

**Definition 2.2.2.** $(i)$ *A function $f$ is said to be rapidly decreasing if the product of $f$ by any polynomial is a bounded function.*

$(ii)$ *A function $f$ is said to belong to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ if $f$ and all its derivatives are rapidly decreasing. It is equivalent to say that, for all $p \in \mathbb{N}$,*

$$\mathcal{N}_p(f) = \sum_{|\alpha| \le p, |\beta| \le p} \left\| x^\alpha \partial_x^\beta f \right\|_{L^\infty} < +\infty.$$

**Remark 2.2.3.** *Note that*

$$C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d).$$

*The basic example of a function of $\mathcal{S}(\mathbb{R}^d)$ that is not an element of $C_0^\infty(\mathbb{R}^d)$ is the gaussian $x \mapsto \exp(-|x|^2)$. This function plays a special role in the study of the Fourier transform. More generally, for all complex numbers $z$ of real part $\operatorname{Re} z > 0$, the function $\exp(-z|x|^2)$ belongs to $\mathcal{S}(\mathbb{R}^d)$.*

Note that $\mathcal{N}_p$ is a norm on $\mathcal{S}(\mathbb{R}^d)$ for all integer $p$. However, if we consider $\mathcal{S}(\mathbb{R}^d)$ as a normed space for this norm, then we do not get a Banach space (a Cauchy sequence for this norm does not converge in general to a $C^\infty$ function). The correct topological notion is that of a topological vector space with a family of semi-norms.

**Proposition 2.2.4.** *The Schwartz class is a graded Fréchet space for the topology induced by the family of semi-norms $\{\mathcal{N}_p\}_{p \in \mathbb{N}}$.*

The following proposition contains several simple properties which are very useful.

**Proposition 2.2.5.** *Suppose that $f$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. Then*

*(i) For all multi-indices $\alpha$ and $\beta$ in $\mathbb{N}^d$, we have $x^\alpha \partial_x^\beta f \in \mathcal{S}(\mathbb{R}^d)$ (and the application $f \mapsto x^\alpha \partial_x^\beta f$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$).*

*(ii) The product of two elements of $\mathcal{S}(\mathbb{R}^d)$ belongs to $\mathcal{S}(\mathbb{R}^d)$ (and the product is continuous from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$).*

*(iii) For all $p \in [1, +\infty]$, we have $f \in L^p(\mathbb{R}^d)$ (and the injection of $\mathcal{S}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ is continuous).*

*(iv) $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$.*

*(v) The product of convolution of two elements of $\mathcal{S}(\mathbb{R}^d)$ belongs to $\mathcal{S}(\mathbb{R}^d)$ (and the convolution product application is continuous).*

*(vi) The Fourier transform $\widehat{f}$ belongs to $C^1(\mathbb{R}^d)$ and, for all $1 \le j \le d$ and all $\xi \in \mathbb{R}^d$,*
$$\partial_{\xi_j} \widehat{f}(\xi) = \mathcal{F}\left((-ix_j) f\right).$$

*(vii) For all $1 \le j \le d$ and all $\xi$ in $\mathbb{R}^d$,*
$$\xi_j \widehat{f}(\xi) = -i\mathcal{F}\left(\partial_{x_j} f\right)(\xi).$$

*Proof.* The first two points are immediate consequences of the definition of $\mathcal{S}(\mathbb{R}^d)$. To show (iii), we begin by observing that
(2.2.1)
$$\|f\|_{L^1} = \int |f(x)|\, \mathrm{d}x \le \sup\left\{(1+|x|)^{d+1} |f(x)|\right\} \int \frac{\mathrm{d}x}{(1+|x|)^{d+1}} \le C\mathcal{N}_{d+1}(f).$$

Then we observe that $f \in L^\infty(\mathbb{R}^d)$ (direct) and we conclude that $f \in L^p(\mathbb{R}^d)$ for all $p \in [1, +\infty]$ because $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is included in $L^p(\mathbb{R}^d)$.

To prove the point (iv), consider a function $\chi \in C_0^\infty(\mathbb{R}^d)$ and show that, for any function $f$ of $\mathcal{S}(\mathbb{R}^d)$, the sequence $\chi(\cdot/k)f$ converges to $f$ in $\mathcal{S}(\mathbb{R}^d)$ when $k$ tends to $+\infty$. For that, it is enough to verify that, for all $p \in \mathbb{N}$, the semi-norms $\mathcal{N}_p(f - \chi(\cdot/k)f)$ tend to 0. This calculation is left as an exercise.

To show (v), consider $f$ and $g$ in $\mathcal{S}(\mathbb{R}^d)$. Then the convolution product $f * g$ is $C^\infty$ on $\mathbb{R}^d$ and, for all multi-index $\beta$ in $\mathbb{N}^d$, we have $\partial_x^\beta (f * g) = (\partial_x^\beta f) * g$. Moreover, for all $m \in \mathbb{N}$, we have

$$|x|^m \leq (|x-y| + |y|)^m \leq (2 \max\{|x-y|, |y|\})^m \leq 2^m (|x-y|^m + |y|^m).$$

So $\left| x^\alpha \partial_x^\beta (f * g)(x) \right|$ is bounded by

$$\int \left( |x-y|^{|\alpha|} \left| \partial_x^\beta f(x-y) \right| |g(y)| + \left| \partial_x^\beta f(x-y) \right| |y|^{|\alpha|} |g(y)| \right) \mathrm{d}y.$$

We then use the obvious inequality$\|F * G\|_{L^\infty} \leq \|F\|_{L^\infty} \|G\|_{L^1}$ and the points (i) and (iii) to bound the $L^1$-norm of $g$ and $y^\alpha g$ by semi-norms of $g$ in $\mathcal{S}(\mathbb{R}^d)$.

To prove the point (vi), it suffices to observe that the hypotheses of the Lebesgue derivation theorem are satisfied and then apply this result. Finally, the point (vii) is obtained by writing that

$$\xi_j e^{-ix\cdot\xi} = i \partial_{x_j} e^{-ix\cdot\xi},$$

then integrating by parts:

$$\xi_j \widehat{f}(\xi) = \int (i\partial_{x_j} e^{-ix\cdot\xi}) f(x) \, \mathrm{d}x = -i \int e^{-ix\cdot\xi} \partial_{x_j} f(x) \, \mathrm{d}x = -i\mathcal{F}\left(\partial_{x_j} f\right)(\xi).$$

This manipulation is justified because $f$ is rapidly decreasing (we can then integrate by parts on a ball $B(0, R)$ and then make $R$ tend to $+\infty$). □

The following proposition shows why $\mathcal{S}(\mathbb{R}^d)$ is the right space to study the Fourier transform.

**Proposition 2.2.6.** *The Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ to itself, and there exist constants $C_p$ such that, for all $f$ in $\mathcal{S}(\mathbb{R}^d)$,*

$$\mathcal{N}_p(\widehat{f}) \leq C_p \, \mathcal{N}_{p+d+1}(f). \tag{2.2.2}$$

*This proves that the Fourier transform is continuous from $\mathcal{S}(\mathbb{R}^d)$ into itself.*

*Proof.* Let $f \in \mathcal{S}(\mathbb{R}^d)$. We can use the previous proposition to get

$$\left| \xi^\alpha \partial_\xi^\beta \widehat{f}(\xi) \right| = \left| \mathcal{F}\left\{ \partial_x^\alpha \left( x^\beta f(x) \right) \right\} \right|.$$

Suppose that $|\alpha| \leq p$ and $|\beta| \leq p$. Using the inequality $\|\widehat{u}\|_{L^\infty} \leq \|u\|_{L^1}$ and the formula of Leibniz, it comes

$$\left|\xi^\alpha \partial_\xi^\beta \widehat{f}(\xi)\right| \leq \left\|\partial_x^\alpha (x^\beta f)\right\|_{L^1} \leq K \sum_{|\alpha'|\leq p, |\beta'|\leq p} \left\|x^{\beta'} \partial_x^{\alpha'} f\right\|_{L^1}.$$

The desired inequality is derived by applying (2.2.1). □

We have said that Gaussian functions play an important role in the study of the Fourier transform. It is because of the following result, which states that the Fourier transform of a Gaussian function is a Gaussian function.

**Proposition 2.2.7.** *For all $a > 0$ and all dimension $d \geq 1$,*

$$\mathcal{F}\left(e^{-a|x|^2}\right) = \left(\frac{\pi}{a}\right)^{d/2} e^{-|\xi|^2/4a}.$$

*Proof.* Let us start with the case of dimension of space $d = 1$ in the special case with $a = 1$. Set $f(x) = e^{-|x|^2}$. The Fourier transform of $f$, denoted $\mathcal{F}(f)(\xi)$, is a regular function regular function which satisfies

$$\begin{aligned}(\mathcal{F}f)'(\xi) &= \int_{\mathbb{R}} (-ix) e^{-ix\xi} e^{-x^2}\,\mathrm{d}x = \frac{i}{2}\int_{\mathbb{R}} e^{-ix\xi} \partial_x e^{-x^2}\,\mathrm{d}x \\ &= \frac{-i}{2}\int_{\mathbb{R}} (-i\xi) e^{-ix\xi} e^{-x^2}\,\mathrm{d}x\end{aligned}$$

so

$$(\mathcal{F}f)'(\xi) = -\frac{1}{2}\xi(\mathcal{F}f)(\xi).$$

By using

$$\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

we deduce that

$$(\mathcal{F}f)(\xi) = e^{-\xi^2/4}(\mathcal{F}f)(0) = \sqrt{\pi} e^{-\xi^2/4}.$$

We get the result by some simple manipulations: if $f(x) \in L^1(\mathbb{R}^d)$ then the Fourier transform of $f(x/\lambda)$ is $|\lambda|^d \widehat{f}(\lambda\xi)$. Moreover the Fourier transform of $f_1(x_1)\cdots f_d(x_d)$ is $\widehat{f_1}(\xi_1)\cdots\widehat{f_d}(\xi_d)$. □

We are then able to prove the following fundamental result.

**Theorem 2.2.8.** *If $u$ belongs to $\mathcal{S}(\mathbb{R}^d)$, then, for all $x$ in $\mathbb{R}^d$,*

$$u(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} \widehat{u}(\xi)\, \mathrm{d}\xi.$$

**Remark 2.2.9.** *We saw that if $u \in \mathcal{S}(\mathbb{R}^d)$ then $widehatu \in \mathcal{S}(\mathbb{R}^d)$. We have also seen that $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ and thus the function $\xi \mapsto e^{ix\cdot\xi}\widehat{u}(\xi)$ is integrable. The previous formula makes sense for all $x \in \mathbb{R}^d$.*

*Proof.* Given $\varepsilon > 0$ let us introduce

$$u_\varepsilon(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} \widehat{u}(\xi) e^{-\frac{1}{2}\varepsilon^2|\xi|^2}\, \mathrm{d}\xi.$$

Using the previous lemma we compute (handling only convergent integrals)

$$\begin{aligned} u_\varepsilon(x) &= \frac{1}{(2\pi)^d} \iint e^{i(x-y)\cdot\xi} u(y) e^{-\frac{1}{2}\varepsilon^2|\xi|^2}\, \mathrm{d}y\, \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^{d/2}} \int u(y) e^{-\frac{1}{2\varepsilon^2}|x-y|^2} \varepsilon^{-d}\, \mathrm{d}y \\ &= \frac{1}{(2\pi)^{d/2}} \int \big(u(x+\varepsilon y) - u(x)\big) e^{-\frac{1}{2}|y|^2}\, \mathrm{d}y + u(x). \end{aligned}$$

Since

$$|u(x+\varepsilon y) - u(x)| \le \varepsilon\, |y|\, \|u'\|_{L^\infty},$$

we obtain the desired result by passing to the limit when $\varepsilon$ tends to 0. □

**Theorem 2.2.10.** *If $f$ and $g$ belong to $\mathcal{S}(\mathbb{R}^d)$, then*

$$\int f(x)\overline{g}(x)\, \mathrm{d}x = \frac{1}{(2\pi)^d} \int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, \mathrm{d}\xi.$$

*In particular, for all $f$ in $\mathcal{S}(\mathbb{R}^d)$, there holds*

$$\|f\|_{L^2}^2 = \frac{1}{(2\pi)^d} \big\|\widehat{f}\big\|_{L^2}^2.$$

*Proof.* We will start by showing that if $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, then

$$\int \widehat{\varphi}(x)\psi(x)\, \mathrm{d}x = \int \varphi(y)\widehat{\psi}(y)\, \mathrm{d}y. \tag{2.2.3}$$

As $\varphi$ and $\psi$ are rapidly decreasing, we can apply Fubini's theorem to obtain

$$\begin{aligned}\int \widehat{\varphi}\psi \,\mathrm{d}x &= \int \left( \int e^{-iy\cdot x}\varphi(y)\,\mathrm{d}y \right) \psi(x)\,\mathrm{d}x \\ &= \int \left( \int e^{-iy\cdot x}\psi(x)\,\mathrm{d}x \right) \varphi(y)\,\mathrm{d}y = \int \varphi\widehat{\psi}\,\mathrm{d}y.\end{aligned}$$

We then apply this identity with $\varphi = f$ and $\overline{g} = \widehat{\psi}$. Then

$$\int f\overline{g} = \int \varphi\widehat{\psi} = \int \widehat{\varphi}\psi = \int \widehat{f}\mathcal{F}^{-1}\overline{g}.$$

Then we verify (using the Fourier inversion theorem) that

$$(\mathcal{F}^{-1}\overline{g})(\xi) = (2\pi)^{-d}\int e^{iy\xi}\overline{g}(y)\,\mathrm{d}y = (2\pi)^{-d}\overline{\int e^{-iy\xi}g(y)\,\mathrm{d}y} = (2\pi)^{-d}\overline{\widehat{g}(\xi)}.$$

The last identity concerning the norm $L^2$ is then an obvious corollary. □

**Corollary 2.2.11.** *The Fourier transform $\mathcal{F}$ is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ to itself, and*

$$\mathcal{F}^{-1}f = (2\pi)^{-d}\overline{\mathcal{F}(\overline{f})}.$$

## 2.3 Tempered distributions

### 2.3.1 Definition of tempered distributions

**Definition 2.3.1.** *By definition, the space of tempered distributions, denoted $\mathcal{S}'(\mathbb{R}^d)$, is the topological dual of $\mathcal{S}(\mathbb{R}^d)$.*

**Notation 2.3.2.** *Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$ be the topological dual. We denote $\langle u, f\rangle_{\mathcal{S}'\times\mathcal{S}}$ the complex number that we obtain by making $u$ act on $f$.*

A linear application $T\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ if and only if there exists $p \in \mathbb{N}$ and $C > 0$ such that

$$\forall f \in \mathcal{S}(\mathbb{R}^d), \qquad |\langle T, f\rangle_{\mathcal{S}'\times\mathcal{S}}| \leq C\mathcal{N}_p(f) = C\sum_{|\alpha|\leq p, |\beta|\leq p} \left\| x^\alpha \partial_x^\beta f \right\|_{L^\infty}.$$

Let us show as a first example that any function $u \in L^\infty(\mathbb{R}^d)$ allows to define a tempered distribution. We define a linear form $U: \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ by

$$\langle U, v\rangle_{\mathcal{S}'\times\mathcal{S}} = \int_{\mathbb{R}^d} u(x)v(x)\,\mathrm{d}x. \tag{2.3.1}$$

We verify that $U$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathbb{C}$ by the following estimate

$$\begin{aligned}|\langle U, v\rangle_{\mathcal{S}'\times\mathcal{S}}| &\leq \|u\|_{L^\infty}\,\|v\|_{L^1}\\ &\leq \|u\|_{L^\infty}\left(\int \frac{dx}{(1+|x|)^{d+1}}\right)\sup_{\mathbb{R}^d}\left|(1+|x|)^{d+1}\,v(x)\right|,\end{aligned}$$

which implies that

$$|\langle U, v\rangle_{\mathcal{S}'\times\mathcal{S}}| \leq C\,\|u\|_{L^\infty}\,\mathcal{N}_{d+1}(v).$$

By reasoning in a similar way, we show that the formula (2.3.1) defines a tempered distribution for all functions $u \in L^p(\mathbb{R}^d)$ with $p \in [1, +\infty]$. This procedure allows us to embed the Lebesgue spaces into $\mathcal{S}'(\mathbb{R}^d)$. In fact, we can embed many spaces and we will see later on the fundamental example of Sobolev spaces.

**Definition 2.3.3.** *We will say that a tempered distribution $U \in \mathcal{S}'(\mathbb{R}^d)$ belongs to a certain space $X$ if there exists $u \in X$ such that*

$$\forall v \in \mathcal{S}(\mathbb{R}^d), \quad \langle U, v\rangle_{\mathcal{S}'\times\mathcal{S}} = \int_{\mathbb{R}^d} u(x)v(x)\,\mathrm{d}x.$$

### 2.3.2 Extension of the calculus to tempered distributions

We have seen that we can embed all Lebesgue spaces in $\mathcal{S}'(\mathbb{R}^d)$. We can also embed the Hölder spaces and the Sobolev spaces (both will be defined later). We think of the space of tempered distributions distributions as the largest space in which we want to work [2]. It is then natural to want to extend the definition of important operators in analysis to $\mathcal{S}'(\mathbb{R}^d)$.

We will see that this can be done very simply.

**Definition 2.3.4.** *Consider a linear application $A: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ which is assumed to be continuous. We will say that $A$ has a continuous adjoint on $\mathcal{S}(\mathbb{R}^d)$ if there exists a continuous linear application $A^*: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ such that*

$$\forall (u,v) \in \mathcal{S}(\mathbb{R}^d)^2, \qquad (Au, v) = (u, A^*v) \quad \textit{where } (f,g) = \int f(x)\overline{g(x)}\,\mathrm{d}x.$$

[2] There are larger spaces, such as the space of distributions, but we will not use those spaces in these lectures

**Example 2.3.5.** *$i$)The Fourier transform satisfies this hypothesis.*

*$ii$) Let $1 \leq j \leq d$. If $A = \partial_{x_j}$, then $A$ is indeed continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$ because $\mathcal{N}_p(Au) \leq \mathcal{N}_{p+1}(u)$ and we have, integrating by parts, $(Au, v) = (u, A^*v)$ with $A^* = -\partial_{x_j}$.*

*$iii$) Let us note $C_b^\infty(\mathbb{R}^d)$ the space of $C^\infty$ functions which are bounded together with all their derivatives. If $c \in C_b^\infty(\mathbb{R}^d)$, then the operator $A_c$ defined by $A_c(f)(x) = c(x)f(x)$ satisfies this property. Then $(A_c)^* = A_{\overline{c}}$.*

*$iv$) If $A$ and $B$ satisfy this property then $A \circ B$ also satisfies $(A \circ B)^* = B^* \circ A^*$. We deduce from the two previous points that any differential operator $A$, of the form $A(f)(x) = \sum_{|\alpha| \leq m} c_\alpha(x)\partial_x^\alpha f(x)$ with $c_\alpha \in C_b^\infty(\mathbb{R}^d)$, satisfies this property.*

*$v$) We will see another example later which generalizes the notion of differential operator (see the section on pseudo-differential operators).*

We will show that there exists $\widetilde{A} \colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ continuous linear which extends the definition of $A$. For that we define

$$\forall u \in \mathcal{S}'(\mathbb{R}^d),\ \forall v \in \mathcal{S}(\mathbb{R}^d), \quad \langle \widetilde{A}u, v\rangle_{\mathcal{S}'\times\mathcal{S}} = \langle u, \overline{A^*\overline{v}}\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

Let us show that the operator thus constructed extends the definition of $A$.

**Proposition 2.3.6.** *Consider the application $\mathcal{T} \colon u \in \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{T}_u \in \mathcal{S}'(\mathbb{R}^d)$ defined by*

$$\mathcal{T}_u(v) = (u, \overline{v}) = \int u(x)v(x)\,\mathrm{d}x.$$

*Then this application is well defined, linear, continuous and injective and moreover*

$$\widetilde{A}\mathcal{T}_u = \mathcal{T}_{Au}, \quad \forall u \in \mathcal{S}(\mathbb{R}^d).$$

**Remark 2.3.7.** *The first part of the result means that $\mathcal{T}$ is an injection of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$; the second part means that $\widetilde{A}$ coincides with $A$ on $\mathcal{S}(\mathbb{R}^d)$.*

*Proof.* For all $u, v \in \mathcal{S}(\mathbb{R}^d)$ we have already seen that

$$|\langle \mathcal{T}_u, v\rangle_{\mathcal{S}'\times\mathcal{S}}| \leq \|u\|_{L^\infty}\|v\|_{L^1} \leq C\mathcal{N}_0(u)\mathcal{N}_{d+1}(v),$$

which shows that $\mathcal{T}_u$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ and that $u \mapsto \mathcal{T}_u$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$. Moreover $\mathcal{T}$ is injective because $\mathcal{T}_{u_1} = \mathcal{T}_{u_2}$ implies $\mathcal{T}_{u_1-u_2}(\overline{u_1 - u_2}) = 0$

so $\|u_1 - u_2\|_{L^2} = 0$ hence $u_1 = u_2$. With the previous definitions, for all $u, v$ in $\mathcal{S}(\mathbb{R}^d)$, we have

$$\langle \widetilde{A}\mathcal{T}_u, v\rangle_{\mathcal{S}'\times\mathcal{S}} = \langle \mathcal{T}_u, \overline{A^*\overline{v}}\rangle_{\mathcal{S}'\times\mathcal{S}} = (u, A^*\overline{v}) = (Au, \overline{v}) = \langle \mathcal{T}_{Au}, v\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

This proves that $\widetilde{A}\mathcal{T}_u = \mathcal{T}_{Au}$. □

In the following we will simply denote $A$ instead of $\widetilde{A}$ the operator extended to $\mathcal{S}'(\mathbb{R}^d)$. Using this construction, we can define the partial derivative $\partial_{x_j}$ of any tempered distribution! By definition, we have

$$\forall u \in \mathcal{S}'(\mathbb{R}^d),\ \forall v \in \mathcal{S}(\mathbb{R}^d), \quad \langle \partial_{x_j} u, v\rangle_{\mathcal{S}'\times\mathcal{S}} = -\langle u, \partial_{x_j} v\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

We deduce that we can define $\partial_x^\alpha u$ for all $\alpha \in \mathbb{N}^d$ and all $u \in \mathcal{S}'(\mathbb{R}^d)$. One can thus derive to any order any distribution (which is of course of course false for functions). Note that if $u$ belongs to the Sobolev space $H^1(\mathbb{R}^d)$ then the derivative in the weak sense of $u$ coincides with the derivative in the sense of tempered distributions.

We will now apply the previous construction with the Fourier transform. Recall that (cf (2.2.3)), for all $\varphi, \psi$ in $\mathcal{S}(\mathbb{R}^d)$, we have

$$\int \widehat{\varphi}(x)\psi(x)\,\mathrm{d}x = \int \varphi(y)\widehat{\psi}(y)\,\mathrm{d}y.$$

Let us consider a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$. We can then apply the previous principle to define its Fourier define its Fourier transform, denoted $\mathcal{F}(u)$, by

$$\langle \mathcal{F}(u), v\rangle_{\mathcal{S}'\times\mathcal{S}} = \langle u, \widehat{v}\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

We denote also $\widehat{u}$ the Fourier transform of a tempered distribution.

**Proposition 2.3.8.** *The Fourier transform $\mathcal{F}$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^d)$ in itself (continuous linear application with continuous inverse). Moreover we have*

$$\mathcal{F}^{-1} f = (2\pi)^{-d}\overline{\mathcal{F}(\overline{f})}.$$

We have already noticed that we can embed the Lebesgue spaces in $\mathcal{S}'(\mathbb{R}^d)$. In particular, we can consider the Fourier transform of a function $L^p(\mathbb{R}^d)$. The most important case in practice is that of $L^2(\mathbb{R}^d)$. In this case we have the following result.

**Proposition 2.3.9.** *If $f$ belongs to $L^2(\mathbb{R}^d)$, then $\mathcal{F}(f)$ belongs to $L^2(\mathbb{R}^d)$ and*

$$\|f\|_{L^2}^2 = \frac{1}{(2\pi)^d}\,\|\mathcal{F}(f)\|_{L^2}^2\,.$$

**Remark 2.3.10.** *With the previous conventions, the fact that $\mathcal{F}(f)$ belongs to $L^2(\mathbb{R}^d)$ means that theure exists a function $h \in L^2(\mathbb{R}^d)$ such that $\langle \mathcal{F}(f), v\rangle_{\mathcal{S}'\times\mathcal{S}} = \int_{\mathbb{R}^d} h(x)v(x)\,\mathrm{d}x$ for all $v \in \mathcal{S}(\mathbb{R}^d)$. Then we have $\|f\|_{L^2}^2 = \frac{1}{(2\pi)^d}\left\|h\right\|_{L^2}^2$.*

**Definition 2.3.11.** *A function $m$ belonging to $C^\infty(\mathbb{R}^d)$ is said to be slowly growing if there exists a polynomial $P$ such that $|m(\xi)| \leq P(\xi)$ for all $\xi \in \mathbb{R}^d$.*

*If $m$ is a slowly growing function and if $v \in \mathcal{S}(\mathbb{R}^d)$, then $mv$ also belongs to the class of Schwartz. We check that we can define an operator, noted $m(D_x)$ on $\mathcal{S}'(\mathbb{R}^d)$ in the following way:*

$$\mathcal{F}(m(D_x)u) = m\mathcal{F}u.$$

These operators are used very often. We will see them again in the chapter on symbolic computation for pseudo-differential operators.

# Chapter 3

# Fourier analysis and Sobolev spaces

## 3.1 Definitions and first properties

Recall the notation

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

Given a real number $s \in [0, +\infty)$, we say that a function $u \in L^2(\mathbb{R}^d)$ belongs to the Sobolev space $H^s(\mathbb{R}^d)$ if

$$\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \, \mathrm{d}\xi < +\infty.$$

**Proposition 3.1.1.** *Let $s \in [0, +\infty)$. Equipped with the scalar product*

$$(u, v)_{H^s} = (2\pi)^{-d} \int (1 + |\xi|^2)^s \widehat{u}(\xi) \, \overline{\widehat{v}(\xi)} \, \mathrm{d}\xi,$$

*and therefore the norm*

$$\|u\|_{H^s} = (2\pi)^{-d/2} \left\| (1 + |\xi|^2)^{s/2} \widehat{u} \right\|_{L^2},$$

*the Sobolev space $H^s(\mathbb{R}^d)$ is a Hilbert space.*

*Proof.* The application $u \mapsto (2\pi)^{-d/2}(1 + |\xi|^2)^{s/2}\widehat{u}$ is by definition an isometric bijection of $H^s(\mathbb{R}^d)$ on $L^2(\mathbb{R}^d)$. This last space being a Banach space, it is the same for $H^s(\mathbb{R}^d)$ with the norm defined above. $\square$

**Proposition 3.1.2.** *The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$ for all $s \geq 0$.*

*Proof.* Let us consider the isometry $u \mapsto (2\pi)^{-d/2}(1+|\xi|^2)^{s/2}\widehat{u}$ from $H^s(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$. The inverse isometry transforms the dense subspace $\mathcal{S}(\mathbb{R}^d)$ of $L^2(\mathbb{R}^d)$ into a dense subspace of $H^s(\mathbb{R}^d)$. Now this application is a bijection of $\mathcal{S}(\mathbb{R}^d)$ onto itself. We deduce that $\mathcal{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$. □

**Proposition 3.1.3.** *For any real number $s > d/2$,*

$$H^s(\mathbb{R}^d) \subset C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),$$

*with continuous injection.*

*Proof.* According to Cauchy-Schwarz inequality, for any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|f\|_{L^\infty} \leq \big\|\hat{f}\big\|_{L^1} \leq \big\|\langle\xi\rangle^{s}\big\|_{L^2}\big\|\langle\xi\rangle^{s}\hat{f}\big\|_{L^2}, \tag{3.1.1}$$

and we deduce the result by density of $\mathcal{S}(\mathbb{R}^d)$ in $H^s(\mathbb{R}^d)$. □

**Theorem 3.1.4.** *For any real number $s > d/2$, the product of two elements of $H^s(\mathbb{R}^d)$ belongs to $H^s(\mathbb{R}^d)$. In addition, there is a constant $C$ such that for any $u, v$ in $H^s(\mathbb{R}^d)$,*

$$\|uv\|_{H^s} \leq C\,\|u\|_{H^s}\,\|v\|_{H^s}\,.$$

*Proof.* The proof rests on the following inequality: for every $\xi, \eta$ in $\mathbb{R}^d$ we have

$$\forall s \geq 0, \qquad (1+|\xi|^2)^{s/2} \leq 2^s\Big\{(1+|\xi-\eta|^2)^{s/2} + (1+|\eta|^2)^{s/2}\Big\},$$

which is deduced from the triangular inequality and the bound $(a+b)^r \leq 2^r(a^r+b^r)$ for any triplet $(a,b,r)$ of positive numbers. Let us write then that for every $u, v$ in $\mathcal{S}(\mathbb{R}^d)$, we have (check the following formula in exercise)

$$\widehat{uv}(\xi) = (2\pi)^{-d}\int \widehat{u}(\xi-\eta)\widehat{v}(\eta)\,\mathrm{d}\eta.$$

Multiplying the two members by $\langle\xi\rangle^s$ and using the previous inequality, we find

$$\begin{aligned}\langle\xi\rangle^s\,|\widehat{uv}(\xi)| &\leq C\int \langle\xi-\eta\rangle^s\,|\widehat{u}(\xi-\eta)|\,|\widehat{v}(\eta)|\,\mathrm{d}\eta \\ &\quad + C\int |\widehat{u}(\xi-\eta)|\,\langle\eta\rangle^s\,|\widehat{v}(\eta)|\,\mathrm{d}\eta.\end{aligned}$$

If $s > d/2$ then $\mathcal{F}(H^s(\mathbb{R}^d)) \subset L^1(\mathbb{R}^d)$ as we have already seen (cf (3.1.1)). We then recognize above two products of convolution between a function of $L^1(\mathbb{R}^d)$ and another of $L^2(\mathbb{R}^d)$, that belong to $L^2(\mathbb{R}^d)$. This implies that $\langle\xi\rangle^s\widehat{uv} \in L^2(\mathbb{R}^d)$, hence the desired result $uv \in H^s(\mathbb{R}^d)$. □

We have seen that, any real number $s > d/2$, the product of two elements of $H^s(\mathbb{R}^d)$ is still in $H^s(\mathbb{R}^d)$. The following proposition shows that we can also define the product $\varphi u$ for everything $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and any $u \in H^s(\mathbb{R}^d)$ with $s \in [0, +\infty[$.

**Proposition 3.1.5.** *For any $s \in \mathbb{R}$, if $u \in H^s(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ then $\varphi u \in H^s(\mathbb{R}^d)$.*

*Proof.* The proof uses an inequality, called Peetre's inequality that states that for every $\xi, \eta$ in $\mathbb{R}^d$, we have

$$\forall s \in \mathbb{R}, \qquad (1 + |\xi|^2)^s \leq 2^{|s|}(1 + |\eta|^2)^s(1 + |\xi - \eta|^2)^{|s|}.$$

Let us assume that $s \geq 0$. To obtain this inequality, just use the triangular inequality

$$1 + |\xi|^2 \leq 1 + (|\eta| + |\xi - \eta|)^2 \leq 1 + 2\,|\eta|^2 + 2\,|\xi - \eta|^2 \leq 2(1 + |\eta|^2)(1 + |\xi - \eta|^2),$$

then raise both sides to the power $s \geq 0$. If $s < 0$, then $-s > 0$ and the previous inequality leads to

$$(1 + |\eta|^2)^{-s} \leq 2^{-s}(1 + |\xi|^2)^{-s}(1 + |\xi - \eta|^2)^{-s}.$$

The desired result is obtained by dividing by $(1 + |\eta|^2)^{-s}(1 + |\xi|^2)^{-s}$.

We then proceed as in the proof of the theorem 3.1.4. Indeed, one can still write for $u \in H^s(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\widehat{\varphi u}(\xi)$ as a convolution product. As $\widehat{\varphi}(\zeta)$ is in Schwartz's class, the previous inequality allows the product of convolution of a function to appear. of $L^1$ and $\langle\eta\rangle^s|\widehat{u}(\eta)|$ which is in $L^2$. □

## 3.2 Sobolev embeddings

We will now study the injection of Sobolev spaces $H^s(\mathbb{R}^d)$ into Lebesgue spaces $L^p(\mathbb{R}^d)$.

**Theorem 3.2.1.** *Let $d \geq 1$ and $s$ be a real such that $0 \leq s < d/2$. Then the Sobolev space $H^s(\mathbb{R}^d)$ is continuously embedded into $L^p(\mathbb{R}^d)$ for any $p$ such that*

$$2 \leq p \leq \frac{2d}{d - 2s}.$$

**Remark 3.2.2.** *The previous theorem states that for any real number s in* $[0, d/2[$, *we have*

$$\|f\|_{L^{\frac{2d}{d-2s}}} \leq C_s \|f\|_{H^s}.$$

*In fact, we will show a stronger result (see* (3.2.1)*) :*

$$\|f\|_{L^{\frac{2d}{d-2s}}} \leq C \|f\|_{\dot{H}^s} := \Big( \int |\xi|^{2s} \big|\widehat{f}(\xi)\big|^2 \, \mathrm{d}\xi \Big)^{\frac{1}{2}}.$$

*In particular, for s =1, this gives another proof of the fact that*

$$q = \frac{2d}{d-2} \quad \Rightarrow \quad \|f\|_{L^q} \leq C \|\nabla f\|_{L^2}.$$

*Proof.* We will show that there is a constant $C$ such as, for any $f \in \mathcal{S}(\mathbb{R}^d)$, we have

(3.2.1) $$p = \frac{2d}{d-2s} \quad \Rightarrow \quad \|f\|_{L^p} \leq C \|f\|_{\dot{H}^s} := \Big( \int |\xi|^{2s} \big|\widehat{f}(\xi)\big|^2 \, \mathrm{d}\xi \Big)^{\frac{1}{2}}.$$

This is a stronger result than the one stated. Indeed, if $p < 2d/(d-2s)$ then there is $s' \in [0, s)$ such that $p = 2d/(d-2s')$ and hence

$$\|f\|_{L^p} \leq C \|f\|_{\dot{H}^{s'}} \leq C \|f\|_{H^s}.$$

(A word of caution: one cannot bound $\|f\|_{\dot{H}^{s'}}$ by $\|f\|_{\dot{H}^s}$ because we do not have $|\xi|^{2s'} \leq |\xi|^{2s}$ for $|\xi| \leq 1$).

We use the proof of Chemin and Xu which is based on the estimate of level sets. We will denote by $\{|f| > \lambda\}$ the set $\big\{x \in \mathbb{R}^d \,:\, |f(x)| > \lambda\big\}$ and $|\{|f| > \lambda\}|$ the Lebesgue measure of this set.

Let us consider a function $f \in \mathcal{S}(\mathbb{R}^d)$. We can assume without loss of generality that $\|f\|_{\dot{H}^s} = 1$. We start from the classical identity

$$\|f\|_{L^p}^p = p \int_0^{+\infty} \lambda^{p-1} \, |\{|f| > \lambda\}| \, \mathrm{d}\lambda.$$

To estimate $|\{|f| > \lambda\}|$, we will use a decomposition in terms of low and high frequencies. For any $\lambda > 0$, we will decompose $f$ into the form

$$f = g_\lambda + h_\lambda$$

where, for a certain constant $A_\lambda$ to be determined,

$$\widehat{g_\lambda}(\xi) = \widehat{f}(\xi) \quad \text{if} \quad |\xi| \leq A_\lambda, \qquad \widehat{g_\lambda}(\xi) = 0 \qquad \text{if} \quad |\xi| > A_\lambda$$

$$\widehat{h_\lambda}(\xi) = 0 \qquad \text{if} \quad |\xi| \leq A_\lambda, \qquad \widehat{h_\lambda}(\xi) = \widehat{f}(\xi) \quad \text{if} \quad |\xi| > A_\lambda.$$

So, according to the triangular inequality,

$$\{|f| > \lambda\} \subset \{|g_\lambda| > \lambda/2\} \cup \{|h_\lambda| > \lambda/2\}\,.$$

We will choose the constant $A_\lambda$ so that $\{|g_\lambda| > \lambda/2\} = \emptyset$. Then we will have

$$|\{|f| > \lambda\}| \leq |\{|h_\lambda| > \lambda/2\}| \leq \frac{4}{\lambda^2}\,\|h_\lambda\|_{L^2}^2\,,$$

because

$$\|h_\lambda\|_{L^2}^2 \geq \int_{\{|h_\lambda|>\lambda/2\}} |h_\lambda|^2\,\mathrm{d}x \geq \frac{\lambda^2}{4}\,|\{|h_\lambda| > \lambda/2\}|\,.$$

Combining the above observations, we conclude

$$\|f\|_{L^p}^p \leq 4p \int_0^{+\infty} \lambda^{p-3}\,\|h_\lambda\|_{L^2}^2\,\mathrm{d}\lambda. \tag{3.2.2}$$

Choice of $A_\lambda$. According to the Fourier inversion theorem, we have

$$|g_\lambda(x)| = \left|\frac{1}{(2\pi)^d}\int e^{ix\cdot\xi}\,\widehat{g_\lambda}(\xi)\,\mathrm{d}\xi\right| = \left|\frac{1}{(2\pi)^d}\int_{|\xi|\leq A_\lambda} e^{ix\cdot\xi}\,\widehat{f}(\xi)\,\mathrm{d}\xi\right|.$$

As $2s < d$, we can use the Cauchy-Schwarz inequality and write that

$$|g_\lambda(x)| \leq \frac{1}{(2\pi)^d}\left(\int_{|\xi|\leq A_\lambda} |\xi|^{-2s}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int |\xi|^{2s}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

If we switch to polar coordinates, we obtain

$$\int_{|\xi|\leq A_\lambda} |\xi|^{-2s}\,\mathrm{d}\xi = \int_0^{A_\lambda}\int_{\mathbb{S}^{d-1}} r^{d-1-2s}\,\mathrm{d}\theta\,\mathrm{d}r = \frac{|\mathbb{S}^{d-1}|A_\lambda^{d-2s}}{d-2s}.$$

As $\|f\|_{\dot{H}^s} = 1$ by assumption, we finally get

$$\|g_\lambda\|_{L^\infty} \leq C_1(s,d)A_\lambda^{\frac{d}{2}-s}.$$

We then define $A_\lambda$ by

$$C_1(s,d)A_\lambda^{\frac{d}{2}-s} = \frac{\lambda}{2}.$$

So $\|g_\lambda\|_{L^\infty} \leq \lambda/2$. Since $g_\lambda$ is a continuous function (it is the Fourier transform of an integrable function), we deduce that $\{|g_\lambda| > \lambda/2\} = \emptyset$, which is the desired result.

End of the proof. By definition of $h_\lambda$, using the identity (3.2.2) and Plancherel's formula, we find

$$\|f\|_{L^p}^p \leq 4p(2\pi)^d \int_0^{+\infty} \int_{|\xi|\geq A_\lambda} \lambda^{p-3} \big|\widehat{f}(\xi)\big|^2 \, \mathrm{d}\xi \, \mathrm{d}\lambda.$$

By definition of $A_\lambda$, if $|\xi| \geq A_\lambda$ then

$$\lambda \leq \Lambda(\xi) := 2C_1(s,d)\, |\xi|^{\frac{d}{2}-s},$$

so, using Fubini's theorem, it comes

$$\|f\|_{L^p}^p \leq 4p(2\pi)^d \int_{\mathbb{R}^d} \left( \int_0^{\Lambda(\xi)} \lambda^{p-3} d\lambda \right) \big|\widehat{f}(\xi)\big|^2 \, \mathrm{d}\xi,$$

from where

$$\|f\|_{L^p}^p \leq C_2(s,d) \int_{\mathbb{R}^d} \Lambda(\xi)^{p-2} \big|\widehat{f}(\xi)\big|^2 \, \mathrm{d}\xi,$$

As $\frac{d}{2} - s = \frac{d}{p}$, we have

$$\Lambda(\xi) \leq C_1(s,d)\, |\xi|^{\frac{d}{p}}.$$

We finally get

$$\|f\|_{L^p}^p \leq C_3(s,d) \int_{\mathbb{R}^d} |\xi|^{\frac{d(p-2)}{p}} \big|\widehat{f}(\xi)\big|^2 \, \mathrm{d}\xi,$$

which is the desired result. $\square$

**Corollary 3.2.3** (Sobolev embeddings)**.** *Let $d \geq 1$ and $p \in (1,d)$. Define $p^*$ by*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}.$$

*Then there exists a constant $C$ such that, for any function $f \in C_0^\infty(\mathbb{R}^d)$,*

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C\, \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

*Proof.* Here we use the following identity

$$f(x) = -\frac{1}{\left|S^{d-1}\right|}\int_{\mathbb{R}^d}\frac{(x-y)\cdot\nabla f(y)}{|x-y|^d}\,\mathrm{d}y.$$

It follows that

$$|f| \le \frac{1}{\left|S^{d-1}\right|}I_1(|\nabla f|),$$

and hence the wanted inequality follows directly from the Hardy-Littlewood-Sobolev inequality. □

**Theorem 3.2.4** (Sobolev embedding)**.** *Consider an integer $d \ge 1$ and two real numbers $s \in (0,1)$ and $p \in (0, d/s)$, then set*

$$p^* = \frac{dp}{d-sp}.$$

*There exists a constant $C$ such that for all $f$ in $C^1_b(\mathbb{R}^d)$ and all $x$ in $\mathbb{R}^d$ we have*

$$|f(x)|^{p^*} \le C\,\|f\|_{L^{p^*}}^{p^*-p}\int_{\mathbb{R}^d}\frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}\,\mathrm{d}y.$$

*It follows that*

$$\|f\|_{L^{p^*}} \le C^{\frac{1}{p}}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

*Proof.* The following nice proof is taken from the book by Ponce [22] where it is credited to Brézis. A similar proof is given by Brué and Nguyen in [7] (see also [8]).

We denote by $C$ several constants that do not have to be depend on $d, p$ or $s$ and whose values can change from one line to another.

*Step 1.* We first check that the integral

$$\int_{\mathbb{R}^d}\frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}\,\mathrm{d}y$$

is well defined for all $f$ in $\mathcal{S}(\mathbb{R}^d)$ and all $x$ in $\mathbb{R}^d$. To do so, we cut the integral on $\mathbb{R}^d$ in two parts: the integral on $B(x,1)$ and that on $\mathbb{R}^d \setminus B(x,1)$. On $B(x,1)$, we use the estimate

$$|f(x)-f(y)| \le K\,|x-y| \quad \text{with} \quad K = \sup_{m\in\mathbb{R}^d}|\nabla f(m)|,$$

while on $\mathbb{R}^d \setminus B(x,1)$ one writes $|f(x) - f(y)| \le 2 \sup |f|$.

*Step 2.* Let us fix $x \in \mathbb{R}^d$ and a real $t > 0$. We denote by $C_t$ the annulus

$$C_t = B(0,2t) - B(0,t) = \{y \in \mathbb{R}^d\,;\, t \le |y| < 2t\},$$

and we denote by $|C_t|$ its Lebesgue measure.

Then

$$|f(x)|^p = \frac{1}{|C_t|} \int_{C_t} |f(x)|^p \,\mathrm{d}h \le \frac{1}{|C_t|} \int_{C_t} \big(\,|f(x+h) - f(h)| + |f(x+h)|\,\big)^p \,\mathrm{d}h.$$

Since

$$|a+b|^p \le (2 \max\{|a|, |b|\})^p \le 2^p \big(|a|^p + |b|^p\big),$$

we deduce that

$$|f(x)|^p \le \frac{2^p}{|C_t|} \int_{C_t} |f(x+h) - f(x)|^p \,\mathrm{d}h + \frac{2^p}{|C_t|} \int_{C_t} |f(x+h)|^p \,\mathrm{d}h.$$

*Step 3.* Hölder's inequality implies that

$$\frac{1}{|C_t|} \int_{C_t} |f(x+h)|^p \,\mathrm{d}h \le C\, \|f\|^p_{L^{p^*}} \frac{1}{t^{d-sp}}.$$

Consequently, we see that there exists $C$ such that

$$|f(x)|^p \frac{1}{t^{sp}} \le \frac{C}{t^d} \|f\|^p_{L^{p^*}} + \frac{C}{t^{sp}\,|C_t|} \int_{C_t} |f(x+h) - f(x)|^p \,\mathrm{d}h.$$

*Step 4.* Note that $|C_t| \sim t^d$. Moreover, on $C_t$ we have $t \sim |h|$. It follows that

$$|f(x)|^p \frac{1}{t^{sp}} \le C\, \|f\|^p_{L^{p^*}} \frac{1}{t^d} + C \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^p}{|x-y|^{d+ps}} \,\mathrm{d}h.$$

Multiplying the two members of the inequality in the previous question by $t^{sp}$, we find

$$|f(x)|^p \le C\, \|f\|^p_{L^{p^*}} t^{sp-d} + C t^{sp} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^p}{|x-y|^{d+ps}} \,\mathrm{d}h.$$

We then choose $t$ such that

$$\|f\|^p_{L^{p^*}} t^{sp-d} = t^{sp} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^p}{|x-y|^{d+ps}} \,\mathrm{d}h$$

and the desired inequality is inferred. □

# Chapter 4

# Littlewood-Paley decomposition

In this chapter we introduce a dyadic decomposition of the unity. This decomposition allows to introduce a parameter (large or small) in a problem which does not have any. It is a simple and extremely fruitful idea. For an introduction to this topic, we refer the reader to Bahouri [3] or Danchin [11]. There are many books which develop a systematic study of this tool, see Coifman and Meyer [10, 21], Métivier [19], Alinhac and Gérard [2], Bahouri, Danchin and Chemin [4], Tao [24] or Taylor [27, 28].

## 4.1 Dyadic decomposition

**Lemma 4.1.1.** *Let $d \geq 1$. There exist $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that the following properties hold:*

*(i) (Support conditions) We have $0 \leq \psi \leq 1$, $0 \leq \varphi \leq 1$ and*

$$\operatorname{supp} \psi \subset \{|\xi| \leq 1\}, \qquad \operatorname{supp} \varphi \subset \left\{\frac{3}{4} \leq |\xi| \leq 2\right\}.$$

*(ii) (Decomposition of the unity) For any $\xi \in \mathbb{R}^d$,*

$$1 = \psi(\xi) + \sum_{p=0}^{\infty} \varphi(2^{-p}\xi). \tag{4.1.1}$$

*(iii) (Almost orthogonality) For any $\xi \in \mathbb{R}^d$,*

$$\frac{1}{3} \leq \psi^2(\xi) + \sum_{p=0}^{+\infty} \varphi^2(2^{-p}\xi) \leq 1. \tag{4.1.2}$$

*Proof.* Let $\psi \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ be a radial function verifying $\psi(\xi) = 1$ for $|\xi| \leq 3/4$, and $\psi(\xi) = 0$ for $|\xi| \geq 1$, and decreasing (if $|\xi| \geq |\eta|$ then $\psi(\xi) \leq \psi(\eta)$). Then, we set $\varphi(\xi) = \psi(\xi/2) - \psi(\xi)$ and notice that $\varphi$ is supported in the annulus $\{3/4 \leq |\xi| \leq 2\}$. For any integer $N$ and any $\xi \in \mathbb{R}^d$, we have

$$\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi) = \psi(2^{-N-1}\xi),$$

which immediately implies (10.1.1) by letting $N$ goes to $+\infty$.

It remains to prove (4.1.2). For any integer $N$ we have

$$\psi^2(\xi) + \sum_{p=0}^{N} \varphi^2(2^{-p}\xi) \leq \Big(\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi)\Big)^2.$$

On the other hand, notice that, for all $\xi \in \mathbb{R}^d$, there are never more than three non-zero terms in the set $\{\psi(\xi),\ \varphi(\xi), \ldots, \varphi(2^{-p}\xi), \ldots\}$. Consequently, using the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we get

$$\Big(\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi)\Big)^2 \leq 3\Big(\psi^2(\xi) + \sum_{p=0}^{N} \varphi^2(2^{-p}\xi)\Big).$$

Then we obtain (4.1.2) by letting $N$ goes to $+\infty$ in the previous inequalities. □

Let us define, for $p \geq -1$, the Fourier multipliers $\Delta_p$ as follows:

$$\Delta_{-1} := \psi(D_x) \quad \text{and} \quad \Delta_p := \varphi\big(2^{-p}D_x\big) \quad (p \geq 0).$$

Let us also introduce, for $p \geq 0$, the Fourier multipliers $S_p$:

$$S_p := \psi\big(2^{-p}D_x\big) = \sum_{k=-1}^{p-1} \Delta_k.$$

The partition of the unity also implies a partition of the identity.

**Proposition 4.1.2.** *We have*

$$I = \sum_{p \geq -1} \Delta_p,$$

*in the sense of distributions: For any $u \in \mathcal{S}'(\mathbb{R}^d)$, the series $\sum u_p$ converges to $u$ in $\mathcal{S}'(\mathbb{R}^d)$, which means that $\sum_p \langle \Delta_p u, \varphi \rangle_{\mathcal{S}' \times \mathcal{S}}$ converges to $\langle u, \varphi \rangle_{\mathcal{S}' \times \mathcal{S}}$ for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$.*

*Proof.* Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\theta \in \mathcal{S}(\mathbb{R}^d)$. The partial sums $S_N u = \sum_{p \geq 0}^{N-1} \Delta_p u$ are well defined and

$$\langle \mathcal{F}(S_N u), \theta \rangle = \langle \psi(2^{-N}\xi)\mathcal{F}(u), \theta \rangle = \langle \mathcal{F}(u), \psi(2^{-N}\xi)\theta \rangle.$$

Now $\lim_{N \to +\infty} \psi(2^{-N}\xi)\theta = \theta$ in $\mathcal{S}(\mathbb{R}^d)$, so

$$\mathcal{F}(S_N u) \underset{p \to +\infty}{\longrightarrow} \mathcal{F}(u) \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

By continuity of $\mathcal{F}^{-1} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ we have $u = \sum_{p \geq -1} \Delta_p u$. □

## 4.2 Characterization of Sobolev spaces

**Proposition 4.2.1.** *(i) For all $u \in L^2(\mathbb{R}^d)$,*

$$\sum_{p \geq -1} \left\|\Delta_p u\right\|_{L^2}^2 \leq \|u\|_{L^2}^2 \leq 3 \sum_{p \geq -1} \left\|\Delta_p u\right\|_{L^2}^2. \tag{4.2.1}$$

*(ii) Consider $s \in \mathbb{R}$. A tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the Sobolev space $H^s(\mathbb{R}^d)$ if and only if*

*(a) $\Delta_{-1} u \in L^2(\mathbb{R}^d)$ and for all $p \geq 0$, $\Delta_p u \in L^2(\mathbb{R}^d)$;*

*(b) the sequence $\delta_p = 2^{ps} \left\|\Delta_p u\right\|_{L^2}$ belongs to $\ell^2(\mathbb{N} \cup \{-1\})$.*

*Moreover, there exists a constant $C$ such that*

$$\frac{1}{C} \|u\|_{H^s} \leq \Big( \sum_{p=-1}^{+\infty} \delta_p^2 \Big)^{\frac{1}{2}} \leq C \|u\|_{H^s}. \tag{4.2.2}$$

*Proof.* The first point follows immediately from (4.1.2) and Plancherel's identity.

Since $\|u\|_{H^s} = \|\langle D_x\rangle^s u\|_{L^2}$, by applying (4.2.1) with $u$ replaced by $\langle D_x\rangle^s u$, we obtain

$$\sum_{p\geq -1} \left\|\Delta_p \langle D_x\rangle^s u\right\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq 3 \sum_{p\geq -1} \left\|\Delta_p \langle D_x\rangle^s u\right\|_{L^2}^2 .$$

Consider $p \geq 0$ and write that

$$\left\|\Delta_p \langle D_x\rangle^s u\right\|_{L^2}^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} (1+|\xi|^2)^s \varphi^2(2^{-p}\xi) \left|\hat{u}(\xi)\right|^2 \,\mathrm{d}\xi.$$

Since $(1+|\xi|^2)^s \varphi^2(2^{-p}\xi) \sim 2^{2ps}$ on the support of $\varphi^2(2^{-p}\xi)$, we see that

$$\frac{1}{C} 2^{2ps} \left\|\Delta_p u\right\|_{L^2}^2 \leq \left\|\Delta_p \langle D_x\rangle^s u\right\|_{L^2}^2 \leq C 2^{2ps} \left\|\Delta_p u\right\|_{L^2}^2 , \tag{4.2.3}$$

for some constant $C$ depending only on $s$. We have a similar estimate for $\Delta_{-1} u$ and the wanted result easily follows. □

**Proposition 4.2.2.** *i) Consider $s \in \mathbb{R}$ and $R \geq 1$. Assume that $(u_j)_{j\geq -1}$ is a sequence of functions in $L^2(\mathbb{R}^d)$ such that*

$$\operatorname{supp} \widehat{u_{-1}} \subset \{|\xi| \leq R\}, \qquad \operatorname{supp} \widehat{u_j} \subset \left\{\frac{1}{R} 2^j \leq |\xi| \leq R 2^j\right\},$$

*and, in addition,*

$$\sum_{j\geq -1} 2^{2js} \left\|u_j\right\|_{L^2}^2 < +\infty. \tag{4.2.4}$$

*Then the series $\sum u_j$ converges to a function $u \in H^s(\mathbb{R}^d)$ and moreover,*

$$\|u\|_{H^s}^2 \leq C \sum_{j\geq -1} 2^{2js} \left\|u_j\right\|_{L^2}^2 ,$$

*for some constant $C$ depending only on $s$ and $R$.*

*ii) If $s > 0$, then the previous result holds under the weaker assumption that* $\operatorname{supp} \widehat{u_j}$ *is included in the ball $B(0, R2^j)$.*

*Proof. i)* We begin by proving that the series $\sum u_j$ is normally convergent in $H^r(\mathbb{R}^d)$ for any $r < s$. Assuming that $\operatorname{supp} \widehat{u_j}$ is included in a ball $\left\{|\xi| \leq R2^j\right\}$, parallel to (4.2.3), we see that $\|u_j\|_{H^r} \lesssim 2^{jr}\|u_j\|_{L^2}$. So, the Cauchy-Schwarz inequality implies that

$$\sum_{j\geq -1}\|u_j\|_{H^r} \leq \Big(\sum_{j\geq -1} 2^{2js}\|u_j\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{j\geq -1} 2^{2j(r-s)}\Big)^{\frac12}$$
$$\lesssim \Big(\sum 2^{2js}\|u_j\|_{L^2}^2\Big)^{\frac12} < +\infty.$$

This shows that the series $\sum u_j$ is normally convergent and hence convergent in $H^r(\mathbb{R}^d)$. Now we can set $u = \sum_{j\geq -1} u_j$. Our goal is then to prove that $u$ belongs to $H^s(\mathbb{R}^d)$.

*i*) Ift $\operatorname{supp}\widehat{u_j}$ is included in an annulus $\left\{\frac{1}{R}2^j \leq |\xi| \leq R2^j\right\}$, then there exists some integer $N$ depending only on $R$ such that $\Delta_p u_j = 0$ if $|j-p| > N$. Therefore

$$\|\Delta_p u\|_{L^2} \leq \sum_{|j-p|\leq N}\|\Delta_p u_j\|_{L^2} \leq \sum_{|j-p|\leq N}\|u_j\|_{L^2},$$

whence the result.

*ii*) If one only assumes that $\operatorname{supp}\widehat{u_j}$ is included in a ball $\{|\xi| \leq R2^j\}$, then we just have, for some integer $N$,

$$\Delta_p u = \sum_{j\geq p-N}\Delta_p u_j.$$

It follows from the triangle inequality that

$$2^{ps}\|\Delta_p u\|_{L^2} \leq \sum_{j\geq p-N} 2^{(p-j)s}2^{js}\|u_j\|_{L^2}.$$

Now, since $s > 0$, the sequence $(2^{(p-j)s})_{j\geq p-N}$ belongs to $\ell^1$ and the convolution inequality $\ell^1 * \ell^2 \hookrightarrow \ell^2$ gives the result. □

## 4.3 Characterization of Hölder spaces

In this paragraph we are going to show that we can describe Hölder spaces using the Fourier transform.

**Definition 4.3.1.** *Let $r \in (0,1)$. The Hölder space $C^{0,r}(\mathbb{R}^d)$ consists of those bounded functions $u\colon \mathbb{R}^d \to \mathbb{C}$ satisfying*

$$\exists C > 0 / \ \forall x,y\in\mathbb{R}^d, \quad |u(x)-u(y)| \leq C\,|x-y|^r.$$

*ii*) *Let $k \in \mathbb{N}$ and $\alpha \in (0,1]$. Let $C^{k,\alpha}(\mathbb{R}^d)$ be the space of functions $C^k(\mathbb{R}^d)$ whose derivatives up to order $k$ belong to $C^{0,\alpha}(\mathbb{R}^d)$.*

*iii*) *Let $r \in \mathbb{R}^+ \setminus \mathbb{N}$ so that $r = k+\alpha$ with $k \in \mathbb{N}$ and $\alpha \in (0,1]$ then we simply denote by $C^r(\mathbb{R}^d)$ the space $C^{k,\alpha}(\mathbb{R}^d)$.*

For $r \in (0,1[$, the space $C^r(\mathbb{R}^d)$ is provided with a Banach space structure by the norm

$$\|u\|_{C^r} = \|u\|_{L^\infty} + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^r}.$$

We give below an equivalent norm.

**Proposition 4.3.2.** *Let $r bein ]0,1[$. There exists a constant $A_r > 0$ such that the following two properties hold:*

*i*) *If $u \in C^r(\mathbb{R}^d)$ then, for any $p \geq -1$,*

$$\left\|\Delta_p u\right\|_{L^\infty} \leq A_r \|u\|_{C^r} 2^{-pr}.$$

*ii*) *Conversely, if, for any $p \geq -1$,*

$$\left\|\Delta_p u\right\|_{L^\infty} \leq C 2^{-pr},$$

*then $u \in C^r(\mathbb{R}^d)$ and $\|u\|_{C^r} \leq A_r C$.*

*Proof. i*) Consider $p \geq 0$. To prove the first point, we start by writing $u_p$ in integral form,

$$u_p(x) = 2^{pd} \int \mathcal{F}^{-1}(\varphi)(2^p(x-y))u(y)\,\mathrm{d}y \quad \text{for} \quad p \geq 0. \tag{4.3.1}$$

The rest of the proof, as well as a precise calculation of the constants, just uses the fact that the moments of $\mathcal{F}^{-1}(\varphi)$ are all zero. Thus with the moment of order 0 we get

$$u_p(x) = 2^{pd} \int \mathcal{F}^{-1}(\varphi)(2^p(x-y))\,(u(y)-u(x))\,\mathrm{d}y \quad \text{for} \quad p \geq 0,$$

hence

$$\begin{aligned}\left|u_p(x)\right| &\leq 2^{pn} \|u\|_{C^r} \int \left|\mathcal{F}^{-1}(\varphi)(2^p(x-y))\right| |y-x|^r\,\mathrm{d}y \\ &= 2^{-pr} \|u\|_{C^r} \int \left|\mathcal{F}^{-1}(\varphi)(z)z\right| \mathrm{d}z.\end{aligned}$$

The case $p = -1$ is treated in a similar way. We write

$$u_{-1}(x) = \int \mathcal{F}^{-1}(\psi)(x-y)u(y)\,\mathrm{d}y,$$

then we use that $\mathcal{F}^{-1}(\psi) \in \mathcal{S}(\mathbb{R}^d)$ and thus $\mathcal{F}^{-1}(\psi)$ belongs to $L^1(\mathbb{R}^d)$. We deduce that the $L^\infty$-norm of $u_p$ is controlled by the $L^\infty$-norm of $u$.

Let us show the converse. We verify that $u \in L^\infty$ because $\Delta_p u \in L^\infty$ and $\sum \Delta_p u$ converge normally if $\left\|\Delta_p u\right\|_{L^\infty} \leq C2^{-pr}$. It remains to estimate

$$\sup_{x\neq y,\ |x-y|\leq 1} \frac{|u(x)-u(y)|}{|x-y|^r}.$$

(Note that we can obviously restrict ourselves to $|x-y| \leq 1$.)

To do this, we pose, for an integer $p$ to be determined,

$$u = S_p u + R_p u, \quad S_p u = \sum_{q=-1}^{p-1} u_q \text{ and } R_p u = \sum_{q=p}^{+\infty} u_q.$$

By hypothesis it comes

$$\left\|R_p u\right\|_{L^\infty} \leq \sum_{q\geq p} \left\|u_q\right\|_{L^\infty} \leq \sum_{q\geq p} C2^{-qr} = \frac{C}{1-2^{-r}}2^{-pr}.$$

from which we obviously deduce that $\left|R_p u(y) - R_p u(x)\right| \leq \frac{2C}{1-2^{-r}}2^{-pr}$. On the other hand

$$\left|S_p u(x) - S_p u(y)\right| \leq |x-y| \sum_{q=-1}^{p-1} \left\|\nabla u_q\right\|_{L^\infty}.$$

From the formula recalled at the beginning of the proof we get

$$\left\|\nabla u_q\right\|_{L^\infty} \leq C'2^q \left\|u_q\right\|_{L^\infty} \leq C''C2^{q-qr}.$$

With for $q=-1$, $\|\nabla u_{-1}\|_{L^\infty} \leq C'''C$. Since $r < 1$, we have $1-r > 0$ and thus

$$\sum_{q=0}^{p-1} 2^{q-qr} \leq \frac{1}{2^{1-r}-1}2^{p(1-r)}.$$

Let's put the two estimates together: there are two constants $K_1$ and $K_2$ which depend only on $r$ such that

$$|u(y) - u(x)| \leq K_1 C\,|x - y|\,2^{p-pr} + K_2 2^{-pr}.$$

Let's choose $p$ such that $2^{-1} \leq 2^p\,|x - y| \leq 1$ (which is possible because we assume $|x - y| \leq 1$). Then

$$|u(y) - u(x)| \leq K_3 C\,|x - y|^r\,,$$

which completes the proof. □

### 4.3.1 Zygmund spaces

We have shown that if $r \in (0, 1[$,

$$u \in C^r(\mathbb{R}^d) \iff \sup_p 2^{pr}\,\|u_p\|_{L^\infty} < +\infty.$$

In fact, we have more generally

$$r \in \mathbb{R}^+ \setminus \mathbb{N}, \quad u \in C^r(\mathbb{R}^d) \iff \sup_p 2^{pr}\,\|u_p\|_{L^\infty} < +\infty.$$

**Definition 4.3.3.** *Let $r$ be a real number, we denote by $C^r_*(\mathbb{R}^d)$ the subspace of tempered distributions defined by*

$$u \in C^r_*(\mathbb{R}^d) \iff \sup_{p \geq -1} 2^{pr}\,\|u_p\|_{L^\infty} < +\infty.$$

**Remark 4.3.4.** *We define these spaces for all $r \in \mathbb{R}$ and not only $r \geq 0$. This is convenient because spaces formed by derivatives of functions of $C^r_*$ naturally occur.*

Thus the previous result gives directly

$$r \in \mathbb{R}^+ \setminus \mathbb{N} \Rightarrow C^r_*(\mathbb{R}^d) = C^r(\mathbb{R}^d).$$

Moreover, for $k \in \mathbb{N}$, we easily show that

$$\forall k \in \mathbb{N}, \quad C^k(\mathbb{R}^d) \subset W^{k,\infty}(\mathbb{R}^d) \subset C^k_*(\mathbb{R}^d).$$

However, we can show that

$$r \in \mathbb{N} \Rightarrow C^r_*(\mathbb{R}^d) \neq C^r(\mathbb{R}^d).$$

Let's give an elementary characterization of $C^1_*(\mathbb{R}^d)$.

**Proposition 4.3.5.** *The space $C^1_*(\mathbb{R}^d)$ is the space of bounded functions $u$ such that*

$$\exists C > 0 / \ \forall x, y \in \mathbb{R}^d, \quad |u(x+y) + u(x-y) - 2u(x)| \leq C\,|y|\,.$$

*Proof.* Suppose that the function $u$ belongs to $C^1_*(\mathbb{R}^d)$. Then consider a non-zero point $y \in B(0,1)$. Using the dyadic decomposition of the frequency space and and the Taylor inequality of order 2 between $y$ and 0, we get

$$|u(x+y) + u(x-y) - 2u(x)| \leq C\,\|u\|_{C^1_*} \left( |y|^2 \sum_{q \leq N} 2^q + 4 \sum_{q > N} 2^{-q} \right),$$

where $N$ is any integer. Choosing $N$ such that $2^{qN} = |y|^{-1}$, we obtain

$$|u(x+y) + u(x-y) - 2u(x)| \leq C\,\|u\|_{C^1_*}\,|y|\,.$$

Conversely, let us choose a function $u$ such that, for all $y$ in $xR^n$, we have

$$|u(x+y) + u(x-y) - 2u(x)| \leq C\,|y|\,.$$

It is now a matter of estimating $\left\|\Delta_q u\right\|_{L^\infty}$. The fact that the function $\varphi$ is radial, therefore even, entails that

$$\begin{aligned} u_q(x) &= 2^{qn}\mathcal{F}^{-1}\varphi(2^q\cdot) * u(x) = 2^{qn}\int \mathcal{F}^{-1}\varphi(2^q y)u(x-y)\,\mathrm{d}y \\ &= 2^{qn}\int \mathcal{F}^{-1}\varphi(2^q y)u(x+y)\,\mathrm{d}y. \end{aligned}$$

Let us introduce $h(z) = \mathcal{F}^{-1}\varphi(z)$. Since $\varphi$ is zero near the origin we deduce that $h$ has zero integral, so

$$2^{qn}\mathcal{F}^{-1}\varphi(2^q\cdot) * u(x) = 2^{qn-1}\int \mathcal{F}^{-1}\varphi(2^q y)(u(x+y) + u(x-y) - 2u(x))\,\mathrm{d}y.$$

Since the function $z \mapsto |z|\,h(z)$ is integrable, we have

$$\left\|u_q\right\|_{L^\infty} \leq C 2^{-q} \sup_{y \in \mathbb{R}^d} \frac{|u(x+y) + u(x-y) - 2u(x)|}{|y|},$$

which concludes the proof. □

# Part II

# Pseudo-differential calculus

# Chapter 5

# Definition of Pseudo-differential operators

Consider a differential operator

$$P = \sum_{|\alpha| \le m} p_\alpha(x) \partial_x^\alpha$$

where the coefficients $p_\alpha$ belong to the space $C_b^\infty(\mathbb{R}^d; \mathbb{C})$ of those $C^\infty$ functions which are bounded as well as all their derivatives. The function

$$p \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}, \qquad p(x, \xi) = \sum_{|\alpha| \le m} p_\alpha(x)(i\xi)^\alpha$$

is called the symbol of $P$. With this definition, we have

$$P e^{ix\cdot\xi} = p(x, \xi) e^{ix\cdot\xi}.$$

Consider now a function $u$ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. It follows from the Fourier inversion formula that

$$u(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} \widehat{u}(\xi) \, \mathrm{d}\xi,$$

and hence we see that one can write $Pu$ under the form

$$Pu(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi) \, \mathrm{d}\xi.$$

A pseudo-differential operator is an operator of the previous form, but where the function $p(x, \xi)$ is not necessarily a polynomial function. In this chapter we propose to study the definition of these operators.

## 5.1 Continuity on the Schwartz class

Consider a function $a \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. By definition, this means that, for all multi-indices $\alpha$ and $\beta$ in $\mathbb{N}^d$, we have

$$\sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| < +\infty.$$

Given any function $u \in \mathcal{S}(\mathbb{R}^d)$ in the Schwartz class and a fixed $x \in \mathbb{R}^d$, the function $\xi \mapsto a(x,\xi)\widehat{u}(\xi)$ belongs to $\mathcal{S}(\mathbb{R}^d_\xi)$. In particular, it is integrable and we may define the function $\operatorname{Op}(a)u$ by

$$\operatorname{Op}(a)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\, \mathrm{d}\xi.$$

We say that $\operatorname{Op}(a)$ is a pseudo-differential operator and we call $a$ its symbol.

**Proposition 5.1.1.** *For any $a \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and any $u \in \mathcal{S}(\mathbb{R}^d)$, the function* $\operatorname{Op}(a)u$ *is well-defined and belongs to $\mathcal{S}(\mathbb{R}^d)$. Moreover* $\operatorname{Op}(a)$ *is continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$.*

*Proof.* Since $\widehat{u} \in \mathcal{S}(\mathbb{R}^d)$, we can apply Lebesgue's differentiation theorem to check easily that $\operatorname{Op}(a)u \in C^\infty(\mathbb{R}^d)$. So it will suffice to prove estimates.

Using $\|a\|_{L^\infty} < +\infty$ and $\left\|\langle\xi\rangle^{d+1}\widehat{u}\right\|_{L^\infty} < +\infty$, we get the inequality

$$|\operatorname{Op}(a)u(x)| \le \frac{1}{(2\pi)^d}\int \|a\|_{L^\infty} \left\|\langle\xi\rangle^{d+1}\widehat{u}\right\|_{L^\infty} \langle\xi\rangle^{-d-1}\, \mathrm{d}\xi,$$

which implies that $\operatorname{Op}(a)u$ is bounded together with the estimate

$$\|\operatorname{Op}(a)u\|_{L^\infty} \le C\, \|a\|_{L^\infty}\, \mathcal{N}_{d+1}(\widehat{u})$$

where we used the notation $\mathcal{N}_p(\varphi) = \sum_{|\alpha|\le p, |\beta|\le p} \left\|x^\alpha \partial_x^\beta \varphi\right\|_{L^\infty}$ to denote the canonical semi-norms on the Schwartz space; let us recall that the Fourier transform is continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$ and that, for any integer $p \in \mathbb{N}$,

$$\mathcal{N}_p(\widehat{u}) \le C_p\, \mathcal{N}_{p+d+1}(u).$$

To estimate the other semi-norms in $\mathcal{S}(\mathbb{R}^d)$ of $\operatorname{Op}(a)u$, we use the following formulas (to be checked as an exercise)

$$\begin{aligned}
\partial_{x_j} \operatorname{Op}(a)u &= \operatorname{Op}(a)(\partial_{x_j} u) + \operatorname{Op}(\partial_{x_j} a)u,\\
x_j \operatorname{Op}(a)u &= \operatorname{Op}(a)(x_j u) + i\operatorname{Op}(\partial_{\xi_j} a)u.
\end{aligned}$$

Thus, $x^\alpha \partial_x^\beta \operatorname{Op}(a)u$ can written as a linear combination of terms of the form

$$\operatorname{Op}\big(\partial_x^\gamma \partial_\xi^\delta a\big)(x^{\alpha-\delta}\partial_x^{\beta-\gamma}u).$$

Since $\partial_x^\gamma \partial_\xi^\delta a$ belongs to $C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and since $x^{\alpha-\delta}\partial_x^{\beta-\gamma}u$ belongs to $\mathcal{S}(\mathbb{R}^d)$, we are back to the previous case. This shows that one can estimate the $L^\infty$ norm of $x^\alpha \partial_x^\beta \operatorname{Op}(a)u$ in terms of the semi-norms of $a$ in $C_b^\infty(\mathbb{R}^{2d})$ and the ones of $u$ in $\mathcal{S}(\mathbb{R}^d)$. This implies that $\operatorname{Op}(a)u$ belongs to $\mathcal{S}(\mathbb{R}^d)$ and the previous estimates imply that $\operatorname{Op}(a)$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to itself. □

## 5.2 The Calderón-Vaillancourt theorem

We can now state the main result, which asserts that one can extend $\operatorname{Op}(a)$ as a bounded operator from $L^2(\mathbb{R}^d)$ into itself.

**Theorem 5.2.1.** *For any symbol $a \in C_b^\infty(\mathbb{R}^{2d})$, the operator $\operatorname{Op}(a)$ can be uniquely extended as a bounded linear operator in $\mathcal{L}(L^2(\mathbb{R}^d))$.*

We will demonstrate this result by assuming, to simplify the notations, that the space dimension $d$ is less than or equal to 3 (otherwise just replace the polynomial $P(\zeta)$ below by $(1+|\zeta|^2)^k$ where $k$ is an integer such that $4k > d$).

Let us introduce the polynomial

$$P(\zeta) = 1 + |\zeta|^2 \qquad (\zeta \in \mathbb{R}^d,\ d = 1,2,3).$$

**Lemma 5.2.2.** *Given a function $u \in \mathcal{S}(\mathbb{R}^d)$, we introduce the function*

$$Wu(x,\xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\xi} P(x-y)^{-1} u(y)\,\mathrm{d}y \qquad ((x,\xi) \in \mathbb{R}^{2d}).$$

*i) Then $Wu$ is a function $C_b^\infty(\mathbb{R}^{2d})$ and moreover for any multi-indices $\alpha,\beta,\gamma$,*

$$\sup_{\mathbb{R}^{2d}} P(x)|\xi|^\gamma \left|(\partial_x^\alpha \partial_\xi^\beta Wu)(x,\xi)\right| < +\infty.$$

*ii) There is a constant $A$ such that*

$$\|Wu\|_{L^2(\mathbb{R}^{2d})} = A\,\|u\|_{L^2(\mathbb{R}^d)} \tag{5.2.1}$$

*for any $u$ in $\mathcal{S}(\mathbb{R}^d)$.*

*iii) For any $\gamma \in \mathbb{N}^d$, there exists a positive constant $A_\gamma$ such that*

$$\left\|\partial_x^\gamma Wu\right\|_{L^2(\mathbb{R}^{2d})} \leq A_\gamma \|u\|_{L^2(\mathbb{R}^d)} .$$

*Proof.* *i*) We verify that

$$\xi^\gamma(\partial_x^\alpha \partial_\xi^\beta Wu)(x,\xi) = \int i^{|\gamma|}\partial_y^\gamma\left(e^{-iy\cdot\xi}\right)(-iy)^\beta \partial_x^\alpha(P(x-y)^{-1})u(y)\,\mathrm{d}y,$$

so, by integrating by parts

$$\begin{aligned}&\xi^\gamma(\partial_x^\alpha\partial_\xi^\beta Wu)(x,\xi)\\ &= \sum_{\gamma'+\gamma''=\gamma}\frac{\gamma!}{\gamma'!\gamma''!}\int(-i)^{|\gamma|}\partial_y^{\gamma'}\left(u(y)(-iy)^\beta\right)(-1)^{|\gamma''|}(\partial^{\gamma''+\alpha}1/P)(x-y)e^{-iy\cdot\xi}\,\mathrm{d}y.\end{aligned}$$

We next use the elementary estimates

$$\left|\partial_\zeta^\alpha\langle\zeta\rangle^{-2}\right| \leq C_\alpha\langle\zeta\rangle^{-2-|\alpha|} \leq C_\alpha\langle\zeta\rangle^{-2}$$

to deduce that

$$|\partial^\alpha(1/P)(x-y)| \leq C_\alpha(1+|x-y|^2)^{-1} \leq 2C_\alpha(1+|x|^2)^{-1}(1+|y|^2),$$

where the last inequality comes from the fact that

$$1+|x|^2 = 1+|x-y+y|^2 \leq 1+2|x-y|^2+2|y|^2 \leq 2(1+|x-y|^2)(1+|y|^2).$$

*ii*) For any $x\in\mathbb{R}^d$, $Wu(x,\cdot)$ is the Fourier transform of $y\mapsto u(y)P(x-y)^{-1}$. So

$$\int |W(x,\xi)|^2\,\mathrm{d}\xi = (2\pi)^d\int\left|u(y)P(x-y)^{-1}\right|^2\mathrm{d}y$$

according to Plancherel's theorem. So

$$\iint |W(x,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}x = (2\pi)^d\iint\left|u(y)P(x-y)^{-1}\right|^2\mathrm{d}y dx = A^2\|u\|^2_{L^2(\mathbb{R}^d)},$$

where $A$ is defined by

$$A^2 = (2\pi)^d\int_{\mathbb{R}^d}P(z)^{-2}\,\mathrm{d}z.$$

Notice that this integral is converging by definition of $P$, since we assume that $d\leq 3$.

The statement *iii*) is proved by combining the above observations. $\square$

**Lemma 5.2.3.** *Consider $u, v$ in $\mathcal{S}(\mathbb{R}^d)$. For all $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{R}^d$, there holds*

$$\widehat{u}(\xi) = e^{-ix\cdot\xi}(I - \Delta_\xi)\big(e^{ix\cdot\xi} Wu(x,\xi)\big)$$

*and*

$$\overline{v}(x) = \frac{1}{(2\pi)^d} e^{-ix\cdot\xi}(I - \Delta_x)\big(e^{ix\cdot\xi} W\overline{\widehat{v}}(\xi, x)\big).$$

*Proof.* Write $(I - \Delta_\xi)e^{iX\cdot\xi} = P(X)$ to obtain

$$e^{ix\cdot\xi}\widehat{u}(\xi) = \int e^{i(x-y)\cdot\xi} u(y)\,\mathrm{d}y = (I - \Delta_\xi)\int e^{i(x-y)\cdot\xi} P(x-y)^{-1} u(y)\,\mathrm{d}y.$$

In a dual way, using the inverse Fourier transform, we have

$$\begin{aligned} e^{ix\cdot\xi}\overline{v}(x) &= \frac{1}{(2\pi)^d}\int e^{i(\xi-\eta)\cdot x}\overline{\widehat{v}}(\eta)\,\mathrm{d}\eta \\ &= \frac{1}{(2\pi)^d}(I - \Delta_x)\int e^{i(\xi-\eta)\cdot x} P(\xi-\eta)^{-1}\overline{\widehat{v}}(\eta)\,\mathrm{d}\eta, \end{aligned}$$

which implies the second identity. □

*Proof of Theorem 5.2.1.* Given the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, it is enough to demonstrate the inequality

$$\|\mathrm{Op}(a)u\|_{L^2} \leq C\,\|u\|_{L^2}$$

for any $u$ in $\mathcal{S}(\mathbb{R}^d)$. Let us consider two functions $u, v$ in $\mathcal{S}(\mathbb{R}^d)$ and let us set

$$I := \iint e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\overline{v}(x)\,\mathrm{d}\xi\,\mathrm{d}x.$$

We want to show that $|I| \leq C\,\|u\|_{L^2}\,\|v\|_{L^2}$. For this we will rewrite $I$ as a scalar product in $L^2(\mathbb{R}^{2d})$ of functions involving $Wu$ and $W\overline{\widehat{v}}$.

Let us start by writing $I$ in the form

$$I = \iint a(x,\xi)\Big[(I - \Delta_\xi)\big(e^{ix\cdot\xi} Wu(x,\xi)\Big]\overline{v}(x)\,\mathrm{d}\xi\,\mathrm{d}x.$$

Since $(I - \Delta_\xi)\big(e^{ix\cdot\xi} Wu(x,\xi)\overline{v}(x)\big)$ belongs to $\mathcal{S}(\mathbb{R}^{2d})$, we can integrate by parts in $\xi$ and deduce that

$$I = \iint \Big[(I - \Delta_\xi)a(x,\xi)\Big] Wu(x,\xi)e^{ix\cdot\xi}\overline{v}(x)\,\mathrm{d}\xi\,\mathrm{d}x.$$

Using the identity for $v$ it comes

$$I = \iint \Big[(I-\Delta_\xi)a(x,\xi)\Big] Wu(x,\xi)(I-\Delta_x)\big(e^{ix\cdot\xi}W\overline{\widehat{v}}(\xi,x)\big)\,\mathrm{d}x$$

and integrating by parts in $x$,

$$I = \iint (I-\Delta_x)\Big[\big((I-\Delta_\xi)a(x,\xi)\big)Wu(x,\xi)\Big]e^{ix\cdot\xi}W\overline{\widehat{v}}(\xi,x)\,\mathrm{d}\xi\,\mathrm{d}x$$

so

$$I = \sum_{|\beta|\le 2, |\alpha|+|\gamma|\le 2} C_{\alpha\beta\gamma}\iint (\partial_x^\alpha\partial_\xi^\beta a(x,\xi))\partial_x^\gamma Wu(x,\xi)W\overline{\widehat{v}}(\xi,x)e^{ix\cdot\xi}\,\mathrm{d}x\,\mathrm{d}\xi.$$

We conclude the proof with the Cauchy-Schwarz inequality and the previous results:

$$\begin{aligned}\left\|\partial_x^\gamma Wu\right\|_{L^2(\mathbb{R}^{2d})} &\le A_\gamma \|u\|_{L^2(\mathbb{R}^d)}\,,\\ \left\|W\overline{\widehat{v}}(\xi,x)\right\|_{L^2(\mathbb{R}^{2d})} &= A\left\|\overline{\widehat{v}}\right\|_{L^2} = A(2\pi)^{\frac{d}{2}}\|v\|_{L^2(\mathbb{R}^d)}\,,\end{aligned}$$

where the Plancherel formula was used in the last inequality. □

# Chapter 6

# Symbolic calculus

## 6.1 General symbol classes

**Notation 6.1.1.** *Let $\Omega$ be an open set of a space $\mathbb{R}^d$ with $d \geq 1$. We denote $C_b^\infty(\Omega)$ the set of $C^\infty$ functions on $\Omega$ which are bounded as well as all their derivatives.*

**Definition 6.1.2.** *For $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, the symbol class $S^m_{\rho,\delta}(\mathbb{R}^d)$ is the space of functions $a \in C^\infty(\mathbb{R}^{2d}; \mathbb{C})$ such that, for all multi-indices $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^d$, there exists a constant $C_{\alpha\beta}$ such that*

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right| \leq C_{\alpha\beta}(1 + |\xi|)^{m+\delta|\alpha|-\rho|\beta|}.$$

*We say that $a$ is a symbol of order $m$ and type $(\rho, \delta)$.*

**Remark 6.1.3.** *Notice that $C_b^\infty(\mathbb{R}^{2d}; \mathbb{C}) = S^0_{0,0}(\mathbb{R}^d)$.*

For any real numbers $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, and for any symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^d)$, by using similar arguments to those used to prove Proposition 5.1.1, one can prove that $\mathrm{Op}(a)$ is a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.

Let us state a generalization of Theorem 5.2.1 to the case of general symbol.

**Theorem 6.1.4** (Calderón-Vaillancourt)**.** *Let $a \in S^0_{\rho,\delta}(\mathbb{R}^d)$ with $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Then $\mathrm{Op}(a)$ can be extended as a bounded operator from $L^2(\mathbb{R}^d)$ to itself. Moreover,*

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2)} \leq C \sup_{|\alpha| \leq [\frac{d}{2}]+1} \sup_{|\beta| \leq [\frac{d}{2}]+1} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \left| (1 + |\xi|)^{\delta|\alpha|-\rho|\beta|} \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \right|,$$

*for some absolute constant $C$ depending only on $d, \rho, \delta$.*

*Proof.* We will not use this result and refer to [9] for the proof. The precise bound in terms of a the semi-norms of $p$ is proved for instance by Coifman and Meyer [10]. $\square$

**Remark 6.1.5** (continuity on $L^p$)**.** *We have studied the boundedness of pseudo-differential operators on $L^2(\mathbb{R}^d)$. Let us briefly discuss the boundedness on other functions spaces.*

$(i)$ *Firstly, a pseudo-differential operator of order* $0$ *and type* $(\rho, \delta)$ *is not bounded in general on Lebesgue spaces $L^p(\mathbb{R}^d)$ with $p \neq 2$. Nevertheless, Fefferman proved in [13] that, for any $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$, and any symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^d)$, the operator* $\mathrm{Op}(a)$ *belong to $\mathcal{L}(L^p(\mathbb{R}^d))$ provided that*

$$m \leq -d(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right|.$$

*We also refer to David and Journé (see [12]) for the boundedness of pseudo-differential operators on $L^p(\mathbb{R}^d)$ when $\rho = 1 = \delta$.*

$(ii)$ *We will study later on the case when $a \in S^0_{1,1}(\mathbb{R}^d)$. See also Exercise 11.0.3.*

$(iii)$ *One can also consider the case where $\delta > \rho$, see Hörmander [15].*

It is proved in Exercise 11.0.3 that the statement of Theorem 6.1.4 does not hold for $(\rho, \delta) = (1, 1)$. This means that an operator of 0 and type $(1, 1)$ is not bounded in general from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. However, the following result, due to Stein, states that such an operator is bounded from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for any $s > 0$.

**Theorem 6.1.6** (Stein)**.** *Assume that $a \in S^0_{1,1}(\mathbb{R}^d)$. Then the operator* $\mathrm{Op}(a)$ *is bounded from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for all $s > 0$ and from the Hölder spaces $C^{0,\alpha}(\mathbb{R}^d)$ to itself for any $\alpha \in (0, 1)$.*

*Proof.* We will not use this result and refer to [19] for the proof. $\square$

## 6.2 Classical symbols

**Notation 6.2.1.** *Let us fix some notations which will be used continuously in the sequence. In this course, we will be mainly interested in a particular subclass: the*

$S^m_{1,0}$ *class. We will simply note*

$$S^m(\mathbb{R}^d) = S^m_{1,0}(\mathbb{R}^d).$$

*Notice that*

$$C_b^\infty(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^d).$$

*We will use the bracket notation:*

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

*We also introduce*

$$S^{-\infty} := \bigcap_{m\in\mathbb{R}} S^m \quad \textit{and} \quad S^{+\infty} = \bigcup_{m\in\mathbb{R}} S^m.$$

**Definition 6.2.2** (Elliptic symbols)**.** *Let $m \in \mathbb{R}$. A symbol $a \in S^m(\mathbb{R}^d)$ is elliptic if there exist two strictly positive constants $R$ and $C$ such that,*

$$\forall (x,\xi) \in \mathbb{R}^{2d}, \quad |\xi| \geq R \Rightarrow |a(x,\xi)| \geq C\langle \xi \rangle^m.$$

The elementary rules of differential calculus imply the following proposition.

**Proposition 6.2.3.** *If $a \in S^m$, $b \in S^{m'}$, $\alpha, \beta \in \mathbb{N}^d$ then*

$$\partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|}, \quad ab \in S^{m+m'}.$$

Of course we have $S^0(\mathbb{R}^d) \subset C_b^\infty(\mathbb{R}^{2d})$.

### Examples

1) If $p$ is a function of $x$ only and $p \in C_b^\infty(\mathbb{R}^d)$ then $p \in S^0(\mathbb{R}^d)$.

2) If $p = p(x,\xi)$ belongs to $C_0^\infty(\mathbb{R}^{2d})$ (compact support in $x$ and $\xi$) then $p \in S^{-\infty}$.

3) Suppose that $p(x,\xi)$ is a polynomial in $\xi$ of order $m \in \mathbb{N}$ whose coefficients are functions in $C_b^\infty(\mathbb{R}^d)$, that is

$$p(x,\xi) = \sum_{|\alpha|\leq m} p_\alpha(x)\xi^\alpha \qquad (p_\alpha \in C_b^\infty(\mathbb{R}^d)).$$

Then $p \in S^m(\mathbb{R}^d)$.

4) For all $m \in \mathbb{R}$, the symbol $\langle\xi\rangle^m$ belongs to $S^m(\mathbb{R}^d)$. Indeed, the function $\mathbb{R}\times\mathbb{R}^d \ni (\tau,\xi) \mapsto (\tau^2+|\xi|^2)^{m/2}$ is positively homogeneous of order $m$ on $\mathbb{R}^{d+1}$ and therefore $\partial_\xi^\alpha((\tau^2+|\xi|^2)^{m/2})$ is homogeneous of order $m-|\alpha|$, bounded by $C_\alpha(\tau^2+|\xi|^2)^{(m-|\alpha|)/2}$. As the derivation in $\xi$ and the restriction to $\tau=1$ commute, we deduce the result.

5) The symbol $|\xi|$ is not in $S^1(\mathbb{R}^d)$ because it is not regular in 0.

6) Let $a = a(\xi) \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a homogeneous function of degree $m$, satisfying

$$a(\lambda\xi) = \lambda^m a(\xi) \quad \forall\lambda > 0.$$

For all function $\chi \in C_b^\infty(\mathbb{R}^d_\xi)$ which vanishes in the neighborhood of 0, we have $\chi(\xi)a(\xi) \in S^m(\mathbb{R}^d)$.

7) Let $a = a(x,\xi)$ be an elliptic symbol of order $m$. Then there exists $\chi \in C_0^\infty(\mathbb{R}^d)$ such that

$$\frac{1-\chi(\xi)}{a(x,\xi)} \in S^{-m}(\mathbb{R}^d).$$

8) Let $f = f(x)$ in $C_b^\infty(\mathbb{R})$. The symbol $p(x,\xi) = f(x)\sin(\xi)$ belongs to $C_b^\infty(\mathbb{R}^2)$ but not to $S^0(\mathbb{R})$ because the derivative in $\xi$ of order $\alpha$ does not decrease as $(1+|\xi|)^{-\alpha}$.

## 6.3 Introduction to symbolic calculus

Consider two pseudo-differential operators $A = \mathrm{Op}(a)$ and $B = \mathrm{Op}(b)$ of symbols $a, b \in S^m(\mathbb{R}^d)$. Then $\lambda A + \mu B$ is a pseudo-differential operator of symbol $\lambda a + \mu b \in S^m(\mathbb{R}^d)$. The questions that will interest us in this chapter concern the operators $A \circ B$ and $A^*$. We will see that these are also pseudo-differential operators and that we can compute their symbols. The **symbolic calculation** is precisely the process which allows us to **manipulate** operators by working **at the level of the symbols**.

We will see three very distinct situations in which one can easily study the composition and the transition to the adjoint for pseudo-differential operators.

These situations correspond to the following cases:

A. the **Fourier multipliers** (of symbols not depending on $x$);

B. the **differential operators** (the symbol is a polynomial in $\xi$);

C. the **operators of microlocalization** (of symbols with compact support in $\mathbb{R}^{2d}$).

## A. Fourier multipliers

Let $A = \operatorname{Op}(a)$ with $a = a(\xi)$ independent of $x$. Then $A$ is a special case of Fourier multiplier. Recall that a Fourier multiplier is a linear operator operator which acts on $L^2(\mathbb{R}^d)$ or $\mathcal{S}'(\mathbb{R}^d)$ by multiplying the Fourier transform of a function (or of a tempered distribution) by a given function, called the symbol. Given a function $m = m(\xi)$ with complex values, the Fourier multiplier of symbol $m$ is the operator, denoted $m(D_x)$, defined by

$$\widehat{m(D_x)f}(\xi) = m(\xi)\widehat{f}(\xi).$$

If $m \in L^\infty(\mathbb{R}^d)$ then $m(D_x)$ is well defined on $L^2(\mathbb{R}^d)$ and $m(D_x) \in \mathcal{L}(L^2)$. If $m(D_x) \in C^\infty(\mathbb{R}^d)$ is slowly increasing (there exists $N$ such that for all $\alpha$ we have $\left|\partial_\xi^\alpha m(\xi)\right| \leq C_\alpha \langle \xi \rangle^N$) then $m(D_x)$ is continuous from $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$. We check that

$$m_1(D_x)m_2(D_x) = m(D_x) \quad \text{with } m(\xi) = m_1(\xi)m_2(\xi),$$

$$m(D_x)^* = m^*(D_x) \quad \text{with } m^*(\xi) = \overline{m(\xi)}.$$

**Examples of Fourier multipliers:**

- $\partial_{x_j}$ is the Fourier multiplier of symbol $i\xi_j$.
- The laplacian $\Delta$ is the Fourier multiplier of symbol $-|\xi|^2$.
- The Hilbert transform is the Fourier multiplier of symbol $-i\xi/|\xi|$ ($\xi \in \mathbb{R}$).
- The square root of $-\Delta$ is the Fourier multiplier of symbol $|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}$.
- Let $s \in \mathbb{R}$. The operator which realizes the canonical isomorphism of $H^s$ on $L^2$ is the Fourier multiplier of symbol $\langle \xi \rangle^s$ where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.
- Consider the equation

  $$\partial_t u + i\langle D_x \rangle^s u = 0, \quad u\mathcal{A}rowvert_{t=0} = u_0.$$

  This equation can be solved by the Hille-Yosida theorem or by the Fourier transform. The operator which sends the initial data $u_0$ on the solution at time $t$ is the Fourier multiplier of symbol $\exp(-it\langle \xi \rangle^s)$.

## B. Differential operators

Consider two differential operators

$$A = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha, \quad B = \sum_{|\alpha|\le m'} b_\alpha(x)\partial_x^\alpha$$

where the coefficients $a_\alpha, b_\alpha$ belong to $C_b^\infty(\mathbb{R}^d)$. Let us introduce their symbols

$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)(i\xi)^\alpha, \quad b(x,\xi) = \sum_{|\alpha|\le m'} b_\alpha(x)(i\xi)^\alpha,$$

so that $A = \mathrm{Op}(a)$ and $B = \mathrm{Op}(b)$. Let $e_\xi$ be the exponential function $x \mapsto e^{ix\cdot\xi}$. Then

$$(Ae_\xi)(x) = a(x,\xi)e_\xi(x), \quad (Be_\xi)(x) = b(x,\xi)e_\xi(x).$$

Moreover, for all regular functions $b(x,\xi)$,

$$\begin{aligned} A(be_\xi)(x) &= \sum_\alpha a_\alpha(x)\partial_x^\alpha\big(e^{ix\cdot\xi}b(x,\xi)\big) \\ &= \sum_\alpha a_\alpha(x)\big((i\xi+\partial_x)^\alpha b(x,\xi)\big)e^{ix\cdot\xi} \\ &= e^{ix\cdot\xi}a\Big(x,\xi+\frac{1}{i}\partial_x\Big)b(x,\xi) \\ &= e^{ix\cdot\xi}\sum_{\beta\in\mathbb{N}^d}\frac{1}{i^{|\beta|}\beta!}\big(\partial_\xi^\beta a(x,\xi)\big)\big(\partial_x^\beta b(x,\xi)\big) \end{aligned}$$

where we used the formula of Taylor for a polynomial. We deduce the following result.

**Proposition 6.3.1.** *If $A$ and $B$ are differential operators, then $A\circ B$ is a differential operator of symbol*

$$a\#b(x,\xi) = \sum_{\alpha\in\mathbb{N}^d}\frac{1}{i^{|\alpha|}\alpha!}\big(\partial_\xi^\alpha a(x,\xi)\big)\big(\partial_x^\alpha b(x,\xi)\big).$$

*Note that the sum is finite since $\partial_\xi^\alpha a = 0$ if $|\alpha| > m$.*

*Proof.* The operator $A\circ B$ is of course a differential operator and we have seen that $(A\circ B)e_\xi = (a\#b)e_\xi$. □

**Exercise 6.3.2.** *Let $A$ be a differential operator. Show that $A^*$ is a differential operator of symbol*

$$a^*(x,\xi) = \sum_\alpha \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha\partial_x^\alpha\overline{a}(x,\xi).$$

# C. Microlocalization operators

The localization operators, of the form $u \mapsto \varphi u$ where $\varphi \in C_0^\infty(\mathbb{R}^d)$, are essential in Analysis. In the same way as the frequency localization operators, which are Fourier multipliers $u \mapsto \varphi(D_x)u$ with $\varphi \in C_0^\infty(\mathbb{R}^d)$. The pseudo-differential operators allow a sort of simultaneous localization in $x$ and in $\xi$, by considering an operator $\operatorname{Op}(a)$ with $a \in C_0^\infty(\mathbb{R}^{2d})$ (a word of warning: we recall from Proposition 2.2.1 that it is not possible to localize exactly in both $x$ and $\xi$). If $a \in C_0^\infty(\mathbb{R}^{2d})$ we say that $\operatorname{Op}(a)$ is a microlocalization operator.

We will see that the adjoint of a microlocalization operator is a pseudo-differential operator whose symbol does not necessarily belong to $C_0^\infty(\mathbb{R}^{2d})$ but belongs to all spaces $S^m(\mathbb{R}^d)$ for $m \leq 0$.

**Proposition 6.3.3.** *Let $a = a(x,\xi)$ be a symbol belonging to $C_0^\infty(\mathbb{R}^{2d})$. Then*

$$a^*(x,\xi) = (2\pi)^{-d} \int e^{-iy\cdot\eta}\overline{a}(x-y,\xi-\eta)\,\mathrm{d}y\,\mathrm{d}\eta$$

*defines a symbol $a^*$ belonging to $S^{-\infty}$ and*

$$(\operatorname{Op}(a)u, v) = (u, \operatorname{Op}(a^*)v)$$

*for all $u, v$ in $\mathcal{S}(\mathbb{R}^d)$.*

**Remark 6.3.4.** *One objective of this chapter will be to prove a result which extends the previous proposition to the case of a general symbol $a \in S^{+\infty}$. We start by looking at the case where $a$ has compact support (hence belongs to $S^{-\infty}$) because the analysis is then much easier. The reader will note in particular that the integrals which appear in the proof below are meaningless if $a$ is a general symbol.*

*Proof.* Let $u \in \mathcal{S}(\mathbb{R}^d)$. Since $a$ is compactly supported we can use Fubini's theorem to write

$$\begin{aligned}
\operatorname{Op}(a)u(x) &= (2\pi)^{-d} \int e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi \\
&= (2\pi)^{-d} \int e^{ix\cdot\xi} a(x,\xi) \left( \int e^{-iy\cdot\xi} u(y)\,\mathrm{d}y \right) \mathrm{d}\xi \\
&= (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= (2\pi)^{-d} \int \left( \int e^{i(x-y)\cdot\xi} a(x,\xi)\,\mathrm{d}\xi \right) u(y)\,\mathrm{d}y.
\end{aligned}$$

Therefore

$$\mathrm{Op}(a)u(x) = \int K(x,y)u(y)\,\mathrm{d}y$$

where $K = K(x,y)$ (called **kernel** of $\mathrm{Op}(a)$) is given by

$$\begin{aligned} K(x,y) &= (2\pi)^{-d} \int e^{i(x-y)\cdot\xi} a(x,\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-d} (\mathcal{F}_\xi a)(x, y-x), \end{aligned}$$

where $\mathcal{F}_\xi a(x,\zeta) = \int e^{-i\xi\cdot\zeta} a(x,\xi)\,\mathrm{d}\xi$ is the Fourier transform of $a$ with respect to the second variable. We deduce that $K \in \mathcal{S}(\mathbb{R}^{2d})$.

Now, if $v$ is also in $\mathcal{S}$ then

$$\begin{aligned} (\mathrm{Op}(a)u, v) &= \int \left( \int K(x,y)u(y)\,\mathrm{d}y \right) \overline{v(x)}\,\mathrm{d}x \\ &= \int u(y) \overline{\left( \int \overline{K(x,y)} v(x)\,\mathrm{d}x \right)}\,\mathrm{d}y, \end{aligned}$$

so $(\mathrm{Op}(a)u, v) = (u, (\mathrm{Op}(a))^* v)$ with

$$(\mathrm{Op}(a))^* v(x) := \int \overline{K(y,x)} v(y)\,\mathrm{d}y.$$

Note that $\mathrm{Op}(a)^*$ is an operator with kernel

$$K^*(x,y) = \overline{K(y,x)} = (2\pi)^{-d} \int e^{i(x-y)\cdot\theta} \overline{a(y,\theta)}\,\mathrm{d}\theta.$$

We want to write $K^*(x,y)$ in the form $K^*(x,y) = (2\pi)^{-d} (\mathcal{F}_\xi a^*)(x, y-x)$. Then

$$\begin{aligned} a^*(x,\xi) &= (2\pi)^{-d} \int e^{i\xi\cdot z} (\mathcal{F}_\xi a^*)(x,z)\,\mathrm{d}z \\ &= \int K^*(x, x+z) e^{iz\cdot\xi}\,\mathrm{d}z \\ &= \int K^*(x, x-y) e^{-iy\cdot\xi}\,\mathrm{d}y \\ &= (2\pi)^{-d} \int \left( \int e^{i(x-(x-y))\cdot\theta} \overline{a(x-y,\theta)}\, d\theta \right) e^{-iy\cdot\xi}\,\mathrm{d}y \\ &= (2\pi)^{-d} \iint e^{iy\cdot(\theta-\xi)} \overline{a}(x-y,\theta)\,\mathrm{d}y\,\mathrm{d}\theta \\ &= (2\pi)^{-d} \iint e^{-iy\cdot\eta} \overline{a}(x-y, \xi-\eta)\,\mathrm{d}y\,\mathrm{d}\eta. \end{aligned}$$

Then, the calculations already made at the beginning of the proof lead to the fact that $\mathrm{Op}(a)^*$ is the pseudo-differential operator of symbol $a^*$. $\square$

## 6.4 Oscillating Integrals

In this section we propose to study to study the oscillatory integrals. These integrals, which play a crucial role in micro-local analysis, are of the form

$$\int e^{i\phi(x)} a(x)\,\mathrm{d}x \qquad (x \in \mathbb{R}^N,\ N \geq 1).$$

We say that $\phi$ is a phase and that $a$ is an amplitude. We will always assume that $a$ is a function $C^\infty$ of $\mathbb{R}^N$ in $\mathbb{C}$ and that $\phi$ has real values.

These integrals play a crucial role in micro-local analysis. In particular, they appear naturally to define symbols. For instance, let us notice that for $(x_0, \xi_0)$ fixed in $\mathbb{R}^d \times \mathbb{R}^d$,

$$a^*(x_0, \xi_0) = (2\pi)^{-d} \int e^{-iy\cdot\eta}\overline{a}(x_0 - y, \xi_0 - \eta)\,\mathrm{d}y\,\mathrm{d}\eta$$

is written in the form

$$a^*(x_0, \xi_0) = \int e^{i\phi(x)} A(x)\,\mathrm{d}x$$

where $N = 2d$, $x = (y, \eta)$ and $\phi(x) = -y \cdot \eta$.

If $a$ is the symbol of a differential operator operator, polynomial in $x$, the integral is obviously obviously divergent in the classical sense. To give a meaning to the integral $\int e^{i\phi(x)} a(x)\,\mathrm{d}x$ and to show results of calculations on these integrals, the idea is that, under a hypothesis of strong oscillation of the term $e^{i\phi(x)}$, we can compensate for the growth of $a$.

### A. Principle of non-stationary phase

The analysis of oscillatory integrals is based on the so-called *principle of non-stationary phase*, which expresses the decay of an oscillatory integral as a function of a large parameter.

**Lemma 6.4.1** (Non-stationary phase lemma)**.** *Let $N \geq 1$, $\varphi \in C_b^\infty(\mathbb{R}^N)$ a function with real values and $f \in C_0^\infty(\mathbb{R}^N)$. Let $V$ be a neighbourhood of the support of $f$. It is assumed that*

$$\inf_V |\nabla\varphi(x)| > 0.$$

*Then, for all integers $k$ and for all $\lambda \geq 1$,*

$$\left| \int e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x \right| \leq C_k \lambda^{-k} \sup_{|\alpha|\leq k} \left\| \partial_x^\alpha f \right\|_{L^1(\mathbb{R}^d)} .$$

*where $C_k$ is a constant independent of $\lambda$ and $f$.*

*Proof.* Let us introduce the differential operator

$$L := -i\frac{\nabla\varphi\cdot\nabla}{|\nabla\varphi|^2} \quad \text{where} \quad \nabla\varphi\cdot\nabla = \sum_{1\leq j\leq N} \frac{\partial\varphi}{\partial x_j}\frac{\partial}{\partial x_j}.$$

which is well defined because the differential of the phase does not vanish above $V$. Moreover, $L$ satisfies, for all $\lambda \in \mathbb{R}$,

$$L(e^{i\lambda\varphi}) = \lambda e^{i\lambda\varphi},$$

and therefore $L^k(e^{i\lambda\varphi(x)}) = \lambda^k e^{i\lambda\varphi(x)}$. Then, by doing successive *integrations by parts*, we deduce that

$$\lambda^k \int e^{i\lambda\varphi(x)} f(x)\,\mathrm{d}x = \int e^{i\lambda\varphi(x)} ({}^tL)^k f(x)\,\mathrm{d}x,$$

where

$${}^tLf = i \sum_{1\leq j\leq N} \frac{\partial}{\partial x_j}\left(\frac{1}{|\nabla\varphi|^2}\left(f\frac{\partial\varphi}{\partial x_j}\right)\right).$$

Let us note that $({}^tL)^k$ is a differential operator of order $k$ whose coefficients are $C^\infty$ (and depend on $\varphi$). We thus obtain the desired result by bounding the last integral by $\left\|({}^tL)^k f\right\|_{L^1}$. We also obtain that the constant $C_k$ depends only on $k$, $\inf |\nabla\varphi|^2$ and $\sup_{|\alpha|\leq k+1} \|\partial^\alpha\varphi\|_{L^\infty}$. □

## B. Definition of an oscillatory integral

**Definition 6.4.2.** *Let $m$ be a real number. The space $A^m$ of amplitudes of order $m$ consists of those functions $a \in C^\infty(\mathbb{R}^N;\mathbb{C})$ such that*

$$\forall\alpha \in \mathbb{N}^N, \qquad \sup_{x\in\mathbb{R}^N} \left|(1+|x|)^{-m}\partial_x^\alpha a(x)\right| < +\infty.$$

*We introduce the norms*

$$\|a\|_{m,k} := \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^N} \left|(1+|x|)^{-m}\partial_x^\alpha a(x)\right|.$$

We will assume that $\phi$ is a non-degenerate quadratic form, of the form

$$\phi(x) = (Ax)\cdot x \qquad (x \in \mathbb{R}^N)$$

where $A \in M_N(\mathbb{R})$ is an invertible symmetric matrix. Then $\nabla\phi(x) = 2Ax$ and we can apply the principle of non-stationary phase.

**Theorem 6.4.3.** *Let $m \ge 0$, $\phi$ be a non-degenerate quadratic form on $\mathbb{R}^N$, $a \in A^m$ and $\psi \in \mathcal{S}(\mathbb{R}^N)$ such that $\psi(0) = 1$. Then the integral*

$$I(\varepsilon) := \int e^{i\phi(x)}a(x)\psi(\varepsilon x)\,\mathrm{d}x,$$

*which is well-defined for $\varepsilon > 0$, converges when $\varepsilon$ tends towards $0$ towards a limit independent of $\psi$, which is equal to $\int e^{i(x)}a(x)\,\mathrm{d}x$ if $a$ belongs to $L^1(\mathbb{R}^N)$. When $a \notin L^1$, we continue to denote the limit $\int e^{i\phi(x)}a(x)\,\mathrm{d}x$ and we have*

$$\left|\int e^{i\phi(x)}a(x)\,\mathrm{d}x\right| \le C_{\phi,m}\,\|a\|_{m,m+N+1}\,. \tag{6.4.1}$$

*Proof.* We want to use the principle of non-stationary phase, which requires to make a large parameter appear. To do this, we will use a dyadic decomposition. Let us recall how to obtain such a decomposition. Let $\chi_0 \in C_0^\infty(\mathbb{R}^N;\mathbb{R})$ be a radial function satisfying $\chi_0(x) = 1$ for $|x| \le 1/2$, and $\chi_0(x) = 0$ for $|x| \ge 1$. We pose $\chi(x) = \chi_0(x/2) - \chi_0(x)$. Then the function $\chi$ is supported in the annulus $\{x \in \mathbb{R}^N\,;\, 1/2 \le |x| \le 2\}$ and, for all $x \in \mathbb{R}^N$, we have the equality

$$1 = \chi_0(x) + \sum_{j=0}^{\infty}\chi(2^{-j}x).$$

The convergence of this series is not a problem because, for all $x \in \mathbb{R}^N$, we have $\chi(2^{-p}x) = 0$ for all integer $p$ large enough. Let us set

$$S_p(x) = \chi_0(x) + \sum_{j=0}^{p}\chi(2^{-j}x) = \chi_0(2^{-p-1}x),$$

and introduce the well-defined integrals

$$I_p := \int e^{i\phi(x)} a(x) S_p(x)\,\mathrm{d}x, \quad R_p(\varepsilon) := \int e^{i\phi(x)} a(x)(1-\psi(\varepsilon x)) S_p(x)\,\mathrm{d}x.$$

Notice that, by dominated convergence,

$$I(\varepsilon) = \int e^{i\phi(x)} a(x)\psi(\varepsilon x)\,\mathrm{d}x = \lim_{p\to+\infty} \int e^{i\phi(x)} a(x)\psi(\varepsilon x) S_p(x)\,\mathrm{d}x.$$

Also, by definition,

$$\int e^{i\phi(x)} a(x)\psi(\varepsilon x) S_p(x)\,\mathrm{d}x = I_p - R_p(\varepsilon).$$

Hence, it will suffice to prove that $\lim_{p\to+\infty} I_p$ exists and that $\lim_{p\to+\infty} R_p(\varepsilon) = O(\varepsilon)$. This will prove that $I(\varepsilon)$ has a limit when $\varepsilon$ tends towards 0 and that this limit is independent of $\psi$.

After changing variables $z = 2^{-p}x$,

$$I_p - I_{p-1} = \int e^{i2^{2p}\phi(z)} a(2^p z)\chi(z) 2^{Np}\,\mathrm{d}z$$

where we used the fact that $\phi$ is quadratic to write $\phi(tz) = t^2\phi(z)$.

On the support of $\chi$ we have $|z| \geq 1/2$ so

$$\inf_{z\in\operatorname{supp}\chi} |\nabla\phi(z)| \geq c_0 > 0$$

and we can apply the principle of non-stationary phase. More precisely, it follows from Lemma 6.4.1 applied with $f(x) = a(2^p x)\chi(x)$ and $\lambda = 2^{2p}$ that, for all $k \in \mathbb{N}$,

$$\int e^{i2^{2p}\phi(z)} a(2^p z)\chi(z) 2^{Np}\,\mathrm{d}z \leq C_k 2^{Np-2pk} \max_{|\alpha|\leq k} \int_{|x|\leq 2} \left|\partial_x^\alpha \left(a(2^p x)\chi(x)\right)\right| \mathrm{d}x,$$

where we used the fact that $\operatorname{supp}\chi$ is contained in the ball $B(0,2)$. The assumption that $a$ is an amplitude of order $m$ implies that there exists a constant $C > 0$ such that, for all $p \geq 1$,

$$\int_{|x|\leq 2} \left|\partial_x^\alpha \left(a(2^p x)\chi(x)\right)\right| \mathrm{d}x \leq C 2^{p(|\alpha|+m)} \left\|a\right\|_{m,|\alpha|}.$$

From this we deduce that

$$\int e^{i2^{2p}\phi(z)} a(2^p z)\chi(z) 2^{pN} \,\mathrm{d}z \le CC_k 2^{p(N+k+m-2k)} \|a\|_{m,k} .$$

We choose $k = N + m + 1$ so that

$$\left|I_p - I_{p-1}\right| \le CC_{N+m+1} 2^{-p} \|a\|_{m,N+m+1} .$$

In the same way we obtain that

$$\left|R_p(\varepsilon) - R_{p-1}(\varepsilon)\right| \le \varepsilon C 2^{-p}.$$

This completes the proof. □

## C. An inequality of Hörmander

The aim of this paragraph is to demonstrate a nice result, the proof of which allows us to implement several ideas which are very useful in practice.

Consider a family of operators $T_h$, depending on a small parameter $h$, of the form

$$(T_h f)(\xi) := \int e^{i\phi(x,\xi)/h} a(x,\xi) f(x) \,\mathrm{d}x \qquad (x, \xi \in \mathbb{R}^d).$$

Let us suppose that the phase $\phi$ is with real values and that the amplitude $a$ is compactly supported in $x$ and in $\xi$. Then, we easily satisfy that, for all $h > 0$, $T_h$ is a continuous linear from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. We will prove an estimate, due to Hörmander, which states that if the mixed Hessian $\phi''_{x\xi}$ is not singular on the amplitude support, then $h^{-d/2}T_h$ is uniformly bounded in $\mathcal{L}(L^2)$.

**Theorem 6.4.4.** *Let $a \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. If $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is real and satisfies*

$$(x,\xi) \in \operatorname{supp} a \Rightarrow \det\left[\frac{\partial^2 \phi}{\partial x \partial \xi}(x,\xi)\right] \neq 0,$$

*then there exists a constant $C$ such that, for all $h \in (0, 1]$ and all $f \in L^2(\mathbb{R}^d)$,*

$$\|T_h f\|_{L^2} \le C h^{\frac{d}{2}} \|f\|_{L^2} .$$

**Remark 6.4.5** (Hausdorff-Young inequality)**.** *In view of the obvious estimate*

$$\|T_h f\|_{L^\infty} \leq \|f\|_{L^1},$$

*the Riesz convexity theorem implies that, if* $p \in [1,2]$ *and* $1/p + 1/p' = 1$*, then*

$$\|T_h f\|_{L^{p'}} \lesssim h^{d/p'} \|f\|_{L^p}, \quad f \in C_0^\infty(\mathbb{R}^d).$$

*By taking* $\Phi(x,\xi) = x \cdot \xi$ *and* $a$ *with* $a(0,0) = 1$*, we obtain the Hausdorff-Young inequality by a scaling argument.*

*Proof.* We will use classical results on bounded operators on $\mathcal{L}(L^2)$. First, let $\|T\|^2_{\mathcal{L}(L^2)} = \|T^*\|^2_{\mathcal{L}(L^2)}$. We deduce that $\|TT^*\|^2_{\mathcal{L}(L^2)}$. As furthermore

$$\|T^* f\|_{L^2} = \langle T^* f, T^* f\rangle = \langle TT^* f, f\rangle \leq \|TT^*\|_{\mathcal{L}(L^2)} \|f\|^2_{L^2},$$

we check that

$$\|T^*\|^2_{\mathcal{L}(L^2)} = \|T^*\|^2_{\mathcal{L}(L^2)} = \|TT^*\|_{\mathcal{L}(L^2)}.$$

Therefore, it is sufficient to prove that the norm [1] of $T_h T_h^*$ is bounded by $Ch^d$. Let us write

$$(T_h T_h^* f)(\xi) = \int K_h(\xi,\eta) f(\eta)\,\mathrm{d}\eta$$

where

$$K_h(\xi,\eta) = \int e^{i(\Phi(x,\xi)-\Phi(x,\eta))/h} a(x,\xi)\bar{a}(x,\eta)\,\mathrm{d}x.$$

We then use Schur's lemma (proved at the end of this proof) which states that a kernel operator, of the form

$$(Tf)(x) = \int K(x,y) f(y)\,\mathrm{d}y,$$

satisfies

$$2\,\|T\|_{L^2\to L^2} \leq \sup_y \int |K(x,y)|\,\mathrm{d}x + \sup_x \int |K(x,y)|\,\mathrm{d}y.$$

It thus remains to estimate the kernel $K_h$. If we introduce a partition of the unit, we can always suppose that the support of $a$ is included in a ball of diameter $\delta$ small. One can thus limit oneself to consider the case where $\xi$ and $\eta$ are close. Then

$$|\partial_x\big(\Phi(x,\xi) - \Phi(x,\eta)\big)| = |\Phi''_{x\xi}(x,\eta)(\xi-\eta)| + O(|\xi-\eta|^2) \geq c\,|\xi-\eta|,$$

[1]It is frequently used that it is more convenient to estimate the operator norm of $TT^*$ than that of $T$; we then say that we use the $TT^*$ argument.

and one is able to use the lemma of the non-stationary phase to obtain the majoration

$$|K_h(\xi,\eta)| \leq C_N \left(\frac{|\xi-\eta|}{h}\right)^{-N},$$

for all $N \in \mathbb{N}$. As moreover $K_h$ is bounded, we deduce that

$$|K_h(\xi,\eta)| \leq C'_N(1+|\xi-\eta|/h)^{-N},$$

from which

$$\sup_\eta \int |K_h(\xi,\eta)|\,\mathrm{d}\xi \leq Ch^d, \quad \sup_\xi \int |K_h(\xi,\eta)|\,\mathrm{d}\eta \leq Ch^d,$$

which concludes the proof. □

**Lemma 6.4.6** (Schur's Lemma)**.** *Let $K(x,y)$ be a continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ such that*

$$\sup_y \int |K(x,y)|\,\mathrm{d}x \leq A_1, \quad \sup_x \int |K(x,y)|\,\mathrm{d}y \leq A_2.$$

*Then the operator $P$ of kernel $K$, defined for $u \in C_0^0(\mathbb{R}^d)$ by*

$$Pu(x) = \int K(x,y)u(y)\,\mathrm{d}y$$

*extends uniquely into a continuous operator of $L^2(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ and*

$$\|Pu\|_{L^2} \leq \sqrt{A_1A_2}\,|u\|_{L^2}.$$

*Proof.* According to the Cauchy-Schwarz inequality

$$|Pu(x)|^2 \leq \int |K(x,y)|\,|u(y)|^2\,\mathrm{d}y \int |K(x,y)|\,\mathrm{d}y \leq A_2 \int |K(x,y)|\,|u(y)|^2\,\mathrm{d}y,$$

from which

$$\begin{aligned}\int |Pu(x)|^2\,\mathrm{d}x &\leq A_2 \iint |K(x,y)|\,|u(y)|^2\,\mathrm{d}y\,\mathrm{d}x \\ &\leq A_2 \int |u(y)|^2 \Big(\int |K(x,y)|\,\mathrm{d}x\Big)\,\mathrm{d}y \\ &\leq A_1A_2 \int |u(y)|^2\,\mathrm{d}y,\end{aligned}$$

which implies the desired inequality. □

**Remark 6.4.7.** *Schur's lemma implies that if $f \in L^1$ and $g \in L^2$ then $\|f * g\|_{L^2} \leq \|f\|_{L^1} \|g\|_{L^2}$. To see this it is sufficient to observe that*

$$f * g(x) = \int K(x, y) g(y) \, \mathrm{d}y \quad \textit{where} \quad K(x, y) = f(x - y).$$

*The desired inequality comes from Schur's lemma with $A = \|f\|_{L^1}$.*

## 6.5 Adjoint and composition

To state the main result of this chapter, it is convenient to use the following definition.

**Definition 6.5.1.** *i) Let $s \in \mathbb{R}$. The Sobolev space $H^s(\mathbb{R}^d)$ is the space of tempered distributions $f$ such that $(1+|\xi|^2)^{s/2} \widehat{f}(\xi)$ belongs to $L^2(\mathbb{R}^d)$. This space is equipped with the norm*

$$\|f\|_{H^s}^2 := \frac{1}{(2\pi)^d} \int (1 + |\xi|^2)^s \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi.$$

*ii) Let $m \in \mathbb{R}$. An operator is said to be of order $m$ if it is bounded from $H^\mu(\mathbb{R}^d)$ to $H^{\mu-m}(\mathbb{R}^d)$ for all $m \in \mathbb{R}$.*

**Example 6.5.2.**

- *The identity is an operator of order $0$ and the Laplacian is an operator of order $2$;*
- *A differential operator $P = \sum_{|\alpha| \leq k} p_\alpha(x) \partial_x^\alpha$ with $k \in \mathbb{N}$ and $p_\alpha \in C_b^\infty(\mathbb{R}^d)$ is an operator of order $k$ (non-trivial to prove starting from the definition of Sobolev spaces for $\mu \in \mathbb{R} \setminus \mathbb{N}$);*
- *The convolution operator by a function in the Schwartz class is an operator of order $-\infty$ (which means that it is of order $-l$ for all $l \in \mathbb{N}$ or that it sends $H^{-\infty}(\mathbb{R}^d) = \cup_{s \in \mathbb{R}} H^s(\mathbb{R}^d)$ into $H^\infty(\mathbb{R}^d) = \cap_{s \in \mathbb{R}} H^s(\mathbb{R}^d)$).*

In this chapter, we will prove (and make sense of) the following statement.

**Theorem 6.5.3.** *i) If $a \in S^m(\mathbb{R}^d)$ then $\mathrm{Op}(a)$ can be extended on the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distribution as an operator of order $m$.*

*ii) Suppose $a \in S^m(\mathbb{R}^d)$ and $b \in S^{m'}(\mathbb{R}^d)$. Then $\mathrm{Op}(a) \circ \mathrm{Op}(b)$ is a pseudo-differential operator of symbol denoted $a\#b$ and defined by*

$$a\#b(x, \xi) = (2\pi)^{-d} \iint e^{i(x-y)\cdot(\xi-\eta)} a(x, \xi) b(y, \eta) \, \mathrm{d}\xi \, \mathrm{d}y.$$

*In addition* $\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Op}(ab) + R$ *where* $R$ *is of order* $m + m' - 1$ *and more generally, the operator*

$$\mathrm{Op}(a) \circ \mathrm{Op}(b) - \mathrm{Op}\left( \sum_{|\alpha| \le k} \frac{1}{i^{|\alpha|}\alpha!} \big(\partial_\xi^\alpha a(x,\xi)\big)\big(\partial_x^\alpha b(x,\xi)\big) \right)$$

*is of order* $m + m' - k - 1$*, for all integer* $k \in \mathbb{N}$*.*

*iii) The adjoint* $\mathrm{Op}(a)^*$ *is a pseudo-differential operator of symbol* $a^*$ *defined by*

$$a^*(x,\xi) = (2\pi)^{-d} \iint e^{-iy\cdot\eta} \overline{a}(x-y, \xi-\eta)\,\mathrm{d}y\,\mathrm{d}\eta.$$

*Moreover* $\mathrm{Op}(a)^* = \mathrm{Op}(\overline{a}) + R$ *where* $R$ *is of order* $m - 1$ *and more generally*

$$\mathrm{Op}(a^*) - \mathrm{Op}\left( \sum_{|\alpha| \le k} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a}(x,\xi) \right) \quad \text{is of order } m - k - 1,$$

*for all integer* $k \in \mathbb{N}$*.*

**Corollary 6.5.4.** *Let* $a \in S^m(\mathbb{R}^d)$ *and* $b \in S^{m'}(\mathbb{R}^d)$*. By definition, the Poisson bracket of* $a$ *and* $b$ *is defined by*

$$\{a, b\} = \sum_{1 \le j \le d} \left( \frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} - \frac{\partial b}{\partial \xi_j} \frac{\partial a}{\partial x_j} \right).$$

*Then the commutator*

$$[\mathrm{Op}(a), \mathrm{Op}(b)] = \mathrm{Op}(a) \circ \mathrm{Op}(b) - \mathrm{Op}(b) \circ \mathrm{Op}(a)$$

*is an operator of order* $m + m' - 1$ *whose symbol* $c$ *can be written as*

$$c = \frac{1}{i}\{a, b\} + c' \quad \text{where} \quad c' \in S^{m+m'-2}.$$

To prove Theorem 6.5.3 we start by studying the adjoint with the following proposition.

**Proposition 6.5.5.** *Let* $m \in \mathbb{R}$*. If* $a \in S^m(\mathbb{R}^d)$ *then the oscillatory integral*

$$a^*(x,\xi) = (2\pi)^{-d} \iint e^{-iy\cdot\eta} \overline{a}(x-y, \xi-\eta)\,\mathrm{d}y\,\mathrm{d}\eta$$

*defines a symbol* $a^*$ *that belongs to* $S^m(\mathbb{R}^d)$*.*

*Proof.* Let $\phi(y,\eta) = -y \cdot \eta$. Then $\phi$ is a non-degenerate quadratic form on $\mathbb{R}^{2d}$ (we have $\phi(X) = (AX) \cdot X$ where $A$ is the symmetric invertible matrix $A = -\frac{1}{2}\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$). At $(x,\xi)$ fixed we denote

$$b_{x,\xi}(y,\eta) = \overline{a}(x-y,\xi-\eta).$$

To study $b_{x,\xi}$, we will use the following inequality.

**Lemma 6.5.6** (Peetre's Lemma)**.** *Recall the notation*

$$\langle\xi\rangle = (1+|\xi|^2)^{1/2}.$$

*Let $d \geq 1$. For all $m \in \mathbb{R}$ and all $\xi,\eta$ in $\mathbb{R}^d$, we have*

$$\langle\xi+\eta\rangle^m \leq 2^{|m|}\langle\xi\rangle^{|m|}\langle\eta\rangle^m.$$

*Proof.* According to the triangular inequality

$$1+|\xi+\eta|^2 \leq 1+(|\xi|+|\eta|)^2 \leq 1+2\,|\xi|^2+2\,|\eta|^2 \leq 4(1+|\xi|^2)(1+|\eta|^2)$$

so $\langle\xi+\eta\rangle^2 \leq 2^2\langle\xi\rangle^2\langle\eta\rangle^2$ and we deduce the desired inequality for $m \geq 0$. Now consider $m < 0$ so that $-m > 0$. We can then use the inequality with $-m > 0$ to get

$$\langle\eta\rangle^{-m} \leq 2^{-m}\langle\xi+\eta\rangle^{-m}\langle-\xi\rangle^{-m},$$

which implies the desired result by dividing by $\langle\eta\rangle^{-m}\langle\xi+\eta\rangle^{-m}$. □

The previous lemma implies that

$$\langle\xi-\eta\rangle^m \leq 2^{|m|}\langle\xi\rangle^m\langle\eta\rangle^{|m|} \qquad \forall\xi,\eta \in \mathbb{R}^d.$$

Then, the assumption that $a$ is a symbol implies that

$$\begin{aligned}\left|\partial_y^\alpha\partial_\eta^\beta a(x-y,\xi-\eta)\right| &\leq C_{\alpha\beta}\langle\xi-\eta\rangle^{m-|\beta|} \leq C_{\alpha\beta}\langle\xi-\eta\rangle^m \\ &\leq C_{\alpha\beta}2^{|m|}\langle\xi\rangle^m\langle\eta\rangle^{|m|} \\ &\leq C_{\alpha\beta}2^{|m|}\langle\xi\rangle^m(1+|y|^2+|\eta|^2)^{|m|/2}\end{aligned}$$

for all $\alpha,\beta$ in $\mathbb{N}^d$. By definition of the classes of amplitudes, we deduce that

$$b_{x,\xi} \in A^{|m|}(\mathbb{R}^{2d})$$

and moreover

$$\left\|b_{x,\xi}\right\|_{|m|,|m|+2d+1}=\max_{|\alpha|+|\beta|\leq|m|+2d+1}\left|\langle(y,\eta)\rangle^{-|m|}\partial_y^\alpha\partial_\eta^\beta b_{x,y}(y,\eta)\right|\leq C\langle\xi\rangle^m.$$

Since $a^*(x,\xi)$ is an oscillatory integral given by

$$a^*(x,\xi)=\iint e^{i\phi(y,\eta)}b_{x,\xi}(y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta,$$

the previous estimation and the inequality (6.4.1) imply that $\langle\xi\rangle^{-m}a^*$ is a bounded function. It remains to estimate the derivatives. For that we will prove that $a^*$ is $C^\infty$ and that for all any multi-indices $\alpha,\beta$ we have

$$\partial_x^\alpha\partial_\xi^\beta(a^*)=(\partial_x^\alpha\partial_\xi^\beta a)^*.$$

Let us admit this identity. Then the previous argument applied with the symbol $\partial_x^\alpha\partial_\xi^\beta a\in S^{m-|\beta|}(\mathbb{R}^d)$ instead of $a\in S^m(\mathbb{R}^d)$ implies that $\langle\xi\rangle^{-(m-|\beta|)}\partial_x^\alpha\partial_\xi^\beta(a^*)$ is bounded for all $\alpha,\beta$ in $\mathbb{N}^d$. This will prove that $a^*$ is in $S^m(\mathbb{R}^d)$.

It remains to prove that $\partial_x^\alpha\partial_\xi^\beta(a^*)=(\partial_x^\alpha\partial_\xi^\beta a)^*$. For this we will show that we can differentiate the oscillatory integral which defines $a^*$ under the integral sign. Recall that for all function $\psi\in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ such that $\psi(0,0)=1$ we have

$$a^*(x,\xi)=\lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\bar{a}(x-y,\xi-\eta)\psi(\varepsilon y,\varepsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta.$$

We will again use an integration by parts argument that relies on the identity

$$(1+|y|^2)^{-k}(1+|\eta|^2)^{-k}(I-\Delta_y)^k(I-\Delta_\eta)^k e^{-iy\cdot\eta}=e^{-iy\cdot\eta}.$$

Since we are integrating regular functions with compact support, we can integrate by parts and obtain that

$$\begin{aligned}\iint & e^{-iy\cdot\eta}\bar{a}(x-y,\xi-\eta)\psi(\varepsilon y,\varepsilon\eta)\,\mathrm{d}y\,\mathrm{d}\eta\\ &=\iint e^{-iy\cdot\eta}(I-\Delta_y)^k(I-\Delta_\eta)^k\left[\frac{\bar{a}(x-y,\xi-\eta)\psi(\varepsilon y,\varepsilon\eta)}{(1+|y|^2)^k(1+|\eta|^2)^k}\right]\mathrm{d}y\,\mathrm{d}\eta.\end{aligned}$$

Recall that we have shown that

$$\left|\partial_y^\gamma\partial_\eta^\delta a(x-y,\xi-\eta)\right|\leq C_{\gamma\delta}2^{|m|}\langle\xi\rangle^m(1+|\eta|^2)^{|m|/2}.$$

On the other hand

$$\left|\partial_y^\gamma (1+|y|^2)^{-k}\right| \leq C_{k,\gamma}(1+|y|^2)^{-k}, \quad \left|\partial_\eta^\delta (1+|\eta|^2)^{-k}\right| \leq C_{k,\delta}(1+|\eta|^2)^{-k}.$$

We then easily verify that, if $k > (d+|m|)/2$, then we can use the dominated convergence theorem and deduce that

$$a^*(x,\xi) = \iint e^{-iy\cdot\eta}(I-\Delta_y)^k(I-\Delta_\eta)^k\left[\frac{\bar{a}(x-y,\xi-\eta)}{(1+|y|^2)^k(1+|\eta|^2)^k}\right] \mathrm{d}y\,\mathrm{d}\eta. \tag{6.5.1}$$

The key point is that we have written $a^*(x,\xi)$ in the form of a convergent integral in the Lebesgue sense, and whose integrand depends in a smooth way on the parameters $x,\xi$. We then check that we can apply the derivation theorem under the integral sign for convergent integrals in the usual Lebesgue sense. It follows that $a^* \in C^\infty(\mathbb{R}^{2d})$ and

$$\partial_x^\alpha\partial_\xi^\beta(a^*)(x,\xi) = \iint e^{-iy\cdot\eta}(I-\Delta_y)^k(I-\Delta_\eta)^k\left[\frac{\partial_x^\alpha\partial_\xi^\beta\bar{a}(x-y,\xi-\eta)}{(1+|y|^2)^k(1+|\eta|^2)^k}\right] \mathrm{d}y\,\mathrm{d}\eta.$$

And then observe that the latter integral is equal to $(\partial_x^\alpha\partial_\xi^\beta a)^*$ by applying (6.5.1) with $a$ replaced by $\partial_x^\alpha\partial_\xi^\beta a$. This proves that $\partial_x^\alpha\partial_\xi^\beta(a^*) = (\partial_x^\alpha\partial_\xi^\beta a)^*$, which concludes the proof. □

**Proposition 6.5.7.** *Let $m \in \mathbb{R}$ and $a \in S^m(\mathbb{R}^d)$. Then, for all $u, v$ in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$(\mathrm{Op}(a)u, v) = (u, \mathrm{Op}(a^*)v)\,,$$

*where $(f,g) = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,\mathrm{d}x$.*

*Proof.* To prove this result we will make the additional assumption that $a$ is compactly supported in $x$.

The proof is based on continuity arguments. Recall that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space whose topology is induced by the following family of semi-norms, indexed by $p \in \mathbb{N}$,

$$\mathcal{N}_p(\varphi) = \sum_{|\alpha|\leq p, |\beta|\leq p} \sup_{x\in\mathbb{R}^d}\left|x^\alpha\partial_x^\beta\varphi(x)\right|.$$

The convergence of a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R}^d)$ to a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is therefore equivalent to

$$\forall p \in \mathbb{N}, \quad \lim_{k\to\infty}\mathcal{N}_p(\varphi_k - \varphi) = 0.$$

Similarly, the topology on the symbol class $S^m(\mathbb{R}^d)$ is induced by the following family of semi-norms, indexed by $p \in \mathbb{N}$,

$$\mathcal{M}_p^m(a) = \sum_{|\alpha|\leq p, |\beta|\leq p} \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \left\{ \langle\xi\rangle^{-(m-|\beta|)} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \right\}.$$

The convergence of a sequence of symbols is equivalent to the convergence in the sense of the semi-norms : for $m \in \mathbb{R}$ we say that a sequence $(a_k)$ of symbols belonging to $S^m(\mathbb{R}^d)$ converges to $a$ in $S^m(\mathbb{R}^d)$ if and only if

$$\forall p \in \mathbb{N}, \quad \lim_{k\to+\infty} \mathcal{M}_p^m(a_k - a) = 0.$$

**Lemma 6.5.8.** *Let $m \in \mathbb{R}$, $a \in S^m(\mathbb{R}^d)$ and $(a_k)$ be a sequence of symbols belonging to $S^m(\mathbb{R}^d)$ and converging to $a$ in $S^m(\mathbb{R}^d)$.*

*i) For all $u \in \mathcal{S}(\mathbb{R}^d)$, the sequence $(\mathrm{Op}(a_k)u)$ converges to $\mathrm{Op}(a)u$ in $\mathcal{S}(\mathbb{R}^d)$.*

*ii) The sequence $(a_k^*)$ converges to $a^*$ in $S^m(\mathbb{R}^d)$.*

*iii) Let $\ell \in \mathbb{R}$, $b \in S^\ell(\mathbb{R}^d)$ and $(b_k)$ be a sequence of symbols belonging to $S^\ell(\mathbb{R}^d)$ and converging to $b$ in $S^\ell(\mathbb{R}^d)$. Then $(a_k b_k)$ converges to $ab$ in $S^{m+\ell}(\mathbb{R}^d)$.*

*Proof.* *i*) We have already seen that if $a \in S^m(\mathbb{R}^d)$ and $u \in \mathcal{S}(\mathbb{R}^d)$ then $\mathrm{Op}(a)u \in \mathcal{S}(\mathbb{R}^d)$. The proof of this result leads directly to the result stated in point *i*). In the same way the previous proposition proves the continuity result stated at point *ii*). Finally, the result stated at point *iii*) is a direct consequence of Leibniz' rule. □

**Lemma 6.5.9.** *Let $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi(0) = 1$. Let us introduce $r_\varepsilon(\xi) = \chi(\varepsilon\xi) - 1$. Then $r_\varepsilon$ converges to $0$ in $S^1(\mathbb{R}^d)$.*

*Proof.* We will show that $\left|\partial_\xi^\alpha r_\varepsilon(\xi)\right| \leq C_\alpha \varepsilon \langle\xi\rangle^{1-|\alpha|}$ for all multi-index $\alpha \in \mathbb{N}^d$. For $\alpha = 0$ we write

$$r_\varepsilon(\xi) = \varepsilon \int_0^1 \chi'(t\varepsilon\xi)\cdot\xi \,\mathrm{d}t$$

and we deduce $\langle\xi\rangle^{-1} r_\varepsilon(\xi) = O(\varepsilon)$ because $\chi'$ is bounded. For $|\alpha| > 0$, we check that

$$\left|\langle\xi\rangle^{|\alpha|-1}\partial_\xi^\alpha r_\varepsilon(\xi)\right| = \varepsilon \left|\varepsilon^{|\alpha|-1}\langle\xi\rangle^{|\alpha|-1}(\partial_\xi^\alpha\chi)(\varepsilon\xi)\right|$$

then we use the majoration

$$\left|\varepsilon^{|\alpha|-1}\langle\xi\rangle^{|\alpha|-1}\partial_\xi^\alpha\chi(\varepsilon\xi)\right| \leq \left|\langle\varepsilon\xi\rangle^{|\alpha|-1}(\partial_\xi^\alpha\chi)(\varepsilon\xi)\right| \leq \sup_{\mathbb{R}^d}\left|\langle\zeta\rangle^{|\alpha|-1}\partial_\xi^\alpha\chi(\zeta)\right|$$

to obtain the desired result. □

We are now able to prove the theorem. Consider a symbol $a = a(x, \xi) \in S^m(\mathbb{R}^d)$. We fix $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfying $\chi(0) = 1$ and we introduce, for all $k \in \mathbb{N}^*$,

$$a_k(x, \xi) = \chi\left(\frac{\xi}{k}\right) a(x, \xi).$$

Since $a_k$ is compactly supported in $\xi$ and also in $x$ (by additional hypothesis on $a$) we can apply the proposition 6.3.3 to write that

$$(\mathrm{Op}(a_k)u, v) = \left(u, \mathrm{Op}(a_k^*)v\right). \tag{6.5.2}$$

To prove the theorem, we have to see that we can pass to the limit in this equality. To do this we start by combining Lemma 6.5.9 with the point *iii*) of Lemma 6.5.8 to obtain that $(a_k)$ converges to $a$ in $S^{m'}(\mathbb{R}^d)$ for all $m' > m$. The point *ii*) of Lemma 6.5.8 then implies that $(a_k^*)$ converges to $a^*$ in $S^{m'}(\mathbb{R}^d)$. We can then apply point *i*) of this lemma to obtain that $\mathrm{Op}(a_k)u$ converges to $\mathrm{Op}(a)u$ in $\mathcal{S}(\mathbb{R}^d)$ and similarly we get that $\mathrm{Op}(a_k^*)u$ converges to $\mathrm{Op}(a^*)u$ in $\mathcal{S}(\mathbb{R}^d)$. We can then pass to the limit in the identity (6.5.2), which concludes the proof. □

We can now define the action of $\mathrm{Op}(a)$ on a tempered distribution. To do this, let us recall the principle we saw in the chapter on the Fourier transform. Let $a \in S^m(\mathbb{R}^d)$ with $m \in \mathbb{R}$. Then $\mathrm{Op}(a)\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ a continuous linear application. We define then an operator $A$ of $\mathcal{S}'(\mathbb{R}^d)$ in $\mathcal{S}'(\mathbb{R}^d)$ by

$$\forall (u, v) \in \mathcal{S}(\mathbb{R}^d)^2, \qquad \langle Au, v\rangle_{\mathcal{S}'\times\mathcal{S}} = \langle u, \overline{\mathrm{Op}(a^*)\overline{v}}\rangle.$$

Then the proposition 2.3.6 shows that the operator $A$ thus defined extends the definition of $\mathrm{Op}(a)$. We denote it again as $\mathrm{Op}(a)$.

To conclude this paragraph, we will consider the composition of pseudo-differential operators.

Let $A_1 = \mathrm{Op}(a_1)$ and $A_2 = \mathrm{Op}(a_2)$ be two pseudo-differential operators. Suppose that $a_1$ and $a_2$ belong to $C_0^\infty(\mathbb{R}^{2d})$ and consider $u \in \mathcal{S}(\mathbb{R}^d)$. Then

$$A_1A_2u(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} a_1(x, \xi)\widehat{A_2u}(\xi)\, \mathrm{d}\xi$$

and

$$\widehat{A_2 u}(\xi) = \int e^{-iy\cdot\xi} A_2 u(y)\,\mathrm{d}y$$
$$= (2\pi)^{-d} \iint e^{-iy\cdot(\xi-\eta)} a_2(y,\eta)\widehat{u}(\eta)\,\mathrm{d}\eta\,\mathrm{d}y$$

so

$$A_1 A_2 u(x) = (2\pi)^{-2d} \iiint e^{iy\cdot\eta+i\xi\cdot(x-y)} a_1(x,\xi)a_2(y,\eta)\widehat{u}(\eta)\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\eta.$$

Thus we obtain that $A_1 A_2 u(x)$ is equal to

$$(2\pi)^{-d} \int e^{ix\cdot\eta} \left( (2\pi)^{-d} \iint e^{i(x-y)\cdot(\xi-\eta)} a_1(x,\xi)a_2(y,\eta)\,\mathrm{d}\xi\,\mathrm{d}y \right) \widehat{u}(\eta)\,\mathrm{d}\eta.$$

Formally $A_1 A_2 = \mathrm{Op}(b)$ where

$$b(x,\eta) = (2\pi)^{-d} \iint e^{i(x-y)\cdot(\xi-\eta)} a_1(x,\xi)a_2(y,\eta)\,\mathrm{d}\xi\,\mathrm{d}y.$$

The formula which defines $b$ is still a convolution in the variables $(y,\xi)$ (at $(x,\eta)$ fixed), this is why we can apply arguments parallel to those used to study the symbol of the adjoint.

**Proposition 6.5.10.** *If $a_1 \in S^{m_1}(\mathbb{R}^d)$ and $a_2 \in S^{m_2}(\mathbb{R}^d)$, then* $\mathrm{Op}(a_1) \circ \mathrm{Op}(a_2) = \mathrm{Op}(b)$*, where $b = a_1\#a_2 \in S^{m_1+m_2}(\mathbb{R}^d)$ is given by the oscillatory integral*

$$b(x,\eta) = (2\pi)^{-d} \iint e^{i(x-y)\cdot(\xi-\eta)} a_1(x,\xi)a_2(y,\eta)\,\mathrm{d}\xi\,\mathrm{d}y.$$

We will not discuss the proof, analogous to the one concerning the adjoint.

We have seen in this section that, for all $m \in \mathbb{R}$ and all $a \in S^m(\mathbb{R}^d)$, we can define $\mathrm{Op}(a)$ on the space of tempered distributions. In particular we can define $\mathrm{Op}(a)u$ for all $u$ in a Sobolev space $H^s(\mathbb{R}^d)$ with any $s \in \mathbb{R}$. Thanks to the previous proposition on composition, we will now see that $\mathrm{Op}(a)$ is an operator of order $m$ as was claimed in point $i$) of the theorem 6.5.3.

**Proposition 6.5.11.** *Let $m \in \mathbb{R}$ and $a \in S^m(\mathbb{R}^d)$. The operator* $\mathrm{Op}(a)$ *is bounded from $H^s(\mathbb{R}^d)$ to $H^{s-m}(\mathbb{R}^d)$ for all $d \geq 1$ and all $s \in \mathbb{R}$.*

*Proof.* For $\mu \in \mathbb{R}$, we denote by $(I-\Delta)^{\mu/2}$ the Fourier multiplier of symbol $\langle\xi\rangle^{\mu} = (1+|\xi|^2)^{\mu/2}$. Then $(1-\Delta)^{\mu/2}$ is an isomorphism of $H^{\mu}(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$. It is therefore sufficient to show that the operator

$$A_{s,m} := (I-\Delta)^{(s-m)/2} \circ \operatorname{Op}(a) \circ (I-\Delta)^{-s/2}$$

is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Note that if $b = b(\xi)$ then

$$\operatorname{Op}(a) \circ \operatorname{Op}(b) = \operatorname{Op}(ab)$$

so the symbol of $\operatorname{Op}(a) \circ (1-\Delta)^{-s/2}$ is $a(x,\xi)\langle\xi\rangle^{-s}$. As $a \in S^m(\mathbb{R}^d)$ and $\langle\xi\rangle^{-s} \in S^{-s}(\mathbb{R}^d)$, the product of these two symbols belongs to $S^{m-s}(\mathbb{R}^d)$. On the other hand, to manipulate $(I-\Delta)^{(s-m)/2} \circ \operatorname{Op}\left(a\langle\xi\rangle^{-s/2}\right)$, we use the composition theorem which implies that $A_{s,m}$ is a pseudo-differential operator whose symbol belongs to $S^0(\mathbb{R}^d)$. It is therefore a bounded operator on $L^2(\mathbb{R}^d)$ according to the continuity theorem proved in the previous chapter. □

To conclude, it remains to prove the part concerning the symbolic calculation of pseudo-differential operators. To do this, let us introduce the notion of asymptotic sum of symbols. This notion allows us to give a rigorous meaning to claims such as: $a$ is the sum of a term (usually its so-called principal symbol) and a "better" remainder.

**Definition 6.5.12.** *Let $a_j \in S^{m_j}(\mathbb{R}^d)$ be a sequence indexed by $j \in \mathbb{N}$ of symbols, such that $m_j$ decreases towards $-\infty$. We will say that $a \in S^{m_0}(\mathbb{R}^d)$ is the asymptotic sum of $a_j$ if*

$$\forall k \in \mathbb{N}, \quad a - \sum_{j=0}^{k} a_j \in S^{m_{k+1}}(\mathbb{R}^d).$$

*We then denote $a \sim \sum a_j$.*

**Proposition 6.5.13.** *i) Let $m \in \mathbb{R}$ and $a \in S^m(\mathbb{R}^d)$. Then*

$$a^* \sim \sum_j A_j \quad \text{with } A_j = \sum_{||\alpha|=j} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a}.$$

*ii) Let $m_1, m_2 \in \mathbb{R}$. If $a_1 \in S^{m_1}(\mathbb{R}^d)$ and $a_2 \in S^{m_2}(\mathbb{R}^d)$, then*

$$a_1 \# a_2 \sim \sum_j A_j \quad \text{with } A_j = \sum_{|\alpha|=j} \frac{1}{i^{|\alpha|}\alpha!} \left(\partial_\xi^\alpha a_1\right)\left(\partial_x^\alpha a_2\right).$$

**Remark 6.5.14.** *In practice, by abuse of notation, we write simply*

$$a^* \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a}$$

*and*

$$a_1 \# a_2 \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \big(\partial_\xi^\alpha a_1\big)\big(\partial_x^\alpha a_2\big).$$

*Proof.* We will limit ourselves to proving the point $i$). We use Taylor's formula (whose statement is recalled at the end of the proof of this proposition)

$$\overline{a}(x-y,\xi-\eta) = \sum_{|\alpha+\beta|<2k} \frac{(-y)^\alpha}{\alpha!} \frac{(-\eta)^\beta}{\beta!} \partial_x^\alpha \partial_\xi^\beta \overline{a}(x,\xi) + r_k(x,\xi,y,\eta)$$

with

$$r_k(x,\xi,y,\eta) = \sum_{|\alpha+\beta|=2k} 2k \frac{(-y)^\alpha}{\alpha!} \frac{(-\eta)^\beta}{\beta!} r_{\alpha\beta}(x,\xi,y,\eta)$$

and

$$r_{\alpha\beta}(x,\xi,y,\eta) = \int_0^1 (1-t)^{2k-1} \partial_x^\alpha \partial_\xi^\beta \overline{a}(x-ty,\xi-t\eta)\,\mathrm{d}t.$$

Earlier we proposed to prove as an exercise that, for all $\alpha$ and $\beta$ in $xN^n$,

$$(2\pi)^{-d} \int e^{-iy\cdot x} \frac{y^\alpha}{\alpha!} \frac{x^\beta}{\beta!}\,\mathrm{d}y\,\mathrm{d}x = \begin{cases} 0 \text{ if } \alpha \neq \beta, \\ (-i)^{|\alpha|}/\alpha! \text{ if } \alpha = \beta. \end{cases}$$

This result implies that the sum over $|\alpha+\beta| < 2k$ corresponds to the asymptotic expansion sought for $a^*$. It then only remains to prove that

$$\int e^{-iy\cdot\eta} r_k(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \in S^{m-k}.$$

We will integrate by parts to deal with $r_{\alpha\beta}$. By noting simply $*$ different numerical constants, we obtain [2]

$$\begin{aligned}
&\int e^{-iy\cdot\eta} y^\alpha \eta^\beta r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\
&\qquad = * \int \partial_\eta^\alpha \big(e^{-iy\cdot\eta}\big)\eta^\beta r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\
&\qquad = * \int e^{-iy\cdot\eta} \sum_\gamma (\partial_\eta^\gamma \eta^\beta)\partial_\eta^{\alpha-\gamma} r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\
&\qquad = \sum_\gamma * \int e^{-iy\cdot\eta} \eta^{\beta-\gamma}\partial_\eta^{\alpha-\gamma} r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta \\
&\qquad = \sum_\gamma * \int e^{-iy\cdot\eta} \partial_y^{\beta-\gamma}\partial_\eta^{\alpha-\gamma} r_{\alpha\beta}(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta.
\end{aligned}$$

By definition of $r_{\alpha\beta}$, we have

$$\partial_y^{\beta-\gamma}\partial_\eta^{\alpha-\gamma} r_{\alpha\beta}(x,\xi,y,\eta) = * \int_0^1 (1-t)^{2k-1} t^{2k-2|\gamma|} \partial_x^{\alpha+\beta-\gamma}\partial_\xi^{\alpha+\beta-\gamma}\overline{a}(x-ty,\xi-t\eta)\,\mathrm{d}t.$$

As $\gamma \le \alpha$ and $\gamma \le \beta$ we have $|\gamma| \le k$ and $|a+\beta-\gamma| \ge k$, so

$$\partial_x^{\alpha+\beta-\gamma}\partial_\xi^{\alpha+\beta-\gamma}\overline{a} \in S^{m-k}.$$

Then

$$\int e^{-iy\cdot\eta} r_k(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta = \int e^{-iy\cdot\eta} s_k(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta$$

where $s_k$ is an amplitude $s_k \in A^{|m-k|}$ with

$$\|s_k\|_{|m-k|,|m-k|+2d+1} \le C_k \langle\xi\rangle^{m-k}.$$

We deduce that

$$\langle\xi\rangle^{k-m} \int e^{-iy\cdot\eta} r_k(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta$$

is bounded and then that $\int e^{-iy\cdot\eta} r_k(x,\xi,y,\eta)\,\mathrm{d}y\,\mathrm{d}\eta$ belongs to $S^{m-k}$. □

[2]If $a \in A^m$ and $b \in A^\ell$, for $\alpha \in \mathbb{N}^N$ we have,

$$\int e^{i\phi(x)} a(x)\partial^\alpha b(x)\,\mathrm{d}x = \int b(x)(-\partial)^\alpha\big(e^{i\phi(x)}a(x)\big)\,\mathrm{d}x.$$

In order to be complete, we prove the version of the Taylor formula that was used above.

**Theorem 6.5.15.** *Let $u$ be a function of class $C^k$ on $\mathbb{R}^d$. Then for all $x$ and $y$ in $\mathbb{R}^d$ we have*

$$u(x+y) = \sum_{|\alpha|<k} \frac{1}{\alpha!} y^\alpha \partial_x^\alpha \frac{\mathrm{d}}{\mathrm{d}t} u(x) + \sum_{|\alpha|=k} \frac{k}{\alpha!} y^\alpha \int_0^1 (1-t)^{k-1} (\partial_x^\alpha u)(x+ty)\,\mathrm{d}t.$$

*Proof.* We check that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\sum_{|\alpha|=k-1} \frac{1}{\alpha!} y^\alpha \partial_x^\alpha u(x+ty)\Bigg) &= \sum_{|\beta|=k} \Bigg(\sum_{\alpha\leq\beta, |\alpha|=k-1} \frac{1}{\alpha!}\Bigg) y^\beta (\partial_x^\beta u)(x+ty) \\ &= \sum_{|\beta|=k} \Bigg(\sum_{1\leq j\leq d} \frac{\beta_j}{\beta!}\Bigg) y^\beta (\partial_x^\beta u)(x+ty) \\ &= \sum_{|\beta|=k} \frac{k}{\beta!} y^\beta (\partial_x^\beta u)(x+ty).\end{aligned}$$

So the function

$$v(t) = \sum_{|\alpha|<k} \frac{1}{\alpha!} (1-t)^{|\alpha|} y^\alpha (\partial_x^\alpha u)(x+ty)$$

satisfies $v(1) = u(x+y)$ and

$$v(0) = \sum_{|\alpha|<k} \frac{1}{\alpha!} y^\alpha \partial^\alpha u(x), \quad \partial_t v_k = \sum_{|\alpha|=k} \frac{k}{\alpha!} y^\alpha (1-t)^{k-1} (\partial_x^\alpha u)(x+ty)$$

so that the Taylor formula is a consequence of the fundamental theorem of integral calculus. □

## 6.6 Applications of the symbolic calculus

### 6.6.1 Action on Sobolev spaces

We will give another proof of the following result that we have already seen in the previous chapter.

**Theorem 6.6.1.** *If $a \in S^0(\mathbb{R}^d)$ then* $\mathrm{Op}(a)$ *is bounded on* $L^2(\mathbb{R}^d)$.

*Proof.* Let us set $A = \mathrm{Op}(a)$. The idea is the following, as

$$\|Au\|_{L^2}^2 = (Au, Au) = (A^*Au, u),$$

to show the inequality $\|Au\|_{L^2}^2 \le M\,\|u\|_{L^2}^2$ for some $M > 0$, it is enough to show that suffice to show that $(Bu, u) \ge 0$ where $B = M\,\mathrm{Id} - A^*A$. Note that $B$ is a self-adjoint operator. To prove that $B$ is positive for $M$ large enough, we will show that we can write, approximately, $B$ in the form of a square. Precisely, we will show that we can write $B$ in the form

$$B = C^*C + R,$$

where $C = \mathrm{Op}(c)$ with $c \in S^0(\mathbb{R}^d)$ and $R = \mathrm{Op}(r)$, $r \in S^{-1}$.

Let us choose $M = 2\sup|a(x,\xi)|^2$ and then let us take :

$$c(x,\xi) = (M - |a(x,\xi)|^2)^{1/2}.$$

We check that $c$ belongs to $S^0(\mathbb{R}^d)$. The theorem of composition of operators implies that $C^*C = M\,\mathrm{Id} - A^*A + R$ where $R = \mathrm{Op}(r)$ with $r \in S^{-1}$. Thus

$$\|Au\|_{L^2}^2 \le M\,\|u\|_{L^2}^2 + (Ru, u).$$

Now we have to increase the error $(Ru, u)$. Since $\|Ru\|_{L^2}^2 = (Ru, Ru) = (R^*Ru, u)$, $R$ will be continuous on $L^2$ if $R^*R$ is, with

$$\|R\|_{L^2\to L^2} \le \|R^*R\|_{L^2\to L^2}^{1/2}.$$

Now $r^*\#r \in S^{-2}$ : by iterating the argument we see that it is sufficient to show that, for $k$ large enough, any operator of symbol $r \in S^{-k}$ is continuous on $L^2$. We will show this result by using Schur's lemma and the following remark: if $r \in S^{-n-1}$ then the kernel $K(x,y)$ of $\mathrm{Op}(r)$ is a bounded continuous function, because

$$|K(x,y)| \le (2\pi)^{-n}\int |r(x,\xi)|\,\mathrm{d}\xi \le \frac{C_0}{(2\pi)^n}\int \frac{\mathrm{d}\xi}{(1+|\xi|)^{n+1}} \le C.$$

Moreover, $(x_j - y_j)K(x,y)$ is the kernel of $\mathrm{Op}(i\partial_{\xi_j} r) \in \mathrm{Op}\,S^{-n-2} \subset \mathrm{Op}\,S^{-n-1}$ so by iterating $(n+1)$ times, we finally find $(1 + |x-y|^{n+1})K(x,y) \le C$. The decay of $K$ to infinity implies in particular:

$$\int |K(x,y)|\,\mathrm{d}x \le A, \qquad \int |K(x,y)|\,\mathrm{d}y \le A.$$

We conclude the proof with Schur's lemma (see Lemma 6.4.6). □

### 6.6.2 Subelliptic problems

**Proposition 6.6.2.** *Let $\mu \in \mathbb{R}$ and $e \in S^\mu$ be a symbol such that*

$$|e(x,\xi)| \geq c(1+|\xi|)^\mu \qquad \forall (x,\xi) \in \mathbb{R}^{2d}.$$

*Then $e^{-1}$ belongs to $S^{-\mu}$. Moreover, for all $s \in \mathbb{R}$ there exist constants $K_0, K_1 > 0$ such that, for all $u \in H^s$,*

$$\|u\|_{H^s} \leq K_0 \|\mathrm{Op}(e)u\|_{H^{s-\mu}} + K_1 \|u\|_{H^{s-1}}.$$

*Proof.* The fact that $e^{-1} \in S^{-\mu}$ can be proven directly. Then $e \# e^{-1} = 1 + b$ with $b \in S^{-1}$ and we deduce $\mathrm{Op}(e^{-1})\,\mathrm{Op}(e)u = \mathrm{Op}(1)u + \mathrm{Op}(b)u = u + \mathrm{Op}(b)u$ so

$$\|u\|_{H^s} \leq \left\|\mathrm{Op}(e^{-1})\right\|_{\mathcal{L}(H^{s-\mu};H^s)} \|\mathrm{Op}(e)u\|_{H^{s-\mu}} + \|\mathrm{Op}(b)\|_{\mathcal{L}(H^{s-1};H^s)} \|u\|_{H^{s-1}},$$

which gives the desired result. □

**Proposition 6.6.3** (Gårding's inequality). *Let $m \in \mathbb{R}$ and $a \in S^m(\mathbb{R}^d)$ be a symbol such that*

$$\exists c > 0 / \ \forall (x,\xi) \in \mathbb{R}^{2d}, \quad \mathrm{Re}\, a(x,\xi) \geq c(1+|\xi|)^m. \tag{6.6.1}$$

*Then there exist constants $C_0, C_1 > 0$ such that, for all $u \in \mathcal{S}(\mathbb{R}^d)$,*

$$\mathrm{Re}(\mathrm{Op}(a)u, u) \geq C_0 \|u\|^2_{H^{m/2}} - C_1 \|u\|^2_{H^{(m-1)/2}}.$$

**Remark 6.6.4.** *One can improve this inequality in two directions.*

*i) Firstly, as it is explained in Exercise 11.0.9, one can prove that, for all $N$ there exists a constant $C_N$ such that,*

$$\mathrm{Re}(Au, u) \geq \frac{c}{2} \|u\|^2_{H^{m/2}} - C_N \|u\|^2_{H^{-N}},$$

*for all $u \in \mathcal{S}(\mathbb{R}^d)$.*

*ii) The previous proposition remains true if a is a matrix-valued symbol (in this case* $\mathrm{Re}\, a = a + a^*$*). The proof reduces to showing that, for all symbol $a \in S^0$ such that $a(x,\xi)$ is Hermitian definite positive uniformly for $(x,\xi) \in \mathbb{R}^d$, there exists $b \in S^0$ such that $b(x,\xi)^* b(x,\xi) = a(x,\xi)$.*

*Proof.* To prove this inequality, which is a relation between the positivity of a symbol and that of the associated operator, we will use the symbolic calculus to write $A = \mathrm{Op}(a)$ as a square (i.e. $P^*P$) plus an operator of order $m-1$.

Let us set

$$B := \operatorname{Re} A = \frac{1}{2}(A + A^*),$$

so that $\operatorname{Re}(Au, u) = (Bu, u)$. As $A^* \in \mathrm{Op}(\bar{a}) + \mathrm{Op}\, S^{m-1}$, we have $B = \mathrm{Op}(b)$ with $b = \frac{1}{2}(a + a^*) = \operatorname{Re} a + d$ where $d \in S^{m-1}$.

We then denote $e$ the positive square root of $\operatorname{Re} a$, which is a symbol belonging to $S^{m/2}$ by assumption (6.6.1). Moreover, the composition of the symbols is such that

$$f := e^* \# e - \operatorname{Re} a \in S^{m-1}.$$

We deduce that $b = e^* \# e + g$ where $g = d - f \in S^{m-1}$. We can then write

$$\begin{aligned}
\operatorname{Re}(Au, u) &= (\mathrm{Op}(b)u, u) \\
&= (\mathrm{Op}(e)^* \mathrm{Op}(e)u, u) + (\mathrm{Op}(g)u, u) \\
&= \|\mathrm{Op}(e)u\|_{L^2}^2 + (\mathrm{Op}(g)u, u) \\
&\geq \|\mathrm{Op}(e)u\|_{L^2}^2 - \|\mathrm{Op}(g)u\|_{H^{\frac{1-m}{2}}} \|u\|_{H^{\frac{m-1}{2}}}.
\end{aligned}$$

The previous proposition implies that

$$\|u\|_{H^{\frac{m}{2}}} \leq K_0 \|\mathrm{Op}(e)u\|_{L^2} + K_1 \|u\|_{H^{\frac{m}{2}-1}},$$

and the theorem about the continuity of ΨDOs on Sobolev spaces implies that

$$\|\mathrm{Op}(g)u\|_{H^{\frac{1-m}{2}}} \leq K_2 \|u\|_{H^{\frac{m-1}{2}}}.$$

Combining the previous inequalities we get the desired result. □

Recall the Poisson bracket notation:

$$\{a, b\} = \sum_{1 \leq j \leq n} \frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} - \frac{\partial b}{\partial \xi_j} \frac{\partial a}{\partial x_j}.$$

**Theorem 6.6.5.** *Let $P = \mathrm{Op}(p)$ be a pseudo-differential operator such that $p = p_1 + p_0$ with $p_1 \in S^1(\mathbb{R}^d)$ and $p_0 \in S^0(\mathbb{R}^d)$. Suppose that there exists a constant $c$ such that,*

$$i\{p_1, \overline{p_1}\} > c(1 + |\xi|).$$

*Then there exists a constant $C$ such that*

$$\|u\|_{H^{1/2}} \leq C \|Pu\|_{L^2} + C \|u\|_{L^2}.$$

**Remark 6.6.6.** *The previous condition on $i\{p_1, \overline{p_1}\}$ is called the Hörmander hypoellipticity condition. We check that for all $p \in C^1(\mathbb{R}^d)$ with complex values, the Poisson bracket $i\{p, \overline{p}\}$ is a real-valued function.*

*Proof.* Let us introduce the operator $Q = P^*P - PP^*$. Then

$$\begin{aligned}
\|Pu\|_{L^2}^2 &= (P^*Pu, u) \\
&= (PP^*u, u) + \big((P^*P - PP^*)u, u\big) \\
&= \|P^*u\|_{L^2}^2 + (Qu, u) \\
&\geq (Qu, u).
\end{aligned}$$

Thus, any estimate of positivity of $Q$ will give an estimate on $\|Pu\|_{L^2}^2$.

Recall first that if $A = \mathrm{Op}(a) \in \mathrm{Op}\, S^{m_1}$ and $B = \mathrm{Op}(b) \in \mathrm{Op}\, S^{m_2}$, are two pseudo-differential operators, then $A^* \in \mathrm{Op}\, S^{m_1}$ and $[A, B] \in \mathrm{Op}\, S^{m_1+m_2-1}$. Furthermore

$$A^* \in \mathrm{Op}(\overline{a}) + \mathrm{Op}\, S^{m_1-1}, \qquad [A, B] \in \mathrm{Op}\left(\frac{1}{i}\{a, b\}\right) + \mathrm{Op}\, S^{m_1+m_2-2}.$$

Therefore $Q = \mathrm{Op}(q)$ with $q = q_1 + q_0$ where $q_1 \in S^1(\mathbb{R}^d)$, $q_0 \in S^0(\mathbb{R}^d)$ and

$$q_1 = \frac{1}{i}\{\overline{p_1}, p_1\}.$$

By hypothesis we deduce that $\mathrm{Re}\, q_1 > c\,|\xi|$ if $|\xi| \geq R$. The Gårding's inequality implies that

$$\mathrm{Re}(Qu, u) \geq \frac{1}{C}\, \|u\|_{H^{1/2}}^2 - C\, \|u\|_{L^2}^2 .$$

This concludes the proof. $\square$

# Part III

# Propagation of singularities

# Chapter 7

# The Cauchy-Lipschitz theorem

In this chapter, we recall several fundamental results : the Banach fixed point theorem, the local inversion theorem and the Cauchy-Lipschitz theorem.

## 7.1 Reminders of differential calculus

Let $E$ and $F$ be two real normed vector spaces and let $U$ be an open set of $E$. Consider an application $f\colon U \longrightarrow F$ and a point $a \in U$. We say that $f$ is differentiable at point $a$ in the Fréchet sense if there exists a continuous linear application $L\colon E \longrightarrow F$ and an application $\varepsilon\colon E \longrightarrow F$ such that

$$f(x) = f(a) + L(x-a) + \|x-a\|_E\, \varepsilon(x-a) \quad \text{with} \quad \lim_{\|h\|_E \to 0} \varepsilon(h) = 0.$$

The existence of $L$ depends on the choice of the norm $\|\cdot\|_E$. Such a linear application $L$ is necessarily unique and is called the differential of $f$ in $a$, denoted $\mathrm{d}(a)$ (or $\mathrm{d}_a f$ or $f'(a)$). By abuse, we will simply say differentiable instead of differentiable in the Fréchet sense. Recall the following result.

**Theorem 7.1.1.** *Let $f\colon U \subset E \longrightarrow F$ be a differentiable application on an open convex open $U$. Suppose that there exists a constant $C$ such that*

$$\forall a \in U, \quad \|\mathrm{d}f(a)\|_{\mathcal{L}(E,F)} \leq C.$$

*Then, for all $(x, y)$ in $U \times U$, we have*

$$\|f(x) - f(y)\|_F \leq C\, \|x - y\|_E\,.$$

Suppose that $f$ is differentiable at any point of $U$. Then we denote $\mathrm{d}f$ the application $a \mapsto \mathrm{d}f(a)$, called the differential of $f$. If the application $\mathrm{d}f$ is continuous from $U$ into $\mathcal{L}(E, F)$, then we say that $f$ is of class $C^1$ on $U$ and we denote $f \in C^1(U)$. If the application $\mathrm{d}f$ is differentiable at any point $a$ of $U$, then we say that $f$ is twice differentiable on $U$ and we denote $\mathrm{d}^2 f$ the resulting application. If this application is continuous from $U$ into $\mathcal{L}(E, \mathcal{L}(E, F))$, then we then we say that $f$ belongs to the space $C^2(U)$ of functions of class $C^2$ on $U$. By induction, we define more generally the notion of a function of class $C^k$ for all integer $k \in \mathbb{N}$. We say that $f$ belongs to the space $C^\infty(U)$ of functions of class $C^\infty$ on $U$ if $f$ is of class $C^k$ for all $k$. Finally, given a closed set $K \subset U$, we will say that $f$ is of class $C^k$ on $K$ if there exists an open set $V$ such that $K \subset V \subset U$ and such that $f$ belongs to $C^k(V)$.

## 7.2 Banach fixed point theorem

Let us start with the fundamental example of solving an equation $\Phi(u) = 0$ in the case where $\Phi - \mathrm{Id}$ is a contracting application, in the sense of the following definition.

**Definition 7.2.1.** *Let $(E, d)$ be a metric space and a positive real number $k$. We say that an application $f\colon E \to E$ is $k$-Lipschitzian if, for all pair $(x, y)$ in $E \times E$,*

$$d(f(x), f(y)) \leq k d(x, y).$$

*We say that $f$ is contracting if it is $k$-Lipschitzian for some $k \in [0, 1)$.*

**Theorem 7.2.2.** *Let $E$ be a complete metric space and $f\colon E \to E$ a contracting application. There exists a unique fixed point $x^*$ of $f$ in $E$, such that $f(x^*) = x^*$. Moreover any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $E$ satisfying $x_{n+1} = f(x_n)$ converges to $x^*$.*

*Proof.* Let $x_0 \in E$ and let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by $x_{n+1} = f(x_n)$. Then $d(x_{m+1}, x_m) \leq k d(x_m, x_{m-1})$ so

$$d(x_{m+1}, x_m) \leq k^m d(x_1, x_0).$$

Since $x_{n+p} - x_n = x_{n+1} - x_n + \cdots + x_{n+p} - x_{n+p-1}$ we deduce

$$d(x_{n+p}, x_n) \leq (k^n + \cdots + k^{n+p-1}) d(x_1, x_0) \leq k^n \frac{1}{1-k} d(x_1, x_0),$$

so the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $E$ is a complete space, this sequence converges to an element denoted $x^*$. To show that $x^*$ is a fixed point of $f$ we will use the previous inequality $d(x_{m+1}, x_m) \leq k^m d(x_1, x_0)$ which leads to $d(f(x_m), x_m) \leq k^m d(x_1, x_0)$. Since $k < 1$ and $f$ is continuous, we can pass to the limit in this inequality to deduce that $d(f(x^*), x^*) = 0$, which shows that $x^*$ is a fixed point of $f$. □

## 7.3 Inverse function theorem

In this section we will see the proof of the inverse function theorem in Banach spaces.

**Definition 7.3.1.** *Consider two normed spaces $B_1, B_2$ and open sets $U \subset B_1$ and $V \subset B_2$. We say that an application $f\colon U \to V$ is a $C^k$-diffeomorphism, with $k \in \mathbb{N} \cup \{\infty\}$, if:*

- *$f$ is of class $C^k$,*
- *$f$ is a bijection from $U$ to $V$,*
- *the inverse $f^{-1}$ is of class $C^k$.*

**Theorem 7.3.2.** *Let $f\colon U \to B_2$ be an application $C^1$ from an open set $U$ of a Banach space $B_1$ to a Banach space $B_2$. If $\mathrm{d}f(x_0)$ is an isomorphism from $B_1$ to $B_2$ then $f$ is a $C^1$ diffeomorphism of a neighborhood of $x_0$ on a neighborhood of $f(x_0)$.*

**Remark 7.3.3.** *Moreover, we can prove that, if $f$ is injective and if for all $x$ of $U$ the differential $\mathrm{d}f(x)$ is a bi-continuous isomorphism, then $f(U)$ is an open set and the inverse bijection, from $f(U)$ to $U$, is of class $C^1$.*

*Proof.* We begin by observing that it is sufficient to consider the case $B_1 = B_2$, $x_0 = 0$, $f(x_0) = x_0$ and $\mathrm{d}f(x_0) = \mathrm{Id}$. To see this, we replace $U$ by the set $\tilde{U}$ of elements $x$ such that $x_0 + x$ belongs to $U$, and $f$ by the application

$$\tilde{f}(x) = (\mathrm{d}f(x_0))^{-1}(f(x_0 + x) - f(x_0)).$$

Now let us introduce $\varphi(x) = x - f(x)$. The differential $\mathrm{d}\varphi(0)$ of $\varphi$ at 0 vanishes so there exists $r > 0$ such that $\overline{B_r}$ is included in $U$ and such that the norm of the differential of $\varphi$ is always less than $1/2$ on this ball. We introduce $W = B_{r/2}$ and $V = B_r \cap f^{-1}(W)$. Let us show that $f$ is bijective from $V$ to $W$.

*Surjectivity*. Let $y \in W$. We look for $x \in V$ such that $y = f(x)$. To do this we write the equation $y = f(x)$ in the form

$$x = h(x) \quad \text{with} \quad h(x) = y + \varphi(x) = x + y - f(x),$$

and we are looking for a fixed point of $h$.

Theorem 7.1.1 implies that $\varphi$ is $1/2$-lipschitzian on $\overline{B_r}$. Thus $\|\varphi(x)\| \leq r/2$ for all $x \in \overline{B_r}$. For all $y \in W = B_{r/2}$ we have $\|y\| < r/2$ so $h$ sends $\overline{B_r}$ in $B_r$ by the triangular inequality. Moreover $h$, like $\varphi$, is $1/2$-lipschitzian. Therefore $h$ has a fixed point $x$ in $\overline{B_r}$ (according to the fixed point theorem). We check that $x$ belongs to $B_r$ because $x = h(x)$. Similarly we have $x \in f^{-1}(W)$ because $f(x) = y$. This proves that for all $y \in W$, we can find $x \in V$ such that $f(x) = y$.

*Injectivity*. For all $(x_1, x_2) \in V \times V$,

$$\begin{aligned} \|x_1 - x_2\| &= \|\varphi(x_1) + f(x_1) - \varphi(x_2) - f(x_2)\| \\ &\leq \frac{1}{2} \|x_1 - x_2\| + \|f(x_1) - f(x_2)\| \end{aligned}$$

so

$$\|x_1 - x_2\| \leq 2 \|f(x_1) - f(x_2)\|, \tag{7.3.1}$$

which implies that $f\colon V \to W$ is injective.

*Regularity*. First we observe that $\mathrm{d}f(x) = \mathrm{Id} - \mathrm{d}\varphi(x)$ and $\mathrm{d}\varphi(x)$ has norm less than $1/2 < 1$ for all $x$ in $V$. Therefore $\mathrm{d}f(x)$ has bounded inverse according to a classical result proved below, and its inverse is given by $\sum_{n\in\mathbb{N}}(\mathrm{d}\varphi(x))^n$, of norm smaller than 2. Let us then show that $f^{-1}$ is differentiable and that its differential is the inverse of $\mathrm{d}f(f^{-1}(x))$. To do this, let $y \in W$, let $x = f^{-1}(y)$ and set $L = (\mathrm{d}f(f^{-1}(y)))^{-1}$. We want to show that

$$\left\|f^{-1}(y+z) - f^{-1}(y) - Lz\right\| = o(\|z\|). \tag{7.3.2}$$

For this let us introduce $h$ such that $x + h = f^{-1}(y+z)$. Then

$$\begin{aligned} \left\|f^{-1}(y+z) - f^{-1}(y) - Lz\right\| &= \|x + h - x - L(f(x+h) - f(x))\| \\ &= \left\|L\big(f(x+h) - f(x) - L^{-1}h\big)\right\| \\ &\leq 2\left\|f(x+h) - f(x) - L^{-1}h\right\|, \end{aligned}$$

because $\|L\|_{\mathcal{L}(E)}$ is bounded by 2. Now we use $L^{-1} = \mathrm{d}f(x)$ to conclude

$$\left\|f^{-1}(y+z) - f^{-1}(y) - Lz\right\| \leq 4\,\|h\|\,.$$

We can then use the inequality (7.3.1) to conclude that $\|h\| \leq 2\,\|z\|$, which finally proves (7.3.2).

Since $f^{-1}$ is differentiable it is continuous and $x \mapsto (\mathrm{d}f(f^{-1}(x)))^{-1}$ is continuous by composition of continuous functions. $\square$

**Lemma 7.3.4** (Neumann series)**.** *Let $B$ be a Banach space and $T \in \mathcal{L}(B)$ satisfy $\|T\| < 1$. Then $\mathrm{Id}-T$ is invertible and its inverse is given by*

$$(\mathrm{Id}-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

*Proof.* The proof relies on the fact that $\mathcal{L}(B)$ is a Banach space (because $B$ is one) and on the fact that the operator norm on $\mathcal{L}(B)$ satisfies the following inequality: $\|T_1T_2\|_{\mathcal{L}(B)} \leq \|T_1\|_{\mathcal{L}(B)}\,\|T_2\|_{\mathcal{L}(B)}$.

Let us consider the partial sum $S_n = T^0 + T + \cdots + T^n$. Then $(\mathrm{Id}-T)S_n = \mathrm{Id}-T^{n+1}$ converges to Id because $\left\|T^{n+1}\right\|_{\mathcal{L}(B)} \leq \|T\|^{n+1}_{\mathcal{L}(B)}$ and $\|T\|_{\mathcal{L}(B)} < 1$ by hypothesis. Moreover the series $S_n$ converges normally so it converges because $\mathcal{L}(B)$ is a Banach space. This classical result can be proved directly in the following way: since

$$\|S_{n+m} - S_n\|_{\mathcal{L}(B)} \leq \sum_{j=n+1}^{n+m} \|T\|^j_{\mathcal{L}(B)} \leq \sum_{j=n+1}^{+\infty} \|T\|^j_{\mathcal{L}(B)}\,,$$

and since the right-hand side of the previous inequality converges to 0, the sequence $(S_n)$ is a Cauchy sequence and therefore it has a limit in the Banach space $\mathcal{L}(B)$. $\square$

## 7.4 Cauchy-Lipschitz theorem

**Theorem 7.4.1.** *Let $n \geq 1$ and consider an application $f \in C^1(\mathbb{R}\times\mathbb{R}^d;\mathbb{R}^d)$. Then, for all $y_0$ in $\mathbb{R}^d$, there exists $T > 0$ such that the system of differential equations $y' = f(t,y)$ has a unique solution $y \in C^1([-T,T];\mathbb{R}^d)$ satisfying $y(0) = y_0$.*

*Proof.* Let us fix a parameter $T > 0$. Note that $y$ satisfies

$$y' = f(t,y(t)), \quad y|_{t=0} = y_0, \tag{7.4.1}$$

if and only if the function $z(\tau) = y(T\tau) - y_0$ satisfies

$$z'(\tau) = Tf(T\tau, z(\tau) + y_0), \quad z|_{\tau=0} = 0. \tag{7.4.2}$$

So if we can find $T > 0$ such that there exists a solution $z$ defined on a time interval $[-1, 1]$ we will have a solution of the initial problem, defined on the time interval $[-T, T]$.

Let us introduce the spaces

$$B_0 = C^0([-1,1];\mathbb{R}^d) \quad \text{and} \quad B_1 = \left\{ z \in C^1([-1,1];\mathbb{R}^d) \,;\, z(0) = 0 \right\}.$$

These are Banach spaces for the norms

$$\|u\|_{B_0} := \sup_{[-1,1]} |u(\tau)| \,, \quad \|u\|_{B_1} := \sup_{[-1,1]} |u(\tau)| + \sup_{[-1,1]} |u'(\tau)| \,,$$

where $|\cdot|$ denotes any norm on $\mathbb{R}^d$. Let us also introduce the functional $\Phi \colon \mathbb{R} \times B_1 \to \mathbb{R} \times B_0$ defined by

$$\Phi(T, z) = (T, v) \quad \text{where} \quad v(\tau) = z'(\tau) - Tf\big(T\tau, z(\tau) + y_0\big).$$

Then $\phi$ is a $C^1$ application and its differential at the origin $(0, 0)$ is given by the application $(\mathrm{Id}_{\mathbb{R}}, \mathrm{d}/\mathrm{d}\tau)$, that is

$$\mathrm{d}\,\Phi(0,0) \cdot (T, h) = (T, u) \quad \text{with} \quad u(\tau) = h'(\tau) - Tf(0, y_0).$$

This application is a linear isomorphism of $\mathbb{R} \times B_1$ onto $\mathbb{R} \times B_0$, whose inverse is the application $L$ defined by $L(T, v) = (T, w)$ where $w(\tau) = T\tau f(0, y_0) + \int_0^\tau v(s)\,\mathrm{d}s$.

We can then apply the inverse function theorem. We deduce that $\Phi$ is a $C^1$-diffeomorphism of a neighborhood $U \subset \mathbb{R} \times B_1$ of $(0, 0)$ on $\Phi(U)$. Since $(0, 0)$ belongs to $U$, $\Phi((0, 0))$ belongs to $\Phi(U)$. Now, by definition of $\Phi$, we have $\Phi((0, 0)) = (0, 0)$. Moreover $\Phi(U)$ is an open set, because it is the preimage of the open set $U$ by the continuous application $\Phi^{-1}$. In particular, the pair $(T, 0)$ belongs to $\Phi(U)$ for $T$ small enough. There exists a pair $(T', z)$ in $\mathbb{R} \times B_1$ such that $\Phi((T', z)) = (T, 0)$. We deduce that $T' = T$ and that $z$ is a solution of the equation (7.4.2). Then, as we explained at the beginning of the proof, the function $y(t) = z(t/T)$ is a solution of (7.4.1).

Let us finally notice that the uniqueness of the solution comes from the uniqueness result for the inverse function theorem. □

## 7.5 Propagation along bicharacteristic curves

**Definition 7.5.1.** *Consider a function $b = b(x,\xi) \in C^2(\mathbb{R}^{2d})$ with real values. We denote $H_b \colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ the vector field defined by*

$$H_b(x,\xi) = \Big(\frac{\partial b}{\partial \xi_1}(x,\xi), \dots, \frac{\partial b}{\partial \xi_n}(x,\xi), -\frac{\partial b}{\partial x_1}(x,\xi), \dots, -\frac{\partial b}{\partial x_n}(x,\xi)\Big). \tag{7.5.1}$$

*We say that $H_b$ is the Hamiltonian field of $b$. Its integral curves are called bicharacteristics. For $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, we denote $t \mapsto \Phi^t_{H_b}(x,\xi) = (x(t), \xi(t))$ the unique maximal solution of the system*

$$\begin{aligned} &\frac{dx}{dt} = \frac{\partial b}{\partial \xi}(x(t),\xi(t)), \quad \frac{d\xi}{dt} = -\frac{\partial b}{\partial x}(x(t),\xi(t)), \\ &x(0) = x, \quad \xi(0) = \xi. \end{aligned} \tag{7.5.2}$$

**Proposition 7.5.2.** *Suppose $b$ is a real-valued symbol with $b \in S^1(\mathbb{R}^d)$. Then the flow $\Phi^t_{H_b} \colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is defined for all time $t \in \mathbb{R}$. Moreover, if $p \in S^0(\mathbb{R}^d)$, then $p(\Phi^t_{H_b}(x,\xi))$ defines a symbol that belongs to $S^0(\mathbb{R}^d)$ uniformly in $t$.*

*Proof.* Given $(x,\xi) \in \mathbb{R}^{2d}$, the Cauchy problem (7.5.2) can be written under the form

$$\begin{cases} M'(t) = H_b(M(t)) \quad \text{where} \quad M(t) = (x(t),\xi(t)), \\ M(0) = (x,\xi). \end{cases}$$

Since $b$ is a $C^\infty$ function, the vector field $H_b$ is $C^1$ and hence the Cauchy-Lipschitz theorem implies that there exists a unique maximal solution $m \colon [0, T^*) \to \mathbb{R}^{2d}$. Let us prove that this solution is globally defined, which means that $T^* = +\infty$. Recall the following alternative: either $T^* = +\infty$ or $\limsup_{t\to T^*} |m(t)| = +\infty$. To prove that the latter condition is impossible, we will estimate $y(t) = |m(t)|^2$. Since $b$ belongs to $S^1$, there exists a constant $C > 0$ such that $|H_b(m)| \le C + C\,|m|$ for all $m \in \mathbb{R}^{2d}$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t)^2 = 2m'(t)\cdot m(t) \le 2C(1+|m|)\,|m| \le C + 3Cy(t)^2.$$

Then it follows from Gronwall's lemma that

$$y(t)^2 \le y(0)^2 e^{3Ct} + \frac{1}{3}(e^{3Ct} - 1),$$

from which we deduce that $T^* = +\infty$.

In addition, the smooth dependence of the solution of an ordinary differential equation with respect to the initial data implies that, for all $t \geq 0$, the flow $(x,\xi) \mapsto \Phi^t_{H_b}(x,\xi)$ is $C^\infty$. Denote $\Phi^t(x,\xi) = (X^t(x,\xi), \Xi^t(x,\xi))$. We claim that, for all multi-indices $\alpha$ and $\beta$ in $\mathbb{N}^d$, there exist constant $C_{\alpha\beta}$ and $C'_{\alpha\beta}$ such that

$$\forall (x,\xi) \in \mathbb{R}^{2d}, \qquad \left|\partial_x^\alpha \partial_\xi^\beta X^t(x,\xi)\right| \leq C_{\alpha\beta}\langle\xi\rangle^{-|\beta|} \quad \text{if } |\alpha|+|\beta|>0, \tag{7.5.3}$$

$$\forall (x,\xi) \in \mathbb{R}^{2d}, \qquad \left|\partial_x^\alpha \partial_\xi^\beta \Xi^t(x,\xi)\right| \leq C'_{\alpha\beta}\langle\xi\rangle^{1-|\beta|} \quad \text{for any } \alpha,\beta \in \mathbb{N}^d. \tag{7.5.4}$$

We begin by studying $\Xi^t(x,\xi)$. Since $\partial_x b$ is a symbol of order 1, as above we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Xi^t(x,\xi)\right| \leq C + C\left|\Xi^t(x,\xi)\right|, \tag{7.5.5}$$

and since $\Xi^0(x,\xi) = \xi$, the same argument as above implies that

$$\left|\Xi^t(x,\xi)\right|^2 \leq |\xi|^2 e^{3Ct} + \frac{1}{3}(e^{3Ct}-1).$$

This proves (7.5.4) when $\alpha = \beta = 0$. In addition, this implies that there exists $t_1$ small enough (namely for $e^{3Ct_1} < 4$), for all $t \in [0,t_1]$, we have $\left|\Xi^t(x,\xi)\right| \leq 2(1+|\xi|)$. Now set $t_0 = 1/(6C)$ (notice that $t_0 \leq t_1$). Then, by plugging the estimate $\left|\Xi^t(x,\xi)\right| \leq 2(1+|\xi|)$ in (7.5.5) it follows that, for all $t \in [0,t_0]$,

$$\left|\Xi^t(x,\xi) - \xi\right| \leq \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\Xi^s(x,\xi)\right| \mathrm{d}s \leq \frac{1}{2}(1+|\xi|).$$

Therefore, for all $(x,\xi) \in \mathbb{R}^{2d}$ and all time $t \in [0,t_0]$,

$$\frac{1}{2}|\xi| - \frac{1}{2} \leq \left|\Xi^t(x,\xi)\right| \leq \frac{1}{2} + \frac{3}{2}|\xi|\,. \tag{7.5.6}$$

Set

$$S_t(x,\xi) = \begin{pmatrix} \mathrm{d}_x X^t & \langle\xi\rangle\, \mathrm{d}_\xi X^t \\ \langle\xi\rangle^{-1}\, \mathrm{d}_x \Xi^t & \mathrm{d}_\xi \Xi^t \end{pmatrix}$$

where the differentials $\mathrm{d}_x X^t$, $\mathrm{d}_\xi X^t$, $\mathrm{d}_x \Xi^t$ and $\mathrm{d}_\xi \Xi^t$ are identified with matrices. One can form an evolution equation on $S_t$, namely

$$\frac{\partial}{\partial t} S_t(x,\xi) = A(t,x,\xi) S_t(x,\xi) \quad ; \quad S_0(x,\xi) = \mathrm{Id}_{\mathbb{R}^{2d}}$$

where

$$A = \begin{pmatrix} \mathrm{d}_x \nabla_\xi b \circ \Phi^t(x,\xi) & \langle \xi \rangle \, \mathrm{d}_\xi \nabla_\xi b \circ \Phi^t(x,\xi) \\ -\langle \xi \rangle^{-1} \, \mathrm{d}_x \nabla_x b \circ \Phi^t(x,\xi) & -\, \mathrm{d}_\xi \nabla_x b \circ \Phi^t(x,\xi) \end{pmatrix}.$$

Let assume that (7.5.3) and (7.5.4) hold for all multi-indices $\alpha, \beta$ such that $|\alpha|+|\beta| \le k$ with $k \in \mathbb{N}^*$. It follows that, if $|\alpha| + |\beta| \le k$ then

$$\sup_{\mathbb{R}^{2d}} \langle \xi \rangle^{|\beta|} \left| \partial_x^\alpha \partial_\xi^\beta S_t(x,\xi) \right| < +\infty. \tag{7.5.7}$$

We want to prove a similar estimate for $A$. We claim that, if $|\alpha| + |\beta| \le k$ then

$$\forall t \in [0,t_0], \quad \sup_{\mathbb{R}^{2d}} \langle \xi \rangle^{|\beta|} \left| \partial_x^\alpha \partial_\xi^\beta A(t,x,\xi) \right| < +\infty. \tag{7.5.8}$$

To see this, we first observe that, for any function $F \in C^\infty(\mathbb{R}^{2d})$, and for all multi-indices $\alpha, \beta$, $\partial_x^\alpha \partial_\xi^b F(\Phi^t(x,\xi))$ is a linear combination of terms of the form

$$(\partial_x^{\alpha'} \partial_\xi^{\beta'} F)(\Phi^t(x,\xi)) \big(\partial_x^{a_1} \partial_\xi^{b_1} X_{i_1}^t\big) \cdots \big(\partial_x^{a_{|\alpha'|}} \partial_\xi^{b_{|\alpha'|}} X_{i_{|\alpha'|}}^t\big) \big(\partial_x^{a'_1} \partial_\xi^{b'_1} X_{j_1}^t\big) \cdots \big(\partial_x^{a'_{|\alpha'|}} \partial_\xi^{b'_{|\alpha'|}} \Xi_{j_{|\beta'|}}^t\big)$$

where

$$a_1 + \cdots + a_{|\alpha'|} + a'_1 + \cdots + a'_{|\beta'|} = \alpha, \quad b_1 + \cdots + b_{|\alpha'|} + b'_1 + \cdots + b'_{|\beta'|} = \beta.$$

In addition, for any symbol $r \in S^0$, we have

$$\left| (\partial_x^{\alpha'} \partial_\xi^{\beta'} r)(\Phi^t(x,\xi)) \right| \le C \langle \Xi^t(x,\xi) \rangle^{-|\beta'|} \le C' \langle \xi \rangle^{-|\beta'|},$$

where we used (7.5.6). By using the previous inequality with

$$r = \partial_{\xi_j} \partial_{x_k} b, \quad r = \langle \xi \rangle^{-1} \partial^2_{x_j x_k} b \quad \text{or} \quad r = \langle \xi \rangle \partial^2_{\xi_j \xi_k} b,$$

(which are symbols of order 0) and combining this with the induction hypothesis, we get the wanted result (7.5.9).

It follows that

$$\frac{\partial}{\partial t} \partial_x^\alpha \partial_\xi^\beta S_t(x,\xi) = R(t,x,\xi) + A(t,x,\xi) \partial_x^\alpha \partial_\xi^\beta S_t(x,\xi) \quad ; \quad \partial_x^\alpha \partial_\xi^\beta S_0(x,\xi) = \delta_\alpha^0 \delta_b^0 \, \mathrm{Id}_{\mathbb{R}^{2d}},$$

where $R(t,x,\xi)$ is a linear combination of terms of the form $\partial_x^{\alpha-\alpha'} \partial_\xi^{\beta-\beta'} A \partial_x^{\alpha'} \partial_\xi^{\beta'} S_t$ where $|\alpha'| + |\beta'| < |\alpha| + |\beta| \le k$. In particular, it follows from (7.5.10) and (7.5.9) that

$$\forall t \in [0,t_0], \qquad \sup_{\mathbb{R}^{2d}} \langle \xi \rangle^{|\beta|} \left| R(t,x,\xi) \right| < +\infty. \tag{7.5.9}$$

Then the Gronwall lemma implies that

$$\sup \langle \xi \rangle^{|\beta|} \left| \partial_x^\alpha \partial_\xi^\beta S_t(x,\xi) \right| < +\infty. \tag{7.5.10}$$

Now, directly from the definition of $S_t$, we deduce that the induction hypothesis holds at rank $k+1$.

Now, consider a symbol $p \in S^0$, a time $t \in [0, t_0]$ and set $q(x,\xi) = p(\Phi^t(x,\xi))$. It follows from the estimates (7.5.3) and (7.5.4) and the arguments used to estimate $A$ above, that $q$ is a symbol of order 0. The above argument holds for all time $t$ small enough, namely for $t \in [0, t_0]$. To conclude that the result holds for all times, we will see that it suffices to iterate. To begin, let us first prove the desired result on the time interval $[0, 2t_0]$. To do so, set $q = p \circ \Phi^{t_0}$ and let $\tau \in [0, t_0]$. Since $\Phi^{t_0+\tau} = \Phi^{t_0} \circ \Phi^\tau$, we have $p \circ \Phi^{t_0+\tau} = q \circ \Phi^\tau$. The previous result applies twice implies successively that $q$ is a symbol of order 0 and then that $q \circ \Phi^\tau$ is also a symbol of order 0, uniformly in $\tau$. By induction we successively prove that, for all integer $N$, $p \circ \Phi^t$ belongs to $S^0$ for all $t \in [0, Nt_0]$, uniformly in time. This concludes the proof. □

# Chapter 8

# Sobolev energy estimates for hyperbolic equations

## 8.1 Introduction

Let $d \geq 1$ and $v = (v_1, \dots, v_d) \in \mathbb{R}^d$. The transport equation is the prototype of a first order hyperbolic equation. It is the equation

$$\partial_t u + v \cdot \nabla u = 0$$

where the unknown $u = u(t, x)$ is a real function of class $C^1$, defined on $\mathbb{R} \times \mathbb{R}^d$ and

$$v \cdot \nabla u = \sum_{j=1}^{d} v_j \partial_{x_j} u.$$

**Proposition 8.1.1.** *Let $u_0 \in C^1(\mathbb{R}^d)$. There exists a unique function $u \in C^1(\mathbb{R} \times \mathbb{R}^d)$ which is a solution of the Cauchy problem*

$$\begin{cases} \partial_t u + v \cdot \nabla u = 0, \\ u_{|t=0} = u_0. \end{cases}$$

*This solution is given by the formula $u(t, x) = u_0(x - tv)$.*

*Proof.* The idea is to introduce a family of functions $t \mapsto X(t, x)$ indexed by $x \in \mathbb{R}^d$ such that, if $u$ is a solution of $\partial_t u + v \cdot \nabla u = 0$ then $t \mapsto u(t, X(t, x))$ is a constant function. Here, this amounts to introducing $X(t,x) = x + vt$. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t, x + vt) = (\partial_t u + v \cdot \nabla u)(t, x + vt).$$

So if $u$ is a solution of the Cauchy problem then

$$u(t, x + vt) = u(t, X(t,x)) = u(0, X(0,x)) = u(0,x) = u_0(x),$$

hence $u(t,x) = u_0(x - vt)$.

Conversely, we directly verify that $(t,x) \mapsto u_0(x - tv)$ is a $C^1$ function which is a solution of the Cauchy problem. □

In the case where the constant vector $v$ is replaced by a function with variable coefficients, we still have a formula for representing the solution based on the use of the characteristics curves. We will not study use this approach. Instead we will study an approach based on *a priori* energy estimates, which is a powerful tool to study PDE of different natures. In the general theory of partial differential equations, an *a priori* estimate is an inequality for the solution or its derivatives of a partial differential equation. *A priori* means "from before" in Latin and is used to refer to the fact that one proves an estimate about the possible solutions of an equation *before* one knows that these solutions exist. This is a fundamental method from which it is often possible to prove that solutions do exist using arguments from functional analysis (see for instance Exercise 11.0.11). In this chapter we will see an example of this principle.

Let us recall Gronwall's lemma which plays a fundamental role in the study of evolution equations.

**Lemma 8.1.2** (Gronwall's Lemma)**.** *Let $A, B \geq 0$ and $b, \phi \colon \mathbb{R}_+ \to \mathbb{R}_+$ be two continuous continuous functions such that*

$$\forall t \geq 0, \qquad \phi(t) \leq A + B \int_0^t \phi(s)\,\mathrm{d}s + \int_0^t b(s)\,\mathrm{d}s.$$

*Then, for all $t \geq 0$,*

$$\phi(t) \leq Ae^{Bt} + \int_0^t b(s) e^{B(t-s)}\,\mathrm{d}s.$$

*Proof.* Let us introduce

$$w(t) = A + B \int_0^t \phi(s)\,\mathrm{d}s + \int_0^t b(s)\,\mathrm{d}s.$$

By hypothesis, this function is of class $C^1$ on $\mathbb{R}_+$ and

$$w'(t) = B\phi(t) + b(t) \leq Bw(t) + b(t).$$

Therefore

$$\left(w(t)e^{-Bt}\right)' \leq b(t)e^{-Bt},$$

and we deduce the desired result by integrating this inequality and noting that $\phi(t) \leq w(t)$. □

Let us now consider $V \in C_b^\infty(\mathbb{R} \times \mathbb{R}^d)$ with real values and a solution $u \in C^1(\mathbb{R}_+; L^2(\mathbb{R}^d))$ of the equation

$$\partial_t u + V(t,x) \cdot \nabla u = 0.$$

By multiplying the equation by $u$ and integrating we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u(t,x)^2 \, \mathrm{d}x = 2 \int u \partial_t u \, \mathrm{d}x = -2 \int u(V \cdot \nabla u) \, \mathrm{d}x$$

and by integrating by parts we deduce that

$$\int u(V \cdot \nabla u) \, \mathrm{d}x = \frac{1}{2} \int V \cdot \nabla u^2 \, \mathrm{d}x = -\frac{1}{2} \int (\operatorname{div} V) u^2 \, \mathrm{d}x,$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u(t,x)^2 \, \mathrm{d}x \leq \|\operatorname{div} V\|_{L^\infty} \int u^2 \, \mathrm{d}x.$$

The Gronwall lemma then gives

$$\forall t \geq 0, \quad \|u(t)\|_{L^2}^2 \leq e^{t\|\operatorname{div} V\|_{L^\infty}} \|u_0\|_{L^2}^2 .$$

Note that if $\operatorname{div} V = 0$ then the $L^2(\mathbb{R}^d, \mathrm{d}x)$-norm is preserved.

The aim of this chapter is to derive similar estimates in Sobolev spaces for general hyperbolic equations and then to deduce from these estimates that the Cauchy problem for the latter equations has a unique solution.

## 8.2 Pseudo-differential hyperbolic equations

We consider complex valued symbols $a(x, \xi)$ depending on a the variables $x, \xi$ in $\mathbb{R}^d$ where $d \geq 1$ is a fixed integer.

**Definition 8.2.1.** *Consider a complex valued symbol $a = a(x, \xi)$ in $S^1$. We say that a is hyperbolic if a can be written as $a = a_1 + a_0$ where $a_1 \in S^1$ is purely imaginary and $a_0$ belongs to $S^0$.*

**Example 8.2.2.** *The symbol $a(x, \xi) = iV(x) \cdot \xi$ is hyperbolic, then* $\mathrm{Op}(a)u = V \cdot \nabla u$.

We consider in addition:

- a time $T > 0$ and a real number $s$;
- an initial data $u_0 \in H^s(\mathbb{R}^d)$;
- a source term $f \in C^0([0, T]; H^s(\mathbb{R}^d))$.

We are interested in the following Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{Op}(a)u = f, \\ u_{|t=0} = u_0, \end{cases} \tag{8.2.1}$$

where the unknown is the function $u = u(t, x)$, the variable $t \in \mathbb{R}_+$ corresponds to time and the variable $x \in \mathbb{R}^d$ $(d \geq 1)$ corresponds to the space variable.

**Theorem 8.2.3.** *Let $T > 0$, $d \geq 1$ and $s \in \mathbb{R}$. For any initial $u_0 \in H^s(\mathbb{R}^d)$ and any $f \in C^0([0, T]; H^s(\mathbb{R}^d))$ there exists a unique function*

$$u \in C^0([0, T]; H^s(\mathbb{R}^d)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^d))$$

*which verifies*

$$\partial_t u + \mathrm{Op}(a)u = f$$

*and which is such that $u(0) = u_0$.*

## 8.3 A priori estimate

As explained in the introduction, the key point is to obtain an *a priori* estimate. We will deduce Theorem 8.2.3 from the following lemma.

**Lemma 8.3.1.** *Let $s \in \mathbb{R}$, $T > 0$. There exists a constant $C$ such that, for all $u \in C^1([0,T]; H^s) \cap C^0([0,T]; H^{s+1})$, all $f \in C^0([0,T]; H^s)$, all $u_0 \in H^s$ and all $t \in [0,T]$, if $u$ is a solution of (8.2.1) then*

$$\|u(t)\|_{H^s} \le e^{Ct} \|u_0\|_{H^s} + \int_0^t e^{C(t-t')} \|f(t')\|_{H^s} \, \mathrm{d}t'. \tag{8.3.1}$$

*Moreover there are two constants $K$ and $N$ which depend only on $s$ such that*

$$C \le K \sum_{|\alpha|+|\beta| \le N} \sup_{x,\xi} \left| \langle \xi \rangle^{-|\beta|} \partial_x^\alpha \partial_\xi^\beta b(x,\xi) \right| \quad \text{where } b := a + a^* = \left(a^* - \overline{a}\right) + 2 \operatorname{Re} a.$$

*(Here $a^*$ denotes the symbol for the adjoint of* $\operatorname{Op}(a)$.*)*

*Proof.* We start with the case $s = 0$. Since $u$ is $C^1$ with values in $L^2$ we can write that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_{L^2}^2 &= \frac{\mathrm{d}}{\mathrm{d}t} (u(t), u(t)) \\ &= 2 \operatorname{Re} (\partial_t u(t), u(t)) \\ &= -2 \operatorname{Re} (\operatorname{Op}(a) u(t), u(t)) + 2 \operatorname{Re} (f(t), u(t)) . \end{aligned} \tag{8.3.2}$$

Now write that

$$(\operatorname{Op}(a) u(t), u(t)) = (u(t), (\operatorname{Op}(a))^* u(t)) = (u(t), \operatorname{Op}(a^*) u(t)) ,$$

to get

$$2 \operatorname{Re} (\operatorname{Op}(a) u(t), u(t)) = (\operatorname{Op}(a + a^*) u(t), u(t)) .$$

The assumption that the symbol $a$ is hyperbolic means that $\overline{a} = -a + 2 \operatorname{Re} a$ with $\operatorname{Re} a \in S^0(\mathbb{R}^d)$. Since in addition we have $a^* - \overline{a} \in S^0$, we deduce that $a^* = -a + b$ where $b := \left(a^* - \overline{a}\right) + 2 \operatorname{Re} a$ belongs to $S^0$. Then we deduce from the Cauchy-Schwarz inequality and the continuity theorem of ΨDO of order 0 on $L^2$ that

$$|(\operatorname{Op}(b) u(t), u(t))| \le \|\operatorname{Op}(b)\|_{\mathcal{L}(L^2)} \|u(t)\|_{L^2}^2 \le C_0 \|u(t)\|_{L^2}^2$$

where $C_0$ is a constant that does not depend on $t$. By plugging this inequality into (8.3.2) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_{L^2}^2 \le C_0 \|u(t)\|_{L^2}^2 + 2 \|f(t)\|_{L^2} \|u(t)\|_{L^2} . \tag{8.3.3}$$

To conclude the proof, we would like to write that $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 = 2\|u(t)\|_{L^2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}$ and simplify the inequality by dividing by $\|u(t)\|_{L^2}$. To do so in a rigorous way, we proceed as follows: Given $\delta > 0$, we deduce from (8.3.3) that the function $y(t) = \sqrt{\|u(t)\|_{L^2}^2 + \delta}$ verifies

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)^2 = \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 \leq C_0 y(t)^2 + 2\|f(t)\|_{L^2} y(t),$$

and since $\|u(t)\|_{L^2}^2 + \delta > 0$, the function $y(t)$ is $C^1$ and then it is possible to write

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)^2 = 2y(t)\frac{\mathrm{d}}{\mathrm{d}t}y(t),$$

and then to infer that

$$2\frac{\mathrm{d}}{\mathrm{d}t}y(t) \leq C_0 y(t) + 2\|f(t)\|_{L^2}.$$

Gronwall's lemma implies that

$$\|u(t)\|_{L^2} \leq y(t) \leq y(0)e^{C_0 t/2} + \int_0^t \|f(t')\|_{L^2} e^{C_0(t-t')/2}\,\mathrm{d}t',$$

for all $\delta > 0$. By making $\delta$ tend to 0 we obtain that

$$\|u(t)\|_{L^2} \leq \|u(0)\|_{L^2} e^{C_0 t/2} + \int_0^t \|f(t')\|_{L^2} e^{C_0(t-t')/2}\,\mathrm{d}t',$$

which concludes the proof of the lemma in the case $s = 0$.

Now for any $s \in \mathbb{R}$ we commute $L = \partial_t + \mathrm{Op}(a)$ to $\Lambda_s = \langle D_x\rangle^s$, which gives

$$\Lambda_s L u = \widetilde{L}\Lambda_s u, \quad \widetilde{L} = \partial_t + \widetilde{A}, \quad \widetilde{A} = \Lambda_s \mathrm{Op}(a)\Lambda_{-s}.$$

Note that $\widetilde{A}$ is a ΨDO operator of hyperbolic symbol. We conclude the proof by applying the previous $L^2$ estimate to $\widetilde{L}$ (that is the estimate (8.3.1) with $s = 0$). □

## 8.4 Proof of Theorem 8.2.3

### 8.4.1 Step 1: uniqueness

The first consequence that we can draw from Lemma 8.3.1 is the part of Theorem 8.2.3 which has to do with uniqueness. Indeed, if $u_j$, $j = 1, 2$ are two different solutions of the Cauchy problem

$$\partial_t u_j + \mathrm{Op}(a) u_j = f \quad ; \quad u_j|_{t=0} = u_0,$$

with $u_j \in C^0([0,T]; H^s(\mathbb{R}^d)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^d))$, then the difference $u = u_1 - u_2$ belongs to $C^1([0,T]; H^{s-1}(\mathbb{R}^d))$. Then one can use the energy estimate (8.3.1) applied with $s$ replaced by $s-1$ to obtain that $u_1 = u_2$.

### 8.4.2 Step 2: construction of approximated solutions

So it remains only to prove the existence. We will construct a solution of the Cauchy problem as the limit of solutions of approximate problems. To do so, we introduce for $\varepsilon > 0$ the following Cauchy problem

$$\partial_t u + \mathrm{Op}(a) J_\varepsilon u = f, \quad u(0) = u_0 \tag{8.4.1}$$

where $J_\varepsilon$, called Friedrichs mollifier, is defined by

$$\widehat{J_\varepsilon v}(\xi) = \chi(\varepsilon\xi)\widehat{v}(\xi),$$

where $\chi$ is a function $C^\infty$ on $\mathbb{R}^d$, with support in the ball of center 0 and radius 2, and value 1 on the ball of center 0 and radius 1.

The next statement contains all the properties that we are going to prove about the approximated Cauchy problems (8.4.1). It will imply immediately Theorem 8.2.3.

**Proposition 8.4.1.** *Let $T > 0$ and $s \in \mathbb{R}$. For all $\varepsilon \in (0,1]$, all $u_0 \in H^s$ and all $f \in C^0([0,T]; H^s)$, there exists a unique solution $u_\varepsilon$ belonging to $C^1([0,T]; H^s)$ of the Cauchy problem (8.4.1). Moreover, for all $\sigma < s$, the sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is a Cauchy sequence in $C^0([0,T]; H^\sigma) \cap C^1([0,T]; H^{\sigma-1})$ and converges in this space to the unique solution $u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ of the Cauchy problem*

$$\partial_t u + \mathrm{Op}(a) u = f, \quad u(0) = u_0.$$

*Proof.* We will note $C$ several constants (whose value can vary from one expression to another) which depend only on $T$ and $s$.

The main difference between the Cauchy problem (8.4.1) and the same problem without the operator $J_\varepsilon$ is that it is very easy to show that the problem (8.4.1) has a solution.

**Lemma 8.4.2.** *For any $u_0 \in H^s$ and all $f \in C([0,T]; H^s)$, there exists a unique solution $u_\varepsilon$ belonging to $C^1([0,T]; H^s)$ of the Cauchy problem (8.4.1).*

*Proof.* We have already seen that if $a = a(x,\xi)$ and $b = b(\xi)$ (that is $b$ is a symbol independent of $x$) then $\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Op}(ab)$. We deduce that, for all $\varepsilon > 0$,

$$\mathrm{Op}(a) J_\varepsilon = \mathrm{Op}(a^\varepsilon)$$

where

$$a^\varepsilon(x,\xi) = a(x,\xi)\chi(\varepsilon\xi).$$

For all $\varepsilon > 0$, the symbol $a^\varepsilon$ is compactly supported in $\xi$ and in particular it belongs to $\Gamma_0^0(\mathbb{R}^d)$. The continuity theorem for pseudo-differential operators implies that $\mathrm{Op}(a^\varepsilon)$ is a bounded from $H^s(\mathbb{R}^d)$ to itself. Then the equation $\partial_t u + \mathrm{Op}(a^\varepsilon)u = f$ is an ordinary differential equation in Banach spaces, for which the Cauchy-Lipschitz theorem applies (one can also use the conclusion of Exercise 11.0.10). □

In the following we use the notation $a^\varepsilon(x,\xi) = a(x,\xi)\chi(\varepsilon\xi)$.

**Lemma 8.4.3.** *There exists a constant $C$ such that for any $\varepsilon > 0$ and any $t \in [0,T]$, and any function $v \in C^1([0,T]; H^s(\mathbb{R}^d))$,*

$$\|v(t)\|_{H^s} \le C\,\|v(0)\|_{H^s} + C\int_0^t \|(\partial_t v + \mathrm{Op}(a^\varepsilon)v(\tau)\|_{H^s}\, \mathrm{d}\tau.$$

*Proof.* Note that the symbol $a^\varepsilon(x,\xi) = a(x,\xi)\chi(\varepsilon\xi)$ is bounded in $S^1$ uniformly in $\varepsilon$, in the sense that $\{\, a_1(x,\xi)\chi(\varepsilon\xi) \,:\, \varepsilon \in ]0,1]\,\}$ is a bounded in $S^1$. Moreover $\mathrm{Re}\, a^\varepsilon$ is uniformly bounded in $S^0$. The desired inequality is therefore a consequence of (8.3.1). □

Applying the previous inequality to $v = u_\varepsilon$, we obtain that there is a constant $C$ such that, for all $\varepsilon > 0$,

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{H^s} \le C\,\|u_0\|_{H^s} + C\int_0^T \|f(t)\|_{H^s}\, \mathrm{d}t. \tag{8.4.2}$$

This implies that $(u_\varepsilon)_{\varepsilon\in]0,1]}$ is a bounded family in $C^0([0,T]; H^s(\mathbb{R}^d))$. Then, using the equation, we verify that $(u_\varepsilon)_{\varepsilon\in]0,1]}$ is a bounded family in $C^1([0,T]; H^{s-1}(\mathbb{R}^d))$. Our goal is to show that $u_\varepsilon$ converges when $\varepsilon$ tends to 0 to a solution of the Cauchy problem. For this we will show the following lemma.

**Lemma 8.4.4.** *The family $(u_\varepsilon)_{\varepsilon\in]0,1]}$ is a Cauchy sequence in $C^0([0,T];H^{s-2}(\mathbb{R}^d))$.*

*Proof.* Let $\varepsilon$ and $\varepsilon'$ in $]0,1]$. Starting from

$$\partial_t u_\varepsilon + \mathrm{Op}(a)J_\varepsilon u_\varepsilon = f,$$
$$\partial_t u_{\varepsilon'} + \mathrm{Op}(a)J_{\varepsilon'} u_{\varepsilon'} = f,$$

we deduce that $v = u_\varepsilon - u_{\varepsilon'}$ verifies

$$\partial_t v + \mathrm{Op}(a^\varepsilon)v = f_\varepsilon \quad \text{avec} \quad f_\varepsilon = \mathrm{Op}(a)\big(J_{\varepsilon'} - J_\varepsilon\big)u_{\varepsilon'}.$$

Since $u_\varepsilon$ and $u_{\varepsilon'}$ coincide for $t = 0$, we have $v(0) = 0$ and we can then use the inequality of the previous lemma to obtain that

$$\|v\|_{C^0([0,T];H^{s-2})} \leq C\int_0^T \|f_\varepsilon(t)\|_{H^{s-2}}\,\mathrm{d}t.$$

Now

$$\|f^\varepsilon(t)\|_{H^{s-2}} = \left\|\mathrm{Op}(a)\big(J_{\varepsilon'} - J_\varepsilon\big)u_{\varepsilon'}(t)\right\|_{H^{s-2}} \leq K\left\|\big(J_{\varepsilon'} - J_\varepsilon\big)u_{\varepsilon'}(t)\right\|_{H^{s-1}}.$$

By definition

$$\left\|\big(J_{\varepsilon'} - J_\varepsilon\big)u_{\varepsilon'}(t)\right\|_{H^{s-1}}^2 = (2\pi)^{-2d}\int \langle\xi\rangle^{2(s-2)}\,|\chi(\varepsilon\xi) - \chi(\varepsilon'\xi)|^2\,|\widehat{u}_{\varepsilon'}(t,\xi)|^2\,\mathrm{d}\xi.$$

We use the elementary inequality $|\chi(\varepsilon\xi) - \chi(\varepsilon'\xi)| \leq K\,|\varepsilon - \varepsilon'|\,|\xi|$ to conclude that

$$\int_0^T \|f_\varepsilon(t)\|_{H^{s-2}}\,\mathrm{d}t \leq K'\,|\varepsilon - \varepsilon'|\int_0^T \|u_{\varepsilon'}(t)\|_{H^s}\,\mathrm{d}t.$$

Since $\|u_{\varepsilon'}\|_{C^0([0,T];H^s)}$ is uniformly bounded according to (8.4.2), we obtain

$$\|v\|_{C^0([0,T];H^{s-2})} = O(|\varepsilon - \varepsilon'|),$$

which is the desired result. □

We thus have seen that the family $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is bounded in $C^0([0,T];H^s)$ and that it is also a Cauchy sequence in $C^0([0,T];H^{s-2})$. The following lemma will allow us to deduce that $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is a Cauchy sequence in $C^0([0,T];H^\sigma)$ for any $\sigma < s$.

**Lemma 8.4.5** (Interpolation in Sobolev spaces)**.** *Consider three real numbers*

$$s_1 < \sigma < s_2 \quad \textit{with} \quad \sigma = \alpha s_1 + (1-\alpha)s_2 \quad \textit{where} \quad \alpha \in [0,1].$$

*Then, for all $u \in H^{s_2}(\mathbb{R}^d)$,*

$$\|u\|_{H^\sigma} \leq \|u\|_{H^{s_1}}^{\alpha} \, \|u\|_{H^{s_2}}^{1-\alpha} \, . \tag{8.4.3}$$

*Proof.* Let us write that

$$\begin{aligned}
\|u\|_{H^s}^2 &= (2\pi)^{-d} \int \langle \xi \rangle^{2s} \, |\widehat{u}(\xi)|^2 \, \mathrm{d}\xi \\
&= (2\pi)^{-d} \int \langle \xi \rangle^{2\alpha s_1} \, |\widehat{u}(\xi)|^{2\alpha} \, \langle \xi \rangle^{2(1-\alpha)s_2} \, |\widehat{u}(\xi)|^{2(1-\alpha)} \, \mathrm{d}\xi
\end{aligned}$$

so that the desired inequality is a consequence of the Hölder inequality. □

As mentioned above, for any $\sigma \in [s-2, s)$, the previous inequality applied with $(s_1, s_2, \sigma) = (s-2, s, \sigma)$ leads to the fact that $(u_\varepsilon)_{\varepsilon \in (0,1]}$ is a Cauchy sequence in $C^0([0,T]; H^\sigma)$. Using the equation, we further obtain that $(\partial_t u_\varepsilon)_{\varepsilon \in ]0,1]}$ is Cauchy in $C^1([0,T]; H^{\sigma-1})$. So $u_\varepsilon$ converges in $C^0([0,T]; H^\sigma) \cap C^1([0,T]; H^{\sigma-1})$ to a limit denoted $u$. By passing to the limit, we find that $u$ is a solution of the Cauchy problem

$$\partial_t u + \mathrm{Op}(a)u = f, \quad u(0) = u_0.$$

To conclude the proof, we just have to show that $u$ belongs to $C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$. To do so, we regularize $f$ and $u_0$, construct a regular solution sequence a sequence of regular solutions and pass to the limit. More precisely, we introduce two sequences $(f^n)_{n \in \mathbb{N}}$ and $(u_0^n)_{n \in \mathbb{N}}$ with $f^n \in C^0([0,T]; H^{s+2})$ and $u_0^n \in H^{s+2}$ which converges to $f$ in $C^0([0,T]; H^s)$ and to $u$ in $H^s$, respectively. The previous work gives a sequence of solutions $u^n$ belonging to $C^1([0,T]; H^s)$. The energy estimate (8.3.1) then shows that $(u^n)$ a is a Cauchy sequence in $C([0,T]; H^s)$ and that $(\partial_t u_n)$ is a Cauchy sequence in $C([0,T]; H^{s-1})$ , so it converges in $C^0([0,T]; H^s)$ towards a solution of the Cauchy problem. By uniqueness of the solution of the Cauchy problem, we deduce that $u$ belongs to $C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$. □

# Chapter 9

# The wave front set

In this chapter, we give an introduction to the study of microlocal singularities.

## 9.1 Local properties

Let $u \in C_0^\infty(\mathbb{R}^d)$ and $a \in S^m(\mathbb{R}^d)$ with $m$ any real. The continuity theorem for pseudo-differential operators implies that $\mathrm{Op}(a)$ is continuous from $H^s(\mathbb{R}^d)$ into $H^{s-m}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$. Since $C_0^\infty(\mathbb{R}^d)$ is included in $H^s(\mathbb{R}^d)$ whatever $s \in \mathbb{R}$, we deduce that

$$\mathrm{Op}(a)\big(C_0^\infty(\mathbb{R}^d)\big) \subset H^\infty(\mathbb{R}^d) \subset C_b^\infty(\mathbb{R}^d),$$

where the second inclusion comes from the Sobolev injection theorem. One may wonder if we have better. For instance is it true that $\mathrm{Op}(a)u$ is a function with compact support? This result is true, trivially, if $a$ is a polynomial in $\xi$ (with coefficients depending on $x$). Indeed, in this case, $\mathrm{Op}(a)$ is a differential operator and $\mathrm{Op}(a)u$ is supported in $\operatorname{supp} u$. Conversely, a classical result of differential calculus states that local operators (which do not increase the support) are necessarily differential operators (see Problem 23 in [1]). This means that, given a pseudo-differential operator, it is false in general that if $u$ belongs to $C_0^\infty(\mathbb{R}^d)$ then $\mathrm{Op}(a)u \in C_0^\infty(\mathbb{R}^d)$. However, we have several results concerning the local theory of pseudo-differential operators and we will describe them. Among these results, the simplest one is given by the following proposition.

**Proposition 9.1.1.** *Let $a \in S^m(\mathbb{R}^d)$ and $u \in L^2(\mathbb{R}^d)$ be a function with compact support. Consider a function $\varphi \in C_b^\infty(\mathbb{R}^d)$ which vanishes on a neighborhood of the* *support of $u$. Then $\varphi\,\mathrm{Op}(a)u$ belongs to $C_b^\infty(\mathbb{R}^d)$.*

*Proof.* Let us consider a function $\psi \in C_0^\infty(\mathbb{R}^d)$ which is 1 on the support of $u$ and whose support is included in $\varphi^{-1}(\{0\})$.

Since $\varphi = \varphi(x)$ is a function of the variable $x$ only, we have $\varphi\,\mathrm{Op}(a)u = \mathrm{Op}(\varphi a)u$. Moreover, by construction we have $u = \psi u$. The composition theorem implies that

$$\varphi\,\mathrm{Op}(a)u = \mathrm{Op}(\varphi a)\{\psi u\} = \mathrm{Op}\big((\varphi a)\#\psi\big)u.$$

In addition

$$(\varphi a)\#\psi \sim \sum_\alpha \frac{1}{i^{|\alpha|}\alpha!}\varphi(\partial_\xi^\alpha a)(\partial_x^\alpha \psi).$$

By hypothesis on $\varphi$ and $\psi$ we have $\varphi(\partial_x^\alpha\psi) = 0$ for all $\alpha \in \mathbb{N}^n$ therefore $(\varphi a)\#\psi \sim 0$. We deduce that $\mathrm{Op}((\varphi a)\#\psi)$ is a regularizing operator, bounded from $H^{s_1}(\mathbb{R}^d)$ into $H^{s_2}(\mathbb{R}^d)$ for all real numbers $s_1, s_2$. This concludes the proof. □

**Remark 9.1.2.** *The result remains true, with the same proof, if we only suppose that $u$ belongs to the space $\mathcal{E}'(\mathbb{R}^d)$ of distributions with compact support. To see this it is enough to know that any element $u$ of $\mathcal{E}'(\mathbb{R}^d)$ belongs to a Sobolev space $H^s(\mathbb{R}^d)$ for a certain $s \in \mathbb{R}$.*

Recall that

$$S^{+\infty}(\mathbb{R}^d) = \bigcup_{m\in\mathbb{R}} S^m(\mathbb{R}^d), \qquad S^{-\infty}(\mathbb{R}^d) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^d).$$

Thus $S^\infty(\mathbb{R}^d)$ is the space of all the symbols while $S^{-\infty}(\mathbb{R}^d)$ is the space of regularizing symbols. We have of course $S^{-\infty}(\mathbb{R}^d) \subset S^{+\infty}(\mathbb{R}^d)$.

We denote

$$H^{-\infty}(\mathbb{R}^d) = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^d), \qquad H^{+\infty}(\mathbb{R}^d) = \bigcap_{s\in\mathbb{R}} H^m(\mathbb{R}^d),$$

and we have this time $H^{+\infty}(\mathbb{R}^d) \subset H^{-\infty}(\mathbb{R}^d)$.

We saw that if $a \in S^m(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $u \in H^s(\mathbb{R}^d)$ with $s \in \mathbb{R}$ then $\mathrm{Op}(a)u \in H^{s-m}(\mathbb{R}^d)$. If $m \leq 0$ then $\mathrm{Op}(a)u$ is more regular than $u$. In particular

$$a \in S^{-\infty}(\mathbb{R}^d),\ u \in H^{-\infty}(\mathbb{R}^d) \implies \mathrm{Op}(a)u \in H^\infty(\mathbb{R}^d) \subset C_b^\infty(\mathbb{R}^d).$$

**Proposition 9.1.3.** *Consider a regularizing pseudo-differential operator* $\mathrm{Op}(a)$*, such that* $a \in S^{-\infty}(\mathbb{R}^d)$*. Then* $\mathrm{Op}(a)$ *is continuous from* $\mathcal{E}'(\mathbb{R}^d)$ *into* $\mathcal{S}(\mathbb{R}^d)$*.*

*Proof.* Let $a \in S^{-\infty}(\mathbb{R}^d)$ and $u \in \mathcal{E}'(\mathbb{R}^d)$. Since $\mathcal{E}'(\mathbb{R}^d) \subset H^{-\infty}(\mathbb{R}^d)$, the above proves that $\mathrm{Op}(a)u$ belongs to $H^{\infty}(\mathbb{R}^d)$ and thus to $C_b^{\infty}(\mathbb{R}^d)$. Then we apply a reasoning already met which tells us that, for all $\alpha \in \mathbb{N}^n$, $x^{\alpha}\,\mathrm{Op}(a)u$ is a linear combination of terms of the form $\mathrm{Op}(\partial_{\xi}^{\delta} a)(x^{\alpha-\delta}u)$, which belong to $C_b^{\infty}(\mathbb{R}^d)$ for the same reasons ($\mathcal{E}'(\mathbb{R}^d)$ is stable by derivation and by multiplication by a smooth function). Thus we conclude that $\mathrm{Op}(a)u \in \mathcal{S}(\mathbb{R}^d)$. $\square$

We recall the definition of the singular support of a distribution.

**Definition 9.1.4.** *We say that a distribution* $f \in \mathcal{S}'(\mathbb{R}^d)$ *is of class* $C^{\infty}$ *in the neighborhood of* $x_0$*, if there exists a neighborhood* $\omega$ *of* $x_0$ *such that for all functions* $\varphi \in C_0^{\infty}(\omega)$ *we have* $\varphi f \in C^{\infty}(\mathbb{R}^d)$*.*

*The singular support of* $f$*, denoted* $\operatorname{supp\,sing} f$*, is the complementary of the set of points in the neighborhood of which* $f$ *is* $C^{\infty}$*.*

This notion allows to generalize Proposition 9.1.1.

**Proposition 9.1.5.** *For all* $a \in S^{+\infty}(\mathbb{R}^d)$ *and all* $u \in H^{-\infty}(\mathbb{R}^d)$ *we have*

$$\operatorname{supp\,sing} \mathrm{Op}(a)u \subset \operatorname{supp\,sing} u.$$

*Proof.* Let $a \in S^{+\infty}(\mathbb{R}^d)$, $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\Omega = \mathbb{R}^d \setminus \operatorname{supp\,sing} u$. Thus $\psi u \in C_0^{\infty}(\mathbb{R}^d)$ for all $\psi \in C_0^{\infty}(\Omega)$. Moreover for all $\varphi \in C_0^{\infty}(\Omega)$ we can find $\psi \in C_0^{\infty}(\Omega)$ with $\psi = 1$ on the support of $\varphi$ (because $\operatorname{dist}(\operatorname{supp}(\varphi), \partial\Omega) > 0$), and

$$\varphi\,\mathrm{Op}(a)u = \varphi\,\mathrm{Op}(a)\,(\psi u) + \varphi\,\mathrm{Op}(a)\,((1-\psi)u)\,.$$

The first term is in $\mathcal{S}(\mathbb{R}^d)$ since $\psi u \in C_0^{\infty}(\mathbb{R}^d) \subset \mathcal{S}$, and the second term can be written $\mathrm{Op}(b)u$ where $b = \varphi a\#(1-\psi)$. As we have already seen, the symbol $b$ satisfies $b \sim 0$ since $\operatorname{supp}(\varphi) \cap \operatorname{supp}(1-\psi) = \emptyset$ by construction of $\psi$. Moreover, if $u \in H^{-\infty}(\mathbb{R}^d)$, then we also have $(1-\psi)u \in H^{-\infty}(\mathbb{R}^d)$. We deduce that $\varphi\,\mathrm{Op}(a)\,((1-\psi)u) \in H^{+\infty}(\mathbb{R}^d)$. So, for all $\varphi \in C_0^{\infty}(\Omega)$, we have $\varphi\,\mathrm{Op}(a)u \in C_0^{\infty}(\Omega)$. We deduce that $\mathrm{Op}(a)u \in C^{\infty}(\Omega)$ (regularity is a local notion) which is the desired property. $\square$

## 9.2 Wave front set

The wave front set of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, denoted $\mathrm{WF}(f)$, is a subset of $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, which describes not only the points where $f$ is singular, but but also the co-directions in which it is singular. This set is defined by its complementary.

**Definition 9.2.1.** *Let $f \in \mathcal{S}'(\mathbb{R}^d)$.*

*i) We say that $f$ is microlocally of class $C^\infty$ at a point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ if there exists an open set $\omega \subset \mathbb{R}^d$ containing $x_0$ and an open cone $\Gamma$ of $\mathbb{R}^d \setminus \{0\}$ containing $\xi_0$ such that we have*

$$\forall \varphi \in C_0^\infty(\omega), \quad \forall N \in \mathbb{N}, \exists C_N > 0 : \forall \xi \in \Gamma, \quad \left|\widehat{\varphi f}(\xi)\right| \le C_N (1+|\xi|)^{-N}. \tag{9.2.1}$$

*ii) The set of points $(x_0, \xi_0)$ where $f$ is not microlocally $C^\infty$ is called the wave front set of $f$ and noted* $\mathrm{WF}(f)$.

The wave front set is a conical subset of $\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, which means that for all $t > 0$,

$$(x,\xi) \in \mathrm{WF}(f) \iff (x, t\xi) \in \mathrm{WF}(f).$$

The wave front set allows to specify the notion of singular support. Indeed, we have the following proposition.

**Proposition 9.2.2.** *The projection on $\mathbb{R}^d$ of* $\mathrm{WF}(u)$ *is* $\mathrm{supp\,sing}(u)$.

*Proof.* Consider a point $x_0 \in \mathbb{R}^d$ that does not belong to $\mathrm{supp\,sing}(u)$. If $\varphi \in C_0^\infty(\mathbb{R}^d)$ is supported in a sufficiently small ball centered at $x_0$, then $\varphi u$ is a $C^\infty$ function with compact support and therefore belongs to the Schwartz class. As the Fourier transform of a function of $\mathcal{S}(\mathbb{R}^d)$ belongs to $\mathcal{S}(\mathbb{R}^d)$, we deduce that $\widehat{\varphi u}$ is rapidly decaying in all directions. In particular, no $(x_0, \xi_0)$ belongs to $\mathrm{WF}(u)$.

Conversely, suppose that $x_0$ is such that no $(x_0, \xi_0)$ belongs to $\mathrm{WF}(u)$. For each $\xi_0$ we can find an open set $\omega$ containing $x_0$ and a cone $\Gamma$ containing $\xi_0$ such that (9.2.1) is valid. By compactness of the sphere, we can find a finite number of such couples $(\omega_j, \Gamma_j)$ so that the $\Gamma_j$ cover $\mathbb{R}^d \setminus \{0\}$. For $\varphi \in C_0^\infty(\mathbb{R}^d)$ whose support is contained in $\cap_j \omega_j$ we deduce that the function $\widehat{\varphi u}$ is rapidly decreasing, which completes the proof. □

If $P$ is a differential operator of order $m$ whose coefficients $p_\alpha$ are real and $C^\infty$,

$$P = \sum_{|\alpha|\leq m} p_\alpha(x)\partial_x^\alpha.$$

An important question in PDE is to determine the wave front set of the distributions of the equation $Pf = 0$. The basic results relate the geometry of the operator to the geometry of the singularities of its solutions. The two simplest geometrical objects that we associate with the PDE $P(f) = 0$ are the following.

i) The **principal symbol**

$$p_m(x,\xi) = i^m \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha,$$

which is a homogeneous polynomial of degree $m$ in $\xi$.

ii) The **characteristic variety** of $P$, denoted by $\mathrm{Car}(P)$, which is the closed set (homogeneous in $\xi$) defined by

$$\mathrm{Car}(P) = \big\{ (x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})\,;\; p_m(x,\xi) = 0 \big\}.$$

The first important result of the theory is the following.

**Theorem 9.2.3** (Sato-Hörmander)**.** *Singularities are contained in the characteristic variety: If $P$ is a differential operator whose coefficients belong to $C_b^\infty(\mathbb{R}^d)$, then for all $u \in \mathcal{S}'(\mathbb{R}^d)$,*

$$P(u) = 0 \Longrightarrow \mathrm{WF}(u) \subset \mathrm{Car}(P).$$

*Proof.* We start with a technical lemma. Given a differential operator $Q$ and a function $\varphi \in C_0^\infty(\mathbb{R}^d)$, we look for $\psi \in C_0^\infty(\mathbb{R}^d)$ which satisfies, approximately, the equation

$$Q\big(\psi e^{ix\cdot\xi}\big) = \varphi e^{ix\cdot\xi}.$$

Solving approximately means that we will have an error and that this error is measured in function of the natural parameter which is the frequency $|\xi|$ (here $|\xi|$ is large). Let us also observe that $e^{-ix\cdot\xi}Q(fe^{ix\cdot\xi}) = q_m(x,\xi)f + \cdots$ where the dotted lines hide a polynomial in $\xi$ of degree less than $m-1$. Thus, as a first approximation, we look for $\psi$ as a perturbation of $\dfrac{\varphi}{q_m(x,\xi)}$.

**Lemma 9.2.4.** *Consider a differential operator* $Q = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha$ *of order* $m$ *and let us note* $q_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha$ *its principal symbol. Let* $\omega$ *be an open set of* $\mathbb{R}^d$ *and* $V \subset \mathbb{R}^d \setminus \{0\}$ *an open cone such that*

$$\exists C > 0 / \ \forall (x,\xi) \in \omega \times V, \quad |q_m(x,\xi)| \ge C\,|\xi|^m .$$

*For any integer* $N$, *for any* $\varphi \in C_0^\infty(\omega)$ *and any* $\xi \in V$, *there exists* $\psi_{\xi,N} \in C_0^\infty(\omega)$ *and* $r_{\xi,N} \in C_0^\infty(\omega)$ *such that*

$$Q\Big(\frac{\psi_{\xi,N}(x)}{q_m(x,\xi)} e^{ix\cdot\xi}\Big) = \varphi(x)e^{ix\cdot\xi} + r_{\xi,N}(x)e^{ix\cdot\xi}$$

*with* $\sup_{x\in\mathbb{R}^d} \langle\xi\rangle^N \left|\partial_x^\alpha r_{\xi,N}(x)\right| < +\infty$ *for all* $\alpha \in \mathbb{N}^d$.

*Proof.* Let us introduce an operator $R_\xi$ (which depends on $\xi$) by posing

$$Q\Big(\frac{\psi}{q_m(x,\xi)} e^{ix\cdot\xi}\Big) = (\psi + R_\xi(\psi))e^{ix\cdot\xi}.$$

It is then a question of solving, approximately, the equation $\psi + R_\xi(\psi) = \varphi$. Let us start by giving an expression of $R_\xi(\psi)$. For that we compute $e^{-ix\cdot\xi}Q\Big(\frac{\psi}{q_m(x,\xi)} e^{ix\cdot\xi}\Big)$ where $Q = \sum_\alpha q_\alpha(x)D_x^\alpha$, directly with Leibniz's rule, by separating the expression into several terms: the first term corresponds to the case where all derivatives are of order $|\alpha| = m$ and act on the oscillatory factor $e^{ix\cdot\xi}$ (the contribution of this term is $\psi$); the sum of the other terms corresponds to $R_\xi(\psi)$, it is the sum of terms for which either $|\alpha| \le m-1$ and all derivatives act on $e^{ix\cdot\xi}$, or at least one derivative acts on the factor $\psi/q_m(x,\xi)$. We find

$$e^{-ix\cdot\xi}Q\Big(\frac{\psi}{q_m(x,\xi)} e^{-ix\cdot\xi}\Big) = (I) + R_\xi(\psi) \qquad \text{where}$$

$$(I) = e^{-ix\cdot\xi} \sum_{|\alpha|=m} \frac{\psi}{q_m(x,\xi)} a_\alpha(x)\partial_x^\alpha(e^{ix\cdot\xi}),$$

$$R_\xi(\psi) = e^{-ix\cdot\xi} \sum_{|\alpha|\le m-1} \frac{\psi}{q_m(x,\xi)} a_\alpha(x)\partial_x^\alpha(e^{ix\cdot\xi}),$$

$$+ e^{-ix\cdot\xi} \sum_{\beta+\gamma=\alpha,\ |\beta|>0} a_\alpha(x)\partial_x^\beta\Big(\frac{\psi}{q_m(x,\xi)}\Big)\partial_x^\gamma e^{ix\cdot\xi}.$$

Then $(I) = \psi$ because

$$\sum_{|\alpha|=m} a_\alpha(x)\partial_x^\alpha(e^{ix\cdot\xi}) = q_m(x,\xi)e^{ix\cdot\xi},$$

by definition of $q_m$.

Let us set

$$\psi_{\xi,N} := \sum_{n=0}^{N-1} (-R_\xi)^n(\varphi), \quad r_{\xi,N} = (-1)^{N+1} R_\xi^N(\varphi).$$

Then $\psi_{\xi,N} + R_\xi(\psi_{\xi,N}) = \varphi + r_{\xi,N}$ and we check that $r_{\xi,N}$ satisfies the desired properties. □

Let us now prove the theorem. Let $(x_0, \xi_0) \notin \mathrm{Car}(P)$. Then there exist an open set $\omega$ of $\mathbb{R}^d$ and a cone $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ such that

$$\exists C > 0 \,/ (x, \xi) \in \omega \times \Gamma \;\Rightarrow\; |p_m(x, \xi)| \geq C\,|\xi|^m .$$

Then, with $Q = {}^tP$, we have

$$\exists C > 0 \,/ \forall (x, \xi) \in \omega \times \Gamma, \quad |q_m(x, -\xi)| \geq C\,|\xi|^m .$$

Let us fix a function $\varphi \in C_0^\infty(\omega)$. To show that $(x_0, \xi_0) \notin \mathrm{WF}(u)$, we will estimate $\widehat{\varphi u}(\xi)$. The previous lemma implies that, for all integers $N$ and all $\xi \in \Gamma$, there exist $\psi_{\xi,N} \in C_0^\infty(\mathbb{R}^d)$ and $r_{\xi,N}$ such that

$$Q\left(\frac{\psi_{\xi,N}}{q_m(x, -\xi)} e^{-ix\cdot\xi}\right) = \varphi e^{-ix\cdot\xi} + r_{\xi,N} e^{-ix\cdot\xi}$$

with $\sup \left|\partial_x^\alpha r_{\xi,N}\right| = O(|\xi|^{-N})$.

Then we can write

$$\begin{aligned}
\widehat{\varphi u}(\xi) &= \langle u, \varphi e^{-ix\cdot\xi}\rangle = \langle u, {}^tP\big((\psi_{\xi,N}/q_m(x,-\xi))e^{-ix\cdot\xi}\big) - r_{\xi,N}e^{-ix\cdot\xi}\rangle \\
&= \langle Pu, (\psi_{\xi,N}/q_m(x,-\xi))e^{-ix\cdot\xi}\rangle - \langle u, r_{\xi,N}e^{-ix\cdot\xi}\rangle \\
&= -\langle u, r_{\xi,N}e^{-ix\cdot\xi}\rangle,
\end{aligned}$$

where we used that $Pu = 0$.

Recall that by definition of tempered distributions, there exists an integer $p$ and a constant $C$ such that

$$\forall \kappa \in \mathcal{S}(\mathbb{R}^d), \quad |\langle u, \kappa\rangle| \leq C \sum_{|\alpha|+|\beta|\leq p} \sup_{x\in\mathbb{R}^d} \left| x^\alpha \partial_x^\beta \kappa(x)\right|.$$

As $r_{\xi,N}$ is a $C^\infty$ function with compact support in $\omega$,

$$\left|\langle u, r_{\xi,N} e^{-ix\cdot\xi}\rangle\right| \le C \sum_{|\alpha|\le p} \sup \left|\partial_x^\beta \left(r_{\xi,N} e^{-ix\cdot\xi}\right)\right|$$

but $\sup_x \left|\partial_x^\beta \left(r_{\xi,N} e^{-ix\cdot\xi}\right)\right| = O(|\xi|^{|\beta|-N})$ so $\left|\langle u, re^{-ix\cdot\xi}\rangle\right| \le C_N \langle\xi\rangle^{p-N}$. (The constant $C_N$ depends on $\omega$ and $\varphi$, but this is not a problem.) Taking $N$ large enough, we conclude the proof. □

## 9.3 Theorem of propagation of singularities

The theorem of propagation of singularities says that not only the wavefront set is contained in the characteristic variety, but it is also necessarily a union of trajectories for a natural dynamical system.

Let us recall one result proved in §7.5.

**Proposition 9.3.1.** *Suppose $b$ is a real-valued symbol with $b \in S^1(\mathbb{R}^d)$ and denote by $H_b$ the hamiltonian vector field*

$$H_b = \sum_{j=1}^d \left(\frac{\partial b}{\partial \xi_j}\frac{\partial}{\partial_{x_j}} - \frac{\partial b}{\partial x_j}\frac{\partial}{\partial \xi_j}\right).$$

*(See (7.5.1).) Then the flow $\Phi^t_{H_b} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is defined for all time $t \in \mathbb{R}$. Moreover, if $p \in S^0(\mathbb{R}^d)$, then $p(\Phi^t_{H_b}(x,\xi))$ defines a symbol that belongs to $S^0(\mathbb{R}^d)$ uniformly in $t$.*

Consider a symbol $a \in S^1(\mathbb{R}^d)$. It is assumed that $a$ can be written as $a^1 + a^0$ where

1. $a^0 \in S^0$;
2. $a^1 \in S^1$ is a symbol with purely imaginary values.

For instance, $a(x,\xi) = iV(x)\xi$ with $V \in C_b^\infty(\mathbb{R}^d)$ a real valued function.

We have shown in the previous chapter how to solve the Cauchy problem for the equation

$$\partial_t u + \mathrm{Op}(a)u = 0.$$

We denote $S(t,s) = e^{(s-t)\operatorname{Op}(a)}\colon L^2 \to L^2$ the solution operator which to a given function $u_0 \in L^2(\mathbb{R}^d)$ associates the value at time $t$ of the unique solution of the Cauchy problem which is $u_0$ at time $s$. That is: $u(t) = S(t,s)u_0$ is the unique function $u \in C^0(\mathbb{R}; L^2(\mathbb{R}^d))$ such that

$$\partial_t u + \operatorname{Op}(a)u = 0, \quad u(s) = u_0.$$

**Theorem 9.3.2.** *Consider a symbol $p_0 \in S^0$ and set $P_0 = \operatorname{Op}(p_0)$. Then, for all $t \in \mathbb{R}$, modulo a regularizing operator, $S(t,0)P_0S(0,t)$ is a pseudo-differential operator: there exists a symbol $q_t \in S^0(\mathbb{R}^d)$ such that, for all $u_0 \in L^2(\mathbb{R}^d)$,*

$$S(t,0)P_0S(0,t)u_0 - \operatorname{Op}(q_t)u_0 \in H^{+\infty}(\mathbb{R}^d).$$

*In addition*

$$q_t(x,\xi) - p_0(\Phi_H^{-t}(x,\xi)) \in S^{-1}(\mathbb{R}^d)$$

*where $\Phi_H^t$ is the flow associated to the vector field*

$$H = \frac{1}{i}\sum_{j=1}^d \Big(\frac{\partial a^1}{\partial \xi_j}\frac{\partial}{\partial_{x_j}} - \frac{\partial a^1}{\partial x_j}\frac{\partial}{\partial \xi_j}\Big).$$

*Proof.* Deriving with respect to $t$ we find that $P(t) = S(t,0)P_0S(0,t)$ satisfies

$$P'(t) + \big[\operatorname{Op}(a), P(t)\big] = 0, \quad P(0) = P_0.$$

We will construct an approximate solution $Q(t)$ of this equation and show that $P(t) - Q(t)$ is a regularizing operator. So we look for $Q(t) = \operatorname{Op}(q_t)$ with $q \in S^0$ solution of

$$Q'(t) + \big[\operatorname{Op}(a_t), Q(t)\big] = R(t), \quad Q(0) = P_0,$$

where $R(t)$ is a family of regularizing operators. We will construct $q$ of the form

$$q(t,x,\xi) \sim q^{(0)}(t,x,\xi) + q^{(-1)}(t,x,\xi) + \cdots$$

where $q^{(-k)}$ is a symbol of order $-k$. Then the commutator $[\operatorname{Op}(a_t), Q(t)]$ is a pseudo-differential operator of order 0, and in addition its symbol satisfies

$$a\#q - q\#a \sim Hq + \frac{1}{i}\{\operatorname{Op}(a^0), q\} + \sum_{|\alpha|=2+k}\frac{1}{i^{|\alpha|}\alpha!}\Big[(\partial_\xi^\alpha a)(\partial_x^\alpha q) - (\partial_\xi^\alpha q)(\partial_x^\alpha a)\Big].$$

This suggests defining $q^{(0)}$ by

$$\Big(\frac{\partial}{\partial t} + H\Big)q^{(0)}(t,x,\xi) = 0, \quad q^{(0)}(0,x,\xi) = p_0(x,\xi).$$

Thus $q^{(0)}(t,x,\xi) = p_0(\Phi_H^{-t}(x,\xi))$, the symbol given by the theorem statement. We have $q^{(0)}(t,x,\xi) \in S^0(\mathbb{R}^d)$. By induction we solve

$$\Big(\frac{\partial}{\partial t} + H\Big) q^{(-j)}(t,x,\xi) = b^{(-j)}(t,x,\xi), \quad q^{(-j)}(0,x,\xi) = 0,$$

where $b^{-j}$ is determined by induction, so as to obtain a solution of (**??**).

Finally, it remains to prove that $P(t) - Q(t)$ is a regularizing operator. Equivalently, we will show that $v(t) - w(t) = S(t,0)P_0 f - Q(t)S(t,0)f \in H^\infty(\mathbb{R}^d)$. Note that

$$\frac{\partial v}{\partial t} + \mathrm{Op}(a)v = 0, \quad v(0) = P_0 f,$$

and

$$\frac{\partial w}{\partial t} + \mathrm{Op}(a)w = g, \quad w(0) = P_0 f,$$

where $g := R(t)S(t,0)f \in C^0(\mathbb{R}; H^\infty(\mathbb{R}^d))$. Taking the difference of the two equations we find

$$\frac{\partial}{\partial t}(v - w) + \mathrm{Op}(a)(v - w) = -g, \quad v(0) - w(0) = 0.$$

Then the theorem about the Cauchy problem for hyperbolic equations implies that $v(t) - w(t) \in H^\infty(\mathbb{R}^d)$ for all $t$ and all $f \in H^{-\infty}(\mathbb{R}^d)$. This completes the proof. $\square$

We can now calculate the action of the solution operator $\exp(t\,\mathrm{Op}(a))$ on the wave front set of the initial data.

Let us recall that in the previous section we proved the following

**Proposition 9.3.3.** *Let $m \in \mathbb{R}$, $a \in S^m$ and $u \in \mathcal{S}'(\mathbb{R}^d)$. Assume that* $\mathrm{Op}(a)u \in C^\infty(\mathbb{R}^d)$ *and that $|a(x,\xi)| \geq |\xi|^m$ for all $(x,\xi) \in \omega \times \Gamma$ where $\omega$ is a neighborhood of $x_0$ and $\Gamma$ is a cone containing $\xi_0$. Then $(x_0,\xi_0) \notin$* $\mathrm{WF}(u)$.

**Theorem 9.3.4.** *If $u$ satisfies*

$$\partial_t u + \mathrm{Op}(a)u = 0, \quad u_{|t=0} = u_0$$

*with $u_0 \in L^2(\mathbb{R}^d)$ then $u \in C^0([0,T]; L^2(\mathbb{R}^d))$ and, for all $0 \leq t \leq T$,*

$$\mathrm{WF}(u(t,\cdot)) = \Phi_H^t(\mathrm{WF}(u_0)).$$

*Proof.* Suppose that $(x_0, \xi_0) \notin \mathrm{WF}(u_0)$. Then there exists a neighborhood $\omega$ of $x_0$, a cone $\Gamma$ containing $\xi_0$ and a symbol $p_0 \in S^0$ such that $|p_0(x_0, \xi_0)| \geq 1$ for all $(x, \xi) \in \omega \times \Gamma$ such that $\mathrm{Op}(p_0)u_0 \in \mathcal{S}(\mathbb{R}^d)$.

Using the operator $Q$ introduced in the proof of the previous theorem we obtain that

$$(\partial_t + \mathrm{Op}(a))Qu = Ru \in C^0([0,T]; H^\infty), \quad Qu|_{t=0} \in \mathcal{S}(\mathbb{R}^d).$$

We deduce that $Q(t)u(t) \in H^\infty(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$.

Since $Q(t)$ is a pseudo-differential operator whose principal symbol is $p_0(\Phi_H^{-t}(x, \xi))$, we deduce that $\Phi_H^t(x_0, \xi_0) \notin \mathrm{WF}\, u(t, \cdot)$. Since we can reverse the direction of time, we find the desired result. $\square$

# Chapter 10

# Paradifferential operators

In this chapter we introduce Bony's paradifferential calculus. It allows to study the regularity of the solutions of nonlinear partial differential equations. This theory lies at the interface between harmonic analysis and microlocal analysis. It has a long history that owes a lot to Calderón and Zygmund, Coifman and Meyer, Kohn and Nirenberg, as well as Hörmander.

We refer to [6, 16, 19, 20, 26] for the general theory. Here we follow the presentation by Métivier in [19]. We refer also to the recent book of Benzoni-Gavage and Serre [5] for applications of paradifferential calculus to hyperbolic systems.

## 10.1 Spectral localization

**Theorem 10.1.1.** *Let* $\varepsilon \in [0,1)$ *and consider a function* $a \in C^\infty(\mathbb{R}^{2d};\mathbb{C})$ *such that*

$$M := \sup_{|\beta| \le [\frac{d}{2}]+1} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \left| (1+|\xi|)^{|\beta|} \partial_\xi^\beta a(x,\xi) \right| < +\infty.$$

*Assume in addition that, for all* $\xi \in \mathbb{R}^d$*, the partial Fourier transform*

$$\hat{a}(\eta,\xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\eta} a(y,\xi)\,\mathrm{d}y$$

*is supported in the ball* $\{\eta \in \mathbb{R}^d \,;\, |\eta| \le \varepsilon\,|\xi|\}$*. Then* $\mathrm{Op}(a) \in \mathcal{L}(L^2(\mathbb{R}^d))$ *and*

$$\|\mathrm{Op}(a)\|_{L^2\to L^2} \le CM,$$

*for some constant $C$ depending only on $\varepsilon$.*

**Remark 10.1.2.** *We will see in the proof that $a$ belongs to $S^0_{1,1}(\mathbb{R}^d)$.*

*Proof.* Set $N = 1 + [d/2]$.

*Step 1: Littlewood-Paley decomposition.* We use the decomposition of the unity introduced in Lemma 4.1.1. Write

$$a(x,\xi) = a(x,\xi)\psi(\xi) + \sum_{p=0}^{\infty} a(x,\xi)\varphi(2^{-p}\xi), \tag{10.1.1}$$

and then set

$$a_{-1}(x,\xi) = a(x,\xi)\psi(\xi) \quad ; \quad a_p(x,\xi) = a(x,\xi)\varphi(2^{-p}\xi) \quad \text{for } p \geq 0.$$

*Step 2: Bernstein Lemma.* We claim that, for any multi-indices $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^d$ with $|\beta| \leq N$, there exists a positive constant $C$ such that,

$$\left|\partial_x^\alpha \partial_\xi^\beta a_{-1}(x,\xi)\right| \leq CM, \tag{10.1.2}$$

and, for $p \geq 0$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a_p(x,\xi)\right| \leq CM2^{p(|\alpha|-|\beta|)}.$$

Since $|\xi| \sim 2^p$ on the support of $\varphi(2^{-p}\xi)$ (resp. $|\xi| \lesssim 1$ for $p = -1$), this follows from the assumption that the partial Fourier transform $\hat{a}(\eta,\xi)$ is supported in the ball $\{|\eta| \leq \varepsilon\,|\xi|\}$, by using the following

**Lemma 10.1.3.** *Consider a function $f \in L^\infty(\mathbb{R}^d)$ whose Fourier transform is included in the ball $\{|\xi| \leq \lambda\}$. Then $f \in C^\infty(\mathbb{R}^d)$ and, for all $\alpha \in \mathbb{N}^d$, there exists a constant $C = C(d,\alpha)$ such that*

$$\left\|\partial_x^\alpha f\right\|_{L^\infty} \leq C\lambda^{|\alpha|} \|f\|_{L^\infty}.$$

*Proof.* Introduce $\theta \in C_0^\infty(\mathbb{R}^d)$ such that $\theta(\xi) = 1$ for $|\xi| \leq 1$ and set $\theta_\lambda(\xi) = \theta(\xi/\lambda)$. Then $\theta_\lambda \hat{f} = \hat{f}$, which implies that

$$f = \kappa_\lambda * f, \quad \text{where} \quad \kappa_\lambda = \mathcal{F}^{-1}(\theta_\lambda).$$

We are now in position to estimate the derivatives of $f$ by exploiting the relation

$$\partial_x^\alpha f = (\partial_x^\alpha \kappa_\lambda) * f.$$

Observing that $\kappa_\lambda(x) = \lambda^d \kappa(\lambda x)$ with $\kappa = \mathcal{F}^{-1}(\theta)$, we obtain that

$$\left\| \partial_x^\alpha \kappa_\lambda \right\|_{L^1(\mathbb{R}^d)} = \lambda^{|\alpha|} \left\| \partial_x^\alpha \kappa \right\|_{L^1(\mathbb{R}^d)},$$

and the result follows. $\square$

*Step 3: low frequency component.* In view of (10.1.2), it follows directly from the Calderón-Vaillancourt theorem (see Theorem 5.2.1), that $\mathrm{Op}(a_{-1})$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, and satisfies the estimate

$$\|\mathrm{Op}(a_{-1})\|_{L^2 \to L^2} \leq CM.$$

*Step 4: rescaling.* We want to prove that the operators $\mathrm{Op}(a_p)$ are also bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. To do so, we use a rescaling argument. More precisely, given a positive real-number $\lambda > 0$, introduce the operator $H_\lambda$ defined by

$$(H_\lambda u)(x) = \lambda^{\frac{d}{2}} u(\lambda x).$$

Then

$$\|H_\lambda u\|_{L^2} = \|u\|_{L^2}.$$

In addition, for any symbol $p = p(x, \xi)$, we have

$$\mathrm{Op}(p)(H_\lambda u) = H_\lambda(\mathrm{Op}(p_\lambda)u) \quad \text{where} \quad p_\lambda(x,\xi) = p\Big(\frac{x}{\lambda}, \lambda \xi\Big).$$

This implies that $\mathrm{Op}(a_p) \in \mathcal{L}(L^2(\mathbb{R}^d))$ if and only if $\mathrm{Op}(b_p) \in \mathcal{L}(L^2(\mathbb{R}^d))$ where

$$b_p(x,\xi) = a_p(2^{-p}x, 2^p\xi),$$

and then $\left\|\mathrm{Op}(a_p)\right\|_{L^2 \to L^2} = \left\|\mathrm{Op}(b_p)\right\|_{L^2 \to L^2}$.

*Step 5: boundedness of the rescaled operators.* Notice that, for any multi-indices $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^d$ with $|\beta| \leq N$, there holds

$$\left| \partial_x^\alpha \partial_\xi^\beta b_p(x,\xi) \right| \leq CM.$$

Then, as already explained above, it follows from the Calderón-Vaillancourt theorem (see Theorem 5.2.1) that

$$\left\|\mathrm{Op}(a_p)\right\|_{L^2 \to L^2} = \left\|\mathrm{Op}(b_p)\right\|_{L^2 \to L^2} \leq CM.$$

*Step 6: spectral localization.* Notice that

$$\widehat{\mathrm{Op}(a_p)u}(\eta) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{a}(\eta-\xi,\xi)\varphi(2^{-p}\xi)\hat{u}(\xi)\,\mathrm{d}\xi.$$

Introduce the function $u_p$ defined by

$$\hat{u_p}(\xi) = \hat{u}(\xi) \quad \text{if} \quad \xi \in \Gamma_p := \{3^{-1}\cdot 2^p \leq |\xi| \leq 3\cdot 2^p\},$$

and $\hat{u_p}(\xi) = 0$ whenever $\xi \notin \Gamma_p$. Then $\varphi(2^{-p}\xi)\hat{u}(\xi) = \varphi(2^{-p}\xi)\hat{u_p}(\xi)$, which in turn implies that

$$\mathrm{Op}(a_p)u = \mathrm{Op}(a_p)u_p.$$

Exploiting again that the partial Fourier transform $\hat{a}(\eta,\xi)$ is supported in the ball $\{|\eta| \leq \varepsilon\,|\xi|\}$, we verify that the support of $\mathcal{F}(\mathrm{Op}(a)u_p)$ is included in the larger shell

$$\Gamma'_p = \Big\{\eta \in \mathbb{R}^d\,;\, \frac{1-\epsilon}{3}2^p \leq |\eta| \leq 3\cdot(1+\epsilon)2^p\Big\}.$$

Now, since any $\eta$ is included in at most $2\log(3/(1-\varepsilon))/\log(2)$ dyadic shells $\Gamma'_p$, we deduce from the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$ that

$$\Big|\sum_p \widehat{\mathrm{Op}(a_p)u}(\eta)\Big|^2 \leq C(\varepsilon)\sum_p\Big|\widehat{\mathrm{Op}(a_p)u}(\eta)\Big|^2. \tag{10.1.3}$$

It follows from Plancherel's theorem that

$$\|\mathrm{Op}(a)u\|_{L^2}^2 \sim \Big\|\sum_p \widehat{\mathrm{Op}(a_p)u}\Big\|_{L^2}^2 \lesssim \sum_p \Big\|\widehat{\mathrm{Op}(a_p)u}\Big\|_{L^2}^2 \sim \sum_p \|\mathrm{Op}(a_p)u\|_{L^2}^2.$$

Remembering that $\mathrm{Op}(a_p)u = \mathrm{Op}(a_p)u_p$ and using the fact that $\mathrm{Op}(a_p) \in \mathcal{L}(L^2(\mathbb{R}^d))$, we get from (10.1.3) that

$$\|\mathrm{Op}(a_p)u\|_{L^2}^2 = \|\mathrm{Op}(a_p)u_p\|_{L^2}^2 \lesssim M^2\,\|u_p\|_{L^2}^2,$$

we conclude that

$$\sum_p \|\mathrm{Op}(a_p)u\|_{L^2}^2 \lesssim M^2\sum_p \|u_p\|_{L^2}^2.$$

Eventually, since each $\xi$ is contained in at most a fix number of dyadic shells $\Gamma_p$, we have

$$\sum_p \|u_p\|_{L^2}^2 \lesssim \|u\|_{L^2}^2.$$

This concludes the proof. □

## 10.2 Notations

Given an integer $k \in \mathbb{N}$, we note $W^{k,\infty}(\mathbb{R}^d)$ the Sobolev space of distributions $f$ such that $\partial_x^\alpha f \in L^\infty(\mathbb{R}^d)$ for $|\alpha| \leq k$. This space is equipped with the norm

$$\|u\|_{W^{k,\infty}} = \sum_{|\alpha|\leq k} \left\|\partial_x^\alpha u\right\|_{L^\infty}.$$

Given $\rho \in ]0,+\infty[\backslash\mathbb{N}$, $W^{\rho,\infty}(\mathbb{R}^d)$ is the space of bounded functions whose derivatives of order $[\rho] \in \mathbb{N}$ are uniformly Hölder continuous with exponent $\rho - [\rho]$. This space is provided with the norm

$$\|u\|_{W^{\rho,\infty}} = \|u\|_{W^{[\rho],\infty}} + \sum_{|\alpha|=[\rho]} \frac{\left|\partial_x^\alpha u(x) - \partial_x^\alpha u(y)\right|}{|x-y|^{\rho-[\rho]}}.$$

**Definition 10.2.1.** *Consider $\rho$ in $[0,+\infty)$ and $m$ in $\mathbb{R}$. One denotes by $\Gamma_\rho^m(\mathbb{R}^d)$ the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^d\times\mathbb{R}^d$, which are $C^\infty$ functions of $\xi$ and such that, for any $\alpha \in \mathbb{N}^d$ and any $\xi \in \mathbb{R}^d$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbb{R}^d)$ and there exists a constant $C_\alpha$ such that,*

$$\forall \xi \in \mathbb{R}^d, \quad \left\|\partial_\xi^\alpha a(\cdot,\xi)\right\|_{W^{\rho,\infty}} \leq C_\alpha(1+|\xi|)^{m-|\alpha|}. \tag{10.2.1}$$

Given a symbol $a$, to define the paradifferential operator $T_a$ we need to introduce a cutoff function $\theta$.

**Definition 10.2.2.** *A function $\theta \in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ is said to be an admissible cut-iff function if it satisfies the three following properties:*

*(i) There exists $\varepsilon_1, \varepsilon_2$ satisfying $0 < 2\varepsilon_1 < \varepsilon_2 < 1/2$ such that*

$$\begin{aligned} &\theta(\eta,\xi) = 1 \quad \textit{if} \quad |\eta| \leq \varepsilon_1|\xi| \quad \textit{and} \ |\xi| \geq 2, \\ &\theta(\eta,\xi) = 0 \quad \textit{if} \quad |\eta| \geq \varepsilon_2|\xi| \quad \textit{or} \quad |\xi| \leq 1. \end{aligned}$$

*(ii) For all $(\alpha,\beta) \in \mathbb{N}^d\times\mathbb{N}^d$, there is $C_{\alpha,\beta}$ such that*

$$\forall (\eta,\xi) \in \mathbb{R}^d\times\mathbb{R}^d, \quad \left|\partial_\eta^\alpha \partial_\xi^\beta \theta(\eta,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|}.$$

*(iii) $\theta$ satisfies the following symmetry conditions:*

$$\theta(\eta,\xi) = \theta(-\eta,-\xi) = \theta(-\eta,\xi). \tag{10.2.2}$$

**Proposition 10.2.3.** *For any $0 < \varepsilon_1 < \varepsilon_2 < 1$, there exists an admissible cut-off function $\theta$ satisfying the three properties above.*

*Proof.* Let $F \in C^\infty(\mathbb{R}^{2d} \setminus \{0\})$ be positively homogeneous function of order 0 and such that

$$\begin{aligned} F(\eta,\xi) &= 1 \quad \text{for} \quad |\eta| \leq \varepsilon_1 |\xi|, \\ F(\eta,\xi) &= 0 \quad \text{for} \quad |\eta| \geq \varepsilon_2 |\xi|. \end{aligned}$$

By homogeneity, we have

$$\left|\partial_\eta^\alpha \partial_\xi^\beta F(\eta,\xi)\right| \leq C_{\alpha,\beta}(1+|\eta|+|\xi|)^{-|\alpha|-|\beta|} \leq C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|}.$$

Then consider $\psi \in C^\infty(\mathbb{R}^d)$ such that

$$\begin{aligned} \psi(\xi) &= 1 \quad \text{for} \quad |\xi| \geq 2, \\ \psi(\xi) &= 0 \quad \text{for} \quad |\xi| \leq 1, \end{aligned}$$

and set $\theta(\eta,\xi) = F(\eta,\xi)\psi(\xi)$. The symmetry condition (10.2.2) is satisfied if one assumes that $F$ and $\psi$ are even in $\eta$ and $\xi$. □

**Example 10.2.4.** *As an example, fix $d = 1$ and $\varepsilon_1, \varepsilon_2$ such that $0 < 2\varepsilon_1 < \varepsilon_2 < 1/2$ and a function $f$ in $C_0^\infty(\mathbb{R})$ satisfying $f(t) = f(-t)$, $f(t) = 1$ for $|t| \leq 2\varepsilon_1$ and $f(t) = 0$ for $|t| \geq \varepsilon_2$. Then set*

$$\theta(\eta,\xi) = (1 - f(\xi)) f\left(\frac{\eta}{\xi}\right).$$

*Properties $(i)$, $(ii)$ and $(iii)$ are clearly satisfied.*

**Lemma 10.2.5.** *Consider an admissible cut-off function $\theta$ and set*

$$G(y,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iy\cdot\eta} \theta(\eta,\xi)\, \mathrm{d}\eta.$$

*Then, for all $\xi \in \mathbb{R}^d$ and all $\beta \in \mathbb{N}^d$,*

$$\left\|\partial_\xi^\beta G(\cdot,\xi)\right\|_{L^1(\mathbb{R}^d_y)} \leq C_\beta (1+|\xi|)^{-|\beta|}.$$

*Proof.* We use the identity

$$y^\alpha e^{iy\cdot\eta} = \left(\frac{1}{i}\partial_\eta\right)^\alpha e^{iy\cdot\eta},$$

and integrate by parts to obtain

$$\begin{aligned}\left|y^\alpha \partial_\xi^\beta G(y,\xi)\right| &= \frac{1}{(2\pi)^d}\left|\int \left(\frac{1}{i}\partial_\eta\right)^\alpha e^{iy\cdot\eta}\partial_\xi^\beta\theta\right| \\ &= \frac{1}{(2\pi)^d}\left|\int_{|\eta|\le\varepsilon_2|\xi|} e^{iy\cdot\eta}\partial_\eta^\alpha\partial_\xi^\beta\theta\right| \\ &\le C\left|\xi\right|^d(1+|\xi|)^{-|\alpha|-|\beta|}.\end{aligned}$$

It follows that

$$\left|y^\alpha\left|\xi\right|^{|\alpha|}\partial_\xi^\beta G(y,\xi)\right| \le C\left|\xi\right|^d(1+|\xi|)^{-|\beta|}.$$

We then use the previous inequality with $\alpha = 0$ and $|\alpha| = d+1$ to get that

$$\left|y^\alpha\left|\xi\right|^{|\alpha|}\partial_\xi^\beta G(y,\xi)\right| \le C(1+|\xi|)^{-|\beta|}\frac{(1+|\xi|)^d}{(1+|x|\,(1+|\xi|))^{d+1}}.$$

By integrating in $x$, we obtain the desired estimate for the $L^1$-norm of $\partial_\xi^\beta G(\cdot,\xi)$. □

**Proposition 10.2.6.** *Let $a \in \Gamma_\mu^m$ with $m \in \mathbb{R}$ and $\mu \in [0,+\infty)$. Then the following three definitions of $\sigma = \sigma(x,\xi)$ are equivalent:*

*(i)* $\sigma(\cdot,\xi) = \theta(D_x,\xi)a(\cdot,\xi)$,

*(ii)* $\sigma(x,\xi) = \int G(x-y,\xi)a(y,\xi)\,\mathrm{d}y$,

*(iii)* $(\mathcal{F}_x\sigma)(\eta,\xi) = \theta(\eta,\xi)(\mathcal{F}_x a)(\eta,\xi)$.

*In addition,*

$$\sigma \in \Sigma_\mu^m.$$

Consider a symbol $a \in \Gamma_0^m$ for some $m \in \mathbb{R}$. Then the paradifferential operator $T_a$ with symbol $a$ is defined by

$$T_a = \mathrm{Op}(\sigma),$$

where $\sigma$ is given by Proposition 10.2.6. It follows that

$$\widehat{T_a u}(\xi) = (2\pi)^{-d}\int \theta(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\widehat{u}(\eta)\,\mathrm{d}\eta, \tag{10.2.3}$$

where $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta}a(x,\xi)\,\mathrm{d}x$ is the Fourier transform of $a$ with respect to $x$.

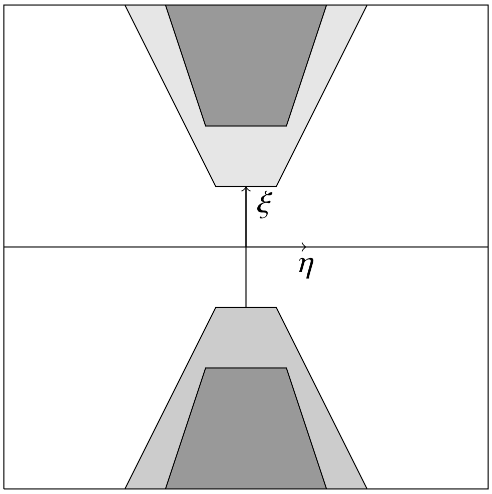


Figure 10.1: The support of the cut-off function $\theta(\eta, \xi)$ is in grey. The set of points $(\eta, \xi)$ where $\theta(\eta, \xi) = 1$ is in darker grey.

**Remark 10.2.7.** *For a pseudo-differential operator* $\operatorname{Op}(a)$*, notice that*

$$\widehat{\operatorname{Op}(a)u}(\xi) = (2\pi)^{-d} \int \widehat{a}(\xi - \eta, \eta)\widehat{u}(\eta)\, \mathrm{d}\eta. \tag{10.2.4}$$

*where again* $\widehat{a}(\theta, \xi) = \int e^{-ix\cdot\theta} a(x, \xi)\, \mathrm{d}x$ *is the Fourier transform of* $a$ *with respect to the first variable. Note that the only difference betwenn* (10.2.3) *and* (10.2.4) *is the cut-off function* $\theta$*; this cut-off allows to define operators for non smooth symbols by means of symbol smoothing.*

## 10.3 Symbolic calculus

In this paragraph, we gather quantitative results about paradifferential operators from [19].

Introduce the following semi-norms.

**Definition 10.3.1.** *For* $m \in \mathbb{R}$*,* $\rho \geq 0$ *and* $a \in \Gamma^m_\rho(\mathbb{R}^d)$*, we set*

$$M^m_\rho(a) = \sup_{|\alpha| \leq \frac{d}{2}+1+\rho} \sup_{\xi \in \mathbb{R}^d} \left\| (1 + |\xi|)^{|\alpha|-m} \partial^\alpha_\xi a(\cdot, \xi) \right\|_{W^{\rho,\infty}(\mathbb{R}^d)}. \tag{10.3.1}$$

Let $m \in \mathbb{R}$. Recall that an operator $T$ is said of order $m$ if, for all $\mu \in \mathbb{R}$, it is bounded from $H^\mu$ to $H^{\mu-m}$.

**Theorem 10.3.2.** *Let $m \in \mathbb{R}$. If $a \in \Gamma_0^m(\mathbb{R}^d)$, then $T_a$ is of order $m$. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \leq K M_0^m(a). \tag{10.3.2}$$

The main features of symbolic calculus for paradifferential operators are given by the following theorems.

**Theorem 10.3.3** (Composition)**.** *Let $m \in \mathbb{R}$ and $\rho > 0$. If $a \in \Gamma_\rho^m(\mathbb{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbb{R}^d)$ then $T_a T_b - T_{a\#b}$ is of order $m + m' - \rho$ where*

$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \left(\partial_\xi^\alpha a\right)\left(\partial_x^\alpha b\right).$$

*Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that*

$$\|T_a T_b - T_{a\#b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K M_\rho^m(a) M_\rho^{m'}(b). \tag{10.3.3}$$

**Remark 10.3.4.** *Note that the definition of the symbol $a\#b$ depends on the regularity of the symbols at stake. To clarify possible confusion, we will sometimes use a notation with an index $\rho$ and write*

$$a\#_\rho b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \left(\partial_\xi^\alpha a\right)\left(\partial_x^\alpha b\right).$$

**Theorem 10.3.5** (Adjoint)**.** *Let $m \in \mathbb{R}$, $\rho > 0$ and $a \in \Gamma_\rho^m(\mathbb{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of $T_a$ and by $\overline{a}$ the complex conjugate of $a$. Then $(T_a)^* - T_{a^*}$ is of order $m - \rho$ where*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a}.$$

*Moreover, for all $\mu$ there exists a constant $K$ such that*

$$\|(T_a)^* - T_{a^*}\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M_\rho^m(a). \tag{10.3.4}$$

## 10.4 Paraproducts

If $a = a(x)$ is a function of $x$ only, then $T_a$ is a called a paraproduct.

If $a \in L^\infty(\mathbb{R})$ then $T_a$ is an operator of order 0, together with the estimate

$$\forall \sigma \in \mathbb{R}, \quad \|T_a u\|_{H^\sigma} \lesssim \|a\|_{L^\infty} \|u\|_{H^\sigma}. \tag{10.4.1}$$

A paraproduct with an $L^\infty$-function acts on any Hölder space $W^{\rho,\infty}(\mathbb{R})$ with $\rho$ in $\mathbb{R}_+^* \setminus \mathbb{N}$,

$$\forall \rho \in \mathbb{R}_+^* \setminus \mathbb{N}, \quad \|T_a u\|_{W^{\rho,\infty}} \lesssim \|a\|_{L^\infty} \|u\|_{W^{\rho,\infty}}. \tag{10.4.2}$$

If $a = a(x)$ and $b = b(x)$ are two functions then $a\sharp b = ab$ and hence (10.3.3) implies that, for any $\rho > 0$,

$$\|T_a T_b - T_{ab}\|_{H^\mu \to H^{\mu+\rho}} \leq K \|a\|_{W^{\rho,\infty}} \|b\|_{W^{\rho,\infty}}, \tag{10.4.3}$$

provided that $a$ and $b$ belong to $W^{\rho,\infty}(\mathbb{R})$.

A key feature of paraproducts is that one can replace nonlinear expressions by paradifferential expressions, to the price of error terms which are smoother than the main terms. As an illustration, we give the following result of Bony [6].

**Definition 10.4.1.** *Given two functions $a, b$, we define the remainder*

$$R_{\mathcal{B}}(a,u) = au - T_a u - T_u a. \tag{10.4.4}$$

We record here two estimates about the remainder $R_{\mathcal{B}}(a,b)$ (see chapter 2 in [4]).

**Theorem 10.4.2.** *Let $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ be such that $\alpha + \beta > 0$. Then*

$$\|R_{\mathcal{B}}(a,u)\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbb{R})} \leq K \|a\|_{H^\alpha(\mathbb{R})} \|u\|_{H^\beta(\mathbb{R})}, \tag{10.4.5}$$

$$\|R_{\mathcal{B}}(a,u)\|_{H^{\alpha+\beta}(\mathbb{R})} \leq K \|a\|_{W^{\alpha,\infty}(\mathbb{R})} \|u\|_{H^\beta(\mathbb{R})}. \tag{10.4.6}$$

We next recall a well-known property of products of functions in Sobolev spaces (see chapter 8 in [16]) that can be obtained from (10.4.1) and (10.4.6): If $u_1, u_2 \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $s > 0$ then

$$\|u_1 u_2\|_{H^s} \leq K \|u_1\|_{L^\infty} \|u_2\|_{H^s} + K \|u_2\|_{L^\infty} \|u_1\|_{H^s}. \tag{10.4.7}$$

Similarly, recall that, for $s > 0$ and $F \in C^\infty(\mathbb{C}^N)$ such that $F(0) = 0$, there exists a non-decreasing function $C \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|F(U)\|_{H^s} \leq C\big(\|U\|_{L^\infty}\big) \|U\|_{H^s}, \tag{10.4.8}$$

for any $U \in (H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$.

**Theorem 10.4.3.** *For all $C^\infty$ function $F$, if $a \in H^\alpha(\mathbb{R}^d)$ then*

$$F(a) - F(0) - T_{F'(a)}a \in H^{2\alpha - \frac{d}{2}}(\mathbb{R}^d).$$

*Moreover,*

$$\left\| F(a) - F(0) - T_{F'(a)}a \right\|_{H^{2\alpha-\frac{d}{2}}(\mathbb{R}^d)} \leq C\left( \|a\|_{H^\alpha(\mathbb{R}^d)} \right),$$

*for some non-decreasing function $C$ depending only on $F$.*

# Part IV

# Exercises and Problems

# Chapter 11

# Exercises

**Exercise 11.0.1** (Semi-classical operators)**.** *Consider a real number $h \in (0,1]$ and a symbol $a = a(x,\xi)$ which belongs to $C_b^\infty(\mathbb{R}^{2d})$. We define*

$$\mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} a(x, h\xi)\widehat{u}(\xi)\,\mathrm{d}\xi.$$

*We want to show that*

$$\left\|\mathrm{Op}_h(a)\right\|_{\mathcal{L}(L^2)} \leq C \sup_{\mathbb{R}^{2d}} |a| + O(h^{\frac{1}{2}}).$$

1. *Show that*
$$\mathrm{Op}_h(a)u(x) = \big(\mathrm{Op}(a_h)u_h\big)(h^{-\frac{1}{2}}x)$$
*where*
$$a_h(x,\xi) = a(h^{\frac{1}{2}}x, h^{\frac{1}{2}}\xi),\ u_h(y) = u(h^{\frac{1}{2}}y).$$

2. *Deduce that there is a constant $C$ and an integer $M$ such that for all $a \in C_b^\infty(\mathbb{R}^{2d})$ and all $h \in (0,1]$,*
$$\begin{aligned}\left\|\mathrm{Op}_h(a)\right\|_{\mathcal{L}(L^2)} \leq C &\sup_{(x,\xi)\in\mathbb{R}^{2d}} |a(x,\xi)| \\ &+ C \sup_{1\leq|\alpha|+|\beta|\leq M}\ \sup_{(x,\xi)\in\mathbb{R}^{2d}} h^{\frac{1}{2}(|\alpha|+|\beta|)} \left|\partial_x^\alpha \partial_\xi^\beta a\right|.\end{aligned}$$

**Exercise 11.0.2** (Wave packet transformation)**.** *Let $u\colon \mathbb{R} \to \mathbb{C}$ in the class of Schwartz $\mathcal{S}(\mathbb{R})$. The wave packet transform of $u$ is the function $Wu\colon \mathbb{R} \to \mathbb{C}$*

*defined by*

$$Wu(x,\xi) = \int_{\mathbb{R}} e^{i(x-y)\xi - \frac{1}{2}(x-y)^2} u(y)\,\mathrm{d}y.$$

*1. Show that* $(x,\xi) \mapsto xWu(x,\xi)$ *and* $(x,\xi) \mapsto \xi Wu(x,\xi)$ *are bounded on* $\mathbb{R}^2$. *Show more generally that* $Wu$ *belongs to the Schwartz class* $\mathcal{S}(\mathbb{R}^2)$*.*

*2. Show that, for any* $x \in \mathbb{R}$*,*

$$\int |Wu(x,\xi)|^2\,\mathrm{d}\xi = 2\pi \int e^{-(x-y)^2/2}\,|u(y)|^2\,\mathrm{d}y.$$

*Deduce that there is a constant* $A > 0$ *such that, for every* $u$ *in* $\mathcal{S}(\mathbb{R})$*, we have*

$$\iint |Wu(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi = A \int |u(y)|^2\,\mathrm{d}y.$$

*(It is not required to calculate* $A$*.)*

*3. Show that for any function* $u$ *in the Schwartz class* $\mathcal{S}(\mathbb{R})$*,*

$$Wu(x,\xi) = c\,e^{ix\xi}\,(W\widehat{u})(\xi,-x),$$

*for a certain constant* $c$ *(it is not required to calculate* $c$*).*

*4. Let* $\varepsilon \in (0,1]$ *and* $u$ *in Schwartz's class* $\mathcal{S}(\mathbb{R}^2)$*. We introduce*

$$W^\varepsilon u(x,\xi) = \varepsilon^{-3/4} \int_{\mathbb{R}} e^{i(x-y)\cdot\xi/\varepsilon - (x-y)^2/2\varepsilon} u(y)\,\mathrm{d}y.$$

*Check that* $A^{-1/2}W^\varepsilon$ *is an isometry and then show that there is* $K$ *such that for all* $\varepsilon \in (0,1]$ *and all functions* $u$ *and* $v$ *in the Schwartz class* $\mathcal{S}(\mathbb{R})$*,*

$$\left\| vW^\varepsilon u - W^\varepsilon(vu) \right\|_{L^2(\mathbb{R}^2)} \leq K\varepsilon^{1/2} \|\partial_x v\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}.$$

*5. Show that there is* $K'$ *such that, for all* $\varepsilon \in (0,1]$ *and for all function* $u$ *in the Schwartz class* $\mathcal{S}(\mathbb{R})$*,*

$$\left\| i\xi W^\varepsilon u - W^\varepsilon(\varepsilon\partial_x u) \right\|_{L^2(\mathbb{R}^2)} \leq K'\varepsilon^{1/2} \|u\|_{L^2(\mathbb{R})}.$$

**Exercise 11.0.3.** *Let* $\chi \in C_0^\infty(\mathbb{R})$ *be such that*

$$\operatorname{supp}\chi \subset \{\xi \in \mathbb{R},\, 2^{-1/2} \leq |\xi| \leq 2^{1/2}\}, \quad \chi(\xi) = 1 \quad \textit{for} \quad 2^{-1/4} \leq |\xi| \leq 2^{1/4}.$$

*Set*

$$a(x,\xi) = \sum_{j=1}^{+\infty} \exp(-i2^j x)\chi(2^{-j}\xi).$$

1. *Show that $a \in C^\infty(\mathbb{R}^2)$ satisfies*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{|\alpha|-|\beta|} \quad \forall \alpha,\beta \in \mathbb{N}^2,\ \forall (x,\xi) \in \mathbb{R}^2.$$

*Does a belong to $S^0$ or $C_b^\infty(\mathbb{R}^2)$?*

2. *Let $f_0$ be a function in the Schwartz space whose Fourier transform $\widehat{f_0}$ is supported in the interval $[-1/2, 1/2]$. Given $N \in \mathbb{N}$ we set*

$$f_N(x) = \sum_{j=2}^{N} \frac{1}{j} \exp(i2^j x) f_0(x).$$

*Using Plancherel's theorem, prove that*

$$\|f_N\|_{L^2}^2 = \Big( \sum_{j=2}^{N} j^{-2} \Big) \|f_0\|_{L^2}^2 \leq c.$$

3. *Show that*

$$\mathrm{Op}(a) f_N = \Big( \sum_{j=2}^{N} j^{-1} \Big) f_0.$$

4. *Conclude.*

**Exercise 11.0.4.** *Let $a \in A^m(\mathbb{R}^N)$. Show that*

$$(2\pi)^{-N} \int e^{-iy\cdot x} a(y)\, \mathrm{d}y\, \mathrm{d}x = (2\pi)^{-N} \int e^{-iy\cdot x} a(x)\, \mathrm{d}y\, \mathrm{d}x = a(0).$$

**Exercise 11.0.5.** *Let $\alpha$ and $\beta$ be in $\mathbb{N}^N$. Show that*

$$(2\pi)^{-N} \int e^{-iy\cdot x} \frac{y^\alpha}{\alpha!} \frac{x^\beta}{\beta!}\, \mathrm{d}y\, \mathrm{d}x = \begin{cases} 0 \text{ if } \alpha \neq \beta, \\ (-i)^{|\alpha|}/\alpha! \text{ if } \alpha = \beta. \end{cases}$$

**Exercise 11.0.6** (Van der Corput's Lemma)**.** *We are interested in the behavior, when the parameter $\lambda$ tends to $+\infty$, of the oscillatory integrals*

$$I_{a,b,\phi}(\lambda) = \int_a^b e^{i\lambda\phi(x)}\, \mathrm{d}x,$$

*where $(a,b) \in \mathbb{R}^2$ and $phi \in C^2(\mathbb{R})$ is a real-valued function.*

*1) Suppose that there exist two constants $c, C > 0$ such that,*

$$\forall x \in [a,b], \quad |\phi'(x)| \geq c \quad et \quad |\phi''(x)| \leq C.$$

*Using the relation*

$$e^{i\lambda\phi} = \frac{1}{i\lambda\phi'}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{i\lambda\phi}\right),$$

*show that, for all $\lambda > 0$, we have*

$$\left|I_{a,b,\phi}(\lambda)\right| \leq \frac{1}{\lambda}\left(\frac{2}{c} + \frac{C(b-a)}{c^2}\right).$$

*2) Show that for all $(a,b) \in \mathbb{R}^2$, for all $\lambda > 0$ and for all functions $\phi \in C^2(\mathbb{R})$ such that $\phi'$ is monotone and does not vanish on $[a,b]$,*

$$\left|I_{a,b,\phi}(\lambda)\right| \leq \frac{4}{\inf_{a\leq x\leq b}|\phi'(x)|}\frac{1}{\lambda}.$$

*(One can use that $\int |(f(g(x))'|\,\mathrm{d}x = |\int (f(g(x)))'\,\mathrm{d}x|$ if $f$ and $g$ are two monotone functions).*

*3) Show that, for all $(a,b) \in \mathbb{R}^2$, all $\lambda > 0$ and any function $\phi \in C^2(\mathbb{R})$ satisfying $\phi'' \geq 1$ on $[a,b]$, we have*

$$\left|I_{a,b,\phi}(\lambda)\right| \leq \frac{10}{\lambda^{1/2}}.$$

*(One can use that $\{x \in [a,b] \,:\, |\phi'(x)| \leq \lambda^{-1/2}\}$ is an interval of length at most equal to $2\lambda^{-1/2}$.)*

**Exercise 11.0.7.** *1) Let $\psi \in C^1(\mathbb{R})$ be a real or complex valued function. Show that*

$$\int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx = \psi(b)I_{a,b,\phi}(\lambda) - \int_a^b \psi'(x)I_{a,x,\phi}(\lambda)\,dx.$$

*2) Deduce that, for all $(a,b) \in \mathbb{R}^2$, all $\lambda > 0$, any phase such that $\phi'' \geq 1$ on $[a,b]$, and any $\psi \in C^1(\mathbb{R})$,*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)\,\mathrm{d}x\right| \leq \frac{10}{\lambda^{1/2}}\left(|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right).$$

*3) (Application) Show that there exists a constant $C$ such that, for all $t \in \mathbb{R}$ and for all all $R > 0$,*

$$\left|\int_{-R}^{R} e^{i(\xi+t\xi^2)}\,|\xi|^{-1/2}\,\mathrm{d}\xi\right| \leq C.$$

**Exercise 11.0.8** (An inequality of Kenig, Ponce and Vega)**.** *The aim of this exercise is to prove the inequality*

$$\left\| \int_{\mathbb{R}} |D_x|^{-1/4} e^{-it\partial_x^2} g(t,x)\,\mathrm{d}t \right\|_{L^2_x} \leq C \|g\|_{L_x^{4/3}L_t^1}.$$

*In order to simplify, we can suppose that $g \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$ and that its Fourier transform $\widehat{g}(s,\xi)$ with respect to $x$ is supported in a compact $A \subset \mathbb{R}$ independent of $s$.*

*1) Show that it is sufficient to prove that*

$$\left\| \int_{\mathbb{R}} |D_x|^{-1/4} e^{i(t-s)\partial_x^2} g(s,x)\,\mathrm{d}s \right\|_{L^4_x L^\infty_t} \leq C \|g\|_{L_x^{4/3}L_t^1}.$$

*2) Show that*

$$\int_{\mathbb{R}} |D_x|^{-1/4} e^{i(t-s)\partial_x^2} g(s,x)\,\mathrm{d}s = \iint K(s-t,x-y) g(s,y)\,\mathrm{d}y\,\mathrm{d}s,$$

*with*

$$K(t,x) = \int_A e^{i(x\xi + t\xi^2)} |\xi|^{-1/2}\,\mathrm{d}\xi.$$

*3) Conclude by using the exercise 11.0.7 and the theorem of Hardy–Littlewood–Sobolev.*

*4) Show that the inequality we have proved implies that the solution $u = u(t,x)$ of*

$$i\partial_t u + \partial_x^2 u = 0, \quad u_{\mathcal{A}rowvert t=0} = u_0 \in \mathcal{S}(\mathbb{R}),$$

*satisfies*

$$(*) \qquad \|u\|_{L^4_x L^\infty_t} \leq C \left\| |D_x|^{1/4} u_0 \right\|_{L^2}.$$

*Using the $TT^*$ argument, deduce the previous estimate from the one established in question* 1.

*5) Compare (*$*$*) with that obtained by an energy estimate.*

**Exercise 11.0.9.** *Show the following improvement of Proposition 6.6.3. Let $m \in \mathbb{R}_+$ and $a \in S^m(\mathbb{R}^d)$. Suppose that there exist two constants $c, R$ such that,*

$$|\xi| \geq R \Rightarrow \operatorname{Re} a(x,\xi) \geq c\,|\xi|^m.$$

*Then, for all N there exists a constant $C_N$ such that,*

$$\operatorname{Re}(\operatorname{Op}(a)u, u) \geq \frac{c}{2} \|u\|^2_{H^{m/2}} - C_N \|u\|^2_{H^{-N}} ,$$

*for all $u \in \mathcal{S}(\mathbb{R}^d)$. [We will use the following inequalities: i) $2xy \leq \eta x^2 + (1/\eta)y^2$ and ii), for all $\varepsilon > 0$ and all $N > 0$, there exists $C_{\varepsilon,N} > 0$ such that*

$$\|u\|_{H^{-1}} \leq \varepsilon \|u\|_{L^2} + C_{\varepsilon,N} \|u\|_{H^{-N}} .$$

*Which results from the easy inequality $\langle \xi \rangle^{-2} \leq \varepsilon^2 + C_{\varepsilon,N} \langle \xi \rangle^{-2N}$.*

**Exercise 11.0.10.** *Let E be a Banach space and let $F\colon [0, +\infty[ \times E \to E$ be a continuous application satisfying the following property: There exists $C > 0$ such that,*

$$\forall t \in [0, +\infty[, \quad \forall (x, y) \in E \times E, \quad \|F(t, x) - F(t, y)\|_E \leq C \|x - y\|_E .$$

*The aim of this problem is to give two proofs of the fact that, for all $u_0$ in E, there exists a unique function $u \in C^1([0, +\infty[; E)$ solution of*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(t, u), \quad u|_{t=0} = u_0.$$

1. *We are looking for a solution u of the equation $\Phi(u) = u$ with*

$$\Phi(u) = u_0 + \int_0^t F(s, u(s))\, \mathrm{d}s.$$

   *Given $T > 0$, we denote $X_T = C^0([0, T]; E)$. Show that $\Phi$ is a contraction of $X_T$ in $X_T$ for T small enough.*

2. *Deduce that, if T is small enough, then for all $u_0$ in E, there exists a unique function $u \in C^1([0, T]; E)$ solution of*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(t, u), \quad u|_{t=0} = u_0.$$

   *Then deduce the existence of a solution defined for all time by fitting together solutions defined on time intervals of length T.*

3. *We want to give another argument which allows us to obtain directly a global existence result. Given a parameter $\lambda > 0$, let us introduce the space of functions with at most exponential growth of factor $\lambda$:*

$$X = \left\{ u \in C^0([0,+\infty[;E)\,,\ \sup_{t\in[0,+\infty[} e^{-\lambda t}\,\|u(t)\|_E < +\infty \right\}.$$

*Verify that it is a Banach space for the norm*

$$\|u\|_X = \sup_{t\in[0,+\infty[} e^{-\lambda t}\,\|u(t)\|_E\,.$$

*Let $u$ belong to $X$. Show that $\Phi(u)$ also belongs to $X$. Show furthermore that for all $u$ and all $v$ in $X$, we have*

$$\|\Phi(u) - \Phi(v)\|_X \le \frac{C}{\lambda}\,\|u - v\|_X\,.$$

*Conclude.*

**Exercise 11.0.11.** *Consider a Banach space $(B, \|\cdot\|_B)$ and a normed vector space $(V, \|\cdot\|_V)$. Let us consider two continuous linear operators $L_0\colon B \mapsto V$ and $L_1\colon B \mapsto V$. For $t$ in $[0,1]$, we define*

$$L_t = (1-t)L_0 + tL_1.$$

*It is assumed that there exists a constant $A > 0$ such that*

$$\forall t \in [0,1], \quad \forall u \in B, \quad \|u\|_B \le A\,\|L_t u\|_V\,.$$

1. *Suppose that $L_s$ is surjective for some $s \in [0,1]$. Show that $L_s$ is bijective and that its inverse is a continuous linear application satisfying*

$$\left\|L_s^{-1}\right\|_{V\to B} \le A.$$

2. *Let $f \in V$ and let $s \in [0,1]$ be such that $L_s$ is surjective. Observe that for all $t \in [0,1]$,*

$$f = L_t u \quad \Leftrightarrow \quad f = L_s u + (t-s)(L_1 - L_0)u.$$

*Introduce an application $T_{s,t} colon B \to B$ depending on $f$ and $s, t$ and verifying the two properties:*

$$(i) \qquad f = L_t u \quad \Leftrightarrow \quad u = T_{s,t}(u),$$

*and*

$$(ii) \qquad T_{s,t} \text{ is a contraction if } \quad |t-s| < \delta = \frac{1}{A(\|L_0\|_{B\to V} + \|L_1\|_{B\to V})}.$$

3. *Deduce that $L_t$ is surjective for all $tin[0, 1]$ such that $|t - s| \in [0, \delta[$. Then show that if $L_0$ is surjective then $L_1$ is surjective.*

# Chapter 12

# Problems

## 12.1 The div-curl lemma of Murat and Tartar

Notations : Let $\Omega$ be an open of $\mathbb{R}^d$. Let $C^\infty(\overline{\Omega})$ be the space of functions $u\colon \Omega \to \mathbb{R}$ which are the restriction to $\Omega$ of functions $C^\infty$ on $\mathbb{R}^d$. We denote by $C_0^\infty(\Omega)$ the set of those functions $C^\infty$ with compact support in $\Omega$.

Let us consider a bounded open $\Omega \subset \mathbb{R}^2$ and two sequences of vector fields vectors, $E_n\colon \Omega \to \mathbb{R}^2$ and $B_n\colon \Omega \to \mathbb{R}^2$. We note $(E_n^1, E_n^2)$ and $(B_n^1, B_n^2)$ the coordinates of $E_n$ and $B_n$. It is assumed that:

(H1) $E_n$ and $B_n$ belong to $C^\infty(\overline{\Omega})^2$ for any integer $n$.

(H2) We have

$$\sup_{n\in\mathbb{N}} \Big( \|E_n\|_{L^2(\Omega)^2} + \|B_n\|_{L^2(\Omega)^2} + \|E_n\|_{L^2(\Omega)} + \|\operatorname{curl} B_n\|_{L^2(\Omega)} \Big) < +\infty,$$

where $\|E_n\|_{L^2(\Omega)^2} = \sqrt{\left\|E_n^1\right\|_{L^2(\Omega)}^2 + \left\|E_n^2\right\|_{L^2(\Omega)}^2}$, and where $\operatorname{div} E_n$ and $\operatorname{curl} B_n$ are functions with values in $\mathbb{R}$ defined by

$$\operatorname{div} E_n = \partial_{x_1} E_n^1 + \partial_{x_2} E_n^2, \qquad \operatorname{curl} B_n = \partial_{x_2} B_n^1 - \partial_{x_1} B_n^2.$$

(H3) There exist $E \in L^2(\Omega)^2$ and $B \in L^2(\Omega)^2$ such that $E_n \rightharpoonup E$ and $B_n \rightharpoonup B$ in $L^2(\Omega)^2$, which means that each coordinate converges weakly ($E_n^j \rightharpoonup E^j$ for $j = 1, 2$ and similarly for $B_n$).

The goal of this exercise is to show that, for all $\varphi \in C_0^\infty(\Omega)$,

$$(*) \qquad \int_\Omega \varphi(x) E_n(x) \cdot B_n(x)\,\mathrm{d}x \underset{n\to+\infty}{\longrightarrow} \int_\Omega \varphi(x) E(x) \cdot B(x)\,\mathrm{d}x,$$

where $y \cdot y'$ is the scalar product of $\mathbb{R}^2$.

1. Let us fix a function $\varphi \in C_0^\infty(\Omega)$ and consider $\chi \in C_0^\infty(\Omega)$ with $\chi \equiv 1$ on the support of $\varphi$, so that $\chi\varphi = \varphi$. We introduce

$$v_n = \varphi E_n, \quad w_n = \chi B_n, \quad v = \varphi E, \quad w = \chi B.$$

   We extend these functions by 0 on $\mathbb{R}^2 \backslash \Omega$ (and we always note them $v_n, w_n, v, w$). Show that $v_n$ and $w_n$ belong to $H^1(\mathbb{R}^2)^2$. Show that $v_n$ converges weakly to $v$ in $L^2(\mathbb{R}^2)^2$ and that similarly $w_n$ converges weakly to $w$ in $L^2(\mathbb{R}^2)^2$.

2. If $f = (f^1, f^2)$ is a function with values in $\mathbb{R}^2$, we note $\widehat{f} = (\widehat{f^1}, \widehat{f^2})$ its Fourier transform. Show that $(\widehat{v}_n)$ and $(\widehat{w}_n)$ are bounded in $L^2(\mathbb{R}^2)^2$ and in $L^\infty(\mathbb{R}^2)^2$.

3. Show that $(*)$ is equivalent to

$$(**) \qquad \int_{\mathbb{R}^2} \widehat{v}_n(\xi) \cdot \overline{\widehat{w}_n(\xi)}\,\mathrm{d}\xi \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2} \widehat{v}(\xi) \cdot \overline{\widehat{w}(\xi)}\,\mathrm{d}\xi.$$

4. Show that, for all $\xi \in \mathbb{R}^2$, the sequences $(\widehat{v}_n(\xi))_{n\in\mathbb{N}}$ and $(\widehat{w}_n(\xi))_{n\in\mathbb{N}}$ converge to $\widehat{v}(\xi)$ and $\widehat{w}(\xi)$, respectively.

5. Let $R > 0$. Let $B(0, R)$ be the ball of center 0 and radius $R$ in $\mathbb{R}^2$. Show that

$$\int_{B(0,R)} \widehat{v}_n(\xi) \cdot \overline{\widehat{w}_n(\xi)}\,\mathrm{d}\xi \underset{n\to+\infty}{\longrightarrow} \int_{B(0,R)} \widehat{v}(\xi) \cdot \overline{\widehat{w}(\xi)}\,\mathrm{d}\xi.$$

6. Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ different from zero. Set $\xi^\perp = (\xi_2, -\xi_1)$. Show that, for all $X \in \mathbb{R}^2$, we have

$$X = \left(X \cdot \frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|} + \left(X \cdot \frac{\xi^\perp}{|\xi|}\right)\frac{\xi^\perp}{|\xi|}.$$

   Then show that for all $X, Y$ in $\mathbb{R}^2$ we have

$$|X \cdot Y| \leq \frac{1}{|\xi|}\Big(|Y|\,|X \cdot \xi| + |X|\,\big|Y \cdot \xi^\perp\big|\Big).$$

7. Show that the sequences of functions $\xi \mapsto \xi \cdot \widehat{v}_n(\xi)$ and $\xi \mapsto \xi^\perp \cdot \widehat{w}_n(\xi)$ are bounded in $L^2(\mathbb{R}^2)$.

8. Deduce that for all $R > 0$ we have

$$\sup_{n\in\mathbb{N}} \int_{|\xi|>R} \widehat{v}_n(\xi) \cdot \overline{\widehat{w}_n(\xi)}\, \mathrm{d}\xi \underset{R\to+\infty}{\longrightarrow} 0,$$

and conclude the demonstration of ($*$).

## 12.2 Continuity on Hölder spaces

We denote $C, C_\alpha, C_{\alpha,\beta}, \ldots$ absolute constants (where $C_\alpha$ depends on the multi-index $\alpha$...) which do not depend on the symbols, nor on the unknowns.

The aim of this problem is to study the action of a pseudo-differential operator

$$\mathrm{Op}(p)u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} p(x,\xi)\widehat{u}(\xi)\, \mathrm{d}\xi,$$

on the Hölder spaces $C^{0,r}(\mathbb{R}^d)$ with $r \in ]0,1[$.

$*$ $*$ $*$ **Preliminary** $*$ $*$ $*$

Let $M > 0$ be fixed. Let $q(x,\xi)$ be a function $C^\infty$ on $\mathbb{R}^d \times \mathbb{R}^d$, supported in $\{(x,\xi) : |\xi| \le 3\}$. We suppose that, for all $\beta \in \mathbb{N}^n$ such that $|\beta| \le \frac{d}{2} + 2$, we have

$$\forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \qquad \left|\partial_\xi^\beta q(x,\xi)\right| \le M.$$

1. Let us introduce $Q(x,z) = (2\pi)^{-n} \displaystyle\int_{\mathbb{R}^d} e^{iz\cdot\xi} q(x,\xi)\, \mathrm{d}\xi$. Using the relation $\partial_x i e^{iz\cdot\xi} = ize^{iz\cdot\xi}$, show that, for $\alpha \in \mathbb{N}^n$, we can write the function $z \mapsto z^\alpha Q(x,z)$ in the form of a Fourier transform of a function that we will specify.

2. Deduce that, for all $|\alpha| \le (d/2) + 2$, there exists a constant $C_\alpha$ such that

$$\int |z|^{2\alpha} |Q(x,z)|^2 \, \mathrm{d}z \le C_\alpha M^2.$$

   Then deduce that $\int |Q(x,z)|\,, dz \le CM$.

3. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Show that $\mathrm{Op}(q)f(x) = \displaystyle\int Q(x, x-y)f(y)\,\mathrm{d}y$ then deduce

$$\|\mathrm{Op}(q)f\|_{L^\infty} \leq CM\,\|f\|_{L^\infty}.$$

$*\,*\,*$ **Stein's theorem** $*\,*\,*$

Consider a symbol $p = p(x,\xi)$ which is $C^\infty$ on $\mathbb{R}^d \times \mathbb{R}^d$ and such that

$$\forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \left|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{-|\beta|}.$$

We showed in class that $\mathrm{Op}(p)$ is bounded from $L^2(\mathbb{R}^d)$ in itself. We will show in this problem that $\mathrm{Op}(p)$ is bounded by $C^{0,r}(\mathbb{R}^d)$ in itself for all $r \in (0,1[$.

We recall that Hölder spaces can be studied using the Littlewood-Paley decomposition. To fix the notations, let us recall that there exist two functions $\chi_0$ and $\chi$, $C^\infty$ on $\mathbb{R}^d$, supported respectively in the ball $\{|\xi| \leq 1\}$ and in the annulus $\{1/3 \leq |\xi| \leq 3\}$ and such that:

$$\forall \xi \in \mathbb{R}^d, \quad \chi_0(\xi) + \sum_{j=0}^{+\infty} \chi(2^{-j}\xi) = 1.$$

Let's introduce

$$p_{-1}(x,\xi) = p(x,\xi)\chi_0(\xi), \quad p_j(x,\xi) = p(x,\xi)\chi(2^{-j}\xi) \quad \text{for} \quad j \in \mathbb{N}.$$

1. Show that there exists $M > 0$ such that, for all $\beta \in \mathbb{N}^n$ satisfying $|\beta| \leq (d/2)+2$, we have

$$\begin{aligned} &\left|\partial_\xi^\beta p_{-1}(x,\xi)\right| \leq M, \\ &\left|\partial_\xi^\beta p_j(x,\xi)\right| \leq M 2^{-j|\beta|} \qquad (\forall j \in \mathbb{N}). \end{aligned}$$

2. Show that

$$\|\mathrm{Op}(p_{-1})f\|_{L^\infty} \leq C\,\|f\|_{L^\infty}.$$

3. Let $a = a(x,\xi)$ be a symbol and $\lambda \in \mathbb{R}_*^+$. Let us set $b(x,\xi) = a\left(\dfrac{x}{\lambda}, \lambda\xi\right)$. Let us note $H_\lambda$ the application which to the function $u = u(x)$ associates

$$(H_\lambda u)(x) = u(\lambda x).$$

Show that $\mathrm{Op}(a) = H_\lambda \circ \mathrm{Op}(b) \circ H_\lambda^{-1}$ and deduce that, if $\mathrm{Op}(b)$ is bounded from $L^\infty$ to $L^\infty$, then $\mathrm{Op}(a)$ is also bounded and then they have the same norm.

4. For all $j \in \mathbb{N}$, show that we can choose $\lambda_j$ so that $\tilde{p}_j(x,\xi) = p_j(\lambda_j^{-1}x, \lambda_j\xi)$ is supported in $\{(x,\xi) \,:\, |\xi| \leq 3\}$. Deduce that

$$\left\|\mathrm{Op}(p_j)f\right\|_{L^\infty} \leq C \left\|f\right\|_{L^\infty}.$$

5. Let us introduce $f_{-1} = \chi_0(D_x)f$ and $f_k = \chi(2^{-k}D_x)f$ for $k \geq 0$. Show that

$$\mathrm{Op}(p_j)f = \sum_{|j-k|\leq 3} \mathrm{Op}(p_j)f_k.$$

6. Show that there exists $C > 0$ such that for all $j \in \mathbb{N}$ and all $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\left\|\mathrm{Op}(p_j)f\right\|_{L^\infty} \leq C \left\|f\right\|_{C^{0,r}} 2^{-jr}.$$

7. Let $\alpha \in \mathbb{N}^n$ satisfy $|\alpha| \leq 1$. Show that for all $j \in \mathbb{N} \cup \{-1\}$, we have

$$\partial_x^\alpha \mathrm{Op}(p_j) = \mathrm{Op}(q_j) \quad \text{where} \quad q_j(x,\xi) = \big((i\xi + \partial_x)^\alpha p_j\big)(x,\xi).$$

By repeating the previous steps, show that

$$\left\|\partial_x^\alpha \mathrm{Op}(p_j)f\right\|_{L^\infty} \leq C_\alpha 2^{j(|\alpha|-r)} \left\|f\right\|_{C^{0,r}}.$$

8. $(*)$ Let $(f_j)$ be a sequence of functions $C^1(\mathbb{R}^d)$ which satisfy

$$\left\|\partial_x^\alpha f_j\right\|_{L^\infty} \leq M 2^{j(|\alpha|-r)} \quad \text{pour tout } |\alpha| \leq 1.$$

Show that $f = \sum f_j$ belongs to $C^{0,r}(\mathbb{R}^d)$ and that its norm is bounded by $CM$.

9. Conclude: $\mathrm{Op}(p)$ is bounded from $C^{0,r}(\mathbb{R}^d)$ in itself for all $r \in (0,1[$. What can we say for $r = 0$?

### $* * *$ Relation between $L^\infty$ and $C^0_*$ $* * *$

i) Show that there exists a constant $C$ such that, for all $\varepsilon \in (0,1)$ and any function $f$ belonging to the Hölder space $C^{0,\varepsilon}(\mathbb{R}^d)$, we have

$$\left\|f\right\|_{L^\infty} \leq \frac{C}{\varepsilon} \left\|f\right\|_{C^0_*} \log\left(e + \frac{\left\|f\right\|_{C^{0,\varepsilon}}}{\left\|f\right\|_{C^0_*}}\right).$$

Hint: use the Littlewood-Paley decomposition and, for $N \in \mathbb{N}$ to choose, write that

$$\|f\|_{L^\infty} \leq \sum_{q \leq N-1} \left\|\Delta_q f\right\|_{L^\infty} + \sum_{q \geq N} \left\|\Delta_q f\right\|_{L^\infty}.$$

ii) Consider the distribution

$$u = \sum_{q=0}^{\infty} e^{i2^q x}.$$

Show that $u \in C^0_* \setminus L^\infty$.

## 12.3 Sums of squares of vector fields

Warning : this problem is very difficult. The goal is to show a famous result of Lars Hörmander on the hypoellipticity of some sums of squares of vector fields.

$* * *$ **Notations** $* * *$

We consider only real-valued functions defined on an open set of $\mathbb{R}^d$ with $n \geq 1$ an arbitrary integer. Given $s \in \mathbb{R}$, we denote $H^s(\mathbb{R}^d)$ the Sobolev space of order $s$ and $\langle D_x \rangle^s$ the Fourier multiplier of symbol $\langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$. We denote $\langle u, v \rangle$ the scalar product on $L^2(\mathbb{R}^d)$, $\langle u, v \rangle = \int_{\mathbb{R}^d} u(x)v(x)\,dx$.

Given two operators $A$ and $B$, we denote $AB$ the compound $A \circ B$ (so $A^2 = A \circ A$) and $[A, B] = AB - BA$ their commutator.

In this problem we are interested in the second order operator

$$L = \sum_{1 \leq j \leq m} X_j^2,$$

or $X_1, \ldots, X_m$ are differential operators of order 1 : for $1 \leq j \leq m$, $X_j$ is defined by

$$(X_j u)(x) = \sum_{1 \leq i \leq n} a_{i,j}(x) \frac{\partial u}{\partial x_i}(x),$$

where $a_{i,j}$ is $C^\infty$ on $\mathbb{R}^d$ and has values in $\mathbb{R}$ for all $1 \leq i \leq n$. Note that we only assume that functions $a_{i,j}$ are $C^\infty$ (and not $C^\infty$ and bounded as well as their derivatives). For instance, we wish to study the case $L = \partial_x^2 + x^2\partial_y^2 = X_1^2 + X_2^2$ with $X_1 = \partial_x$ and $X_2 = x\partial_y$.

### $* * *$ Preliminary questions $* * *$

1. Show that the adjoint $X_j^*$ of $X_j$ satisfies $X_j^* u = -X_j u + c_j u$ where $c_j \in C^\infty(\mathbb{R}^d)$ is a function that we will determine. That is, show that for all $u, v \in C_0^\infty(\mathbb{R}^d)$,
$$\int_{\mathbb{R}^d} (X_j u)(x)v(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \big(-u(x)(X_j v)(x) + c_j(x)u(x)v(x)\big)\,\mathrm{d}x.$$
2. Show that there exists a constant $C > 0$ such that, for all $u \in C_0^\infty(\mathbb{R}^d)$,
$$\sum_{1\le j\le m} \left\|X_j u\right\|_{L^2}^2 \le C\left\|Lu\right\|_{L^2}^2 + C\left\|u\right\|_{L^2}^2.$$

### $* * *$ Study a class of operators $* * *$

Let us fix a bounded open set $V$.

We denote $Psi_V^0$ the set of operators $P \in \mathcal{L}(L^2(\mathbb{R}^d))$ which can be written as
$$Pu = \varphi_1 \operatorname{Op}(a)\big(\varphi_2 u\big)$$
with

- $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^d)$ and $\operatorname{supp}\varphi_k \subset V$ for $k = 1, 2$;
- $a$ is a symbol with complex values belonging to $S^0$.

Let $\varepsilon \in (0, 1/2]$. We denote $\mathcal{A}_\varepsilon$ the set of operators $P \in \Psi_V^0$ such that
$$\exists C > 0 / \ \forall u \in C_0^\infty(V), \quad \left\|Pu\right\|_{H^\varepsilon}^2 \le C\left\|Lu\right\|_{L^2}^2 + C\left\|u\right\|_{L^2}^2.$$

1. Show that if $P_1$ and $P_2$ belong to $\Psi_V^0$ then $P_1P_2 \in \Psi_V^0$ and $P_1^* \in \Psi_V^0$.
2. Show that if $P \in \mathcal{A}_\varepsilon$ then $P^* \in \mathcal{A}_\varepsilon$.

3. Show that for all $\varepsilon \in (0, 1/2]$, $\mathcal{A}_\varepsilon$ is stable by composition on the left or on the right by a pseudo of $\Psi^0_V$: this means that if $P \in \mathcal{A}_\varepsilon$ and $Q \in \Psi^0_V$, then
$$QP \in \mathcal{A}_\varepsilon, \quad PQ \in \mathcal{A}_\varepsilon.$$

4. Let $\theta_1$ and $\theta_2$ be two functions $C^\infty$ with compact support such that $\operatorname{supp}\theta_k \subset V$ for $k = 1, 2$ and $\theta_1 \equiv 1$ on the support of $\theta_2$. Let us introduce the operator $S$ defined by
$$Su = \theta_1 \langle D_x \rangle^{-1} (\theta_2 u).$$
Show that $X_j S \in \Psi^0_V$. Show moreover that, for all $1 \le j \le n$ and $\varepsilon \in [0, 1/2]$, we have
$$X_j S \in \mathcal{A}_\varepsilon.$$

5. Let $\varepsilon, \delta \in (0, 1/2]$ with $\delta \le \varepsilon/2$. Let us consider $P \in \mathcal{A}_\varepsilon$. We want to show in this question that
$$[X_j, P] \in \mathcal{A}_\delta$$
for all $1 \le j \le n$.

   $(a)$ Write $\left\|[X_j, P]u\right\|^2_{H^\delta}$ in the form $\langle [X_j, P]u, Tu\rangle$ where $T = \operatorname{Op}(\tau)$ is a pseudo-differential operator with $\tau \in S^{2\delta}$.

   $(b)$ Show that there exists a constant $C > 0$ such that
$$\left|\langle PX_j u, Tu\rangle\right| \le \left\|X_j u\right\|^2_{L^2} + \|TP^* u\|^2_{L^2} + C\,\|u\|^2_{H^{2\delta-1}}.$$

   $(c)$ Obtain a similar estimate for $\left|\langle X_j Pu, Tu\rangle\right|$ and conclude.

6. We denote $\mathcal{A}$ the set of operators $P \in \Psi^0_V$ such that $P \in \mathcal{A}_\varepsilon$ for a certain $\varepsilon \in (0, 1/2]$. That is $P \in \mathcal{A}_\varepsilon$ if and only if
$$\exists \varepsilon \in ]0, 1/2],\ \exists C > 0/\ \forall u \in C^\infty_0(V), \quad \|Pu\|^2_{H^\varepsilon} \le C\,\|Lu\|^2_{L^2} + C\,\|u\|^2_{L^2}.$$
Let $1 \le i, j \le m$. Show that the commutator $[X_i, X_j] = X_i X_j - X_j X_i$ is a differential operator of order 1.

   Show that for all $1 \le j, k \le m$, we have
$$[X_j, X_k]S \in \mathcal{A}.$$
(Hint: observe that $[X_j, X_k S]$ belongs to $\mathcal{A}$.)

**∗ ∗ ∗ The sublaplacian on the Heisenberg group ∗ ∗ ∗**

Let us consider the case of the space dimension $n = 3$. Consider the operator

$$L = X^2 + Y^2$$

with

$$X = \partial_{x_2} + 2x_1\partial_{x_3}, \quad Y = \partial_{x_1} - 2x_2\partial_{x_3}.$$

1. Let $V$ be a bounded open set. Show that, for all $1 \leq k \leq 3$,

   $$\exists \varepsilon \in ]0, 1/2], \ \exists C > 0 / \ \forall u \in C_0^\infty(V), \quad \left\| \partial_{x_k}(Su) \right\|_{H^\varepsilon}^2 \leq C \left\| Lu \right\|_{L^2}^2 + C \left\| u \right\|_{L^2}^2 .$$

2. Deduce that for all compact $K \subset \mathbb{R}^d$ there exists $\varepsilon > 0$ and a constant $C > 0$ such that, for all $u \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} u \subset K$,

   $$\|u\|_{H^\varepsilon}^2 \leq C \|Lu\|_{L^2}^2 + C \|u\|_{L^2}^2 .$$

$* * *$ **General case** $* * *$

We identify a differential operator $X = \sum_{1 \leq i \leq n} a_i \partial_{x_i}$, with the vector field $a = (a_1, \ldots, a_n) \colon \mathbb{R}^d \to \mathbb{R}^d$. Given $x \in \mathbb{R}^d$, we denote $X(x)$ the vector $a(x) \in \mathbb{R}^d$.

We consider again a general operator $L = \sum_{1 \leq j \leq m} X_j^2$ and we suppose that there exists $r \in \mathbb{N}^*$ such that for all $x \in \mathbb{R}^d$,

$$\operatorname{vect}\left\{ \left( [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{p-1}}, X_{i_p}] \ldots ]](x) \ : \ p \leq r, \ i_k \in \{1, \ldots, m\} \right\} = \mathbb{R}^n.$$

$(a)$ Show that this condition is satisfied for the following two examples:

- $n = 2$, $X_1 = \partial_x$ and $X_2 = x\partial_y$.
- $n = 4$ (we denote $(x, y, z, t)$ the coordinates of a point of $\mathbb{R}^4$), $X_1 = \partial_x$, $X_2 = \frac{1}{2}x^2\partial_t + x\partial_z + \partial_y$.

$(b)$ Show that

$$[X_{i_1}, [X_{i_2}, \ldots [X_{i_{p-1}}, X_{i_p}] \ldots ]]S \in \mathcal{A}$$

for all $p$-uple of indices.

$(c)$ Show that for all compact $K \subset \mathbb{R}^d$ there exists $\varepsilon > 0$ and a constant $C > 0$ such that, for all $u \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} u \subset K$,

$$\|u\|_{H^\varepsilon}^2 \leq C \|Lu\|_{L^2}^2 + C \|u\|_{L^2}^2 .$$

$* * *$ **Complement** $* * *$

1) We now consider the operator

$$\mathcal{L} = \sum_{1\leq j\leq m} X_j^2 + X_0,$$

where $X_0, X_1, \ldots, X_m$ are $m+1$ vector fields as before.

Show that there exists a constant $C$ such that, for all $u \in C_0^\infty(\mathbb{R}^d)$,

$$\sum_{1\leq i\leq m} \|X_i u\|_{L^2}^2 \leq C\,\|\mathcal{L}u\|_{L^2}^2 + C\,\|u\|_{L^2}^2\,.$$

We say that $P \in \Psi_V^0$ belongs to $\mathcal{A}_{\mathcal{L}}$ if and only if

$$\exists \varepsilon \in ]0, 1/2],\ \exists C > 0/\ \forall u \in C_0^\infty(V), \quad \|Pu\|_{H^\varepsilon}^2 \leq C\,\|\mathcal{L}u\|_{L^2}^2 + C\,\|u\|_{L^2}^2\,.$$

Show that, with $S$ as before and $P \in \Psi_V^0$,

$$X_0 S \in \mathcal{A}_{\mathcal{L}}, \quad [X_0, P] \in \mathcal{A}_{\mathcal{L}}$$

and that, for all $0 \leq i, j \leq m$, we have $[X_j, X_k]S \in \mathcal{A}_{\mathcal{L}}$.

Deduce that $\mathcal{A}_{\mathcal{L}} = \Psi_V^0$ if there exists $r \in \mathbb{N}$ such that

$$\text{vect}\left\{\big([X_{i_1}, [X_{i_2}, \ldots, [X_{i_{p-1}}, X_{i_p}]\ldots]](x)\ :\ p \leq r,\ i_k \in \{0, \ldots, m\}\right\} = \mathbb{R}^n.$$

2) We have thus shown that for all compact $K \subset \mathbb{R}^d$ there exist $\varepsilon > 0$ and $C > 0$ such that, for all $u \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} u \subset K$,

$$\|u\|_{H^\varepsilon}^2 \leq C\,\|\mathcal{L}u\|_{L^2}^2 + C\,\|u\|_{L^2}^2\,.$$

We say that this is a subelliptic estimate. Using this estimate and the structure of $L$, show that $L$ is hypoelliptic: which means that if $u \in \mathcal{D}'(\Omega)$ satisfies $Lu \in C^\infty(\omega)$ with $\omega \subset \Omega$ then $u \in C^\infty(\omega)$.

Nevertheless, an operator can satisfy a subelliptic estimate without being hypoelliptic. Consider for instance the operator $\Box u = (\partial_t^2 - \partial_x^2)u$. Then this operator is not hypoelliptic (there exist non $C^\infty$ solutions of $\Box u = 0$) but it satisfies the previous estimate. Show (in a direct way) that there exists a constant $C$ such that for all $u \in C_0^\infty(\mathbb{R}^2)$,

$$\|u\|_{H^1}^2 \leq C\,\|\Box u\|_{L^2}^2 + C\,\|u\|_{L^2}^2\,.$$

## 12.4 Regularizing effect for Schrödinger and Airy

Let $n \geq 1$ be. Let $L^2(\mathbb{R}^d)$ be the space of complex valued functions and integrable square, with scalar product

$$(f,g) = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,\mathrm{d}x.$$

Given two operators $A$ and $B$, let $AB$ be the operator $A \circ B$ and $[A,B] = AB - BA$. the commutator of $A$ and $B$.

Let $m \in \mathbb{N}$. Consider a symbol $a \in S^m(\mathbb{R}^d)$ and set $A = \mathrm{Op}(a)$. Let $A^*$ be the adjoint of $A$ and assume that $A - A^*$ is an operator of order 0, so that

$$\forall f \in H^m(\mathbb{R}^d), \quad \|Af - A^*f\|_{L^2(\mathbb{R}^d)} \leq K_0 \|f\|_{L^2(\mathbb{R}^d)}. \tag{12.4.1}$$

We fix a time $T > 0$ and consider a function $u \in C^1([0,T]; H^m(\mathbb{R}^d))$ solution of

$$\partial_t u = iAu. \tag{12.4.2}$$

We admit the existence of such a solution.

**1.** Let us consider the following operators:

- $A_1 = \Delta$;
- $A_2 = \mathrm{div}(\gamma(x)\nabla\cdot)$ where $\gamma \in C_b^\infty(\mathbb{R}^d;\mathbb{R})$ (which means that $\gamma$ is a real-valued function, $C^\infty$ and bounded on $\mathbb{R}^d$, as well as all its derivatives);
- $n = 1$ and $A_3 = i\partial_x^3 + iV(x)\partial_x$ where $V \in C_b^\infty(\mathbb{R};\mathbb{R})$.

Write these operators in the form $A_j = \mathrm{Op}(a_j)$ where $a_j$ is a symbol of order $m_j$ (for a $m_j$ to specify). Then check that these operators verify the hypothesis (12.4.1).

**2.** Let $f, g \in C^1([0,T]; H)$ where $H$ is a Hilbert space with scalar product $(\cdot,\cdot)_H$. Show that the function $(f,g)_H \colon t \mapsto (f(t), g(t))_H$ is $C^1$ and that

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(t),g(t))_H = \left(\frac{\mathrm{d}}{\mathrm{d}t}(t), g(t)\right)_H + \left(f(t), \frac{\mathrm{d}g}{\mathrm{d}t}(t)\right)_H.$$

Consider a symbol $b = b(x,\xi)$ belonging to $S^0(\mathbb{R}^d)$. We pose $B = \mathrm{Op}(b)$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(Bu(t), u(t)) = \big(i[B,A]u(t), u(t)\big) + \big(Bu(t), i(A - A^*)u(t)\big).$$

**3.** Applying this with $b = 1$ show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|^2_{L^2(\mathbb{R}^d)} \leq K_0 \|u(t)\|^2_{L^2(\mathbb{R}^d)}$$

where $K_0$ is defined by (12.4.1). Then deduce that there exists a constant $K_1$ depending only on $T$ and $K_0$ such that

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(\mathbb{R}^d)} \leq K_1 \|u(0)\|^2_{L^2(\mathbb{R}^d)} .$$

**4.** Consider $b \in S^0(\mathbb{R}^d)$ and set

$$C = i[B, A].$$

From the previous questions, it can be deduced that there exists a constant $K_2$ (depending only on $T, A, B$) such that

$$\int_0^T (Cu(t), u(t))\,\mathrm{d}t \leq K_2 \|u(0)\|^2_{L^2(\mathbb{R}^d)} .$$

**5.** Suppose $d = 1$ and $A = \partial_x^2$.

**a.** Write $C$ in the form $\mathrm{Op}(p) + R$ where $p \in S^1(\mathbb{R})$ is a symbol depending on $b$ to be calculated and $R$ is an operator of order 0. Deduce from the above that there exists a constant $K_3$ depending only on $T, A, B$ such that

$$\int_0^T (\mathrm{Op}(p)u(t), u(t))\,\mathrm{d}t \leq K_3 \|u(0)\|^2_{L^2(\mathbb{R})} .$$

**b.** Let us choose

$$b(x,\xi) = -\frac{1}{2}\frac{\xi}{\langle\xi\rangle}\int_0^x \frac{\mathrm{d}y}{\langle y\rangle^2} \quad \text{where} \quad \langle\zeta\rangle = (1 + |\zeta|^2)^{1/2}.$$

Check that $b \in S^0(\mathbb{R})$ and that

$$\mathrm{Op}(p) = -\langle x\rangle^{-2}\Lambda^{-1}\partial_x^2 \quad \text{where} \quad \Lambda^{-1} = \mathrm{Op}\left(\langle\xi\rangle^{-1}\right).$$

**c.** Deduce that there is a constant $K_4$ (depending only on $T, A$) such that

$$\int_0^T \left\|\partial_x\Lambda^{-\frac{1}{2}}\left(\langle x\rangle^{-1}u(t)\right)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}t \leq K_4 \|u(0)\|^2_{L^2(\mathbb{R})} .$$

Then show that

$$\int_0^T \left\| \langle x \rangle^{-1} u(t) \right\|^2_{H^{\frac{1}{2}}(\mathbb{R})} \mathrm{d}t \leq K_4 \left\| u(0) \right\|^2_{L^2(\mathbb{R})} . \tag{12.4.3}$$

**6.** Suppose $n = 1$ and $A = i\partial_x^3 + iV(x)\partial_x$. Let $M > 0$ and consider an increasing function $\varphi \in C^\infty(\mathbb{R};\mathbb{R})$ such that $\varphi(x) = x$ if $|x| \leq M$ and $\varphi'(x) = 0$ if $|x| \geq 2M$. Let $b(x,\xi) = \varphi(x)$ (independent of $\xi$). Write $C$ in the form $\mathrm{Op}(p) + R$ where $R$ is of order 0 (be careful to use the symbolic calculus with the right order) and check that $C = 3\partial_x(\varphi'(x)\partial_x\cdot) + R$ where $R$ is of order 0. Deduce that

$$\int_0^T \int_{-M}^M |\partial_x u(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t \leq K \int_{\mathbb{R}} |u(0,x)|^2 \,\mathrm{d}x,$$

for a constant $K$ depending only on $T$ and $M$.

**7.** $(*)$ Show that the estimate (12.4.3) is true for $A = \Delta$ in any dimension $d$.

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