Tools and Problems in Partial Differential Equations

Thomas Alazard and Claude Zuily

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Contents

Introduction

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The aim of this book is to present, through sixty completely solved long problems, various aspects of the current theory of partial differential equations (pde). It is intended for graduate students who would like, through practice, to test their understanding of the theory.

Even though the main purpose of this book is to present these problems, for the reader's convenience, we have recalled some of the main theoretical results concerning each topic. This is why each chapter of problems is preceded by a short introduction recalling without proof the basic facts. This makes the book essentially self-contained for the reader. At the end of each introduction a short bibliography is given where one may find the details of proofs. The solutions of the problems are gathered at the end of the book.

Since the theory of partial differential equations is a very wide subject, it is by no means realistic to hope to describe all the topics in a single volume. Therefore choices have to be made and we have chosen to focus on a few of them.

Let us now describe more precisely the contents of this book.

In the first chapter we have recalled some essential tools which are commonly used in pde: main theorems in functional analysis, distributions, Fourier analysis. Problems follow. The second chapter is devoted to the description of the main function spaces used in pde and it contains problems on Sobolev, Hölder, Zygmund (including their Littlewood-Paley description), weak Lebesgue spaces, space of bounded mean oscillation as well as other spaces. Other tools, such as interpolation theory and paraproducts, are also discussed and used. The third chapter is concerned with the theory of microlocal analysis and contains problems on pseudo-differential operators, para-differential operators, microlocal defect measures etc. The forth chapter is devoted to the main partial differential equations currently discussed in the literature. It contains problems on the Laplace operator and its spectral theory, on the heat equation, on the linear and non linear wave and Schrödinger equations as well as problems on Kinetic, Benjamin-Ono, Burgers and Euler equations.

The detailed solutions to all the problems are gathered in the last and main part of the book.

In an appendix we have gathered several fundamental results concerning the basics of classical analysis, such as Lebesgue integration, differential calculus, differential equations and holomorphic functions.

T. Alazard and C. Zuily