LOW MACH NUMBER FLOWS, AND COMBUSTION

THOMAS ALAZARD

Abstract. We prove uniform existence results for the full Navier-Stokes equations for time intervals which are independent of the Mach number, the Reynolds number and the Péclet number. We consider general equations of state and we give an application for the low Mach number limit combustion problem introduced by Majda in [18].

1. Introduction

For a fluid with density ϱ, velocity v, pressure P, temperature T, internal energy e, Lamé coefficients ζ, η and coefficient of thermal conductivity k, the full Navier-Stokes equations, written in a non-dimensional way, are

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \dfrac{\nabla P}{\varepsilon^2} = \mu\big(2\operatorname{div}(\zeta D v) + \nabla(\eta \operatorname{div} v)\big), \\ \partial_t(\rho e) + \operatorname{div}(\rho v e) + P \operatorname{div} v = \kappa \operatorname{div}(k\nabla T) + Q, \end{cases} \tag{1.1}$$

where $\varepsilon \in (0,1]$, $(\mu,\kappa) \in [0,1]^2$ and Q is a given source term (see [11, 16, 18]). In order to be closed, the system is supplemented with a thermodynamic closure law, so that ρ, P, e, T are completely determined by only two of these variables. Also, it is assumed that ζ, η and k are smooth functions of the temperature.

This paper is devoted to the asymptotic limit where the Mach number ε tends to 0. We are interested in proving results independent of the Reynolds number $1/\mu$ and the Péclet number $1/\kappa$. Our main result asserts that the classical solutions of (1.1) exist and are uniformly bounded on a time interval independent of ε, μ and κ.

This is a continuation of our previous work [1] where the study was restricted to perfect gases and small source terms Q of size $O(\varepsilon)$. We refer to the introduction of [1] for references and a short historical survey of the background of these problems (see also the survey papers of Danchin [9], Desjardins and Lin [10], Gallagher [13], Schochet [24] and Villani [26]).

The case of perfect gases is interesting in its own: first, perfect gases are widely studied in the physical literature; and second, it contains the important analysis of the singular terms. Yet, modeling real gases requires general equations of state (see [4, 19]). Moreover, we shall see that it is interesting to consider large source terms Q for it allows us to answer a question addressed by Majda in [18] concerning the combustion equations.

1.1. **The equations.** To be more precise, we begin by rewriting the equations under the form

$$L(u, \partial_t, \partial_x)u + \frac{1}{\varepsilon} S(u, \partial_x)u = 0,$$

which is the classical framework of a singular limit problem.

Before we proceed, three observations are in order. Firstly, for the low Mach number limit problem, the point is not so much to use the conservative form of the equations, but instead to balance the acoustics components. This is one reason it is interesting to work with the unknowns P, v, T (see [18]). Secondly, the general case must allow for large density and temperature variations as well as very large acceleration of order of the inverse of the Mach number (see Section 5 in [16]). Since $\partial_t v$ is of order of $\varepsilon^{-2}\nabla P$, this suggests that we seek P under the form $P = \mathrm{Cte} + O(\varepsilon)$. As in [20], since P and T are positive functions, it is pleasant to set

$$P = \underline{P} e^{\varepsilon p}, \quad T = \underline{T} e^{\theta}, \tag{1.2}$$

where $\underline{P}$ and $\underline{T}$ are given positive constants, say the reference states at spatial infinity. Finally, the details of the following computations are given in the Appendix.

From now on, the unknown is (p, v, θ) with values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$. We are interested in the general case where p and θ are uniformly bounded in ε (so that $\nabla T = O(1)$ and $\partial_t v = O(\varepsilon^{-1})$).

By assuming that ρ and e are given smooth functions of (P, T), it is found that, for smooth solutions of (1.1), (P, v, T) satisfies a system of the form:

$$\begin{cases} \alpha(\partial_t P + v \cdot \nabla P) + \operatorname{div} v = \kappa\beta \operatorname{div}(k\nabla T) + \beta Q, \\ \rho(\partial_t v + v \cdot \nabla v) + \dfrac{\nabla P}{\varepsilon^2} = \mu\big(2\operatorname{div}(\zeta D v) + \nabla(\eta \operatorname{div} v)\big), \\ \gamma(\partial_t T + v \cdot \nabla T) + \operatorname{div} v = \kappa\delta \operatorname{div}(k\nabla T) + \delta Q, \end{cases} \tag{1.3}$$

where the coefficients α, β, γ and δ are smooth functions of (P, T). Then, by writing $\partial_{t,x} P = \varepsilon P \partial_{t,x} p$, $\partial_{t,x} T = T\partial_{t,x}\theta$ and redefining the functions k, ζ and η, it is found that (p, v, θ) satisfies a system of the form:

$$\begin{cases} g_1(\phi)(\partial_t p + v \cdot \nabla p) + \dfrac{1}{\varepsilon} \operatorname{div} v = \dfrac{\kappa}{\varepsilon}\chi_1(\phi) \operatorname{div}(k(\theta)\nabla\theta) + \dfrac{1}{\varepsilon}\chi_1(\phi) Q, \\ g_2(\phi)(\partial_t v + v \cdot \nabla v) + \dfrac{1}{\varepsilon}\nabla p = \mu B_2(\phi, \partial_x) v, \\ g_3(\phi)(\partial_t \theta + v \cdot \nabla \theta) + \operatorname{div} v = \kappa\chi_3(\phi) \operatorname{div}(k(\theta)\nabla\theta) + \chi_3(\phi) Q, \end{cases} \tag{1.4}$$

where $\phi := (\theta, \varepsilon p)$ and $B_2(\phi, \partial_x) = \chi_2(\phi) \operatorname{div}(\zeta(\theta) D\cdot) + \chi_2(\phi)\nabla(\eta(\theta) \operatorname{div} \cdot)$.

We are now in position to explain the main differences between ideal gases and general gases. Firstly, we note that the source term Q introduces an arbitrary unsigned large term of order of $1/\varepsilon$ in the equations. Secondly, to emphasize the role of the thermodynamics, we suppose now that $Q = 0$ and we mention that, for perfect gases, the coefficient $\chi_1(\phi)$ is a function of εp

alone (see Proposition A.8). Hence, for perfect gases, the limit constraint is linear in the sense that it reads $\operatorname{div} v_e = 0$ with $v_e = v - \kappa\chi_1(0)k(\theta)\nabla\theta$. By contrast, for general equations of state, the limit constraint is nonlinear.

1.2. **Assumptions.** To avoid confusion, we denote by $(\vartheta, \wp) \in \mathbb{R}^2$ the place holder of the unknown $(\theta, \varepsilon p)$. Hereafter, it is assumed that:

(H1) The functions ζ, η and k are C^∞ functions of $\vartheta \in \mathbb{R}$, satisfying $k > 0$, $\zeta > 0$ and $\eta + 2\zeta > 0$.

(H2) The functions g_i and χ_i $(i = 1, 2, 3)$ are C^∞ positive functions of $(\vartheta, \wp) \in \mathbb{R}^2$. Moreover,
$$\chi_1 < \chi_3,$$
and there exist two functions F and G such that $(\vartheta, \wp) \mapsto (F(\vartheta, \wp), \wp)$ and $(\vartheta, \wp) \mapsto (\vartheta, G(\vartheta, \wp))$ are C^∞ diffeomorphisms from $\mathbb{R}^2$ onto $\mathbb{R}^2$, $F(0, 0) = G(0, 0) = 0$ and
$$g_1 \frac{\partial F}{\partial \vartheta} = -g_3 \frac{\partial F}{\partial \wp} > 0, \qquad g_1\chi_3 \frac{\partial G}{\partial \vartheta} = -g_3\chi_1 \frac{\partial G}{\partial \wp} < 0.$$

Remark 1.1. Assumption (H2) is used to prove various energy estimates. The main hypothesis is the inequality $\chi_1 < \chi_3$. In Appendix A, it is proved that the inequality $\chi_1 < \chi_3$ holds whenever the density ρ and the energy e are C^∞ functions of $(P, T) \in (0, +\infty)^2$, such that $\rho > 0$ and

$$P\frac{\partial \rho}{\partial P} + T\frac{\partial \rho}{\partial T} = \rho^2\frac{\partial e}{\partial P}, \quad \frac{\partial \rho}{\partial P} > 0, \quad \frac{\partial \rho}{\partial T} < 0, \quad \frac{\partial e}{\partial T}\frac{\partial \rho}{\partial P} > \frac{\partial e}{\partial P}\frac{\partial \rho}{\partial T}\cdot \tag{1.5}$$

1.3. **Main result.** We are interested in the case without smallness assumption: namely, we consider general initial data, general equations of state and large source terms Q. To get around the above mentioned nonlinear features of the penalization operator, we establish a few new qualitative properties. These properties are enclosed in various uniform stability results, which assert that the classical solutions of (1.4) exist and they are uniformly bounded for a time independent of ε, μ and κ. We concentrate below on the whole space problem or the periodic case and we work in the Sobolev spaces H^σ endowed with the norms $\|u\|_{H^\sigma} := \left\|(I - \Delta)^{\sigma/2}u\right\|_{L^2}$.

The following result is the core of all our other uniform stability results. On the technical side, it contains the idea that one can prove uniform estimates without uniform control of the L^2_x norm of the velocity v.

Theorem 1.2. *Let $d = 1$ or $d \geqslant 3$ and $\mathbb{N} \ni s > 1 + d/2$. For all source term $Q = Q(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$ and all $M_0 > 0$, there exist $T > 0$ and $M > 0$ such that, for all $(\varepsilon, \mu, \kappa) \in (0, 1] \times [0, 1] \times [0, 1]$ and all initial data $(p_0, v_0, \theta_0) \in H^{s+1}(\mathbb{R}^d)$ satisfying*

$$\|(\nabla p_0, \nabla v_0)\|_{H^{s-1}} + \|(\theta_0, \varepsilon p_0, \varepsilon v_0)\|_{H^{s+1}} \leqslant M_0, \tag{1.6}$$

the Cauchy problem for (1.4) has a unique classical solution $(p, v, \theta) \in C^0([0, T]; H^{s+1}(\mathbb{R}^d))$ such that

$$\sup_{t\in[0,T]} \|(\nabla p(t), \nabla v(t))\|_{H^{s-1}} + \|(\theta(t), \varepsilon p(t), \varepsilon v(t))\|_{H^s} \leqslant M. \tag{1.7}$$

A refined statement is proved in Section 3.

A notable corollary of Theorem 1.2 is Theorem 4.1, which is the requested result for application to the low Mach number limit. Detailed discussions of the periodic case and the combustion equations are included in Sections 5 and 6. The assumption $d \neq 2$ is explained in Remark 2.6.

2. Preliminaries

In order not to interrupt the proofs later on, we collect here some estimates. The main result of this section is Proposition 2.4, which complements the Friedrichs-type estimate

$$\left\lVert \nabla v \right\rVert_{H^s} \leqslant \left\lVert \operatorname{div} v \right\rVert_{H^s} + \left\lVert \operatorname{curl} v \right\rVert_{H^s} , \tag{2.1}$$

which is immediate using Fourier transform. We prove a variant where $\operatorname{div} v$ is replaced by $\operatorname{div}(\rho v)$ where ρ is a positive weight.

Notation. The symbol $\lesssim$ stands for $\leqslant$ up to a positive, multiplicative constant, which depends only on parameters that are considered fixed.

2.1. Nonlinear estimates.

Throughout the paper, we will make intensive and often implicit uses of the following estimates.

For all $\sigma \geqslant 0$, there exists K such that, for all $u, v \in L^\infty \cap H^\sigma(\mathbb{R}^d)$,

$$\left\lVert uv \right\rVert_{H^\sigma} \leqslant K \left\lVert u \right\rVert_{L^\infty} \left\lVert v \right\rVert_{H^\sigma} + K \left\lVert u \right\rVert_{H^\sigma} \left\lVert v \right\rVert_{L^\infty} . \tag{2.2}$$

For all $s > d/2$, $\sigma_1 \geqslant 0$, $\sigma_2 \geqslant 0$ such that $\sigma_1 + \sigma_2 \leqslant 2s$, there exists a constant K such that, for all $u \in H^{s-\sigma_1}(\mathbb{R}^d)$ and $v \in H^{s-\sigma_2}(\mathbb{R}^d)$,

$$\left\lVert uv \right\rVert_{H^{s-\sigma_1-\sigma_2}} \leqslant K \left\lVert u \right\rVert_{H^{s-\sigma_1}} \left\lVert v \right\rVert_{H^{s-\sigma_2}} . \tag{2.3}$$

For all $s > d/2$ and for all C^∞ function F vanishing at the origin, there exists a smooth function C_F such that, for all $u \in H^s(\mathbb{R}^d)$,

$$\left\lVert F(u) \right\rVert_{H^s} \leqslant C_F(\left\lVert u \right\rVert_{L^\infty}) \left\lVert u \right\rVert_{H^s} . \tag{2.4}$$

2.2. Estimates in $\mathbb{R}^3$.

Consider the Fourier multiplier $\nabla\Delta^{-1}$ with symbol $-i\xi/\left\lvert \xi \right\rvert^2$. This operator is, at least formally, a right inverse for the divergence operator. The only thing we will use below is that $\nabla\Delta^{-1}u$ is well defined whenever $u = u_1 u_2$ with $u_1, u_2 \in L^\infty \cap H^\sigma(\mathbb{R}^d)$ for some $\sigma \geqslant 0$.

Proposition 2.1. *Given $d \geqslant 3$ and $\sigma \in \mathbb{R}$, the Fourier multiplier $\nabla\Delta^{-1}$ is well defined on $L^1(\mathbb{R}^d) \cap H^\sigma(\mathbb{R}^d)$ with values in $H^{\sigma+1}(\mathbb{R}^d)$. Moreover, there exists a constant K such that, for all $u \in L^1(\mathbb{R}^d) \cap H^\sigma(\mathbb{R}^d)$,*

$$\left\lVert \nabla\Delta^{-1} u \right\rVert_{H^{\sigma+1}} \leqslant K \left\lVert u \right\rVert_{L^1} + K \left\lVert u \right\rVert_{H^\sigma} . \tag{2.5}$$

Proof. Set $\langle \xi \rangle := (1 + \left\lvert \xi \right\rvert^2)^{1/2}$. It suffices to check that the L^2-norm of $(\langle \xi \rangle^{\sigma+1}/\left\lvert \xi \right\rvert) \left\lvert \widehat{u}(\xi) \right\rvert$ is estimated by the right-hand side of (2.5). To do that we write

$$\int_{\left\lvert \xi \right\rvert \leqslant 1} \frac{\langle \xi \rangle^{2\sigma+2}}{\left\lvert \xi \right\rvert^2} \left\lvert \widehat{u}(\xi) \right\rvert^2 \, d\xi \lesssim \left\lVert u \right\rVert_{L^1}^2 , \quad \int_{\left\lvert \xi \right\rvert \geqslant 1} \frac{\langle \xi \rangle^{2\sigma+2}}{\left\lvert \xi \right\rvert^2} \left\lvert \widehat{u}(\xi) \right\rvert^2 \, d\xi \lesssim \left\lVert u \right\rVert_{H^\sigma}^2 ,$$

where we used $1/\left\lvert \xi \right\rvert^2 \in L^1(\{\left\lvert \xi \right\rvert \leqslant 1\})$ for all $d \geqslant 3$. □

The next proposition is well known. Its corollary is a special case of a general estimate established in [5].

Proposition 2.2. *Given $d \geqslant 3$ and $s > d/2$, there exists a constant K such that, for all $u \in H^s(\mathbb{R}^d)$,*

$$\left\| u \right\|_{L^\infty} \leqslant K \left\| \nabla u \right\|_{H^{s-1}} . \tag{2.6}$$

Proof. Since $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$, it suffices to prove the result for u in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. Now, starting from the Fourier inversion theorem, the Cauchy–Schwarz inequality yields the desired estimate:

$$\left\| u \right\|_{L^\infty} \leqslant \left(\int \frac{d\xi}{\left|\xi\right|^2 \langle \xi \rangle^{2(s-1)}} \right)^{1/2} \left(\int \langle \xi \rangle^{2(s-1)} \left| \xi \widehat{u}(\xi) \right|^2 \, d\xi \right)^{1/2} \lesssim \left\| \nabla u \right\|_{H^{s-1}} .$$

Corollary 2.3. *Given $d \geqslant 3$ and $\mathbb{N} \ni s > d/2$, there exists a constant K such that, for all $u_1, u_2 \in H^s(\mathbb{R}^d)$,*

$$\left\| u_1 u_2 \right\|_{H^s} \leqslant K \left\| \nabla u_1 \right\|_{H^{s-1}} \left\| u_2 \right\|_{H^s} . \tag{2.7}$$

Proof. One has to estimate the L^2-norm of $\partial_x^\alpha(u_1 u_2)$, where $\alpha \in \mathbb{N}^d$ satisfies $\left|\alpha\right| \leqslant s$. Rewrite this term as $u_1 \partial_x^\alpha u_2 + [\partial_x^\alpha, u_1] u_2$. Since the commutator is a sum of terms of the form $\partial_x^\beta u_1 \partial_x^\gamma u_2$ with $\beta > 0$, the product rule (2.3) implies that

$$\left\| [\partial_x^\alpha, u_1] u_2 \right\|_{L^2} \lesssim \left\| \nabla u_1 \right\|_{H^{s-1}} \left\| u_2 \right\|_{H^s} . \tag{2.8}$$

Moving to the estimate of the first term, we write

$$\left\| u_1 \partial_x^\alpha u_2 \right\|_{L^2} \leqslant \left\| u_1 \right\|_{L^\infty} \left\| u_2 \right\|_{H^s} \lesssim \left\| \nabla u_1 \right\|_{H^{s-1}} \left\| u_2 \right\|_{H^s}. \qquad \square$$

2.3. **A Friedrichs' Lemma.** With these preliminaries established, we are prepared to prove the following:

Proposition 2.4. *Let $d \geqslant 3$ and $\mathbb{N} \ni s > d/2$. There exists a function $\mathcal{C}$ such that, for all $\varphi \in H^{s+1}(\mathbb{R}^d)$ and all vector field $v \in H^{s+1}(\mathbb{R}^d)$,*

$$\left\| \nabla v \right\|_{H^s} \leqslant C \left\| \operatorname{div}(e^\varphi v) \right\|_{H^s} + C \left\| \operatorname{curl} v \right\|_{H^s} , \tag{2.9}$$

where $C := (1 + \left\| \varphi \right\|_{H^{s+1}}) \mathcal{C}\big(\left\| \varphi \right\|_{H^s}, \left\| \nabla \varphi \right\|_{L^\infty} \big)$.

Proof. For this proof, we use the notation

$$R = \left\| \operatorname{div}(e^\varphi v) \right\|_{H^s} + \left\| \operatorname{curl} v \right\|_{H^s} ,$$

and we denote by C_φ various constants depending only on $\left\| \varphi \right\|_{H^s} + \left\| \nabla \varphi \right\|_{L^\infty}$.

All the computations given below are meaningful since it is sufficient to prove (2.9) for C^∞ functions with compact supports. We begin by setting

$$\widetilde{v} = v + \nabla \Delta^{-1} \big(\nabla \varphi \cdot v \big).$$

The reason to introduce $\widetilde{v}$ is that

$$e^\varphi \operatorname{div} \widetilde{v} = \operatorname{div}(e^\varphi v), \qquad \operatorname{curl} \widetilde{v} = \operatorname{curl} v.$$

Hence, by using (2.1), we have

$$\|\nabla\widetilde{v}\|_{H^s} \leqslant \left\|e^{-\varphi}\operatorname{div}(e^{\varphi}v)\right\|_{H^s} + \|\operatorname{curl} v\|_{H^s} \leqslant C_\varphi R. \tag{2.10}$$

The proof of (2.9) thus reduces to estimating $v_1 := v - \widetilde{v}$, which satisfies

$$\operatorname{div}(e^{\varphi}v_1) = -e^{\varphi}\nabla\varphi\cdot\widetilde{v}, \qquad \operatorname{curl} v_1 = 0.$$

Again, to estimate v_1 we introduce $\widetilde{v}_1 := v_1 + \nabla\Delta^{-1}\big(\nabla\varphi\cdot v_1\big)$, which solves

$$\operatorname{div}\widetilde{v}_1 = -\nabla\varphi\cdot\widetilde{v}, \qquad \operatorname{curl}\widetilde{v}_1 = 0.$$

The estimate (2.1) implies that $\|\nabla\widetilde{v}_1\|_{H^s} \leqslant \|\nabla\varphi\cdot\widetilde{v}\|_{H^s}$. By using (2.10) and the product rule (2.7), applied with $u_1 = \widetilde{v}$ and $u_2 = \nabla\varphi$, we find that

$$\|\nabla\widetilde{v}_1\|_{H^s} \leqslant \|\nabla\varphi\cdot\widetilde{v}\|_{H^s} \lesssim \|\varphi\|_{H^{s+1}}\|\nabla\widetilde{v}\|_{H^{s-1}} \leqslant \|\varphi\|_{H^{s+1}} C_\varphi R. \tag{2.11}$$

Hence, it remains only to estimate $v_2 = v_1 - \widetilde{v}_1$, which satisfies

$$\operatorname{div}(e^{\varphi}v_2) = -e^{\varphi}\nabla\varphi\cdot\widetilde{v_1} \quad \text{and} \quad \operatorname{curl} v_2 = 0.$$

To estimate v_2 the key point is the estimate

$$\|\nabla\varphi\cdot\widetilde{v}_1\|_{L^{d^\star}} \leqslant C_\varphi R, \quad \text{with } d^\star = 2d/(d+2). \tag{2.12}$$

Let us assume (2.12) for a moment and continue the proof.

The constraint $\operatorname{curl} v_2 = 0$ implies that $v_2 = \nabla\Psi$, for some Ψ satisfying

$$\operatorname{div}(e^{\varphi}\nabla\Psi) = -e^{\varphi}\nabla\varphi\cdot\widetilde{v}_1.$$

This allows us to estimate $\nabla\Psi$ by a duality argument. We denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the scalar product in L^2 and write

$$\langle\, e^{\varphi}\nabla\Psi\,,\,\nabla\Psi\,\rangle = \langle\, e^{\varphi}\nabla\varphi\cdot\widetilde{v}_1\,,\,\Psi\,\rangle.$$

Denote by $\overline{d}$ the conjugate exponent of $d^\star$, $\overline{d} = d^\star/(d^\star-1) = 2d/(d-2)$. The Holder's inequality yields

$$\langle\, e^{\varphi}\nabla\Psi\,,\,\nabla\Psi\,\rangle \leqslant \|e^{\varphi}\nabla\varphi\cdot\widetilde{v}_1\|_{L^{d^\star}}\|\Psi\|_{L^{\overline{d}}}.$$

The first factor is estimated by means of the claim (2.12). In view of the Sobolev's inequality $\|\Psi\|_{L^{\overline{d}}} \lesssim \|\nabla\Psi\|_{L^2}$, we obtain

$$\langle\, e^{\varphi}\nabla\Psi\,,\,\nabla\Psi\,\rangle \leqslant C_\varphi R\,\|\nabla\Psi\|_{L^2}.$$

By using the elementary estimate $\|\nabla\Psi\|_{L^2}^2 \leqslant \|e^{-\varphi}\|_{L^\infty}\langle\, e^{\varphi}\nabla\Psi\,,\,\nabla\Psi\,\rangle$, we get

$$\|v_2\|_{L^2} = \|\nabla\Psi\|_{L^2} \leqslant C_\varphi R. \tag{2.13}$$

The end of the proof is straightforward. We write

$$\Delta\Psi = e^{-\varphi}\operatorname{div}(e^{\varphi}\nabla\Psi) - \nabla\varphi\cdot\nabla\Psi = -\nabla\varphi\cdot\widetilde{v}_1 - \nabla\varphi\cdot\nabla\Psi,$$

to obtain, for all $\sigma\in[0,s-1]$,

$$\|\nabla\Psi\|_{H^{\sigma+1}} \lesssim \|\nabla\Psi\|_{L^2} + \|\Delta\Psi\|_{H^\sigma} \lesssim \|\nabla\varphi\cdot\widetilde{v}_1\|_{H^\sigma} + (1+\|\varphi\|_{H^s})\|\nabla\Psi\|_{H^\sigma}.$$

To estimate the first term on the right-hand side, we verify that the analysis establishing (2.7) also yields

$$\|\nabla\varphi\cdot\widetilde{v}_1\|_{H^{s-1}} \lesssim \|\varphi\|_{H^s}\|\nabla\widetilde{v}_1\|_{H^{s-1}} \leqslant C_\varphi R,$$

hence, by induction on σ,

$$\|\nabla\Psi\|_{H^s} \leqslant C_\varphi R + C_\varphi \|\nabla\Psi\|_{L^2}.$$

Exactly as above, one has

$$\|\nabla\Psi\|_{H^{s+1}} \lesssim \|\nabla\Psi\|_{L^2} + \|\Delta\Psi\|_{H^s} \lesssim \|\nabla\varphi\cdot\widetilde{v}_1\|_{H^s} + (1+\|\varphi\|_{H^{s+1}})\|\nabla\Psi\|_{H^s}$$
$$\|\nabla\varphi\cdot\widetilde{v}_1\|_{H^s} \lesssim \|\varphi\|_{H^{s+1}}\|\nabla\widetilde{v}_1\|_{H^{s-1}} \leqslant \|\varphi\|_{H^{s+1}} C_\varphi R.$$

As a consequence, we end up with

$$\|\nabla\Psi\|_{H^{s+1}} \leqslant \|\varphi\|_{H^{s+1}} (C_\varphi R + C_\varphi \|\nabla\Psi\|_{L^2}).$$

Therefore, the L^2 estimate (2.13) implies that

$$\|v_2\|_{H^{s+1}} = \|\nabla\Psi\|_{H^{s+1}} \leqslant \|\varphi\|_{H^{s+1}} C_\varphi R.$$

By combining this estimate with (2.11), we find that

$$\|\nabla v_1\|_{H^s} \leqslant \|\varphi\|_{H^{s+1}} C_\varphi R.$$

From the definition of v_1 and (2.10), we obtain the desired bound (2.9).

We now have to establish the claim (2.12).

With $\overline{d} = 2d/(d-2)$ as above, the Sobolev's inequality and (2.10) imply that

$$\|\widetilde{v}\|_{L^{\overline{d}}} \lesssim \|\nabla\widetilde{v}\|_{L^2} \leqslant C_\varphi R.$$

On the other hand, the Hölder's inequality yields

$$\|\nabla\varphi\cdot\widetilde{v}\|_{L^\delta} \lesssim \|\nabla\varphi\|_{L^2}\|\widetilde{v}\|_{L^{\overline{d}}}, \qquad \text{with } \delta = \frac{2\overline{d}}{2+\overline{d}} = \frac{d}{d-1}\cdot$$

By interpolating this estimate with $\|\nabla\varphi\cdot\widetilde{v}\|_{L^{\overline{d}}} \lesssim \|\nabla\varphi\|_{L^\infty}\|\widetilde{v}\|_{L^{\overline{d}}}$, we obtain

$$\forall p\in[\delta,\overline{d}], \quad \|\nabla\varphi\cdot\widetilde{v}\|_{L^p} \lesssim \|\nabla\varphi\|_{L^2\cap L^\infty}\|\widetilde{v}\|_{L^{\overline{d}}} \leqslant C_\varphi R.$$

Because $\operatorname{curl} v_1 = 0$, one can write $v_1 = \nabla\Psi_1$ for some function Ψ_1 satisfying $\Delta\Psi_1 = -\nabla\varphi\cdot\widetilde{v}$. Hence, the Calderon–Zygmund inequality and the previous estimate imply that

$$\|\nabla v_1\|_{L^\delta} = \left\|\nabla^2\Psi_1\right\|_{L^\delta} \lesssim \|\Delta\Psi_1\|_{L^\delta} \leqslant C_\varphi R.$$

Therefore, the Sobolev's inequality yields

$$\|v_1\|_{L^D} \leqslant C_\varphi R, \qquad \text{with } D = \frac{\delta d}{d-\delta} = \frac{d}{d-2},$$

hence, exactly as above, the Hölder's inequality gives

$$\forall p\in[\underline{d},\overline{d}], \quad \|e^\varphi\nabla\varphi\cdot\widetilde{v}_1\|_{L^p} \leqslant C_\varphi R, \quad \text{with } \underline{d} = \frac{2D}{2+D} = \frac{2d}{3d-4}\cdot \tag{2.14}$$

The key estimate (2.12) is now a consequence of the previous one. Indeed, the estimate (2.14) applies with $p = d^\star = 2d/(d+2)$ since

$$\forall d \geqslant 3, \qquad \underline{d} = \frac{2d}{3d-4} \leqslant \frac{2d}{d+2} \leqslant \frac{2d}{d-2} = \overline{d}.$$

This completes the proof of (2.9). □

For later references, we will need the following version of (2.9).

Corollary 2.5. *Let $d = 1$ or $d \geqslant 3$ and $\mathbb{N} \ni s > d/2$. There exists a function $\mathcal{C}$ such that, for all $\varphi \in H^{s+1}(\mathbb{R}^d)$ and all vector field $v \in H^{s+1}(\mathbb{R}^d)$,*

$$\|\nabla v\|_{H^s} \leqslant \mathcal{C}(\|\varphi\|_{H^{s+1}})\big(\|\operatorname{div} v\|_{H^s} + \|\operatorname{curl}(e^{\varphi}v)\|_{H^s}\big). \tag{2.15}$$

Proof. The case $d = 1$ is obvious. If $d \geqslant 3$, Proposition 2.4 (applied with (φ, v) replaced with $(-\varphi, e^{\varphi}v)$) yields

$$\|\nabla(e^{\varphi}v)\|_{H^s} \leqslant \mathcal{C}(\|\varphi\|_{H^{s+1}})\big(\|\operatorname{div} v\|_{H^s} + \|\operatorname{curl}(e^{\varphi}v)\|_{H^s}\big).$$

Hence, to prove (2.15) we need only prove that

$$\|\nabla v\|_{H^s} \leqslant \mathcal{C}(\|\varphi\|_{H^{s+1}})\,\|\nabla(e^{\varphi}v)\|_{H^s}\,. \tag{2.16}$$

To do that we write $\partial_i v = e^{-\varphi}\partial_i(e^{\varphi}v) - (e^{-\varphi}\partial_i\varphi)(e^{\varphi}v)$. The usual product rule (2.3) implies that the H^s norm of the first term is estimated by the right-hand side of (2.16). Moving to the second term, we use the product rule (2.7) to obtain $\|(e^{-\varphi}\partial_i\varphi)(e^{\varphi}v)\|_{H^s} \lesssim (1 + \|\varphi\|_{H^s})\,\|\partial_i\varphi\|_{H^s}\,\|\nabla(e^{\varphi}v)\|_{H^{s-1}}$. This proves the desired bound (2.16). □

Remark 2.6. The fact that Theorem 1.2 precludes the case $d = 2$ is a consequence of the fact that we do not know if (2.15) holds for $d = 2$.

3. Uniform stability

In this section, we prove Theorem 1.2. We follow closely the approach given in [1]: we recall the scheme of the analysis and indicate the points at which the argument must be adapted.

Hereafter, we use the notations

$$a := (\varepsilon, \mu, \kappa) \in A := (0, 1] \times [0, 1] \times [0, 1], \qquad \nu := \sqrt{\mu + \kappa},$$
$$\|u\|_{H^{\sigma+1}_{\alpha}} := \|u\|_{H^{\sigma}} + \alpha\,\|u\|_{H^{\sigma+1}} \qquad (\alpha \geqslant 0,\ \sigma \in \mathbb{R}).$$

Step 1: a refined statement. We first give our main result a refined form where the solutions satisfy the same estimates as the initial data do. Also, to prove estimates independent of μ and κ, an important point is to seek the solutions in spaces which take into account an extra damping effect for the penalized terms.

Definition 3.1. *Let $T > 0$, $a = (\varepsilon, \mu, \kappa) \in [0, 1]^3$ and set $\nu = \sqrt{\mu + \kappa}$. The space $\mathcal{X}^s_a(T)$ consists of these $(p, v, \theta) \in C^0([0, T]; H^s(\mathbb{R}^d))$ such that*

$$\nu(p, v, \theta) \in C^0([0, T]; H^{s+1}(\mathbb{R}^d)), \quad (\mu v, \kappa\theta) \in L^2(0, T; H^{s+2}(\mathbb{R}^d)).$$

The space $\mathcal{X}^s_a(T)$ is given the norm

$$\begin{aligned}\|(p, v, \theta)\|_{\mathcal{X}^s_a(T)} := {}& \|(\nabla p, \nabla v)\|_{L^\infty_T(H^{s-1})} + \|(\theta, \varepsilon p, \varepsilon v)\|_{L^\infty_T(H^{s+1}_{\nu})} \\ & + \sqrt{\mu}\,\|\nabla v\|_{L^2_T(H^{s+1}_{\varepsilon\nu})} + \sqrt{\kappa}\,\|\nabla\theta\|_{L^2_T(H^{s+1}_{\nu})} \\ & + \sqrt{\mu + \kappa}\,\|\nabla p\|_{L^2_T(H^s)} + \sqrt{\kappa}\,\|\operatorname{div} v\|_{L^2_T(H^s)}\,,\end{aligned}$$

with $\|\cdot\|_{L^p_T(X)}$ denoting the norm in $L^p(0, T; X)$.

The hybrid norm $\|\cdot\|_{H^{s+1}_{\varepsilon\nu}}$ was already used by Danchin in [8].

For the study of nonlinear problems, it is important to relax the assumption that $Q \in C_0^\infty$.

Definition 3.2. *The space F^s consists of these function Q such that, for all $\mathbb{N} \ni m \leqslant s$, $\partial_t^m Q \in C_b^0(\mathbb{R}; H^{s+1-2m}(\mathbb{R}^d))$, where C_b^0 stands for $C^0 \cap L^\infty$.*

Given a normed space X, we set $B(X; M) = \{x \in X \,:\, \|x\| \leqslant M\}$.

Theorem 3.3. *Assume that $d = 1$ or $d \geqslant 3$ and let $\mathbb{N} \ni s > 1 + d/2$. Given $M_0 > 0$ and $Q \in F^s$, there exist $T > 0$ and $M > 0$ such that, for all $a = (\varepsilon, \mu, \kappa) \in A$ and all initial data $(p_0, v_0, \theta_0) \in H^{s+1}(\mathbb{R}^d)$ satisfying*

$$\|(\nabla p_0, \nabla v_0)\|_{H^{s-1}} + \|(\theta_0, \varepsilon p_0, \varepsilon v_0)\|_{H^{s+1}} \leqslant M_0, \tag{3.1}$$

the Cauchy problem for (1.4) *has a unique classical solution $(p, v, \theta) \in B(\mathcal{X}_a^s(T); M)$.*

This theorem implies Theorem 1.2.

Remark 3.4. A close inspection of the proof indicates that Theorem 3.3 remains valid with (3.1) replaced by

$$\|(p_0, v_0, \theta_0)\|_{\mathcal{X}_a^s(0)} := \|(\nabla p_0, \nabla v_0)\|_{H^{s-1}} + \|(\theta_0, \varepsilon p_0, \varepsilon v_0)\|_{H^{s+1}_\nu} \leqslant M_0.$$

Step 2: local well posedness. We explain here how to reduce matters to proving uniform bounds. To do so, our first task is to establish the local well posedness of the Cauchy problem for fixed $a = (\varepsilon, \mu, \kappa) \in A$.

Lemma 3.5. *Let $d \geqslant 1$, $s > 1 + d/2$ and $a \in A$. For all initial data $U_0 = (p_0, v_0, \theta_0) \in H^s(\mathbb{R}^d)$, there exists $T > 0$ such that the Cauchy problem for* (1.4) *has a unique classical solution $U = (p, v, \theta) \in C^0([0, T]; H^s)$ such that $U(0) = U_0$. Moreover, $[0, T^\star)$, with $T^\star < +\infty$, is a maximal interval of H^s existence if and only if $\limsup_{t \to T^\star} \|U(t)\|_{W^{1,\infty}(\mathbb{R}^d)} = +\infty$.*

Lemma 3.5 is a special case of Proposition 4.5 established below.

As in [1, 20], on account of the previous local existence result for fixed $a \in A$, Theorem 1.2 is a consequence of the following uniform estimates:

Proposition 3.6. *Let $d = 1$ or $d \geqslant 3$, $\mathbb{N} \ni s > 1 + d/2$ and $M_0 > 0$. Set $H^\infty(\mathbb{R}^d) := \cap_{\sigma \geqslant 0} H^\sigma(\mathbb{R}^d)$. There exist a constant C_0 and a non-negative function $C(\cdot)$ such that, for all $T \in (0, 1]$ and all $a \in A$, if $(p, v, \theta) \in C^\infty([0, T]; H^\infty(\mathbb{R}^d))$ is a solution of* (1.4) *with initial data satisfying* (3.1), *then the norm $\Omega_a(T) := \|U\|_{\mathcal{X}_a^s(T)}$ satisfies*

$$\Omega_a(T) \leqslant C_0 \exp\big((\sqrt{T} + \varepsilon) C(\Omega_a(T))\big). \tag{3.2}$$

To prove Proposition 3.6, as usual, a key step is to study the linearized system. This is the purpose of Theorem 3.10. With this result in hands, to establish the desired nonlinear estimates (3.2), the analysis is divided into four steps. This happens for two reasons. Firstly, on the technical side, most of the work concerns the separation of the estimates into high and low frequency components, where the division occurs at frequencies of order of the inverse of ε (since the second-derivative terms with $O(1)$ coefficients and the first-derivative terms with $O(\varepsilon^{-1})$ coefficients balance there). Secondly, there is a division into terms whose evolution is estimated directly by eliminating large terms of size $O(\varepsilon^{-1})$ (see Lemma 3.18 and 3.19), and terms whose size is estimated by means of Theorem 3.10 and the special structure of the equations (see Lemma 3.16).

This scheme of estimates has two useful properties. Firstly, it avoids estimating the L^2 norm of p and v (to obtain a closed set of estimates, we will use the preliminary estimates from Section 2). Secondly, it allows us to overcome the factor $1/\varepsilon$ in front of the source term Q. Indeed, the linear estimate in Theorem 3.10 is applied only to high-frequencies and weighted time derivatives $(\varepsilon\partial_t)^m$. Hence, the fact that the source term is assumed to be neither of high frequency nor have rapid time oscillations allows us to recover the lost factor of ε in the nonlinear estimates. Note that, in the combustion case, the assumptions on the source term Q may be verified directly from the equations. Also, we mention that the L^2 norm of (p, v) will later be estimated in Section 4 under an additional hypothesis.

Let us fix some notations.

Notation 3.7. From now on, we consider an integer $s > 1 + d/2$, a fixed time $0 < T \leqslant 1$, a fixed triple of parameters $a = (\varepsilon, \mu, \kappa) \in A$, a bound M_0, a fixed smooth solution $U = (p, v, \theta) \in C^\infty([0,T]; H^\infty(\mathbb{R}^d))$ of (1.4) with initial data satisfying (3.1) and we set

$$\Omega := \|U\|_{\mathcal{X}^s_a(T)} \,.$$

With these notations, Proposition 3.6 can be formulated concisely as follows: if $d \neq 2$, there exist constants C_0 depending only on M_0 and C depending only on Ω such that

$$\Omega \leqslant C_0 e^{(\sqrt{T}+\varepsilon)C}.$$

Hereafter, we use the notations $\phi := (\theta, \varepsilon p)$ and $\nu := \sqrt{\mu + \kappa}$.

Notation 3.8. For later application to the nonlinear case when $Q = F(Y)$ for some unknown function Y, we also give precise estimates in terms of norms of Q. For our purposes, the requested norm is the following:

$$\Sigma := \sum_{0\leqslant m\leqslant s} \left\| (I - (\varepsilon\nu)^2\Delta)^{-m/2} \big(\varepsilon(\partial_t + v\cdot\nabla)\big)^m Q \right\|_{L^\infty(0,T;H^{s+1-m}_\nu)} . \tag{3.3}$$

Remark 3.9. To use nonlinear estimates, it is easier to work in Banach algebras. If $d \geqslant 3$, Proposition 2.2 shows that we can supplement the $\mathcal{X}^s_a$ estimates with L^∞ estimates for the velocity: it suffices to prove (3.2) with $C(\Omega_a(T))$ replaced by $C(\Omega_a^+(T))$ where $\Omega_a^+(T) := \Omega_a(T) + \|v\|_{L^\infty((0,T)\times\mathbb{R}^d)}$. Similarly, if $d \geqslant 3$, all the estimates involving the source term Q remain valid with Σ replaced by

$$\sum_{0\leqslant m\leqslant s} \left\| (I - (\varepsilon\nu)^2\Delta)^{-m/2} (\varepsilon\partial_t)^m Q \right\|_{L^\infty(0,T;H^{s+1-m}_\nu)} .$$

Step 3: An energy estimate for linearized equations. A key step in the analysis is to estimate the solution $(\widetilde{p}, \widetilde{v}, \widetilde{\theta})$ of linearized equations. As alluded to above, a notable fact is that we can see unsigned large terms $\varepsilon^{-1} f^\varepsilon(t,x)$ in the equations for p and v as source terms provided that: 1) they do not convey fast oscillations in time: $\partial_t f^\varepsilon = O(1)$; 2) it does not implies a loss of derivatives. To be more precise: in the nonlinear estimates, we will see the term $\varepsilon^{-1}\chi_1(\phi)Q$ as a source term. Similarly, we can see terms of the form $\varepsilon^{-1}F(\varepsilon p, \theta, \sqrt{\kappa}\nabla\theta)$ as source terms. As a result, it is sufficient to consider the following linearized system:

(3.4)

$$\begin{cases} g_1(\phi)(\partial_t \widetilde{p} + v\cdot\nabla\widetilde{p}) + \dfrac{1}{\varepsilon}\operatorname{div}\widetilde{v} - \dfrac{\kappa}{\varepsilon}\operatorname{div}(k_1(\phi)\nabla\widetilde{\theta}) = F_1, \\ g_2(\phi)(\partial_t \widetilde{v} + v\cdot\nabla\widetilde{v}) + \dfrac{1}{\varepsilon}\nabla\widetilde{p} - \mu B_2(\phi,\partial_x)\widetilde{v} = F_2, \\ g_3(\phi)(\partial_t\widetilde{\theta} + v\cdot\nabla\widetilde{\theta}) + G(\phi,\nabla\phi)\cdot\widetilde{v} + \operatorname{div}\widetilde{v} - \kappa\chi_3(\phi)\operatorname{div}(k(\phi)\nabla\widetilde{\theta}) = F_3, \end{cases}$$

where the unknown $(\widetilde{p}, \widetilde{v}, \widetilde{\theta})$ is a smooth function of $(t,x) \in [0,T]\times\mathbb{R}^d$.

The following result establishes estimates on

$$\begin{aligned} \|(\widetilde{p}, \widetilde{v}, \widetilde{\theta})\|_{a,T} := {} & \|(\widetilde{p}, \widetilde{v})\|_{L^\infty_T(H^1_{\varepsilon\nu})} + \|\widetilde{\theta}\|_{L^\infty_T(H^1_\nu)} \\ & + \sqrt{\kappa}\|\nabla\widetilde{\theta}\|_{L^2_T(H^1_\nu)} + \sqrt{\mu}\,\|\nabla\widetilde{v}\|_{L^2_T(H^1_{\varepsilon\nu})} \\ & + \sqrt{\mu+\kappa}\,\|\nabla\widetilde{p}\|_{L^2_T(L^2)} + \sqrt{\kappa}\,\|\operatorname{div}\widetilde{v}\|_{L^2_T(L^2)}, \end{aligned} \tag{3.5}$$

in terms of the norm $\|(\widetilde{p}, \widetilde{v}, \widetilde{\theta})\|_{a,0} := \|(\widetilde{p}, \widetilde{v})(0)\|_{H^1_{\varepsilon\nu}} + \|\widetilde{\theta}(0)\|_{H^1_\nu}$ of the data.

Theorem 3.10. *Let $d \geqslant 1$ and assume that G, k_1 and k_3 are C^∞ functions such that, for all $(\vartheta,\wp)\in\mathbb{R}^2$, $0 < k_1(\vartheta,\wp) < \chi_3(\vartheta,\wp)k(\vartheta)$. Set*

$$R_0 := \|\phi(0)\|_{H^{s-1}}, \quad R := \sup_{t\in[0,T]} \left\|(\phi, \partial_t\phi + v\cdot\nabla\phi, \nabla\phi, \nu\nabla^2\phi, \nabla v)(t)\right\|_{H^{s-1}}.$$

There exist constants C_0 depending only on R_0 and C depending only on R such that,

$$\|(\widetilde{p}, \widetilde{v}, \widetilde{\theta})\|_{a,T} \leqslant C_0 e^{TC}\,\|(\widetilde{p}_0, \widetilde{v}_0, \widetilde{\theta}_0)\|_{a,0} + C\int_0^T \|(F_1,F_2)\|_{H^1_{\varepsilon\nu}} + \|F_3\|_{H^1_\nu}\,dt.$$

In [1] we established the previous theorem with R_0 and R replaced by

$$R_0' = \|\phi(0)\|_{L^\infty}, \quad R' = \sup_{t\in[0,T]}\left\|(\phi, \partial_t\phi, v, \nabla\phi, \nu\nabla^2\phi, \nabla v)(t)\right\|_{L^\infty}.$$

To prove the above variant, we need only check two facts. Firstly, in the proof of Theorem 4.3 in [1], the terms $\partial_t\phi$ and v always come together within terms involving the convective derivative $\partial_t\phi + v\cdot\nabla\phi$.

Secondly, we have to verify that the $L^\infty_{t,x}$ norms of the coefficients $(g_i(\phi),\ldots)$ are estimated by constants of the form $C_0 e^{TC}$. In [1] we used the estimate

$$\sup_{t\in[0,T]}\|F(\phi(t))\|_{L^\infty} \leqslant \|F(\phi(0))\|_{L^\infty} + T\sup_{t\in[0,T]}\|\partial_t F(\phi(t))\|_{L^\infty} \leqslant C_0' + TC',$$

for some constants depending only on R_0' and R'. Here, based on an usual estimate for hyperbolic equations, we can prove a similar bound:

Lemma 3.11. *Let $F \in C^\infty(\mathbb{R}^2)$ be such that $F(0) = 0$. There exist constants C_0 depending only on R_0 and C depending only on R such that, for all $t \in [0,T]$, $\|F(\phi(t))\|_{H^{s-1}} \leqslant C_0 e^{TC}$.*

Proof. Since $s-1 > d/2$, the Moser's estimates (2.2) and (2.4) imply that there exists a function $\mathcal{C}$ depending only on the function F such that

$$\begin{aligned}
&\|(\partial_t + v\cdot\nabla)F(\phi)\|_{H^{s-1}} \\
&\quad\leqslant \left(1 + \left\|F'(\phi) - F'(0)\right\|_{H^{s-1}}\right) \|(\partial_t + v\cdot\nabla)\phi(t)\|_{H^{s-1}}\,, \\
&\quad\leqslant \mathcal{C}(\|(\phi, \partial_t\phi + v\cdot\nabla\phi)\|_{H^{s-1}}) \leqslant \mathcal{C}(R),
\end{aligned}$$

and $\|F(\phi(0))\|_{H^{s-1}} \leqslant \mathcal{C}(\|\phi(0)\|_{H^{s-1}})$.

Hence, the desired estimate follows from the following estimate: there exists a constant V depending only on $\|\nabla v\|_{L^\infty_T H^{s-1}}$ such that

$$\sup_{t\in[0,T]} \|F(\phi(t))\|_{H^{s-1}} \leqslant \|F(\phi(0))\|_{H^{s-1}} + TV \sup_{t\in[0,T]} \|(\partial_t + v\cdot\nabla)F(\phi(t))\|_{H^{s-1}}\,.$$

To prove this result we set $\widetilde{u} := \partial_x^\alpha F(\phi)$ where $\alpha \in \mathbb{N}^d$ is such that $|\alpha| \leqslant s-1$. Then $\widetilde{u}$ solves

$$\partial_t u + v\cdot\nabla u = f := \partial_x^\alpha\big((\partial_t + v\cdot\nabla)F(\phi)\big) + [v, \partial_x^\alpha]\cdot\nabla F(\phi).$$

Since $s-1 > d/2$, the product rule (2.3) implies that

$$\begin{aligned}
\|[v,\partial_x^\alpha]\cdot\nabla F(\phi)\|_{L^2} &\lesssim \sum_{\beta+\gamma=\alpha,\ \beta>0} \|\partial_x^\beta v \partial_x^\gamma \nabla F(\phi)\|_{L^2} \\
&\lesssim \sum_{\beta+\gamma=\alpha,\ \beta>0} \|\partial_x^\beta v\|_{H^{s-1-(|\beta|-1)}} \|\partial_x^\gamma \nabla F\|_{H^{s-1-(|\gamma|+1)}}\,,
\end{aligned}$$

hence, $\|f\|_{L^2} \lesssim \|(\partial_t + v\cdot\nabla)F(\phi)\|_{H^{s-1}} + \|\nabla v\|_{H^{s-1}} \|F(\phi)\|_{H^{s-1}}$.

We next use an integration by parts argument yielding

$$\frac{d}{dt}\|\widetilde{u}\|_{L^2}^2 \leqslant (1 + \|\operatorname{div} v\|_{L^\infty})\|\widetilde{u}\|_{L^2}^2 + \|f\|_{L^2}^2\,.$$

The Gronwall's Lemma concludes the proof. □

Step 4: High frequency estimates. We begin by estimating the high frequency component

$$\Omega^{\mathrm{HF}} := \|(I - J_{\varepsilon\nu})U\|_{\mathcal{X}_a^s(T)}\,,$$

where $\{J_h \,|\, h \in [0,1]\}$ is a Friedrichs mollifiers: $J_h = \jmath(hD_x)$ is the Fourier multiplier with symbol $\jmath(h\xi)$ where $\jmath$ is a C^∞ function of $\xi \in \mathbb{R}^d$, satisfying

$$0 \leqslant \jmath \leqslant 1, \quad \jmath(\xi) = 1 \text{ for } |\xi| \leqslant 1, \quad \jmath(\xi) = 0 \text{ for } |\xi| \geqslant 2, \quad \jmath(\xi) = \jmath(-\xi)\,.$$

Proposition 3.12. *Let $d \geqslant 1$. There exist constants C_0 depending only on M_0 and C depending only on Ω, such that*

$$\Omega^{\mathrm{HF}} \leqslant C_0 e^{\sqrt{T}C} + \sqrt{T} C \left\| Q \right\|_{L^\infty_T(H^{s+1}_\nu)}. \tag{3.6}$$

Proof. Introduce $P := (I - J_{\varepsilon\nu})\Lambda^s$ and $\widetilde{U} := (Pp, Pv, P\theta)$. Then, $\widetilde{U}$ satisfies System (3.4) with

$$k_1(\phi) := \chi_1(\phi) k(\theta), \qquad G(\phi, \nabla\phi) := g_3(\phi)\nabla\theta,$$

and $F = (F_1, F_2, F_3)^T := f_{\mathrm{HF}} + f_Q + f_\chi$, where

$$f_Q := \begin{pmatrix} \varepsilon^{-1} P\big(\chi_1(\phi) Q\big) \\ 0 \\ P\big(\chi_3(\phi) Q\big) \end{pmatrix}, \quad f_\chi := \begin{pmatrix} -\kappa\varepsilon^{-1} \nabla\chi_1(\phi)\cdot(k(\theta)\nabla\widetilde{\theta}) \\ 0 \\ 0 \end{pmatrix},$$

and f_{HF} is given by

$$f_{1,\mathrm{HF}} = \big[g_1(\phi), P\big](\partial_t + v\cdot\nabla)p + g_1(\phi)\big[v, P\big]\cdot\nabla p \; - \frac{\kappa}{\varepsilon}\big[B_1(\phi,\partial_x), P\big]\theta,$$

$$f_{2,\mathrm{HF}} = \big[g_2(\phi), P\big](\partial_t + v\cdot\nabla)v + g_2(\phi)\big[v, P\big]\cdot\nabla v \; - \mu\big[B_2(\phi,\partial_x), P\big]v,$$

$$f_{3,\mathrm{HF}} = \big[g_3(\phi), P\big](\partial_t + v\cdot\nabla)\theta + g_3(\phi)\big\{v; P\big\}\cdot\nabla\theta - \kappa\big[B_3(\phi,\partial_x), P\big]\theta,$$

where $B_i(\phi,\partial_x) = \chi_i(\phi)\operatorname{div}(k(\theta)\nabla\cdot)$ $(i = 1, 3)$, $[A, B] = AB - BA$ and

$$\big\{v; P\big\}\cdot\nabla\theta := v\cdot\nabla P\theta + (Pv)\cdot\nabla\theta - P(v\cdot\nabla\theta).$$

Estimate for f_{HF}. We use the following analogue of Lemma 5.3 in [1]: there exists a constant $K = K(d, s)$ such that

$$\big\| \big[f, P\big]u \big\|_{H^1_{\varepsilon\nu}} \leqslant \varepsilon\nu K \left\|\nabla f\right\|_{L^\infty} \left\|u\right\|_{H^s} + \varepsilon\nu K \left\|\nabla f\right\|_{H^s} \left\|u\right\|_{L^\infty},$$

$$\big\| \big[f, P\big]u \big\|_{H^1_\nu} \lesssim \nu K \left\|\nabla f\right\|_{L^\infty} \left\|u\right\|_{H^s} + \nu K \left\|\nabla f\right\|_{H^s} \left\|u\right\|_{L^\infty}.$$

The fact that the right-hand side only involves ∇f follows from the most simple of all the sharp commutator estimates established in [17]: for all $s > 1 + d/2$ and all Fourier multiplier $A(D_x) \in \operatorname{Op} S^s_{1,0}$, there exists a constant K such that, for all $f \in H^s(\mathbb{R}^d)$ and all $u \in H^s(\mathbb{R}^d)$,

$$\left\| [f, A(D_x)]u \right\|_{L^2} \leqslant K \left\|\nabla f\right\|_{L^\infty} \left\|u\right\|_{H^{s-1}} + K \left\|\nabla f\right\|_{H^{s-1}} \left\|u\right\|_{L^\infty}. \tag{3.7}$$

As in [1], from this and the usual nonlinear estimates (2.2) and (2.4), it can be verified that there exists a generic function $\mathcal{C}$ (depending only on parameters that are considered fixed) such that,

$$\left\| f_{1,\mathrm{HF}} \right\|_{H^1_{\varepsilon\nu}} \leqslant \mathcal{C}\big(\left\|(\theta, \varepsilon p, \varepsilon v)\right\|_{H^{s+1}_\nu}\big)\{1 + \left\|\varepsilon(\partial_t + v\cdot\nabla)p\right\|_{H^s_\nu} + \kappa\left\|\theta\right\|_{H^{s+2}}\},$$

$$\left\| f_{2,\mathrm{HF}} \right\|_{H^1_{\varepsilon\nu}} \leqslant \mathcal{C}\big(\left\|(\theta, \varepsilon p, \varepsilon v)\right\|_{H^{s+1}_\nu}\big)\{1 + \left\|\varepsilon(\partial_t + v\cdot\nabla)v\right\|_{H^s_\nu} + \mu\left\|\varepsilon v\right\|_{H^{s+2}}\},$$

$$\left\| f_{3,\mathrm{HF}} \right\|_{H^1_{\nu}} \leqslant \mathcal{C}\big(\left\|(\theta, \varepsilon p, \varepsilon v)\right\|_{H^{s+1}_\nu}\big)\{1 + \left\|(\partial_t + v\cdot\nabla)\theta\right\|_{H^s_\nu} + \kappa\left\|\theta\right\|_{H^{s+2}}\}.$$

Set $\psi = (\theta, \varepsilon p, \varepsilon v)$. The key point is that

$$\begin{aligned} &\left\|(\partial_t + v\cdot\nabla)\psi\right\|_{H^s_\nu} \\ &\leqslant \mathcal{C}(\left\|\psi\right\|_{H^{s+1}_\nu})\big\{1 + \big\|(\nu\nabla p, \nu\operatorname{div} v, \varepsilon\mu\nabla^2 v, \kappa\nabla^2\theta)\big\|_{H^s} + \left\|Q\right\|_{H^s_\nu}\big\}. \end{aligned} \tag{3.8}$$

This estimate differs from the one that appears in Lemma 5.14 in [1] in that the right-hand side does not involve v itself but only its derivatives. Yet, as the reader can verify, the same proof applies since we do not estimate $\partial_t\psi$ but instead $\partial_t\psi + v\cdot\nabla\psi$.

Estimate for f_Q and f_χ. By using the elementary estimate

$$\left\|(I-J_{\varepsilon\nu})u\right\|_{H^{\sigma+1}_{\varepsilon\nu}} \lesssim \varepsilon\nu \left\|u\right\|_{H^{\sigma+1}},$$

we find that

$$\frac{1}{\varepsilon}\left\|P(\chi_1(\phi)Q)\right\|_{H^1_{\varepsilon\nu}} + \left\|P(\chi_3(\phi)Q)\right\|_{H^1_\nu} \leqslant \left\|\chi_1(\phi)Q\right\|_{H^{s+1}_\nu} + \left\|\chi_3(\phi)Q\right\|_{H^{s+1}_\nu}.$$

The tame estimates (2.2) and (2.4) (see also Lemma 5.5 and 5.6 in [1]) imply

$$\left\|\chi_i(\phi)Q\right\|_{H^{s+1}_\nu} \lesssim (1+\left\|\chi_i(\phi)-\chi_i(0)\right\|_{H^{s+1}_\nu})\left\|Q\right\|_{H^{s+1}_\nu} \lesssim \mathcal{C}(\left\|\phi\right\|_{H^{s+1}_\nu})\left\|Q\right\|_{H^{s+1}_\nu}$$

so that $\left\|f_{1,Q}\right\|_{L^\infty_T(H^1_{\varepsilon\nu})} + \left\|f_{3,Q}\right\|_{L^\infty_T(H^1_\nu)} \leqslant C\left\|Q\right\|_{L^\infty_T(H^{s+1}_\nu)}$. The technique for estimating f_χ is similar; we find that $\left\|f_{1,\chi}\right\|_{L^\infty_T(H^1_{\varepsilon\nu})} \leqslant C$.

By definition of $\left\|\cdot\right\|_{\mathcal{X}^s_a(T)}$, the previous estimates imply that there exists a constant C depending only on Ω such that

$$\begin{aligned}\int_0^T \left\|(F_1,F_2)\right\|_{H^1_{\varepsilon\nu}} + \left\|F_3\right\|_{H^1_\nu}\,dt &\leqslant \sqrt{T}\Big(\int_0^T \left\|(F_1,F_2)\right\|^2_{H^1_{\varepsilon\nu}} + \left\|F_3\right\|^2_{H^1_\nu}\,dt\Big)^{1/2}\\ &\leqslant \sqrt{T}C + \sqrt{T}C\left\|Q\right\|_{L^\infty_T(H^{s+1}_\nu)}.\end{aligned}$$

From here we can parallel the rest of the argument of Section 5 in [1], to prove that $\left\|(Pp,Pv,P\theta)\right\|_{a,T} \leqslant C_0\exp(\sqrt{T}C) + \sqrt{T}C\left\|Q\right\|_{L^\infty_T(H^{s+1}_\nu)}$ where the norm $\left\|\cdot\right\|_{a,T}$ is as defined in (3.5). Since $\Omega^{\mathrm{HF}} \lesssim \left\|(Pp,Pv,P\theta)\right\|_{a,T}$, this completes the proof. □

Step 5: Low frequency estimates. The following step is to estimate the low frequency part of the fast components:

$$\begin{aligned}\Omega^{\mathrm{LF}} := &\left\|\operatorname{div} J_{\varepsilon\nu}v\right\|_{L^\infty_T(H^{s-1})} + \nu\left\|\operatorname{div} J_{\varepsilon\nu}v\right\|_{L^2_T(H^s)}\\ &+ \left\|\nabla J_{\varepsilon\nu}p\right\|_{L^\infty_T(H^{s-1})} + \nu\left\|\nabla J_{\varepsilon\nu}p\right\|_{L^2_T(H^s)}.\end{aligned}$$

Proposition 3.13. *Let $d\geqslant 1$. There exist constants C_0 depending only on M_0, C depending only on Ω and C' depending only on $\Omega+\Sigma$, such that*

$$\Omega^{\mathrm{LF}} \leqslant C_0 e^{(\sqrt{T}+\varepsilon)C} + \sqrt{T}C'. \tag{3.9}$$

By contrast with the high frequency regime, the estimate (3.9) cannot be obtained from the L^2 estimates by an elementary argument using differentiation of the equations (see [20, 24]). To overcome this problem, we first give estimates for the time derivatives, and next we use the special structure of the equations to estimate the spatial derivatives.

For the case of greatest physical interest ($d=3$), the proof given in [1] applies with only minor changes. Indeed, as alluded to in Remark 3.9, it suffices to check that all the estimates involving $\left\|v\right\|_{H^s}$ remain valid with $\left\|v\right\|_{H^s}$ replaced by $\left\|v\right\|_{L^\infty} + \left\|\nabla v\right\|_{H^{s-1}}$. Yet, if $d\leqslant 2$, because of the lack of

L^2 estimates for the velocity, we cannot use the time derivatives. For this problem, we use an idea introduced by Secchi in [25]. Namely, we replace ∂_t by the convective derivative

$$D_v := \partial_t + v \cdot \nabla.$$

For the reader convenience, we indicate how to adapt the three main calculus inequalities in [1] when ∂_t is replaced by D_v.

First, to localize in the low frequency region we use the following commutator estimate. The thing of interest is the gain of an extra factor ε.

Lemma 3.14. *Given $s > 1 + d/2$, there exists a constant K such that for all $\varepsilon \in [0,1]$, all $\nu \in [0,2]$, all $T > 0$, all $m \in \mathbb{N}$ such that $1 \leqslant m \leqslant s$ and all f, u and v in $C^\infty([0,T]; H^\infty(\mathbb{D}))$,*

$$\begin{aligned}
&\left\| \left[f, J_{\varepsilon\nu}(\varepsilon D_v)^m \right] u \right\|_{H^{s-m+1}_{\varepsilon\nu}} \\
&\quad \leqslant K\varepsilon \Big\{ \|f\|_{H^s} + \sum_{\ell=0}^{m-1} \|\Lambda_{\varepsilon\nu}^{-\ell}(\varepsilon D_v)^\ell D_v f\|_{H^{s-1-\ell}} \Big\} \\
&\qquad \times \Big\{ \|\Lambda_{\varepsilon\nu}^{-m}(\varepsilon D_v)^m u\|_{H^{s-m}_\nu} + \sum_{\ell=0}^{m-1} \|\Lambda_{\varepsilon\nu}^{-\ell}(\varepsilon D_v)^\ell u\|_{H^{s-1-\ell}} \Big\},
\end{aligned}$$

where $\Lambda_{\varepsilon\nu}^\sigma := (I - (\varepsilon\nu)^2\Delta)^{\sigma/2}$.

To apply the previous lemma, we need estimates of the coefficients f and $D_v f$. Since, for System (1.4), the coefficients are functions of the slow variable $(\theta, \varepsilon p, \varepsilon v)$, the main estimates are the following.

Lemma 3.15. *Let $s > 1 + d/2$ be an integer. There exists a function $\mathcal{C}(\cdot)$ such that, for all $a = (\varepsilon, \mu, \kappa) \in A$, all $T > 0$ and all smooth solution $(p, v, \theta) \in C^\infty([0,T]; H^\infty(\mathbb{D}))$ of* (1.4)*, if $\nu \in [(\mu+\kappa)/2, 2]$ then the function Ψ defined by*

$$\Psi := \big(\psi, D_v\psi, \nabla\psi\big) \quad \textit{where} \quad \psi := (\theta, \varepsilon p, \varepsilon v),$$

satisfies

$$\sum_{0\leqslant \ell \leqslant s} \|\Lambda_{\varepsilon\nu}^{-\ell}(\varepsilon D_v)^\ell \Psi\|_{H^{s-\ell-1}} \leqslant \mathcal{C}\big(\|\Psi\|_{H^{s-1}} + \Sigma\big), \tag{3.10}$$

$$\sum_{0\leqslant \ell \leqslant s} \|\Lambda_{\varepsilon\nu}^{-\ell}(\varepsilon D_v)^\ell \Psi\|_{H^{s-\ell}_\nu} \leqslant \mathcal{C}\big(\|\Psi\|_{H^{s-1}} + \Sigma\big) \|\Psi\|_{H^s_\nu}, \tag{3.11}$$

where Σ is as defined in (3.3).

Once this is granted, we are in position to estimate the commutator of the equations (1.4) and $\mathcal{P} := J_{\varepsilon\nu}(\varepsilon D_v)^s$:

$$\begin{aligned}
f_{1,\mathrm{LF}} &= \big[g_1(\phi), \mathcal{P}\big] D_v p + g_1(\phi)\big[v, \mathcal{P}\big] \cdot \nabla p - \frac{\kappa}{\varepsilon}\big[B_1(\phi, \partial_x), \mathcal{P}\big]\theta, \\
f_{2,\mathrm{LF}} &= \big[g_2(\phi), \mathcal{P}\big] D_v v + g_2(\phi)\big[v, \mathcal{P}\big] \cdot \nabla v - \mu\big[B_2(\phi, \partial_x), \mathcal{P}\big] v, \\
f_{3,\mathrm{LF}} &= \big[g_3(\phi), \mathcal{P}\big] D_v \theta + g_1(\phi)\big[v, \mathcal{P}\big] \cdot \nabla\theta - \kappa\big[B_3(\phi, \partial_x), \mathcal{P}\big]\theta.
\end{aligned}$$

It is found that

$$\|f_{1,\mathrm{LF}}\|_{H^1_{\varepsilon\nu}} + \|f_{2,\mathrm{LF}}\|_{H^1_{\varepsilon\nu}} + \|f_{3,\mathrm{LF}}\|_{H^1_{\nu}} \leqslant (1+\|\Psi\|_{H^s_\nu})\mathcal{C}(\|\Psi\|_{H^{s-1}}+\Sigma).$$

Note that Ψ is estimated by means of (3.8).

As in the high frequency regime, we have to estimate source terms of the form $\varepsilon^{-1}\mathcal{P}F(\Psi,Q)$. The fact that these large source terms cause no difficulty comes from the fact that $\varepsilon^{-1}J_{\varepsilon\nu}(\varepsilon D_v)^s F(\Psi,Q) = J_{\varepsilon\nu}(\varepsilon D_v)^{s-1}D_vF(\Psi,Q)$ together with $D_vF(\Psi,Q)=O(1)$ (the norm Σ introduced in (3.3) is the requested norm to give this statement a precise meaning).

With these results in hands, one can estimate $J_{\varepsilon\nu}(\varepsilon D_v)^s(p,v,\theta)$ by means of Theorem 3.10. Next, we give estimate for $\operatorname{div} J_{\varepsilon\nu}v$ and $\nabla J_{\varepsilon\nu}p$ from the estimate of $J_{\varepsilon\nu}(\varepsilon D_v)^s(p,v,\theta)$ by means of the following induction argument:

Lemma 3.16. *Set* $\|u\|_{\mathcal{K}^\sigma_\nu(T)} := \|u\|_{L^\infty_T(H^{\sigma-1})} + \nu\|u\|_{L^2_T(H^\sigma)}$.

Let $\widetilde{U} := (\widetilde{p},\widetilde{v},\widetilde{\theta})$ *solve*

$$\left\{\begin{aligned} &g_1(\phi)(\partial_t\widetilde{p}+v\cdot\nabla\widetilde{p}) + \varepsilon^{-1}\operatorname{div}\widetilde{v} - \kappa\varepsilon^{-1}\chi_1(\phi)\operatorname{div}(k(\theta)\nabla\widetilde{\theta}) = f_1,\\ &g_2(\phi)(\partial_t\widetilde{v}+v\cdot\nabla\widetilde{v}) + \varepsilon^{-1}\nabla\widetilde{p} - \mu B_2(\phi,\partial_x)\widetilde{v} = f_2,\\ &g_3(\phi)(\partial_t\widetilde{\theta}+v\cdot\nabla\widetilde{\theta}) + \operatorname{div}\widetilde{v} - \kappa\chi_3(\phi)\operatorname{div}(k(\theta)\nabla\widetilde{\theta}) = f_3.\end{aligned}\right. \tag{3.12}$$

If support of the Fourier transform of $\widetilde{U}$ *is included in the ball* $\{|\xi|\leqslant 2/\varepsilon\nu\}$, *then there exist constant* C_0 *depending only on* M_0 *and* C *depending only on* Ω *such that, for all* $\sigma\in[1,s]$,

$$\begin{aligned} &\|\nabla\widetilde{p}\|_{\mathcal{K}^\sigma_\nu(T)} + \|\operatorname{div}\widetilde{v}\|_{\mathcal{K}^\sigma_\nu(T)}\\ &\quad\leqslant \widetilde{C}\,\|(\varepsilon D_v)\widetilde{p}\|_{\mathcal{K}^\sigma_\nu(T)} + \widetilde{C}\,\|(\varepsilon D_v)\operatorname{div}\widetilde{v}\|_{\mathcal{K}^{\sigma-1}_\nu(T)}\\ &\qquad + \widetilde{C}\,\|\nabla\widetilde{p}\|_{L^\infty_T(L^2)} + \widetilde{C}\,\|\widetilde{\theta}(0)\|_{H^{\sigma+1}_\nu} + \varepsilon C\,\|\mu\widetilde{v}\|_{\mathcal{K}^{\sigma+1}_\nu(T)}\\ &\qquad + \varepsilon C\,\|(f_1,f_2)\|_{\mathcal{K}^\sigma_\nu(T)} + \nu\widetilde{C}\,\|f_3\|_{L^2_T(H^\sigma)}, \end{aligned} \tag{3.13}$$

where $\widetilde{C} := C_0e^{(\sqrt{T}+\varepsilon)C}$.

Step 6: estimates for the slow components. To complete the proof of (3.2), it remains to estimate $\operatorname{curl} v$ and θ. Yet, this is not straightforward. Following Métivier and Schochet [20], we begin by estimating $\operatorname{curl}(\gamma v)$ for some appropriate positive weight $\gamma=\Gamma(\theta,\varepsilon p)$.

Lemma 3.17. *Let* $d\geqslant 1$. *There exist constants* C_0 *depending only on* M_0 *and* C *depending only on* Ω, *and there exists a function* $\Gamma\in C^\infty(\mathbb{R}^2)$ *such that, with* $\gamma=\Gamma(\theta,\varepsilon p)$, *there holds*

$$\|\operatorname{curl}(\gamma v)\|_{L^\infty_T(H^{s-1})} + \sqrt{\mu}\,\|\operatorname{curl}(\gamma v)\|_{L^2_T(H^s)} \leqslant C_0e^{\sqrt{T}C} + \sqrt{T}C\,\|Q\|_{L^\infty_T(H^{s+1}_\nu)}.$$

Lemma 3.18. *Let* $d\geqslant 1$. *There exist constants* C_0 *depending only on* M_0 *and* C *depending only on* Ω, *such that*

$$\|J_{\varepsilon\nu}\theta\|_{L^\infty_T(H^{s+1}_\nu)} + \sqrt{\kappa}\,\|J_{\varepsilon\nu}\theta\|_{L^2_T(H^{s+2}_\nu)} \leqslant C_0e^{\sqrt{T}C} + \sqrt{T}C\,\|Q\|_{L^\infty_T(H^{s+1}_\nu)}.$$

The proofs of Lemma 3.17 and 3.18 follow from a close inspection of the proofs of Lemma 6.25 and 6.26 in [1]. We just mention that this is where we use the function F of Assumption (H2) in §1.2 (γ is related to the fluid entropy).

Lemma 3.19. *Assume $d \geqslant 3$. There exist constants C_0 depending only on M_0 and C depending only on Ω such that, with $\gamma_0 = \Gamma(\theta_0, \varepsilon p_0)$ where Γ is as above, there holds*

$$\|\operatorname{curl}(\gamma_0 v)\|_{L^\infty_T(H^{s-1})} + \sqrt{\mu}\, \|\operatorname{curl}(\gamma_0 v)\|_{L^2_T(H^s)} \leqslant C_0 e^{\sqrt{T}C} + \sqrt{T}C\, \|Q\|_{L^\infty_T(H^{s+1}_\nu)} .$$

Proof. Set $\widetilde{\gamma} := \gamma - \gamma_0$. By Lemma 3.17, all we need to prove is that

$$\|\operatorname{curl}(\widetilde{\gamma} v)\|_{L^\infty_T(H^{s-1})} + \sqrt{\mu}\, \|\operatorname{curl}(\widetilde{\gamma} v)\|_{L^2_T(H^s)} \leqslant \sqrt{T}C + \sqrt{T}C\, \|Q\|_{L^\infty_T(H^{s+1}_\nu)} . \tag{3.14}$$

To do so, we claim that $\widetilde{\gamma}$ is small for small times:

$$\|\widetilde{\gamma}\|_{L^\infty_T(H^s)} + \nu\, \|\widetilde{\gamma}\|_{L^2_T(H^{s+1})} \leqslant \sqrt{T}C + \sqrt{T}C\, \|Q\|_{L^\infty_T(H^{s+1}_\nu)} . \tag{3.15}$$

Let us assume this and continue the proof.

We have to estimate $\operatorname{curl}(\widetilde{\gamma} v) = \widetilde{\gamma} \operatorname{curl} v + (\nabla \widetilde{\gamma}) \times v$. By combining the Cauchy-Schwarz estimate with the usual product rule (2.3) and the product rule (2.7), we find that

$$\begin{aligned}
\|\widetilde{\gamma} \operatorname{curl} v\|_{L^\infty_T(H^{s-1})} &\leqslant \|\widetilde{\gamma}\|_{L^\infty_T(H^{s-1})} \|\nabla v\|_{L^\infty_T(H^{s-1})} ,\\
\sqrt{\mu}\, \|\widetilde{\gamma} \operatorname{curl} v\|_{L^2_T(H^s)} &\leqslant \|\widetilde{\gamma}\|_{L^\infty_T(H^s)} \|\sqrt{\mu} \nabla v\|_{L^2_T(H^s)} ,\\
\|\nabla \widetilde{\gamma} \times v\|_{L^\infty_T(H^{s-1})} &\leqslant \|\widetilde{\gamma}\|_{L^\infty_T(H^s)} \|\nabla v\|_{L^\infty_T(H^{s-1})} ,\\
\sqrt{\mu}\, \|\nabla \widetilde{\gamma} \times v\|_{L^2_T(H^s)} &\leqslant \|\sqrt{\mu} \widetilde{\gamma}\|_{L^2_T(H^{s+1})} \|\nabla v\|_{L^\infty_T(H^{s-1})} .
\end{aligned}$$

The claim (3.15) then yields the desired bound (3.14).

We now have to prove the claim (3.15). We first note that

$$\begin{aligned}
\nu\, \|\widetilde{\gamma}\|_{L^2_T(H^{s+1})} &\leqslant \nu \sqrt{T}\, \|\widetilde{\gamma}\|_{L^\infty_T(H^{s+1})} \\
&\leqslant \nu \sqrt{T} \mathcal{C}(\|(\theta, \varepsilon p)\|_{L^\infty_T(L^\infty_x)})(1 + \|(\theta, \varepsilon p)\|_{L^\infty_T(H^{s+1})}) \\
&\leqslant \sqrt{T} \mathcal{C}(\|(\theta, \varepsilon p)\|_{L^\infty_T(H^{s+1}_\nu)}) \leqslant \sqrt{T} C.
\end{aligned}$$

To prove the second half of (3.15), we verify that, directly from the definitions, $\widetilde{\gamma}$ satisfies an equation of the form $\partial_t \widetilde{\gamma} + v \cdot \nabla \widetilde{\gamma} = f$ with f bounded in $L^2(0, T; H^s(\mathbb{R}^d))$ by a constant depending only on $\Omega + \|Q\|_{L^\infty_T(H^{s+1}_\nu)}$. Then, we apply the above mentioned estimate for hyperbolic equations:

$$\|\widetilde{\gamma}\|_{L^\infty_T(H^s)} \lesssim e^{TV}\, \|\widetilde{\gamma}(0)\|_{H^s} + \int_0^T e^{(T-t)V}\, \|f\|_{H^s}\, dt, \tag{3.16}$$

where $V = K \int_0^T \|\nabla v\|_{H^{s-1}}\, dt$ with $K = K(s, d)$. Since $\widetilde{\gamma}(0) = 0$, by applying the Cauchy-Schwarz inequality, it is found that the $L^\infty_T(H^s)$ norm of $\widetilde{\gamma}$ is estimated by $\sqrt{T} e^{TV}\, \|f\|_{L^2_T(H^s)}$, thereby obtaining the claim. □

Step 7: closed set of estimates. To complete the proof of Proposition 3.6, it remains to check that we have proved a closed set of estimates.

The obvious estimate $\|u\|_{H^\sigma} \leqslant \|J_{\varepsilon\nu}u\|_{H^\sigma} + \|(I - J_{\varepsilon\nu})u\|_{H^\sigma}$ implies that

$$\|(\nabla p, \operatorname{div} v)\|_{L^\infty_T(H^{s-1})} + \sqrt{\mu + \kappa}\, \|(\nabla p, \operatorname{div} v)\|_{L^2_T(H^s)} \lesssim \Omega_{\mathrm{LF}} + \Omega_{\mathrm{HF}},$$

and, similarly, $\|\theta\|_{L^\infty_T(H^{s+1}_\nu)} + \sqrt{\kappa}\, \|\nabla\theta\|_{L^2_T(H^{s+1}_\nu)}$ is estimated by

$$\|J_{\varepsilon\nu}\theta\|_{L^\infty_T(H^{s+1}_\nu)} + \sqrt{\kappa}\, \|J_{\varepsilon\nu}\nabla\theta\|_{L^2_T(H^{s+1}_\nu)} + \Omega_{\mathrm{HF}}.$$

The estimate $\|\varepsilon u\|_{H^{\sigma+1}_\nu} \lesssim \|\varepsilon u\|_{L^2} + \|\nabla u\|_{H^{\sigma-1}} + \|(I - J_{\varepsilon\nu})u\|_{H^{\sigma+1}_\nu}$ yields

$$\begin{aligned} \|(\varepsilon p, \varepsilon v)\|_{L^\infty_T(H^{s+1}_\nu)} + \sqrt{\mu}\, \|\nabla v\|_{L^2_T(H^{s+1}_{\varepsilon\nu})} &\lesssim \|(\varepsilon p, \varepsilon v)\|_{L^\infty_T(L^2)} \\ &\quad + \|(\nabla p, \nabla v)\|_{L^\infty_T(H^{s-1})} + \sqrt{\mu}\, \|\nabla v\|_{L^2_T(H^s)} + \Omega_{\mathrm{HF}}. \end{aligned} \tag{3.17}$$

On the other hand, Corollary 2.5 implies that, if $d \neq 2$, there exists a constant C_0 depending only on M_0 such that

$$\begin{aligned} &\|\nabla v\|_{L^\infty_T(H^{s-1})} + \sqrt{\mu}\, \|\nabla v\|_{L^2_T(H^s)} \\ &\quad \leqslant C_0 \|(\operatorname{div} v, \operatorname{curl}(\gamma_0 v))\|_{L^\infty_T(H^{s-1})} + C_0\sqrt{\mu}\, \|(\operatorname{div} v, \operatorname{curl}(\gamma_0 v))\|_{L^2_T(H^s)}. \end{aligned}$$

By using the estimate (3.8), one can verify that the term $\|(\varepsilon p, \varepsilon v)\|_{L^\infty_T(L^2)}$ (in the left-hand side of (3.17)) can be estimated as in the proof of Lemma 3.11. Therefore, according to Propositions 3.12–3.13 and Lemma 3.18–3.19, we have proved that, if $d \neq 2$, then $\Omega \leqslant \widetilde{C}$ where $\widetilde{C} = C_0 e^{(\sqrt{T}+\varepsilon)C} + \sqrt{T}C'$ for some constants C_0, C and C' depending only on M_0, Ω and $\Omega + \Sigma$, respectively.

This concludes the proof of Proposition 3.6 and hence Theorem 3.3.

4. Uniform estimates in the Sobolev spaces

With regards to the low Mach number limit problem, we mention that the convergence results [1] proved in [1] apply for general systems (not only for perfect gases). To avoid repetition, we only mention that one can rigorously justify the low Mach number limit for general initial data provided that one can prove that the solutions are uniformly bounded in Sobolev spaces (see Proposition 8.2 in [1]). The problem presents itself: Theorem 1.2 only gives uniform estimates for the derivatives of p and v. In this section, we give uniform bounds in Sobolev norms.

Theorem 4.1. *Let $d \geqslant 1$ and $\mathbb{N} \ni s > 1 + d/2$. Assume that $Q = 0$. Also, assume that either $\chi_1 = \chi_1(\vartheta, \wp)$ is independent of ϑ or that $d \geqslant 3$. Then, for all $M_0 > 0$, there exist $T > 0$ and $M > 0$ such that, for all $a = (\varepsilon, \mu, \kappa) \in A$ and all initial data $(p_0, v_0, \theta_0) \in H^{s+1}(\mathbb{R}^d)$ satisfying*

$$\|(p_0, v_0, \theta_0)\|_{H^{s+1}} \leqslant M_0,$$

[1] These results are strongly based on a Theorem of Métivier and Schochet [20] about the decay to zero of the local energy for a class of wave operators with variable coefficients.

the Cauchy problem for (1.4) *has a unique classical solution* (p, v, θ) *in* $C^0([0,T];H^{s+1}(\mathbb{R}^d))$ *such that*

$$\sup_{t\in[0,T]} \|(p(t), v(t), \theta(t))\|_{H^s} \leqslant M.$$

The first half of this result is proved in [1]. Indeed, the assumption that $\chi_1(\vartheta, \wp)$ does not depend on ϑ is satisfied by perfect gases. So we concentrate on the second half ($d \geqslant 3$). In view of Theorem 3.3, it remains only to prove *a posteriori* uniform L^2 estimates. More precisely, the proof of Theorem 4.1 reduces to establishing the following result.

Lemma 4.2. *Let* $d \geqslant 3$. *Consider a family of solutions* (p^a, v^a, θ^a) *of* (1.4) *(for some source terms* Q^a*) uniformly bounded in the sense of the conclusion of Theorem* 3.3*:*

$$\sup_{a\in A} \|(p^a, v^a, \theta^a)\|_{\mathcal{X}^s_a(T)} < +\infty, \tag{4.1}$$

for some $s > 1 + d/2$ *and fixed* $T > 0$. *Assume further that the source terms* Q^a *are uniformly bounded in* $C^1([0,T];L^1 \cap L^2(\mathbb{R}^d))$ *and that the initial data* $(p^a(0), v^a(0))$ *are uniformly bounded in* $L^2(\mathbb{R}^d)$. *Then the solutions* (p^a, v^a, θ^a) *are uniformly bounded in* $C^0([0,T];L^2(\mathbb{R}^d))$.

Remark 4.3. We allow $Q^a \neq 0$ for application to the combustion equations. To clarify matters, we note that one can replace (4.1) by

$$\sup_{a\in A} \sup_{t\in[0,T]} \|(\nabla p^a(t), \nabla v^a(t))\|_{H^s} + \|\theta^a(t)\|_{H^{s+1}} < +\infty,$$

for some $s > 2 + d/2$.

Proof. For this proof, we set

$$R := \sup_{a\in A}\Big\{ \|(p^a, v^a, \theta^a)\|_{\mathcal{X}^s_a(T)} + \|(p^a(0), v^a(0))\|_{L^2} + \|Q^a\|_{C^1([0,T];L^1\cap L^2)}\Big\},$$

and we denote by $C(R)$ various constants depending only on R.

The strategy of the proof consists of transforming the system (1.4) so as to obtain L^2 estimates uniform in ε by a simple integration by parts argument.

To do that we claim that there exist $U^a \in C^1([0,T];L^2(\mathbb{R}^d))$ satisfying the following properties:

$$\sup_{a\in A} \|(p^a, v^a)\|_{L^\infty_T(L^2)} \leqslant \sup_{a\in A} \|U^a\|_{L^\infty_T(L^2)} + C(R), \tag{4.2}$$

$$\sup_{a\in A} \|U^a(0)\|_{L^2} \leqslant C(R), \tag{4.3}$$

and U^a solves a system having the form

$$E^a \partial_t U^a + \varepsilon^{-1} S(\partial_x) U^a = F^a, \tag{4.4}$$

where $S(\partial_x)$ is skew-symmetric, the symmetric matrices $E^a = E^a(t,x)$ are positive definite and one has the uniform bounds

$$\sup_{a\in A} \|\partial_t E^a\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \big\|(E^a)^{-1}\big\|^{-1}_{L^\infty([0,T]\times\mathbb{R}^d)} + \|F^a\|_{L^1_T(L^2)} \leqslant C(R). \tag{4.5}$$

Before we prove the claim, let us prove that it implies Lemma 4.2. To see this, we combine two basic ingredients:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle\, E^aU^a\,,U^a\,\rangle &= -\varepsilon^{-1}\langle\, S(\partial_x)U^a\,,U^a\,\rangle + \langle\, F^a\,,U^a\,\rangle + \frac{1}{2}\langle\,(\partial_t E^a)U^a\,,U^a\,\rangle \\ &\leqslant \|F^a\|_{L^2}^2 + C(R)\,\|U^a\|_{L^2}^2\,,\end{aligned}$$

and $\|U^a\|_{L^2}^2 \leqslant \big\|(E^a)^{-1}\big\|_{L^\infty}^{-1}\langle\, E^aU^a\,,U^a\,\rangle$. Hence, by (4.3) and (4.5), the Gronwall's Lemma implies that $\|U^a\|_{L^\infty_T(L^2)} \leqslant C(R)$. The estimate (4.2) thus implies the desired result.

To prove the claim, we set $U^a := (p^a, v^a - V^a)^T$ where

$$V^a := \kappa\chi_1(\phi^a)k(\theta^a)\nabla\theta^a + \nabla\Delta^{-1}\big(-\kappa\nabla\chi_1(\phi^a)\cdot k(\theta^a)\nabla\theta^a + \chi_1(\phi^a)Q^a\big).$$

The fact that V^a is well defined follows from Proposition 2.1. We next verify that U^a satisfies (4.4) with

$$E^a = \begin{pmatrix} g_1(\phi^a) & 0 \\ 0 & g_2(\phi^a)\end{pmatrix}, \quad S(\partial_x) = \begin{pmatrix} 0 & \operatorname{div} \\ \nabla & 0\end{pmatrix},$$

$$F^a = \begin{pmatrix} -g_1(\phi^a)v^a\cdot\nabla p^a \\ -g_1(\phi^a)v^a\cdot\nabla v^a + \mu B_2(\phi^a,\partial_x)v^a - g_2(\phi^a)\partial_t V^a\end{pmatrix}.$$

By (2.3), (2.4) and (2.6), to prove that the bounds (4.2) and (4.5) hold, it suffices to prove that $\|\partial_t\phi^a\|_{H^{s-1}} \leqslant C(R)$. Yet, this is nothing new. Indeed, we first observe that, directly from the equations,

$$\|\partial_t\phi^a + v^a\cdot\nabla\phi^a\|_{H^{s-1}} \leqslant C(R).$$

On the other hand, the product rule (2.7) implies that $\|v^a\cdot\nabla\phi^a\|_{H^{s-1}}$ is estimated by $\|\nabla v^a\|_{H^{s-1}}\|\phi^a\|_{H^s} \leqslant C(R)$. This completes the proof. □

Remark 4.4. For our purposes, one of the main differences between $\mathbb{R}^3$ and $\mathbb{R}$ is the following. For all $f \in C_0^\infty(\mathbb{R}^3)$, Proposition 2.5 implies that there exists a vector field $u \in H^\infty(\mathbb{R}^3)$ such that $\operatorname{div} u = f$. In sharp contrast, the mean value of the divergence of a smooth vector field $u \in H^\infty(\mathbb{R})$ is zero. This implies that Lemma 4.2 is false with $d = 1$.

The following result contains an analysis of the easy case where initially $\theta_0 = O(\varepsilon)$. This regime is interesting for the incompressible limit (see [3]).

Proposition 4.5. *Let $d \geqslant 1$ and $\mathbb{R} \ni s > 1 + d/2$. For all $M_0 > 0$, there exists $T > 0$ and $M > 0$ such that for all $a \in A$ and all initial data $(p_0, v_0, \theta_0) \in H^s(\mathbb{R}^d)$ satisfying*

$$\|(p_0, v_0)\|_{H^s} + \varepsilon^{-1}\|\theta_0\|_{H^s} \leqslant M_0, \tag{4.6}$$

the Cauchy problem for (1.4) has a unique classical solution (p, v, θ) in $C^0([0,T]; H^s(\mathbb{R}^d))$ such that

$$\sup_{t\in[0,T]}\|(p(t), v(t))\|_{H^s} + \varepsilon^{-1}\|\theta(t)\|_{H^s} \leqslant M. \tag{4.7}$$

Proof. The proof of this result is based on the change of unknown $(p, v, \theta) \mapsto (G(\theta, \varepsilon p), v, \theta)$ where G is as given by Assumption (H2) in §1.2. By setting $\rho = G(\theta, \varepsilon p)$ it is found that (p, v, θ) satisfies (1.4) if and only if

$$\begin{cases} \chi_3(\partial_t \rho + v \cdot \nabla \rho) + (\chi_3 - \chi_1) \operatorname{div} v = 0, \\ g_2(\partial_t v + v \cdot \nabla v) + \varepsilon^{-2}\gamma_1 \nabla \theta + \varepsilon^{-2}\gamma_2 \nabla \rho - \mu B_2 v = 0, \\ g_3(\partial_t \theta + v \cdot \nabla \theta) + \operatorname{div} v - \kappa \chi_3 \operatorname{div}(k \nabla \theta) = 0, \end{cases} \tag{4.8}$$

where $\gamma_1 = (\chi_1 g_3)/(\chi_3 g_1)$ and $\gamma_2 = 1/g_1$. Notice that Assumption (H2) implies that the coefficients g_i, γ_i, χ_3 and $\chi_3 - \chi_1$ are positive.

The key point is that the assumption (4.6) allows us to symmetrize the equations by setting $u := (\tilde{\rho}, v, \tilde{\theta})$, where

$$\tilde{\rho} := \varepsilon^{-1}\rho, \quad \tilde{\theta} := \varepsilon^{-1}\theta.$$

The fact that this change of unknowns is singular in ε causes no difficulty. Indeed, directly from the assumption (4.6), we have $\|\tilde{\theta}(0)\|_{H^s} \leqslant M_0$. On the other hand, the assumption $G(0,0) = 0$ implies that there is a function C_G such that $\|G(u)\|_{H^\sigma} \leqslant C_G(\|u\|_{L^\infty}) \|u\|_{H^\sigma}$ for all $u \in H^\sigma$ with $\sigma > d/2$. Therefore, we have

$$\begin{aligned} \|\tilde{\rho}\|_{H^s} &= \varepsilon^{-1} \|G(\theta, \varepsilon p)\|_{H^s} \leqslant \varepsilon^{-1} C_G(\|(\theta, \varepsilon p)\|_{L^\infty}) \|(\theta, \varepsilon p)\|_{H^s} \\ &\leqslant C_G(\|(\theta, \varepsilon p)\|_{L^\infty}) \|(\tilde{\theta}, p)\|_{H^s}, \end{aligned} \tag{4.9}$$

hence, $\|\tilde{\rho}(0)\|_{H^s} \leqslant C_0$ for some constant depending only on M_0.

Because $(\vartheta, \wp) \mapsto (\vartheta, G(\vartheta, \wp))$ is a C^∞ diffeomorphism with $G(0,0) = 0$, one can write $\varepsilon p = P\big(\theta, G(\theta, \varepsilon p)\big) = P(\theta, \rho)$, for some C^∞ function P vanishing at the origin. Therefore one can see the coefficients $(g_i,\ \chi_i,\ \gamma_i ...)$ as functions of (θ, ρ). Hence, with $u = (\tilde{\rho}, v, \tilde{\theta})$ as above, one can rewrite System (4.8) under the form

$$A_0(\varepsilon u)\partial_t u + \sum_{1 \leqslant j \leqslant d} A_j(u, \varepsilon u)\partial_j u + \frac{1}{\varepsilon} \sum_{1 \leqslant j \leqslant d} S_j(\varepsilon u)\partial_j u - B(\varepsilon u, \partial_x) u = 0, \tag{4.10}$$

where the matrices S_j, A_j are symmetric (with A_0 positive definite) and the viscous perturbation $B(\varepsilon u, \partial_x)$ is as in (4.8).

Note that one can always assume that the matrices S_j have constant coefficients. Furthermore, since the matrix A_0 multiplying the time derivative depends only on the unknown through εu, and since the initial data $u(0)$ are uniformly bounded in H^s, the proof of the uniform existence Theorem of [15] applies. By that proof, we conclude that the solutions of (4.10) exist and are uniformly bounded for a time T independent of ε. Once this is granted, it remains to verify that the solutions (p, v, θ) of System (1.4) exist and are uniformly bounded in the sense of (4.7). To see this, as for $\tilde{\rho}$ in

(4.9), we note that

$$\begin{aligned}\|p\|_{H^s} &= \|P(\theta,\rho)\|_{H^s}\\ &\leqslant \varepsilon^{-1}C_P(\|(\theta,\rho)\|_{L^\infty})\,\|(\theta,\rho)\|_{H^s} = C_P(\|(\theta,\rho)\|_{L^\infty})\|(\tilde\theta,\tilde\rho)\|_{H^s}\\ &\leqslant C(\|(\tilde\theta,\tilde\rho)\|_{H^s}),\end{aligned}$$

so that $\|(p,v)\|_{H^s} + \varepsilon^{-1}\,\|\theta\|_{H^s} \leqslant C(\|u\|_{H^s})$. This completes the proof. □

Remark 4.6. Consider the Euler equations ($\mu = 0 = \kappa$ and $\varepsilon = 1$). By a standard re-scaling, Proposition 4.5 just says that the classical solutions with small initial data of size δ exist for a time of order of $1/\delta$. Following the approach initiated by Alinhac in [2], several much more precise results have been obtained. In particular, the interested reader is referred to the recent advance of Godin [14] (for the 3D non-isentropic Euler equations).

5. Spatially periodic solutions

In this section, we consider the case where x belongs to the torus $\mathbb{T}^d$.

Theorem 5.1. *Let $d \geqslant 1$ and $\mathbb{N} \ni s > 1 + d/2$. For all source term $Q \in C^\infty(\mathbb{R}\times\mathbb{T}^d)$ and for all $M_0 > 0$, there exist $T > 0$ and $M > 0$ such that, for all $a \in A$ and all initial data $(p_0, v_0, \theta_0) \in H^{s+1}(\mathbb{T}^d)$ satisfying*

$$\|(p_0,v_0)\|_{H^s} + \|(\theta_0,\varepsilon p_0,\varepsilon v_0)\|_{H^{s+1}} \leqslant M_0,$$

the Cauchy problem for (1.4) *has a unique classical solution (p, v, θ) in $C^0([0,T]; H^{s+1}(\mathbb{T}^d))$ such that*

$$\sup_{t\in[0,T]} \|\nabla p(t)\|_{H^{s-1}} + \|v(t)\|_{H^s} + \|(\theta(t),\varepsilon p(t))\|_{H^s} \leqslant M.$$

The proof follows from two observations: first, the results proved in Steps 1–6 in section 3 apply *mutatis mutandis* in the periodic case; and second, as proved below, the periodic case is easier in that one can prove uniform L^2 estimates for the velocity. This in turn implies that (as in [1, 20]) one can directly prove a closed set of estimates by means of the estimate:

$$\|v\|_{H^s(\mathbb{T}^d)} \leqslant C\,\|\operatorname{div} v\|_{H^{s-1}(\mathbb{T}^d)} + C\,\|\operatorname{curl}(\gamma v)\|_{H^{s-1}(\mathbb{T}^d)} + C\,\|v\|_{L^2(\mathbb{T}^d)}\,,$$

for some constant C depending only on $\|\log\gamma\|_{H^s(\mathbb{T}^d)}$ (compare with (2.15)).

Let us concentrate on the main new qualitative property:

Lemma 5.2. *Let $d \geqslant 1$. Consider a family of solutions (p^a, v^a, θ^a) of* (1.4) *(for some source terms Q^a) such that*

$$\sup_{a\in A} \|(p^a,v^a,\theta^a)\|_{\mathcal{X}^s_a(T)} < +\infty,$$

for some $s > 1 + d/2$ and fixed $T > 0$. If Q^a is uniformly bounded in $C^1([0,T]; L^2(\mathbb{T}^d))$ and $(p^a(0), v^a(0))$ is uniformly bounded in $L^2(\mathbb{T}^d)$, then v^a is uniformly bounded in $C^0([0,T]; L^2(\mathbb{T}^d))$.

Proof. The main new technical ingredient is, as used by Schochet in [23], an appropriate ansatz for the pressure.

Again, the proof makes use of the Fourier multiplier $\nabla\Delta^{-1}$. Note, that $\nabla\Delta^{-1}$ is bounded from $L^2_\sharp(\mathbb{T}^d)$ to $H^1(\mathbb{T}^d)$ where $L^2_\sharp(\mathbb{T}^d)$ consists of these functions $u \in L^2(\mathbb{T}^d)$ such that $\langle u\rangle := \int_{\mathbb{T}^d} u(x)\,dx = 0$.

Set

$$F^a := \kappa\chi_1(\phi^a)\operatorname{div}(k(\theta^a)\nabla\theta^a) + \chi_1(\phi^a)Q^a,$$

and introduce the functions $V^a = V^a(t,x)$ and $P^a = P^a(t)$ by

$$P^a := \frac{\langle F^a\rangle}{\langle g_1(\phi^a)\rangle} \quad \text{and} \quad V^a := \nabla\Delta^{-1}\big(F^a - g_1(\phi^a)P^a\big),$$

so that

$$F^a = g_1(\phi^a)P^a + \operatorname{div} V^a.$$

This allows us to rewrite the first equation in (1.4) as

$$g_1(\phi^a)(\partial_t p^a + v^a\cdot\nabla p^a) + \varepsilon^{-1}\operatorname{div}(v^a - V^a) = g_1(\phi^a)P^a.$$

Therefore, by introducing

$$U^a := (q^a, v^a - V^a)^T \quad \text{with} \quad q^a(t,x) = p^a(t,x) - P^a(t),$$

we are back in the situation of Lemma 4.2: U^a satisfies

$$E^a(\partial_t U^a + v^a\cdot\nabla U^a) + \varepsilon^{-1}S(\partial_x)U^a = F^a, \tag{5.1}$$

where $S(\partial_x)$ is skew-symmetric, the matrices E^a are positive definite and

$$\|(E^a, \partial_t E^a + v^a\cdot\nabla E^a)\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \big\|(E^a)^{-1}\big\|^{-1}_{L^\infty([0,T]\times\mathbb{R}^d)} + \|F^a\|_{L^1_T(L^2)}$$

is uniformly bounded.

As before, the proof proceeds by multiplying by U^a and integrating on $\mathbb{T}^d$. We find that $\partial_t\langle\, E^aU^a\,,\,U^a\,\rangle$ is given by

$$\langle\,\big((\partial_t + v^a\cdot\nabla)E^a\big)U^a\,,\,U^a\,\rangle + \langle\,E^a(\operatorname{div} v^a)U^a\,,\,U^a\,\rangle + 2\langle\,F^a\,,\,U^a\,\rangle,$$

and hence conclude that U^a is uniformly bounded in $C^0([0,T];L^2(\mathbb{T}^d))$. Since V^a is uniformly bounded in $C^0([0,T];L^2(\mathbb{T}^d))$, this yields the desired result. □

Remark 5.3. In the periodic case, as shown by Métivier and Schochet [21, 22] as well as Bresch, Desjardins, Grenier and Lin [6], the study of the behavior of the solutions when $\varepsilon \to 0$ involved many additional phenomena.

6. Low Mach number combustion

The system (1.1) is relevant whenever all nuclear or chemical reactions are frozen, which is the case in many treatments of fluid mechanics. By contrast, for the combustion, one has to replace the energy evolution equation by

$$\partial_t(\rho e) + \operatorname{div}(\rho v e) + P\operatorname{div} v = \kappa\operatorname{div}(k\nabla T) + F(Y),$$

with $Y := (Y_1, \dots, Y_L)$ where the Y_ℓ's denote the relative concentrations of nuclear or chemical species. The new unknown Y_ℓ satisfies :

$$\partial_t(\rho Y_\ell) + \operatorname{div}(\rho v Y_\ell) = \lambda \operatorname{div}(D_\ell \nabla Y_\ell) + \rho\omega_\ell(t,x), \tag{6.1}$$

where ω_ℓ is a given source term, $D_\ell > 0$ and λ measures the importance of diffusion processes.

Many results have been obtained for the reactive gas equations (see [7] and the references therein). Yet, the previous studies do not include the dimensionless numbers. Here we consider the system:

$$\begin{cases} \alpha(\partial_t P + v\cdot\nabla P) + \operatorname{div} v = \kappa\beta \operatorname{div}(k\nabla T) + F_1(Y,T,P), \\ \rho(\partial_t v + v\cdot\nabla v) + \dfrac{\nabla P}{\varepsilon^2} = \mu\big(2\operatorname{div}(\zeta Dv) + \nabla(\eta \operatorname{div} v)\big), \\ \gamma(\partial_t T + v\cdot\nabla T) + \operatorname{div} v = \kappa\delta \operatorname{div}(k\nabla T) + F_3(Y,T,P), \\ \rho(\partial_t Y + v\cdot\nabla Y) = \lambda \operatorname{div}(D\nabla Y), \end{cases} \tag{6.2}$$

where α, β, γ and δ are given functions of (Y,T,P).

As explained in the introduction, it is convenient to introduce (p,θ,y) by $P = \underline{P}e^{\varepsilon p}$, $T = \underline{T}e^{\theta}$, $Y = \underline{Y}e^{y}$, where $(\underline{P},\underline{T},\underline{Y}) \in [0,+\infty)^{2+L}$. For smooth solutions, (p,v,θ,y) satisfies a system of the form:

(6.3)

$$\begin{cases} g_1(\Phi)(\partial_t p + v\cdot\nabla p) + \dfrac{1}{\varepsilon}\operatorname{div} v = \dfrac{\kappa}{\varepsilon}\chi_1(\Phi)\operatorname{div}(k(\theta)\nabla\theta) + \dfrac{1}{\varepsilon}Q_1(\Phi), \\ g_2(\Phi)(\partial_t v + v\cdot\nabla v) + \dfrac{1}{\varepsilon}\nabla p = \mu\chi_2(\Phi)\big(\operatorname{div}(\zeta(\theta)Dv) + \nabla(\eta(\theta)\operatorname{div} v)\big), \\ g_3(\Phi)(\partial_t\theta + v\cdot\nabla\theta) + \operatorname{div} v = \kappa\chi_3(\Phi)\operatorname{div}(k(\theta)\nabla\theta) + Q_3(\Phi), \\ g_4(\Phi)(\partial_t y + v\cdot\nabla y) = \lambda\chi_4(\Phi)\operatorname{div}(D(\theta)\nabla y), \end{cases}$$

where $\Phi = (y,\theta,\varepsilon p)$.

Assumption 6.1. Denote by $(\mathsf{y},\vartheta,\wp) \in \mathbb{R}^N$ the place holder of the unknown $(y,\theta,\varepsilon p)$. Parallel to Assumption (H2) in §1.2, we suppose that g_i and χ_i $(i = 1,2,3)$ are C^∞ positive functions of $(\mathsf{y},\vartheta,\wp) \in \mathbb{R}^N$, $\chi_1 < \chi_3$ and there exist two functions F and G such that $(\mathsf{y},\vartheta,\wp) \mapsto (\mathsf{y}, F(\mathsf{y},\vartheta,\wp),\wp)$ and $(\mathsf{y},\vartheta,\wp) \mapsto (\mathsf{y},\vartheta,G(\vartheta,\wp))$ are C^∞ diffeomorphisms from $\mathbb{R}^N$ onto $\mathbb{R}^N$, F and G vanish at the origin, and

$$g_1\frac{\partial F}{\partial\vartheta} = -g_3\frac{\partial F}{\partial\wp} > 0, \qquad g_1\chi_3\frac{\partial G}{\partial\vartheta} = -g_3\chi_1\frac{\partial G}{\partial\wp} < 0.$$

Moreover, Q_1 and Q_3 are C^∞ functions of $(\mathsf{y},\vartheta,\wp)$ vanishing at the origin.

Introduce

$$B := \big\{ (\varepsilon,\mu,\kappa,\lambda) \in (0,1]\times[0,1]\times[0,1]\times[0,2] \,|\, \lambda \geqslant \sqrt{\mu+\kappa} \,\big\}\cdot$$

Definition 6.2. *Let $T > 0$, $s \in \mathbb{R}$, $b = (\varepsilon,\mu,\kappa,\lambda) \in B$ and set $a := (\varepsilon,\mu,\kappa)$. The space $\mathcal{Z}^s_b(T)$ consists of these $(p,v,\theta,y) \in C^0([0,T];H^s(\mathbb{R}^d))$ such that*

$$(p,v,\theta) \in \mathcal{X}^s_a(T), \quad \nu y \in C^0([0,T];H^{s+1}(\mathbb{R}^d)), \quad \lambda y \in L^2(0,T;H^{s+2}_\nu(\mathbb{R}^d)),$$

where $\nu := \sqrt{\mu+\kappa}$ *and* $\mathcal{X}^s_a(T)$ *is as defined in Definition 3.1. The space* $\mathcal{Z}^s_b(T)$ *is given the norm*

$$\|(p,v,\theta,y)\|_{\mathcal{Z}^s_b(T)} := \|(p,v,\theta)\|_{\mathcal{X}^s_a(T)} + \|y\|_{L^\infty_T(H^{s+1}_\nu)} + \sqrt{\lambda}\,\|y\|_{L^2_T(H^{s+2}_\nu)}\,.$$

Having proved estimates for the solutions of System (1.4) with precised estimates in terms of the norm Σ of the source term Q (see (3.3)), we are now in position to assert that:

Theorem 6.3. *Assume that* $d \neq 2$*. Given* $M_0 > 0$ *and* $\mathbb{N} \ni s > 1 + d/2$*, there exist* $T > 0$ *and* $M > 0$*, such that for all* $b \in B$ *and all initial data* $(p_0, v_0, \theta_0, y_0) \in H^{s+1}(\mathbb{R}^d)$ *satisfying*

$$\|(\nabla p_0, \nabla v_0)\|_{H^{s-1}} + \|(y_0, \theta_0, \varepsilon p_0, \varepsilon v_0)\|_{H^{s+1}} \leqslant M_0,$$

the Cauchy problem for (1.4) *has a unique classical solution* (p, v, θ, y) *in the ball* $B(\mathcal{Z}^s_b(T); M)$*.*

Remark 6.4. For the case of greatest physical interest ($d = 3$), Theorem 6.3 has two corollaries. As alluded to in Section 4, it allows us to rigorously justify, at least in the whole space case, the computations given by Majda in [18]. By the way, this proves the well posedness of the Cauchy problem for the zero Mach number combustion in the whole space (this was known only in the periodic case [11]). Moreover, note that the solutions given by Theorem 6.3 satisfy uniform estimates recovering in the limit $\varepsilon \to 0$ those obtained by Embid for the limit system. Finally, we mention that the previous analysis seems to apply with $Q_i(\Phi)$ replaced by $\chi_i(\Phi)Q(\Phi, \nabla y, \nabla^2 y)$ for some smooth function Q, yet we will not address this issue.

Appendix A. General equations of state

Recall that, in order to study the full Navier-Stokes equations (1.1), we choose to work with the unknown (P, v, T). In order to close this system, we must relate (ρ, e) to (P, T) by means of two equations of state: $\rho = \rho(P, T)$ and $e = e(P, T)$. The purpose of this section is to show that Assumption (H2) in §1.2 is satisfied under general assumptions on the partial derivatives of ρ and e with respect to P and T.

A.1. Computation of the coefficients.

We begin by expressing the coefficients g_i and χ_i, which appear in (1.4), in terms of the partial derivatives of ρ and e with respect to P and T. To do that it is convenient to introduce the entropy. Here is where the first identity in (1.5) enters.

Assumption A.1. The functions ρ and e are C^∞ functions of $(P, T) \in (0, +\infty)^2$, satisfying

$$P\frac{\partial \rho}{\partial P} + T\frac{\partial \rho}{\partial T} = \rho^2 \frac{\partial e}{\partial P}.$$

Introduce the 1-form ω defined by $T\omega := \mathrm{d}\, e + P\,\mathrm{d}(1/\rho)$, where we started using the notation $\mathrm{d}\, f = (\partial f/\partial T)\,\mathrm{d}\, T + (\partial f/\partial P)\,\mathrm{d}\, P$. Assumption A.1 implies that $\mathrm{d}\,\omega = 0$. Hence, the Poincaré's Lemma implies that there exists a C^∞ function $S = S(P,T)$, defined on $(0,+\infty)^2$, satisfying the second principle of thermodynamics:

$$T\,\mathrm{d}\, S = \mathrm{d}\, e + P\,\mathrm{d}(1/\rho). \tag{A.1}$$

By combining the evolution equations for ρ and e with (A.1) written in the form $\rho T\,\mathrm{d}\, S = \rho\,\mathrm{d}\, e - (p/\rho)\,\mathrm{d}\,\rho$, we get an evolution equation for S, so that

$$(\partial_t + v\cdot\nabla)\begin{pmatrix}\rho\\ S\end{pmatrix} = \begin{pmatrix}-\rho & 0\\ 0 & (\rho T)^{-1}\end{pmatrix}\begin{pmatrix}\operatorname{div} v\\ \kappa\operatorname{div}(k\nabla T) + Q\end{pmatrix}\cdot$$

On the other hand, one has

$$(\partial_t + v\cdot\nabla)\begin{pmatrix}\rho\\ S\end{pmatrix} = J(\partial_t + v\cdot\nabla)\begin{pmatrix}P\\ T\end{pmatrix} \quad \text{with} \quad J = \begin{pmatrix}\partial\rho/\partial P & \partial\rho/\partial T\\ \partial S/\partial P & \partial S/\partial T\end{pmatrix}\cdot$$

Equating both right hand sides and inverting the matrix J, we obtain

$$\begin{cases}(\partial_t P + v\cdot\nabla P) + a\operatorname{div} v - \kappa b\operatorname{div}(k\nabla T) = bQ,\\ (\partial_t T + v\cdot\nabla T) + c\operatorname{div} v - \kappa d\operatorname{div}(k\nabla T) = dQ,\end{cases} \tag{A.2}$$

where

$$a = \frac{\rho(\partial S/\partial T)}{\det(J)},\ b = -\frac{\partial\rho/\partial T}{\rho T\det(J)},\ c = -\frac{\rho(\partial S/\partial P)}{\det(J)},\ d = \frac{\partial\rho/\partial P}{\rho T\det(J)}\cdot$$

To express the coefficients g_i and χ_i in terms of physically relevant quantities, we need some more notations. We introduce

$$\begin{aligned} &K_T := \frac{1}{\rho}\frac{\partial\rho}{\partial P}, \qquad K_P := -\frac{1}{\rho}\frac{\partial\rho}{\partial T}, \qquad \mathcal{R} := -\rho\frac{\partial S/\partial P}{\partial\rho/\partial P},\\ &C_P := T\frac{\partial S}{\partial T}, \qquad C_V := T\frac{(\partial S/\partial T)(\partial\rho/\partial P) - (\partial S/\partial P)(\partial S/\partial T)}{\partial\rho/\partial P}\cdot\end{aligned} \tag{A.3}$$

The functions K_T, K_P, C_V and C_P are known as the coefficient of isothermal compressibility, the coefficient of thermal expansion and the specific heats at constant volume and pressure, respectively (see Section 2 in [12]). The function $\mathcal{R}$ generalizes the usual gas constant: for perfect gases one can check that $\mathcal{R} = R$.

We now have to convert System A.2 into equations for the fluctuations p and θ as defined by (1.2). Performing a little algebra we find that

$$\begin{cases}\dfrac{K_T C_V P}{C_P}(\partial_t p + v\cdot\nabla p) + \dfrac{1}{\varepsilon}\operatorname{div} v - \dfrac{\kappa}{\varepsilon}\dfrac{K_P}{\rho C_P}\operatorname{div}(kT\nabla\theta) = \dfrac{1}{\varepsilon}\dfrac{K_P}{\rho C_P}Q,\\ \rho(\partial_t v + v\cdot\nabla v) + \dfrac{1}{\varepsilon}P\nabla p = \mu\big(2\operatorname{div}(\zeta Dv) + \nabla(\eta\operatorname{div} v)\big),\\ \rho C_V T(\partial_t\theta + v\cdot\nabla\theta) + \mathcal{R}\rho T\operatorname{div} v - \kappa\operatorname{div}(kT\nabla\theta) = Q.\end{cases}$$

Hence, (p, v, θ) satisfies (1.4) with

$$g_1^* = \frac{K_T C_V P}{C_P},\ g_2^* = \frac{\rho}{P},\ g_3^* = \frac{C_V}{\mathcal{R}},\ \chi_1^* = \frac{K_P}{\rho C_P},\ \chi_2^* = \frac{1}{P},\ \chi_3^* = \frac{1}{\mathcal{R}\rho T}, \tag{A.4}$$

where we used the following notation: for all $f\colon (0,+\infty)^2 \to \mathbb{R}$,

$$f^*(\vartheta, \wp) := f(\underline{T}e^{\vartheta}, \underline{P}e^{\wp}). \tag{A.5}$$

A.2. **Properties of the coefficients.**

Assumption A.2. The functions ρ and e are C^∞ functions of $(P, T) \in (0, +\infty)^2$ such that, $\rho > 0$ and

$$\frac{\partial \rho}{\partial P} > 0, \quad \frac{\partial \rho}{\partial T} < 0 \quad \text{and} \quad \frac{\partial e}{\partial T}\frac{\partial \rho}{\partial P} > \frac{\partial e}{\partial P}\frac{\partial \rho}{\partial T}. \tag{A.6}$$

Remark A.3. This assumption is satisfied by general equations of state. Indeed, (A.6) just means that the coefficients K_T, K_P and C_V are positive.

The following result prove that Assumptions A.1 and A.2 imply that our main structural assumption is satisfied.

Proposition A.4. *If Assumptions A.1 and A.2 are satisfied, then* $\chi_1 < \chi_3$ *and* g_i, χ_i $(i = 1, 2, 3)$ *are* C^∞ *positive functions.*

Proof. In view of (A.4), the proof reduces to establishing that

$$0 < K_T, \quad 0 < K_P, \quad 0 < C_V < C_P \quad \text{and} \quad 0 < \mathcal{R} < \frac{C_P}{T K_P}.$$

The first two inequalities follow from the definitions of K_T and K_P. To prove the last two, we first establish the Maxwell's identity $\partial S/\partial P = \rho^{-2}(\partial \rho/\partial T)$. To see this, by (A.1), we compute

$$\frac{\partial S}{\partial P}\,\mathrm{d}\,T \wedge \mathrm{d}\,P = \mathrm{d}\big(T\,\mathrm{d}\,S\big) = \mathrm{d}\Big\{\mathrm{d}\,e + P\,\mathrm{d}\Big(\frac{1}{\rho}\Big)\Big\} = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial T}\,\mathrm{d}\,P \wedge \mathrm{d}\,T.$$

Since $\partial\rho/\partial T < 0$, the Maxwell's identity implies that $\partial S/\partial P < 0$. By combining this inequality with $\partial\rho/\partial P > 0$, we find $\mathcal{R} > 0$. Also, the identity $\partial S/\partial P = \rho^{-2}(\partial\rho/\partial T)$ implies that

$$\frac{C_P}{C_V} = \frac{(\partial S/\partial T)(\partial \rho/\partial P)}{(\partial S/\partial T)(\partial \rho/\partial P) - \rho^{-2}\big(\partial\rho/\partial T\big)^2},$$

which proves $C_V < C_P$.

In view of (A.1), the assumption $\frac{\partial e}{\partial T}\frac{\partial \rho}{\partial P} > \frac{\partial e}{\partial P}\frac{\partial \rho}{\partial T}$ is equivalent to

$$\frac{\partial S}{\partial T}\frac{\partial \rho}{\partial P} > \frac{\partial S}{\partial P}\frac{\partial \rho}{\partial T}.$$

This inequality has two consequences. Firstly, it implies that $C_V > 0$. Secondly, it yields

$$\frac{T K_P \mathcal{R}}{C_P} = \frac{(\partial S/\partial P)(\partial \rho/\partial T)}{(\partial S/\partial T)(\partial \rho/\partial P)} < 1.$$

This concludes the proof. □

We now discuss the physical meaning of the functions F and G introduced in §1.2. These are compatibility conditions between the singular terms and the viscous terms. To see this, suppose (p,v,θ) is a smooth solution of (1.4) and let $\Psi=\Psi(\vartheta,\wp)\in C^\infty(\mathbb{R}^2)$. Then $\psi:=\Psi(\theta,\varepsilon p)$ satisfies

$$\begin{aligned}&g_1g_3\big(\partial_t\psi+v\cdot\nabla\psi\big)\\&\quad+\underbrace{\Big(g_1\frac{\partial\Psi}{\partial\vartheta}+g_3\frac{\partial\Psi}{\partial\wp}\Big)}_{=:\Gamma_1(\Psi)}\operatorname{div}v=\kappa\underbrace{\Big(g_1\chi_3\frac{\partial\Psi}{\partial\vartheta}+g_3\chi_1\frac{\partial\Psi}{\partial\wp}\Big)}_{=:\Gamma_2(\Psi)}\big(\operatorname{div}(k(\theta)\nabla\theta)+Q\big),\end{aligned}$$

where the coefficients g_i, χ_i, $\partial\Psi/\partial\vartheta$ and $\partial\Psi/\partial\wp$ are evaluated at $(\theta,\varepsilon p)$. We next show that for appropriate function Ψ one can impose

$$\text{(A.7)}\qquad [\Gamma_1(\Psi)=0\text{ and }\Gamma_2(\Psi)>0]\quad\text{or}\quad[\Gamma_1(\Psi)>0\text{ and }\Gamma_2(\Psi)=0].$$

Proposition A.5. *Assume that Assumptions A.1 and A.2 are satisfied and use the notation* (A.5). *The functions S^* and ρ^* satisfy*

$$\text{(A.8)}\qquad g_1\frac{\partial S^*}{\partial\vartheta}=-g_3\frac{\partial S^*}{\partial\wp}>0,\qquad g_1\chi_3\frac{\partial\rho^*}{\partial\vartheta}=-g_3\chi_1\frac{\partial\rho^*}{\partial\wp}<0.$$

Remark A.6. The fact that $\Psi=S^*$ (or $\Psi=\rho^*$) satisfies the first (respectively second) set of conditions in (A.7) now follows from $\chi_1<\chi_3$.

Proof. By (A.4) and the definitions given in (A.3), one has

$$\text{(A.9)}\qquad \frac{g_1^*}{g_3^*}=-\frac{P(\partial S/\partial P)}{T(\partial S/\partial T)}\cdot$$

By definition (A.5), $\partial f^*/\partial\vartheta=\big[T(\partial f/\partial T)\big]^*$ and $\partial f^*/\partial\wp=\big[P(\partial f/\partial P)\big]^*$. This proves that S^* satisfies the first identity in (A.8). Next, we compute

$$\frac{\chi_1^*}{\chi_3^*}=\frac{(\partial\rho/\partial T)(\partial S/\partial P)}{(\partial\rho/\partial P)(\partial S/\partial T)}\cdot$$

By (A.9), this yields $\chi_1^*g_3^*P(\partial\rho/\partial P)=-\chi_3^*g_1^*T(\partial\rho/\partial T)$. Which proves that ρ^* satisfies the second identity in (A.8). □

Remark A.7. Assumption (H2) in §1.2 requires, in addition, that $F=S^*$ and $G=\rho^*$ define bijections. This means nothing but the fact that the thermodynamic state is completely determined by (P,T), or (P,S) or (ρ,T).

The following result contains an example of equation of state such that χ_1 depends on ϑ.

Proposition A.8. *Assume that the gas obeys Mariotte's law: $P=R\rho T$, for some positive constant R, and $e=e(T)$ satisfies $C_V:=\partial e/\partial T>0$. Then, Assumptions A.1 and A.2 are satisfied. Moreover,*

$$\chi_1^*=R/((C_V(T)+R)P),$$

so that $\chi_1(\vartheta,\wp)$ is independent of ϑ if and only if C_V is constant.

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