PARALINEARIZATION OF THE MUSKAT EQUATION AND APPLICATION TO THE CAUCHY PROBLEM

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Abstract. We paralinearize the Muskat equation to extract an explicit parabolic evolution equation having a compact form. This result is applied to give a simple proof of the local well-posedness of the Cauchy problem for rough initial data, in homogeneous Sobolev spaces $\dot{H}^1(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$ with $s > 3/2$. This paper is essentially self-contained and does not rely on general results from paradifferential calculus.

1. Introduction

The Muskat equation is a fundamental equation for incompressible fluids in porous media. It describes the evolution of a time-dependent free surface $\Sigma(t)$ separating two fluid domains $\Omega_1(t)$ and $\Omega_2(t)$. A common assumption in this theory is that the motion is in two dimensions so that the interface is a curve. In this introduction, for the sake of simplicity, we assume that the interface is a graph (the analysis is done later on for a general interface). On the supposition that the fluids extend indefinitely in horizontal directions, it results that

$$\begin{aligned}
\Omega_1(t) &= \{(x,y) \in \mathbb{R} \times \mathbb{R}\,;\, y > h(t,x)\}\,,\\
\Omega_2(t) &= \{(x,y) \in \mathbb{R} \times \mathbb{R}\,;\, y < h(t,x)\}\,,\\
\Sigma(t) &= \partial\Omega_1(t) = \partial\Omega_2(t) = \{y = h(t,x)\}.
\end{aligned}$$

Introduce the density ρ_i, the velocity v_i and the pressure P_i in the domain Ω_i $(i = 1,2)$. One assumes that the velocities v_1 and v_2 obey Darcy's law. Then, the equations by which the motion is to be determined are

$$\begin{aligned}
v_i &= \nabla(P_i + \rho_i g y) && \text{in } \Omega_i,\\
\operatorname{div} v_i &= 0 && \text{in } \Omega_i,\\
P_1 &= P_2 && \text{on } \Sigma,\\
v_1 \cdot n &= v_2 \cdot n && \text{on } \Sigma,
\end{aligned}$$

where g is the gravity and n is the outward unit normal to Ω_2 on Σ,

$$n = \frac{1}{\sqrt{1+(\partial_x h)^2}} \begin{pmatrix} -\partial_x h \\ 1 \end{pmatrix}.$$

The first two equations express the classical Darcy's law and the last two equations impose the continuity of the pressure and the normal velocities at the interface. This system is supplemented with an equation for the evolution of the free surface:

$$\partial_t h = \sqrt{1+(\partial_x h)^2}\, v_2 \cdot n.$$

The previous system has been introduced by Muskat in [41] whose main application was in petroleum engineering (see [42, 43] for many historical comments).

In [21], Córdoba and Gancedo discovered a formulation of the previous system based on contour integral, which applies whether the interface is a graph or not. The latter work opened the door to the solution of many important problems concerning the Cauchy problem or blow-up solutions (see [19, 9, 10, 11], more references are given below as well as in the survey papers [30, 31]). This formulation is a compact equation where the unknown is the parametrization of the free surface, namely a function $f = f(t, x)$ depending on time $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, satisfying

$$\partial_t f = \frac{\rho}{2\pi} \partial_x \int \arctan\left(\Delta_\alpha f\right) \mathrm{d}\alpha, \tag{1.1}$$

where $\rho = \rho_2 - \rho_1$ is the difference of the densities, the integral is understood in the principal value sense and $\Delta_\alpha f$ is the slope, namely

$$\Delta_\alpha f(t, x) = \frac{f(x, t) - f(x - \alpha, t)}{\alpha}.$$

The beauty of equation (1.1) lies in its apparent simplicity, which should be compared with the complexity of the equations written in Eulerian formulation. This might suggest that (1.1) is the simplest version of the Muskat equation one may hope for. However, since the equation is highly nonlocal (this means that the nonlinearity enters in the nonlocal terms), even with this formulation the study of the Cauchy problem for (1.1) is a very delicate problem. We refer the reader to the above mentioned papers for the description of the main difficulties one has to cope with.

Our goal in this paper is to continue this line of research. We want to simplify further the study of the Muskat problem by transforming the equation (1.1) into the simplest possible form. We shall prove that one can derive from the formulation (1.1) an explicit parabolic evolution. In particular, we shall see that one can decouple the nonlinear and nonlocal aspects. There are many possible applications that one could work out of this explicit parabolic formulation. Here we shall study the Cauchy problem in homogeneous Sobolev spaces.

The well-posedness of the Cauchy problem was first proved in [21] by Córdoba and Gancedo for initial data in $H^3(\mathbb{R})$ in the stable regime $\rho_2 > \rho_1$ (they also proved that the problem is ill-posed in Sobolev spaces when $\rho_2 < \rho_1$). Several extensions of their results have been obtained by different proofs. In [14], Cheng, Granero-Belinchón, Shkoller proved the well-posedness of the Cauchy problem in $H^2(\mathbb{R})$ (introducing a Lagrangian point of view which can be used in a broad setting, see [34]) and Constantin, Gancedo, Shvydkoy and Vicol ([18]) considered rough initial data which are in $W^{2,p}(\mathbb{R})$ for some $p > 1$, as well they obtained a regularity criteria for the Muskat problem. We refer also to the recent work [31] where a regularity criteria is obtained in terms of a control of some critical quantities. Many recent results are motivated by the fact that, loosely speaking, the Muskat equation has to do with the slope more than with the curvature of the fluid interface. Indeed, one scale invariant norm is the Lipschitz norm $\sup_{x\in\mathbb{R}} |\partial_x f(t, x)|$. We refer the reader to the work [17] of Constantin, Córdoba, Gancedo, Rodríguez-Piazza and Strain for global well-posedness results assuming that the Lipschitz semi-norm is smaller than 1 (see also [15] where time decay of those solutions is proved). In [26], Deng, Lei and Lin proved the existence of global in time solutions with large slopes, assuming some monotonicity assumption on the data. In [8], Cameron was able to prove a global existence result assuming that some critical quantity, namely the product of the maximal and minimal slopes, is smaller than 1. His result allows to consider arbitrary large slopes. By using a new formulation of the Muskat equation involving oscillatory integrals, Córdoba and the second author in [20] proved that the Muskat equation is globally well-posed for sufficiently smooth data provided the critical Sobolev norm $\dot{H}^{\frac{3}{2}}(\mathbb{R})$ is small enough. The latter is a global existence result of a unique strong solution having arbitrarily large slopes.

These observations suggest to study the local in time well-posedness of the Cauchy problem without assuming that any L^p-norm of the curvature is finite. The well-posedness of the Cauchy problem in this case was obtained by Matioc [37, 38]. Using tools from functional analysis, Matioc proved that the Cauchy problem is locally in time well-posed for initial data in Sobolev spaces $H^s(\mathbb{R})$ with $s > 3/2$, without smallness assumption. We shall give a simpler proof which generalizes the latter result to homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R})$. Eventually, let us mention that many recent results focus on different rough solutions, which are important for instance in the unstable regime $\rho_1 > \rho_2$ (see *e.g.* the existence mixing zones in [12, 13, 44] or the dynamic between the two different regimes [23, 24]). We refer also to [22, 47] where uniqueness issues have been studied using the convex integration scheme.

In this paper we assume that the difference between the densities in the two fluids satisfies $\rho > 0$, so, by rescaling in time, we can assume without loss of generality that $\rho = 2$.

A fundamental difference with the above mentioned results is that we shall determine the full structure of the nonlinearity instead of performing energy estimates. To explain this, we begin by identifying the nonlinear terms. Since $\rho = 2$, one can rewrite equation (1.1) as

$$\partial_t f = \frac{1}{\pi}\int \frac{\partial_x \Delta_\alpha f}{1+(\Delta_\alpha f)^2}\,\mathrm{d}\alpha$$

(in this introduction some computations are formal, but we shall rigorously justify them later). Consequently, the linearized Muskat equation reads

$$\partial_t u = \frac{1}{\pi}\,\mathrm{pv}\int \partial_x \Delta_\alpha u\,\mathrm{d}\alpha. \tag{1.2}$$

Consider the singular integral operators

$$\mathcal{H}u = -\frac{1}{\pi}\,\mathrm{pv}\int \Delta_\alpha u\,\mathrm{d}\alpha \quad \text{and} \quad \Lambda = \mathcal{H}\partial_x. \tag{1.3}$$

Then $\mathcal{H}$ is the Hilbert transform (the Fourier multiplier with symbol $-i\,\mathrm{sgn}(\xi)$) and Λ is the square root of $-\partial_{xx}$. With the latter notation, the linearized Muskat equation (1.2) reads

$$\partial_t u + \Lambda u = 0.$$

With this notation, the Muskat equation (1.1) can be written under the form

$$\partial_t f + \Lambda f = \mathcal{T}(f)f, \tag{1.4}$$

where $\mathcal{T}(f)$ is the operator defined by

$$\mathcal{T}(f)g = -\frac{1}{\pi}\int \left(\partial_x \Delta_\alpha g\right)\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}\,\mathrm{d}\alpha.$$

Our first main result will provide a thorough study of this nonlinear operator. Before going any further, let us fix some notations.

Definition 1.1. *i)* Given a real number σ, we denote by Λ^σ the Fourier multiplier with symbol $|\xi|^\sigma$ and by $\dot{H}^\sigma(\mathbb{R})$ the homogeneous Sobolev space of tempered distributions whose Fourier transform $\hat{u}$ belongs to $L^1_{loc}(\mathbb{R})$ and satisfies

$$\|u\|^2_{\dot{H}^\sigma} = \|\Lambda^\sigma u\|^2_{L^2} = \frac{1}{2\pi}\int_{\mathbb{R}} |\xi|^{2\sigma} |\hat{u}(\xi)|^2 \, d\xi < +\infty.$$

ii) We denote by $H^\sigma(\mathbb{R})$ the nonhomogeneous Sobolev space $L^2(\mathbb{R}) \cap \dot{H}^\sigma(\mathbb{R})$. We set $H^\infty(\mathbb{R}) := \cap_{\sigma \geq 0} H^\sigma(\mathbb{R})$ and introduce $X := \cap_{\sigma\geq 1}\dot{H}^\sigma(\mathbb{R})$, the set of tempered distributions whose Fourier transform $\hat{u}$ belongs to $L^1_{loc}(\mathbb{R})$ and whose derivative belongs to $H^\infty(\mathbb{R})$.

iii) Given $0 < s < 1$, the homogeneous Besov space $\dot{B}^s_{2,1}(\mathbb{R})$ consists of those tempered distributions f whose Fourier transform is integrable near the origin and such that

$$\|f\|_{\dot{B}^s_{2,1}} = \int \left(\int \frac{|f(x) - f(x-\alpha)|^2}{|\alpha|^{2s}} \, dx \right)^{\frac{1}{2}} \frac{d\alpha}{|\alpha|} < +\infty.$$

iv) We use the notation $\|\cdot\|_{E\cap F} = \|\cdot\|_E + \|\cdot\|_F$.

Theorem 1.2. *i) (Low frequency estimate) There exists a constant C such that, for all f in $\dot{H}^1(\mathbb{R})$ and all g in $\dot{H}^{\frac{3}{2}}(\mathbb{R})$, $\mathcal{T}(f)g$ belongs to $L^2(\mathbb{R})$ and*

$$\|\mathcal{T}(f)g\|_{L^2} \leq C \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^{\frac{3}{2}}}.$$

Moreover, $f \mapsto \mathcal{T}(f)f$ is locally Lipschitz from $\dot{H}^1(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R})$ to $L^2(\mathbb{R})$.

ii) (High frequency estimate) For all $0 < \nu < \epsilon < 1/2$, there exists a positive constant $C > 0$ such that, for all functions f, g in $X = \cap_{\sigma\geq 1}\dot{H}^\sigma(\mathbb{R})$,

$$\mathcal{T}(f)g = \gamma(f)\Lambda g + V(f)\partial_x g + R(f,g) \tag{1.5}$$

where

$$\gamma(f) := \frac{f_x^2}{1+f_x^2},$$

and $R(f,g)$ and $V(f)$ satisfy

$$\|R(f,g)\|_{L^2} \leq C \|f\|_{\dot{H}^{\frac{3}{2}+\epsilon}} \|g\|_{\dot{B}^{1-\epsilon}_{2,1}}, \tag{1.6}$$

and

$$\|V(f)\|_{C^{0,\nu}} := \|V(f)\|_{L^\infty} + \sup_{y\in\mathbb{R}} \left(\frac{|V(f)(x+y) - V(f)(x)|}{|y|^\nu} \right) \leq C \|f\|^2_{\dot{H}^1 \cap \dot{H}^{\frac{3}{2}+\epsilon}}. \tag{1.7}$$

iii) Let $0 < \epsilon < 1/2$. There exists a non-decreasing function $\mathcal{F}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all functions f, g in X,

$$\left\|\Lambda^{1+\epsilon}\mathcal{T}(f)g - \mathcal{T}(f)\Lambda^{1+\epsilon}g\right\|_{L^2} \leq \mathcal{F}\big(\|f\|_{\dot{H}^1\cap\dot{H}^{\frac{3}{2}+\epsilon}}\big) \|f\|_{\dot{H}^1\cap\dot{H}^{\frac{3}{2}+\epsilon}} \|g\|_{\dot{H}^{\frac{3}{2}+\epsilon}\cap\dot{H}^2}. \tag{1.8}$$

The proof of the first statement follows directly from the definition of fractional Sobolev spaces in terms of finite differences, see Section 2. The proof of the second statement is the most delicate part of the proof, which requires to uncover some symmetries in the nonlinearity, see Section 4. The last statement is proved in Section 3 by using sharp variants of the usual nonlinear estimates in Sobolev spaces. Namely we used for the later proof a version of the classical Kato-Ponce estimate proved recently by Li and also a refinement of the composition rule in Sobolev spaces proved in Section 2.

We deduce from the previous result a paralinearization formula for the nonlinearity. We do not consider paradifferential operators as introduced by Bony ([6, 39]).

Instead, following Shnirelman [46], we consider a simpler version of these operators which is convenient for the analysis of the Muskat equation for rough solutions.

Corollary 1.3. *Consider* $0 < \epsilon < 1/2$ *and, given a bounded function* $a = a(x)$*, denote by* $\tilde{T}_a\colon \dot{H}^{1+\epsilon}(\mathbb{R}) \to L^2(\mathbb{R}) \cap \dot{H}^{1+\epsilon}(\mathbb{R})$ *the paraproduct operator defined by*

$$\tilde{T}_a g = (I + \Lambda^{1+\epsilon})^{-1}(a\Lambda^{1+\epsilon} g).$$

Then, there exists a function $\mathcal{F}\colon \mathbb{R} \to \mathbb{R}$ *such that, for all* $f \in X$*,*

$$\mathcal{T}(f)f = \tilde{T}_{\gamma(f)}\Lambda f + \tilde{T}_{V(f)}\partial_x f + R_\epsilon(f), \tag{1.9}$$

where

$$\|R_\epsilon(f)\|_{H^{1+\epsilon}} \le \mathcal{F}\big(\|f\|_{\dot{H}^1\cap\dot{H}^{\frac{3}{2}+\epsilon}}\big)\|f\|_{\dot{H}^1\cap\dot{H}^{\frac{3}{2}+\epsilon}}\|f\|_{\dot{H}^1\cap\dot{H}^{2+\frac{\epsilon}{2}}}. \tag{1.10}$$

Proof. Writing

$$\mathcal{T}(f)f = (I + \Lambda^{1+\epsilon})^{-1}\mathcal{T}(f)\Lambda^{1+\epsilon}f + (I + \Lambda^{1+\epsilon})^{-1}\big[\Lambda^{1+\epsilon}, \mathcal{T}(f)\big]f,$$

and using the formula (1.5) we find that (1.9) holds with

$$R_\epsilon(f) = (I + \Lambda^{1+\epsilon})^{-1}R\big(f, \Lambda^{1+\epsilon}f\big) + (I + \Lambda^{1+\epsilon})^{-1}\big[\Lambda^{1+\epsilon}, \mathcal{T}(f)\big]f.$$

Then, it follows from (1.6) and (1.8) that

$$\|R_\epsilon(f)\|_{H^{1+\epsilon}} \le \mathcal{F}\big(\|f\|_{\dot{H}^1\cap\dot{H}^{\frac{3}{2}+\epsilon}}\big)\|f\|_{\dot{H}^1\cap\dot{H}^{\frac{3}{2}+\epsilon}}\Big(\|f\|_{\dot{H}^{\frac{3}{2}+\epsilon}\cap\dot{H}^2} + \big\|\Lambda^{1+\epsilon}f\big\|_{\dot{B}^{1-\epsilon}_{2,1}}\Big).$$

So (1.10) follows from $\dot{H}^{1-\frac{3\epsilon}{2}}(\mathbb{R}) \cap \dot{H}^{1-\frac{\epsilon}{2}}(\mathbb{R}) \hookrightarrow \dot{B}^{1-\epsilon}_{2,1}(\mathbb{R})$ (see Lemma 2.2). □

We now consider the Cauchy problem for the Muskat equation. Substituting the above identity for $\mathcal{T}(f)$ in the equation (1.4) and simplifying, we find

$$\Big(\partial_t - V(f)\partial_x + \frac{1}{1+f_x^2}\Lambda\Big)\Lambda^{1+\epsilon}f = \Lambda^{1+\epsilon}R_\epsilon(f).$$

Now, the key point is that the estimates (1.6) and (1.10) mean that the remainder term $R_\epsilon(f)$ and the operator $V\partial_x$ contribute as operators of order stricly less than 1 (namely $1-\epsilon/2$ and $1-\nu$) to an energy estimate, and so they are sub-principal terms for the analysis of the Cauchy problem. We also observe that the Muskat equation is parabolic as long as one controls the L^∞-norm of f_x only. This observation is related to our second goal, which is to solve the Cauchy problem in homogeneous Sobolev spaces instead of nonhomogeneous spaces. This is a natural result since the Muskat equation is invariant by the transformation $f \mapsto f + C$. This allows us to make an assumption only on the L^∞-norm of the slope of the initial data, allowing initial data which are not bounded or not square integrable.

Theorem 1.4. *Consider* $s \in (3/2, 2)$ *and an initial data* f_0 *in* $\dot{H}^1(\mathbb{R})\cap\dot{H}^s(\mathbb{R})$*. Then, there exists a positive time* T *such that the Cauchy problem for* (1.1) *with initial data* f_0 *has a unique solution* f *satisfying* $f(t,x) = f_0(x) + u(t,x)$ *with* $u(0,x) = 0$ *and*

$$u \in C^0\big([0,T]; H^s(\mathbb{R})\big) \cap C^1\big([0,T]; H^{s-1}(\mathbb{R})\big) \cap L^2\big(0,T; H^{s+\frac{1}{2}}(\mathbb{R})\big),$$

where $H^\sigma(\mathbb{R})$ *denotes the nonhomogeneous Sobolev space* $L^2(\mathbb{R}) \cap \dot{H}^\sigma(\mathbb{R})$*.*

The latter result is proved in the last section. We conclude this introduction by fixing some notations.

Notation 1.5. i) We denote by f_x the spatial derivative of f .

ii) $A \lesssim B$ means that there is $C > 0$, depending only on fixed quantities, such that $A \le CB$.

iii) Given $g = g(\alpha, x)$ and Y a space of functions depending only on x, the notation $\|g\|_Y$ is a compact notation for $\alpha \mapsto \|g(\alpha,\cdot)\|_Y$.

2. Preliminaries

In this section, we recall or prove various results about Besov spaces which we will need throughout the article. We use the definition of these spaces originally given by Besov in [5], using integrability properties of finite differences.

Given a real number α, the finite difference operators δ_α and s_α are defined by:

$$\delta_\alpha f(x) = f(x) - f(x-\alpha),$$
$$s_\alpha f(x) = 2f(x) - f(x-\alpha) - f(x+\alpha).$$

Definition 2.1. Consider three real numbers (p,q,s) in $[1,\infty]^2 \times (0,2)$. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$ consists of those tempered distributions f whose Fourier transform is integrable near the origin and such that the following quantity $\|f\|_{\dot{B}^s_{p,q}}$ is finite:

$$\|f\|_{\dot{B}^s_{p,q}} = \left\| \frac{\|\delta_\alpha f\|_{L^p(\mathbb{R},\mathrm{d}x)}}{|\alpha|^s} \right\|_{L^q(\mathbb{R},|\alpha|^{-1}\,\mathrm{d}\alpha)} \qquad \text{for } s\in(0,1), \tag{2.1}$$

$$\|f\|_{\dot{B}^s_{p,q}} = \left\| \frac{\|s_\alpha f\|_{L^p(\mathbb{R};\mathrm{d}x)}}{|\alpha|^s} \right\|_{L^q(\mathbb{R},|\alpha|^{-1}\,\mathrm{d}\alpha)} \qquad \text{for } s\in[1,2). \tag{2.2}$$

We refer the reader to the book of Peetre [45, chapter 8] for the equivalence between these definitions and the one in terms of Littlewood-Paley decomposition (see also [7, Prop. 9] or [4, Theorems 2.36, 2.37] for the case $s\in(0,1]$).

In this paper, we use only Besov spaces of the form

$$\dot{B}^s_{2,2}(\mathbb{R}), \quad \dot{B}^s_{\infty,2}(\mathbb{R}), \quad \dot{B}^s_{2,1}(\mathbb{R}).$$

We will make extensive use of the fact that $\|\cdot\|_{\dot{H}^s}$ and $\|\cdot\|_{\dot{B}^s_{2,2}}$ are equivalent for $s\in(0,2)$. Moreover, for $s\in(0,1)$,

$$\|u\|^2_{\dot{H}^s} = \frac{1}{4\pi c(s)} \|u\|^2_{\dot{B}^s_{2,2}} \quad \text{with} \quad c(s) = \int_{\mathbb{R}} \frac{1-\cos(t)}{|t|^{1+2s}}\,\mathrm{d}t. \tag{2.3}$$

We will also make extensive use of the fact that, for all s in $(0,2)$,

$$\dot{H}^{s+\frac12}(\mathbb{R}) \hookrightarrow \dot{B}^s_{\infty,2}(\mathbb{R}). \tag{2.4}$$

We will also use the following

Lemma 2.2. *For any $s\in(0,1)$ and any $\delta>0$ such that $[s-\delta,s+\delta]\subset(0,1)$,*

$$\dot{H}^{s-\delta}(\mathbb{R}) \cap \dot{H}^{s+\delta}(\mathbb{R}) \hookrightarrow \dot{B}^s_{2,1}(\mathbb{R}).$$

Proof. We have

$$\begin{aligned}
\int_{|\alpha|\le 1} \frac{\|\delta_\alpha f\|_{L^2(\mathbb{R};\mathrm{d}x)}}{|\alpha|^s}\frac{\mathrm{d}\alpha}{|\alpha|} &= \int_{|\alpha|\le 1} |\alpha|^\delta \frac{\|\delta_\alpha f\|_{L^2(\mathbb{R};\mathrm{d}x)}}{|\alpha|^{s+\delta}}\frac{\mathrm{d}\alpha}{|\alpha|} \\
&\le \left(\int_{|\alpha|\le 1} |\alpha|^{2\delta}\frac{\mathrm{d}\alpha}{|\alpha|}\right)^{\frac12} \left(\int_{\mathbb{R}} \frac{\|\delta_\alpha f\|^2_{L^2(\mathbb{R};\mathrm{d}x)}}{|\alpha|^{2(s+\delta)}}\frac{\mathrm{d}\alpha}{|\alpha|}\right)^{\frac12} \\
&\le C(\delta)\,\|f\|_{\dot{B}^s_{2,2}} = C(\delta,s)\,\|f\|_{\dot{H}^{s+\delta}},
\end{aligned}$$

and similarly

$$\int_{|\alpha|\ge 1} \frac{\|\delta_\alpha f\|_{L^2(\mathbb{R};\mathrm{d}x)}}{|\alpha|^s}\frac{\mathrm{d}\alpha}{|\alpha|} \le C'(\delta,s)\,\|f\|_{\dot{H}^{s-\delta}},$$

which gives the result. □

As an example of properties which are very simple to prove using the definition of Besov semi-norms in terms of finite differences, let us prove the first point in Theorem 1.2. Recall that, by notation,

$$\mathcal{T}(f)g = -\frac{1}{\pi}\int \Delta_\alpha g_x \, \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2} \, \mathrm{d}\alpha,$$

where $g_x := \partial_x g$.

Proposition 2.3. *i) For all f in $\dot{H}^1(\mathbb{R})$ and all g in $\dot{H}^{\frac{3}{2}}(\mathbb{R})$, the function*

$$\alpha \mapsto \Delta_\alpha g_x \, \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}$$

belongs to $L^1_\alpha(\mathbb{R}; L^2_x(\mathbb{R}))$. Consequently, $\mathcal{T}(f)g$ belongs to $L^2(\mathbb{R})$. Moreover, there is a constant C such that

$$\|\mathcal{T}(f)g\|_{L^2} \le C \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^{\frac{3}{2}}} . \tag{2.5}$$

ii) For all $\delta \in [0, 1/2)$, there exists a constant $C > 0$ such that, for all functions f_1, f_2 in $\dot{H}^{1-\delta}(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}+\delta}(\mathbb{R})$,

$$\|(\mathcal{T}(f_1) - \mathcal{T}(f_2))f_2\|_{L^2} \le C \|f_1 - f_2\|_{\dot{H}^{1-\delta}} \|f_2\|_{\dot{H}^{\frac{3}{2}+\delta}} .$$

iii) The map $f \mapsto \mathcal{T}(f)f$ is locally Lipschitz from $\dot{H}^1(\mathbb{R}) \cap \dot{H}^{\frac{3}{2}}(\mathbb{R})$ to $L^2(\mathbb{R})$.

Proof. *i*) Since

$$\left\|\Delta_\alpha g_x \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}\right\|_{L^2} \le \|\Delta_\alpha g_x\|_{L^2} \|\Delta_\alpha f\|_{L^\infty} = \frac{\|\delta_\alpha g_x\|_{L^2}}{|\alpha|} \frac{\|\delta_\alpha f\|_{L^\infty}}{|\alpha|},$$

by using the Cauchy-Schwarz inequality and the definition (2.1) of the Besov semi-norms one finds that

$$\begin{aligned}
\|\mathcal{T}(f)g\|_{L^2} &\le \frac{1}{\pi}\int \frac{\|\delta_\alpha g_x\|_{L^2}}{|\alpha|} \frac{\|\delta_\alpha f\|_{L^\infty}}{|\alpha|} \, \mathrm{d}\alpha \\
&\le \frac{1}{\pi}\int \frac{\|\delta_\alpha g_x\|_{L^2}}{|\alpha|^{1/2}} \frac{\|\delta_\alpha f\|_{L^\infty}}{|\alpha|^{1/2}} \frac{\mathrm{d}\alpha}{|\alpha|} \\
&\le \frac{1}{\pi}\left(\int \frac{\|\delta_\alpha g_x\|_{L^2}^2}{|\alpha|} \frac{\mathrm{d}\alpha}{|\alpha|}\right)^{\frac{1}{2}} \left(\int \frac{\|\delta_\alpha f\|_{L^\infty}^2}{|\alpha|} \frac{\mathrm{d}\alpha}{|\alpha|}\right)^{\frac{1}{2}} \\
&\le \frac{1}{\pi} \|g_x\|_{\dot{B}^{\frac{1}{2}}_{2,2}} \|f\|_{\dot{B}^{\frac{1}{2}}_{\infty,2}} .
\end{aligned}$$

Recalling that $\|\cdot\|_{\dot{B}^{\frac{1}{2}}_{2,2}}$ and $\|\cdot\|_{\dot{H}^{\frac{1}{2}}}$ are equivalent semi-norms, and using the Sobolev embedding (2.4), we have

$$\|g_x\|_{\dot{B}^{\frac{1}{2}}_{2,2}} \lesssim \|g\|_{\dot{H}^{\frac{3}{2}}} , \quad \|f\|_{\dot{B}^{\frac{1}{2}}_{\infty,2}} \lesssim \|f\|_{\dot{H}^1} ,$$

and hence we obtain the wanted inequality (2.5).

ii) Write that

$$(\mathcal{T}(f_1) - \mathcal{T}(f_2))f_2 = -\frac{1}{\pi}\int \Delta_\alpha f_{2x} \Delta_\alpha (f_1 - f_2) M(\alpha, x) \, \mathrm{d}\alpha$$

where

$$M(\alpha, x) = \frac{(\Delta_\alpha f_1) + \Delta_\alpha f_2}{(1 + (\Delta_\alpha f_1)^2)(1 + (\Delta_\alpha f_2)^2)}.$$

Since $|M(\alpha,x)|\le 1$, by repeating similar arguments to those used in the first part (balancing the powers of α in a different way), we get

$$\begin{aligned}\|(\mathcal{T}(f_1)-\mathcal{T}(f_2))f_2\|_{L^2} &\le \frac{1}{\pi}\int \frac{\|\delta_\alpha f_{2x}\|_{L^2}}{|\alpha|}\frac{\|\delta_\alpha(f_1-f_2)\|_{L^\infty}}{|\alpha|}\,\mathrm{d}\alpha\\ &\le \frac{1}{\pi}\int \frac{\|\delta_\alpha f_{2x}\|_{L^2}}{|\alpha|^{1/2+\delta}}\frac{\|\delta_\alpha(f_1-f_2)\|_{L^\infty}}{|\alpha|^{1/2-\delta}}\frac{\mathrm{d}\alpha}{|\alpha|}\\ &\le \frac{1}{\pi}\|f_{2x}\|_{\dot{B}^{\frac12+\delta}_{2,2}}\|f_1-f_2\|_{\dot{B}^{\frac12-\delta}_{\infty,2}}.\end{aligned}$$

which implies

$$\|(\mathcal{T}(f_1)-\mathcal{T}(f_2))f_2\|_{L^2}\le C\|f_1-f_2\|_{\dot{H}^{1-\delta}}\|f_2\|_{\dot{H}^{\frac32+\delta}}.$$

iii) Consider f_1 and f_2 in $\dot{H}^1(\mathbb{R})\cap\dot{H}^{\frac32}(\mathbb{R})$. Then

$$\mathcal{T}(f_1)f_1-\mathcal{T}(f_2)f_2=\mathcal{T}(f_1)(f_1-f_2)+(\mathcal{T}(f_1)-\mathcal{T}(f_2))f_2.$$

Then (2.5) implies that the L^2-norm of the first term is bounded by

$$C\|f_1\|_{\dot{H}^1}\|f_1-f_2\|_{\dot{H}^{\frac32}}.$$

We estimate the second term by using *ii*) applied with $\delta=0$. It follows that

$$\|\mathcal{T}(f_1)f_1-\mathcal{T}(f_2)f_2\|_{L^2}\lesssim\big(\|f_1\|_{\dot{H}^1\cap\dot{H}^{\frac32}}+\|f_2\|_{\dot{H}^1\cap\dot{H}^{\frac32}}\big)\|f_1-f_2\|_{\dot{H}^1\cap\dot{H}^{\frac32}},$$

which completes the proof. □

We gather in the following proposition the nonlinear estimates which will be needed.

Proposition 2.4. *i) Let $s\in(0,1)$, then $L^\infty(\mathbb{R})\cap\dot{H}^s(\mathbb{R})$ is an algebra. Moreover, for all u,v in $L^\infty(\mathbb{R})\cap\dot{H}^s(\mathbb{R})$,*

$$\|uv\|_{\dot{H}^s}\le 2\|u\|_{L^\infty}\|v\|_{\dot{H}^s}+2\|v\|_{L^\infty}\|u\|_{\dot{H}^s}. \tag{2.6}$$

ii) Consider a C^∞ function $F\colon\mathbb{R}\to\mathbb{R}$ satisfying

$$\forall(x,y)\in\mathbb{R}^2,\qquad |F(x)-F(y)|\le K|x-y|.$$

Then, for all $s\in(0,1)$ and all $u\in\dot{H}^s(\mathbb{R})$, one has $F(u)\in\dot{H}^s(\mathbb{R})$ together with the estimate

$$\|F(u)\|_{\dot{H}^s}\le K\|u\|_{\dot{H}^s}. \tag{2.7}$$

iii) Consider a C^∞ function $F\colon\mathbb{R}\to\mathbb{R}$ and a real number σ in $(1,2)$. Then, there exists a non-decreasing function $\mathcal{F}\colon\mathbb{R}\to\mathbb{R}$ such that, for all $u\in\dot{H}^{\sigma-1}(\mathbb{R})\cap\dot{H}^\sigma(\mathbb{R})$ one has $F(u)\in\dot{H}^\sigma(\mathbb{R})$ together with the estimate

$$\|F(u)\|_{\dot{H}^\sigma}\le\mathcal{F}(\|u\|_{L^\infty})\Big(\|u\|_{\dot{H}^{\sigma-1}}+\|u\|_{\dot{H}^\sigma}\Big). \tag{2.8}$$

Remark 2.5. *i*) The inequality (2.6) is the classical Kato–Ponce estimate ([29]). We will use it only when $0<s<1$, for which one has a straightforward proof (see below).

ii) Statement *ii*) is also elementary and classical (see [7]). Notice that (2.8) is a sub-linear estimate, which means that the constant K depends only on F and not on u (which is false in general for $s>1$).

iii) The usual estimate for composition implies that

$$\|F(u)\|_{\dot{H}^\sigma}\le\mathcal{F}(\|u\|_{L^\infty})\|u\|_{L^2\cap\dot{H}^\sigma}.$$

The bound (2.8) improves the latter estimate in that one requires less control of the low frequency component. This will play a role in the proof of Lemmas 3.2 and 3.3.

Proof. *i*) Since $\delta_\alpha(uv) = u\delta_\alpha v + (\tau_\alpha v)\delta_\alpha u$ where $\tau_\alpha v(x) = v(x-\alpha)$, we have

$$\left\|\delta_\alpha(uv)\right\|_{L^2} \le \left\|u\right\|_{L^\infty} \left\|\delta_\alpha v\right\|_{L^2} + \left\|v\right\|_{L^\infty} \left\|\delta_\alpha u\right\|_{L^2}.$$

Directly from the definition (2.1), we deduce that

$$\left\|uv\right\|_{\dot B^s_{2,2}} \le 2\left\|u\right\|_{L^\infty} \left\|v\right\|_{\dot B^s_{2,2}} + 2\left\|v\right\|_{L^\infty} \left\|u\right\|_{\dot B^s_{2,2}}.$$

This implies (2.6) by virtue of the identity (2.3) on the equivalence of $\left\|\cdot\right\|_{\dot H^s}$ and $\left\|\cdot\right\|_{\dot B^s_{2,2}}$.

ii) Similarly, the inequality (2.7) follows directly from the fact that

$$\left\|\delta_\alpha F(u)\right\|_{L^2} \le K\left\|\delta_\alpha u\right\|_{L^2}.$$

iii) We adapt the classical proof of the composition rule in nonhomogeneous Sobolev spaces, which is based on the Littlewood-Paley decomposition. Namely, choose a function $\Phi \in C^\infty_0(\{\xi; |\xi| < 1\})$ which is equal to 1 when $|\xi| \le 1/2$ and set $\phi(\xi) = \Phi(\xi/2) - \Phi(\xi)$ which is supported in the annulus $\{\xi; 1/2 \le |\xi| \le 2\}$. Then, for all $\xi \in \mathbb{R}$, one has $\Phi(\xi) + \sum_{j\in\mathbb{N}} \phi(2^{-j}\xi) = 1$, which one can use to decompose tempered distribution. For $u \in \mathcal{S}'(\mathbb{R})$, we set $\Delta_{-1}u = \mathcal{F}^{-1}(\Phi(\xi)\widehat{u})$ and $\Delta_j u = \mathcal{F}^{-1}(\phi(2^{-j}\xi)\widehat{u})$ for $j \in \mathbb{N}$. We also use the notation $S_j u = \sum_{-1\le p\le j-1} \Delta_p u$ for $j \ge 0$ (so that $S_0 u = \Delta_{-1}u = \Phi(D_x)u$).

The classical proof (see [3, 4, 35]) of the composition rule consists in splitting $F(u)$ as

$$\begin{aligned} F(u) &= F(S_0u) + F(S_1u) - F(S_0u) + \cdots + F(S_{j+1}u) - F(S_ju) + \cdots \\ &= F(S_0u) + \sum_{j\in\mathbb{N}} m_j\Delta_j u \qquad \text{with} \quad m_j = \int_0^1 F'(S_ju + y\Delta_ju)\,\mathrm{d}y \\ &= F(S_0u) + \sum_{j\in\mathbb{N}} m_j\Delta_j \tilde{u} \qquad \text{with} \quad \tilde{u} = u - \Phi(2D_x)u, \end{aligned}$$

where we used $\Delta_j \circ \Phi(2D_x) = 0$ for $j \ge 0$. Then, the Meyer's multiplier lemma (see [40, Theorem 2] or [3, Lemma 2.2]) implies that

$$\Big\|\sum_{j\ge 0} m_j\Delta_j\tilde{u}\Big\|_{H^\sigma} \le \mathcal{F}(\left\|u\right\|_{L^\infty})\left\|\tilde{u}\right\|_{H^\sigma},$$

where, to clarify notations, we insist on the fact that above H^σ is the nonhomogeneous Sobolev space. Since $\left\|\tilde{u}\right\|_{H^\sigma} \le \left\|u\right\|_{\dot H^\sigma}$, we see that the contribution of $\sum m_j\Delta_j\tilde{u}$ is bounded by the right-hand side of (2.8). This shows that the only difficulty is to estimate the low frequency component $F(S_0u)$. We claim that

$$\left\|F(S_0u)\right\|_{\dot H^\sigma} \le \mathcal{F}(\left\|u\right\|_{L^\infty})\left\|u\right\|_{\dot H^{\sigma-1}}. \tag{2.9}$$

To see this, we start with

$$\left\|F(S_0u)\right\|_{\dot H^\sigma} = \left\|\partial_x\big(F(S_0u)\big)\right\|_{\dot H^{\sigma-1}} = \left\|F'(S_0u)\partial_xS_0u\right\|_{\dot H^{\sigma-1}},$$

and then use the product rule (2.6) with $s = \sigma - 1 \in (0,1)$,

$$\begin{aligned} \left\|F'(S_0u)\partial_xS_0u\right\|_{\dot H^{\sigma-1}} \le 2\left\|F'(S_0u)\right\|_{L^\infty}\left\|\partial_xS_0u\right\|_{\dot H^{\sigma-1}} \\ + 2\left\|F'(S_0u)\right\|_{\dot H^{\sigma-1}}\left\|\partial_xS_0u\right\|_{L^\infty}. \end{aligned}$$

Since $|\xi\Phi(\xi)| \le 1$ one has the obvious inequality

$$\left\|\partial_xS_0u\right\|_{\dot H^{\sigma-1}} \le \left\|u\right\|_{\dot H^{\sigma-1}}.$$

On the other hand, since the support of the Fourier transform of $S_0 u$ is included in the ball of center 0 and radius 1, it follows from the Bernstein's inequality that

$$\|S_0 u\|_{L^\infty} \le C_1 \|u\|_{L^\infty}, \qquad \|\partial_x S_0 u\|_{L^\infty} \le C_2 \|u\|_{L^\infty}.$$

The first estimate above also implies that

$$\left\|F'(S_0 u)\right\|_{L^\infty} \le \mathcal{F}_1(\|S_0 u\|_{L^\infty}) \le \mathcal{F}_2(\|u\|_{L^\infty})$$

where $\mathcal{F}_1(r) = \sup_{y\in[-r,r]} |F'(y)|$ and $\mathcal{F}_2(r) = \mathcal{F}_1(C_1 r)$. It thus remains only to estimate $\|F'(S_0 u)\|_{\dot{H}^{\sigma-1}}$. Notice that we may apply the composition rule given in statement *ii*) since the index $\sigma - 1$ belongs to $(0,1)$ and since F' is Lipschitz on an open set containing $S_0 u(\mathbb{R})$. The composition rule (2.7) implies that

$$\left\|F'(S_0 u)\right\|_{\dot{H}^{\sigma-1}} \le K \|S_0 u\|_{\dot{H}^{\sigma-1}} \le K \|u\|_{\dot{H}^{\sigma-1}}$$

with

$$K = \sup_{[-2\|S_0 u\|_{L^\infty}, 2\|S_0 u\|_{L^\infty}]} \left|F''\right| \le \mathcal{F}_3(\|u\|_{L^\infty}).$$

This proves that the $\dot{H}^\sigma$-norm of $F(S_0 u)$ satisfies (2.9) and hence it is bounded by the right-hand side of (2.8), which completes the proof of statement *iii*). □

For later purposes, we prove the following commutator estimate with the Hilbert transform.

Lemma 2.6. *Let $0 < \theta < \nu < 1$. There exists a constant K such that for all $f \in C^{0,\nu}(\mathbb{R})$, and all u in the nonhomogeneous space $H^{-\theta}(\mathbb{R})$,*

$$\|\mathcal{H}(fu) - f\mathcal{H}u\|_{L^2} \le K \|f\|_{C^{0,\nu}} \|u\|_{H^{-\theta}}. \tag{2.10}$$

Proof. We establish this estimate by using the para-differential calculus of Bony [6]. We use the Littlewood-Paley decomposition (see the proof of Proposition 2.4) and denote by T_f the operator of para-multiplication by f, so that

$$T_f u = \sum_{j\ge1} S_{j-1}(f)\Delta_j u.$$

Denote by $f^\flat$ the multiplication operator $u \mapsto fu$ and introduce $\mathcal{H}_0$, the Fourier multiplier with symbol $-i(1-\Phi(\xi))\xi/|\xi|$ where $\Phi \in C_0^\infty(\mathbb{R})$ is such that $\Phi(\xi) = 1$ on a neighborhood of the origin. With these notations, one can rewrite the commutator $\left[\mathcal{H}, f^\flat\right]$ as

$$\begin{aligned}\left[\mathcal{H}, f^\flat\right] &= [\mathcal{H}, T_f] + \mathcal{H}(f^\flat - T_f) - (f^\flat - T_f)\mathcal{H}\\ &= [\mathcal{H}_0, T_f] + (\mathcal{H}-\mathcal{H}_0)T_f - T_f(\mathcal{H}-\mathcal{H}_0) + \mathcal{H}(f^\flat - T_f) - (f^\flat - T_f)\mathcal{H}.\end{aligned} \tag{2.11}$$

Notice that $\mathcal{H} - \mathcal{H}_0$ is a smoothing operator (that is an operator bounded from H^σ to $H^{\sigma+t}$ for any real numbers $\sigma, t \in \mathbb{R}$). We then use two classical estimates for paradifferential operators (see [6, 39]). Firstly,

$$\forall \sigma \in \mathbb{R}, \quad \|T_f\|_{H^\sigma \to H^\sigma} \le c(\sigma) \|f\|_{L^\infty},$$

so

$$\begin{aligned}\|(\mathcal{H}-\mathcal{H}_0)T_f\|_{H^{-\theta}\to L^2} &\le \|\mathcal{H}-\mathcal{H}_0\|_{H^{-\theta}\to L^2} \|T_f\|_{H^{-\theta}\to H^{-\theta}} \lesssim \|f\|_{L^\infty},\\ \|T_f(\mathcal{H}-\mathcal{H}_0)\|_{H^{-\theta}\to L^2} &\le \|T_f\|_{L^2\to L^2} \|\mathcal{H}-\mathcal{H}_0\|_{H^{-\theta}\to L^2} \lesssim \|f\|_{L^\infty}.\end{aligned}$$

Secondly, since $\mathcal{H}_0$ is a Fourier multiplier whose symbol is a smooth function of order 0 (which means that its kth derivative is bounded by $C_k(1+|\xi|)^{-k}$), one has

$$\forall\sigma\in\mathbb{R},\quad \left\|[\mathcal{H}_0,T_f]\right\|_{H^\sigma\to H^{\sigma+\nu}}\le c(\nu,\sigma)\left\|f\right\|_{C^{0,\nu}}.$$

In particular,

$$\left\|[\mathcal{H}_0,T_f]\right\|_{H^{-\nu}\to L^2}\lesssim\left\|f\right\|_{C^{0,\nu}}.$$

It remains only to estimate the last two terms in the right-hand side of (2.11). We claim that

$$\left\|\mathcal{H}(f^\flat-T_f)\right\|_{H^{-\theta}\to L^2}+\left\|(f^\flat-T_f)\mathcal{H}\right\|_{H^{-\theta}\to L^2}\lesssim\left\|f\right\|_{C^{0,\nu}}.$$

Since $\mathcal{H}$ is bounded from H^σ to itself for any $\sigma\in\mathbb{R}$, it is enough to prove that

$$\left\|f^\flat-T_f\right\|_{H^{-\theta}\to L^2}\lesssim\left\|f\right\|_{C^{0,\nu}}.$$

To do so, observe that

$$fg-T_fg=\sum_{j,p\ge -1}(\Delta_jf)(\Delta_pg)-\sum_{-1\le j\le p-2}(\Delta_jf)(\Delta_pg)=\sum_{j\ge-1}(S_{j+2}g)\Delta_jf.$$

Then, using the Bernstein's inequality and the characterization of Hölder spaces in terms of Littlewood-Paley decomposition, it follows from the assumption $\theta<\nu$ that the series $\sum 2^{j(\theta-\nu)}$ converges, so

$$\begin{aligned}\left\|fg-T_fg\right\|_{L^2}&\le\sum\left\|S_{j+2}g\right\|_{L^2}\left\|\Delta_jf\right\|_{L^\infty}\\&\lesssim\sum 2^{\theta(j+2)}\left\|g\right\|_{H^{-\theta}}2^{-j\nu}\left\|f\right\|_{C^{0,\nu}}\lesssim\left\|g\right\|_{H^{-\theta}}\left\|f\right\|_{C^{0,\nu}}.\end{aligned}$$

By combining the previous estimates, we have $\left\|\left[\mathcal{H},f^\flat\right]\right\|_{H^{-\theta}\to L^2}\lesssim\left\|f\right\|_{C^{0,\nu}}$, which gives the result. □

3. Commutator estimate

In this section we prove statement *iii*) in Theorem 1.2. Namely, we prove the following proposition.

Proposition 3.1. *Let* $0<\epsilon<1/2$. *There is a non-decreasing function* $\mathcal{F}\colon\mathbb{R}_+\to\mathbb{R}_+$ *such that, for all functions* f,g *in* $\cap_{\sigma\ge1}\dot{H}^\sigma(\mathbb{R})$,

$$\left\|\Lambda^{1+\epsilon}\mathcal{T}(f)g-\mathcal{T}(f)\Lambda^{1+\epsilon}g\right\|_{L^2}\le\mathcal{F}\big(\left\|f\right\|_{\dot{H}^1\cap\dot{H}^{\frac32+\epsilon}}\big)\left\|f\right\|_{\dot{H}^1\cap\dot{H}^{\frac32+\epsilon}}\left\|g\right\|_{\dot{H}^{\frac32+\epsilon}\cap\dot{H}^2}.\tag{3.1}$$

Proof. Recall that

$$\mathcal{T}(f)g=-\frac{1}{\pi}\int\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}\Delta_\alpha g_x\,\mathrm{d}\alpha.$$

Since

$$\Lambda^{1+\epsilon}\Delta_\alpha g_x=\Delta_\alpha\big(\Lambda^{1+\epsilon}g_x\big),$$

we have

$$\Lambda^{1+\epsilon}\mathcal{T}(f)g=\mathcal{T}(f)\Lambda^{1+\epsilon}g+R_1(f)g+R_2(f)g,$$

where

$$R_1(f)g=-\frac{1}{\pi}\int(\Delta_\alpha g_x)\Lambda^{1+\epsilon}\left(\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}\right)\mathrm{d}\alpha,\tag{3.2}$$

and

$$R_2(f)g=-\frac{1}{\pi}\int\big(\Lambda^{1+\epsilon}(u_\alpha v_\alpha)-u_\alpha\Lambda^{1+\epsilon}v_\alpha-v_\alpha\Lambda^{1+\epsilon}u_\alpha\big)\,\mathrm{d}\alpha\quad\text{with}$$

$$u_\alpha=\Delta_\alpha g_x,\qquad v_\alpha=\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}.$$

We shall estimate these two terms separately. Classical results from paradifferential calculus (see [6, 16, 39]) would allow us to estimate them provided that we work in nonhomogeneous Sobolev spaces. In the homogeneous spaces we are considering, we shall see that one can derive similar results by using only elementary nonlinear estimates.

We begin with the study of $R_1(f)g$.

Lemma 3.2. *There exists a non-decreasing function $\mathcal{F}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\left\| R_1(f)g \right\|_{L^2} \le \mathcal{F}\big(\left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}} \big) \left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}} \left\| g \right\|_{\dot H^{\frac32+\epsilon} \cap \dot H^2}.$$

Proof. By definition

$$R_1(f)g = -\frac{1}{\pi} \int (\Delta_\alpha g_x) \Lambda^{1+\epsilon} \bigg(\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2} \bigg) \, \mathrm{d}\alpha,$$

so

$$\left\| R_1(f)g \right\|_{L^2} \lesssim \int \left\| \Delta_\alpha g_x \right\|_{L^\infty} \left\| \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2} \right\|_{\dot H^{1+\epsilon}} \mathrm{d}\alpha. \tag{3.3}$$

The Sobolev embedding $L^2(\mathbb{R}) \cap \dot H^{\frac12+\epsilon}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ implies that, for all α in $\mathbb{R}$,

$$\sup_{x\in\mathbb{R}} \left| \Delta_\alpha f(x) \right| \le \sup_{x\in\mathbb{R}} \left| f_x(x) \right| \lesssim \left\| f_x \right\|_{L^2 \cap \dot H^{\frac12+\epsilon}} = \left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}}, \tag{3.4}$$

so that the composition rule (2.8) implies that

$$\left\| \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2} \right\|_{\dot H^{1+\epsilon}} \le \mathcal{F}\big(\left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}} \big) \Big(\left\| \Delta_\alpha f \right\|_{\dot H^\epsilon} + \left\| \Delta_\alpha f \right\|_{\dot H^{1+\epsilon}} \Big). \tag{3.5}$$

We claim that we have the two following inequalities

$$\int \left\| \Delta_\alpha g_x \right\|_{L^\infty} \left\| \Delta_\alpha f \right\|_{\dot H^\epsilon} \mathrm{d}\alpha \lesssim \left\| g \right\|_{\dot H^{\frac32+\epsilon}} \left\| f \right\|_{\dot H^1}, \tag{3.6}$$

$$\int \left\| \Delta_\alpha g_x \right\|_{L^\infty} \left\| \Delta_\alpha f \right\|_{\dot H^{1+\epsilon}} \mathrm{d}\alpha \lesssim \left\| g \right\|_{\dot H^2} \left\| f \right\|_{\dot H^{\frac32+\epsilon}}. \tag{3.7}$$

Let us prove (3.6). Directly from the definition of Δ_α, we have

$$\begin{aligned} \int \left\| \Delta_\alpha g_x \right\|_{L^\infty} \left\| \Delta_\alpha f \right\|_{\dot H^\epsilon} \mathrm{d}\alpha &= \int \frac{\left\| \delta_\alpha g_x \right\|_{L^\infty}}{|\alpha|} \frac{\left\| \delta_\alpha \Lambda^\epsilon f \right\|_{L^2}}{|\alpha|} \, \mathrm{d}\alpha \\ &= \int \frac{\left\| \delta_\alpha g_x \right\|_{L^\infty}}{|\alpha|^\epsilon} \frac{\left\| \delta_\alpha \Lambda^\epsilon f \right\|_{L^2}}{|\alpha|^{1-\epsilon}} \frac{\mathrm{d}\alpha}{|\alpha|}. \end{aligned}$$

So, using the Cauchy-Schwarz inequality,

$$\int \left\| \Delta_\alpha g_x \right\|_{L^\infty} \left\| \Delta_\alpha f \right\|_{\dot H^\epsilon} \mathrm{d}\alpha \le \bigg(\int \frac{\left\| \delta_\alpha g_x \right\|_{L^\infty}^2}{|\alpha|^{2\epsilon}} \frac{\mathrm{d}\alpha}{|\alpha|} \bigg)^{\frac12} \bigg(\int \frac{\left\| \delta_\alpha \Lambda^\epsilon f \right\|_{L^2}^2}{|\alpha|^{2(1-\epsilon)}} \frac{\mathrm{d}\alpha}{|\alpha|} \bigg)^{\frac12}$$

and hence, using the definition of Besov semi-norms (see (2.1)),

$$\int \left\| \Delta_\alpha g_x \right\|_{L^\infty} \left\| \Delta_\alpha f \right\|_{\dot H^\epsilon} \mathrm{d}\alpha \lesssim \left\| g_x \right\|_{\dot B^\epsilon_{\infty,2}} \left\| \Lambda^\epsilon f \right\|_{\dot B^{1-\epsilon}_{2,2}}.$$

By using (2.3) and (2.4), we obtain that

$$\int \left\| \Delta_\alpha g_x \right\|_{L^\infty} \left\| \Delta_\alpha f \right\|_{\dot H^\epsilon} \mathrm{d}\alpha \lesssim \left\| g \right\|_{\dot H^{\frac32+\epsilon}} \left\| f \right\|_{\dot H^1},$$

which is the first claim (3.6). To prove the second claim (3.7), we repeat the same arguments except that we balance the powers of α in a different way:

$$\begin{aligned}\int \left\| \Delta_\alpha g_x \right\|_{L^\infty} \left\| \Delta_\alpha f \right\|_{\dot H^{1+\epsilon}} \, \mathrm{d}\alpha &= \int \frac{\left\| \delta_\alpha g_x \right\|_{L^\infty}}{|\alpha|^{1/2}} \frac{\left\| \delta_\alpha \Lambda^{1+\epsilon} f \right\|_{L^2}}{|\alpha|^{1/2}} \frac{\mathrm{d}\alpha}{|\alpha|} \\ &\le \left(\int \frac{\left\| \delta_\alpha g_x \right\|_{L^\infty}^2}{|\alpha|} \frac{\mathrm{d}\alpha}{|\alpha|} \right)^{\frac12} \left(\int \frac{\left\| \delta_\alpha \Lambda^{1+\epsilon} f \right\|_{L^2}^2}{|\alpha|} \frac{\mathrm{d}\alpha}{|\alpha|} \right)^{\frac12} \\ &\le \left\| g_x \right\|_{\dot B^{\frac12}_{\infty,2}} \left\| \Lambda^{1+\epsilon} f \right\|_{\dot B^{\frac12}_{2,2}} \\ &\lesssim \left\| g \right\|_{\dot H^2} \left\| f \right\|_{\dot H^{\frac32+\epsilon}},\end{aligned}$$

which proves the claim (3.7). Now, by combining the two claims (3.6), (3.7) with (3.3) and (3.5), we obtain that

$$\left\| R_1(f) g \right\|_{L^2} \le \mathcal{F}\big(\left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}} \big) \left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}} \left\| g \right\|_{\dot H^{\frac32+\epsilon} \cap \dot H^2},$$

which is the desired result. □

We now move to the second remainder term $R_2(f)g$.

Lemma 3.3. *There exists a non-decreasing function $\mathcal{F} \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\left\| R_2(f) g \right\|_{L^2} \le \mathcal{F}\big(\left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}} \big) \left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}} \left\| g \right\|_{\dot H^{\frac32+\epsilon} \cap \dot H^2}.$$

Proof. We use the classical Kenig–Ponce–Vega commutator estimate

$$\left\| \Lambda^s (uv) - u \Lambda^s v - v \Lambda^s u \right\|_{L^r} \le C \left\| \Lambda^{s_1} u \right\|_{L^{p_1}} \left\| \Lambda^{s_2} v \right\|_{L^{p_2}} \tag{3.8}$$

where $s = s_1 + s_2$ and $1/r = 1/p_1 + 1/p_2$. Kenig, Ponce and Vega considered the case $s < 1$. Since, for our purpose we need $s > 1$, we will use the recent improvement by Li [36] (see also D'Ancona [25]) showing that (3.8) holds under the assumptions

$$s = s_1 + s_2 \in (0,2), \quad s_j \in (0,1), \quad \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}, \quad 2 \le p_j < \infty.$$

With $p_1 = 4$, $p_2 = 4$, $r = 2$, $s = 1 + \epsilon$, $s_1 = \frac{3\epsilon}{2}$, $s_2 = 1 - \frac{\epsilon}{2}$, this implies that

$$\left\| R_2(f) g \right\|_{L^2} \lesssim \int \left\| \Lambda^{\frac{3\epsilon}{2}} \Delta_\alpha g_x \right\|_{L^4} \left\| \Lambda^{1-\frac{\epsilon}{2}} \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \right\|_{L^4} \mathrm{d}\alpha.$$

We now use the Sobolev inequality

$$\left\| u \right\|_{L^4} \lesssim \left\| \Lambda^{\frac14} u \right\|_{L^2},$$

to obtain

$$\left\| R_2(f) g \right\|_{L^2} \lesssim \int \left\| \Lambda^{\frac14 + \frac{3\epsilon}{2}} \Delta_\alpha g_x \right\|_{L^2} \left\| \Lambda^{\frac54 - \frac{\epsilon}{2}} \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \right\|_{L^2} \mathrm{d}\alpha.$$

By combining the composition rule (2.8) (applied with $\sigma = \frac54 - \frac{\epsilon}{2} \in (1,2)$) and (3.4), we obtain that

$$\left\| R_2(f) g \right\|_{L^2} \lesssim \mathcal{F}(M) \int \left\| \Lambda^{\frac14 + \frac{3\epsilon}{2}} \Delta_\alpha g_x \right\|_{L^2} \Big(\left\| \Lambda^{\frac14 - \frac{\epsilon}{2}} \Delta_\alpha f \right\|_{L^2} + \left\| \Lambda^{\frac54 - \frac{\epsilon}{2}} \Delta_\alpha f \right\|_{L^2} \Big) \, \mathrm{d}\alpha$$

where $M = \left\| f \right\|_{\dot H^1 \cap \dot H^{\frac32+\epsilon}}$. We now proceed as in the previous proof. More precisely, we balance the powers of α, use the Cauchy-Schwarz inequality, the definition of the

Besov semi-norms (2.1) and the Sobolev embedding to obtain that

$$\begin{aligned}
&\int \left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}\Delta_\alpha g_x\right\|_{L^2}\left\|\Lambda^{\frac14-\frac{\epsilon}{2}}\Delta_\alpha f\right\|_{L^2}\,\mathrm{d}\alpha\\
&\qquad=\int \frac{\left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}\delta_\alpha g_x\right\|_{L^2}}{|\alpha|}\frac{\left\|\Lambda^{\frac14-\frac{\epsilon}{2}}\delta_\alpha f\right\|_{L^2}}{|\alpha|}\,\mathrm{d}\alpha\\
&\qquad=\int \frac{\left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}\delta_\alpha g_x\right\|_{L^2}}{|\alpha|^{1/4-\epsilon/2}}\frac{\left\|\Lambda^{\frac14-\frac{\epsilon}{2}}\delta_\alpha f\right\|_{L^2}}{|\alpha|^{3/4+\epsilon/2}}\frac{\mathrm{d}\alpha}{|\alpha|}\\
&\qquad\le \left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}g_x\right\|_{\dot B^{\frac14-\epsilon/2}_{2,2}}\left\|\Lambda^{\frac14-\frac{\epsilon}{2}}f\right\|_{\dot B^{\frac34+\epsilon/2}_{2,2}}\\
&\qquad\lesssim \|g\|_{\dot H^{\frac32+\epsilon}}\|f\|_{\dot H^1}\,.
\end{aligned}$$

One estimates the second term in a similar way. We begin by writing that

$$\begin{aligned}
&\int \left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}\Delta_\alpha g_x\right\|_{L^2}\left\|\Lambda^{\frac54-\frac{\epsilon}{2}}\Delta_\alpha f\right\|_{L^2}\,\mathrm{d}\alpha\\
&\qquad=\int \frac{\left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}\delta_\alpha g_x\right\|_{L^2}}{|\alpha|}\frac{\left\|\Lambda^{\frac54-\frac{\epsilon}{2}}\delta_\alpha f\right\|_{L^2}}{|\alpha|}\,\mathrm{d}\alpha\\
&\qquad=\int \frac{\left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}\delta_\alpha g_x\right\|_{L^2}}{|\alpha|^{3/4-3\epsilon/2}}\frac{\left\|\Lambda^{\frac54-\frac{\epsilon}{2}}\delta_\alpha f\right\|_{L^2}}{|\alpha|^{1/4+3\epsilon/2}}\frac{\mathrm{d}\alpha}{|\alpha|}.
\end{aligned}$$

Since ϵ belongs to $(0,1/2)$ we have $3/4-3\epsilon/2>0$ and $1/4+3\epsilon/2<1$. Therefore one can use the definition (2.1) of the Besov semi-norms to deduce that

$$\begin{aligned}
\int \left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}\Delta_\alpha g_x\right\|_{L^2}\left\|\Lambda^{\frac54-\frac{\epsilon}{2}}\Delta_\alpha f\right\|_{L^2}\,\mathrm{d}\alpha &\le \left\|\Lambda^{\frac14+\frac{3\epsilon}{2}}g_x\right\|_{\dot B^{\frac34-\frac{3\epsilon}{2}}_{2,2}}\left\|\Lambda^{\frac54-\frac{\epsilon}{2}}f\right\|_{\dot B^{\frac14+\frac{3\epsilon}{2}}_{2,2}}\\
&\lesssim \|g\|_{\dot H^2}\|f\|_{\dot H^{\frac32+\epsilon}}\,.
\end{aligned}$$

By combining the above inequalities, we have proved that

$$\|R_2(f)g\|_{L^2}\le \mathcal{F}\big(\|f\|_{\dot H^1\cap\dot H^{\frac32+\epsilon}}\big)\|f\|_{\dot H^1\cap\dot H^{\frac32+\epsilon}}\|g\|_{\dot H^{\frac32+\epsilon}\cap\dot H^2},$$

which concludes the proof. □

This completes the proof of the proposition. □

4. High frequency estimate

We now prove the second point of Theorem 1.2 whose statement is recalled in the next proposition.

Proposition 4.1. *For all $0<\nu<\epsilon<1/2$, there exists a positive constant $C>0$ such that, for all functions f,g in $\cap_{\sigma\ge1}\dot H^\sigma(\mathbb{R})$,*

$$\mathcal{T}(f)g=\frac{f_x^2}{1+f_x^2}\Lambda g+V(f)\partial_x g+R(f,g)$$

where

$$\|R(f,g)\|_{L^2}\le C\|f\|_{\dot H^{\frac32+\epsilon}}\|g\|_{\dot B^{1-\epsilon}_{2,1}},\qquad \|V(f)\|_{C^{0,\nu}}\le C\|f\|^2_{\dot H^1\cap\dot H^{\frac32+\epsilon}}.$$

We shall prove this proposition in this section by using a symmetrization argument which consists in replacing the finite differences $\delta_\alpha f(x)=f(x)-f(x-\alpha)$ by the symmetric finite differences $2f(x)-f(x-\alpha)-f(x+\alpha)$. To do so, it will be convenient to introduce a few notations.

Notation 4.2. Given a function $f = f(x)$ and a real number α, we define the functions $\bar{\delta}_\alpha f$, $\bar{\Delta}_\alpha f$, $s_\alpha f$, $S_\alpha f$ and $D_\alpha f$ by:

$$\bar{\delta}_\alpha f(x) = f(x) - f(x+\alpha),$$
$$s_\alpha f(x) = \delta_\alpha f(x) + \bar{\delta}_\alpha f(x) = 2f(x) - f(x-\alpha) - f(x+\alpha),$$

and

$$\bar{\Delta}_\alpha f(x) = \frac{f(x) - f(x+\alpha)}{\alpha},$$
$$S_\alpha f(x) = \Delta_\alpha f(x) + \bar{\Delta}_\alpha f(x) = \frac{s_\alpha f(x)}{\alpha} = \frac{2f(x) - f(x+\alpha) - f(x-\alpha)}{\alpha},$$
$$D_\alpha f(x) = \Delta_\alpha f(x) - \bar{\Delta}_\alpha f(x) = \frac{f(x+\alpha) - f(x-\alpha)}{\alpha}.$$

Lemma 4.3. *One has*

$$D_\alpha f = 2f_x - \frac{1}{\alpha}\int_0^\alpha s_\eta f_x \, d\eta, \tag{4.1}$$

where $s_\eta f_x(x) = 2f_x(x) - f_x(x+\eta) - f_x(x-\eta)$. *Furthermore,*

$$\partial_\alpha(D_\alpha f) = -S_\alpha f_x + \frac{1}{\alpha^2}\int_0^\alpha s_\eta f_x \, d\eta, \tag{4.2}$$

and

$$\partial_\alpha(S_\alpha f) = \bar{\Delta}_\alpha f_x - \Delta_\alpha f_x - \frac{S_\alpha f}{\alpha}. \tag{4.3}$$

Proof. The formula (4.1) can be verified by two direct calculations: one is

$$\frac{1}{\alpha}\int_0^\alpha 2f_x(x)\, d\eta = 2f_x(x),$$

and the other is

$$\frac{1}{\alpha}\int_0^\alpha (f_x(x-\eta) + f_x(x+\eta))\, d\eta = \frac{1}{\alpha}\int_0^\alpha \partial_\eta(f(x+\eta) - f(x-\eta))\, d\eta$$
$$= \frac{1}{\alpha}(f(x+\alpha) - f(x-\alpha)).$$

Now, the value for $\partial_\alpha(D_\alpha f)$ in (4.2) follows by differentiating (4.1).

The formula for $\partial_\alpha(S_\alpha f)$ follows from the definition of $S_\alpha f$ and the chain rule. □

Recall that

$$\mathcal{T}(f)g = -\frac{1}{\pi}\int \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}\Delta_\alpha g_x \, d\alpha.$$

The idea is to decompose the factor

$$\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}$$

into its even and odd components with respect to the variable α. We define

$$\mathcal{E}(\alpha,\cdot) = \frac{1}{2}\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2} + \frac{1}{2}\frac{(\bar{\Delta}_\alpha f)^2}{1+(\bar{\Delta}_\alpha f)^2}, \tag{4.4}$$
$$\mathcal{O}(\alpha,\cdot) = \frac{1}{2}\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2} - \frac{1}{2}\frac{(\bar{\Delta}_\alpha f)^2}{1+(\bar{\Delta}_\alpha f)^2}, \tag{4.5}$$

where the dots in the notations $\mathcal{E}(\alpha,\cdot)$ and $\mathcal{O}(\alpha,\cdot)$ are placeholders for the variable x (notice that $(\bar{\Delta}_\alpha f)^2 = (\Delta_{-\alpha} f)^2$ and $\bar{\Delta}_\alpha f = -\Delta_{-\alpha} f$). Then,

$$\mathcal{T}(f)g = -\frac{1}{\pi}\int \Delta_\alpha g_x \, \mathcal{E}(\alpha,\cdot)\, d\alpha - \frac{1}{\pi}\int \Delta_\alpha g_x \, \mathcal{O}(\alpha,\cdot)\, d\alpha,$$

and hence, since $\alpha \mapsto \mathcal{E}(\alpha,\cdot)$ is even, this yields $\mathcal{T}(f)g = \mathcal{T}_e(f)g + \mathcal{T}_o(f)g$ with

$$\mathcal{T}_e(f)g = -\frac{1}{2\pi}\int \left(\Delta_\alpha g_x - \bar{\Delta}_\alpha g_x\right)\,\mathcal{E}(\alpha,\cdot)\,\mathrm{d}\alpha,$$
$$\mathcal{T}_o(f)g = -\frac{1}{\pi}\int \Delta_\alpha g_x\;\mathcal{O}(\alpha,\cdot)\,\mathrm{d}\alpha.$$

The following result is the key point of the proof where we identify the main contribution of the nonlinearity.

Proposition 4.4. *There exists a constant C such that*

$$\begin{aligned}\left\Vert \mathcal{T}_e(f)g - \frac{f_x^2}{1+f_x^2}\Lambda g\right\Vert_{L^2} &\le C\left\Vert f\right\Vert_{\dot H^{\frac32+\epsilon}}\left\Vert g\right\Vert_{\dot B^{1-\epsilon}_{2,1}},\\ \left\Vert \mathcal{T}_o(f)g - V\partial_x g\right\Vert_{L^2} &\le C\left\Vert f\right\Vert_{\dot H^{\frac32+\epsilon}}\left\Vert g\right\Vert_{\dot B^{1-\epsilon}_{2,1}},\end{aligned}\tag{4.6}$$

where

$$V(x) = -\frac{1}{\pi}\int_{\mathbb{R}}\frac{\mathcal{O}(\alpha,x)}{\alpha}\,\mathrm{d}\alpha.\tag{4.7}$$

Proof. i) The main difficulty is to extract the elliptic component from $\mathcal{T}_e(f)g$. To uncover it, we shall perform an integration by parts in α. The first key point is that

$$\begin{aligned}\Delta_\alpha g_x - \bar{\Delta}_\alpha g_x &= \frac{g_x(\cdot+\alpha) - g_x(\cdot-\alpha)}{\alpha}\\ &= \frac{\partial_\alpha\left(g(\cdot+\alpha)+g(\cdot-\alpha)-2g(\cdot)\right)}{\alpha}\\ &= -\frac{\partial_\alpha(s_\alpha g)}{\alpha}.\end{aligned}$$

Consequently, directly from the definition of $\mathcal{T}_e(f)g$, by integrating by parts in α, we obtain that

$$\begin{aligned}\mathcal{T}_e(f)g &= \frac{1}{2\pi}\int\frac{\partial_\alpha(s_\alpha g)}{\alpha}\mathcal{E}(\alpha,\cdot)\,\mathrm{d}\alpha\\ &= \frac{1}{2\pi}\int\frac{s_\alpha g}{\alpha^2}\mathcal{E}(\alpha,\cdot)\,\mathrm{d}\alpha - \frac{1}{2\pi}\int\frac{s_\alpha g}{\alpha}\partial_\alpha\mathcal{E}(\alpha,\cdot)\,\mathrm{d}\alpha.\end{aligned}\tag{4.8}$$

We now have to estimate the coefficients $\mathcal{E}(\alpha,\cdot)$ and $\partial_\alpha\mathcal{E}(\alpha,\cdot)$.

Lemma 4.5. *i) We have*

$$\mathcal{E}(\alpha,x) = \frac{f_x(x)^2}{1+f_x(x)^2} + Q(\alpha,x)\tag{4.9}$$

for some function Q satisfying

$$\left|Q(\alpha,x)\right| \lesssim \frac{\left|s_\alpha f(x)\right|}{|\alpha|} + \left|\frac{1}{\alpha}\int_0^\alpha s_\eta f_x(x)\,\mathrm{d}\eta\right|.\tag{4.10}$$

ii) Furthermore,

$$\left|\partial_\alpha\mathcal{E}(\alpha,x)\right| \le C\left\{\frac{\left|\bar{\delta}_\alpha f_x(x)\right|}{|\alpha|} + \frac{\left|\delta_\alpha f_x(x)\right|}{|\alpha|} + \frac{\left|s_\alpha f(x)\right|}{|\alpha|^2} + \left|\frac{1}{\alpha^2}\int_0^\alpha s_\eta f_x(x)\,\mathrm{d}\eta\right|\right\}\tag{4.11}$$

for some fixed constant C.

Proof. i) We introduce the function

$$F(a) = \frac{a^2}{1+a^2}.$$

Then we have the identity (4.9) with

$$Q := \frac{1}{2}\left(F(\Delta_\alpha f) + F(\bar{\Delta}_\alpha f)\right) - F(f_x). \tag{4.12}$$

Since F'' is bounded, the Taylor formula implies that, for all $(a,b) \in \mathbb{R}^2$,

$$\left|\frac{1}{2}(F(a)+F(b)) - F\left(\frac{a+b}{2}\right)\right| \le \frac{\|F''\|_{L^\infty}}{8}|a-b|^2.$$

On the other hand, since F is bounded, one has the obvious inequality

$$\left|\frac{1}{2}(F(a)+F(b)) - F\left(\frac{a+b}{2}\right)\right| \le 2\|F\|_{L^\infty}.$$

By combining these two inequalities, we find that

$$\left|\frac{1}{2}(F(a)+F(b)) - F\left(\frac{a+b}{2}\right)\right| \lesssim |a-b|.$$

Since F is even we have $F(b) = F(-b)$ and hence

$$\left|\frac{1}{2}(F(a)+F(b)) - F\left(\frac{a-b}{2}\right)\right| \lesssim |a+b|.$$

We now apply this inequality with $a = \Delta_\alpha f$ and $b = \bar{\Delta}_\alpha f$. Since, by definition, $D_\alpha f = \Delta_\alpha f - \bar{\Delta}_\alpha f$, $S_\alpha f = \Delta_\alpha f + \bar{\Delta}_\alpha f$, we conclude that

$$\left|\frac{1}{2}(F(\Delta_\alpha f) + F(\bar{\Delta}_\alpha f)) - F\left(\frac{1}{2}D_\alpha f\right)\right| \lesssim |S_\alpha f|. \tag{4.13}$$

We now use the fact that F is Lipschitz to infer from (4.1) that

$$\left|F\left(\frac{1}{2}D_\alpha f\right) - F(f_x)\right| \lesssim \left|\frac{1}{\alpha}\int_0^\alpha s_\eta f_x \,\mathrm{d}\eta\right|. \tag{4.14}$$

In light of (4.12), by using the triangle inequality, it follows from (4.13) and (4.14) that

$$|Q| \lesssim |S_\alpha f| + \left|\frac{1}{\alpha}\int_0^\alpha s_\eta f_x \,\mathrm{d}\eta\right|,$$

which gives the result (4.10).

ii) Since

$$\mathcal{E}(\alpha,\cdot) = \frac{1}{2}F(\Delta_\alpha f) + \frac{1}{2}F(\bar{\Delta}_\alpha f),$$

and since F' is bounded, the chain rule implies that

$$|\partial_\alpha \mathcal{E}(\alpha,\cdot)| \lesssim |\partial_\alpha \Delta_\alpha f| + \left|\partial_\alpha \bar{\Delta}_\alpha f\right|.$$

By combining this estimate with the identities

$$2\Delta_\alpha f = S_\alpha f + D_\alpha f, \quad 2\bar{\Delta}_\alpha f = S_\alpha f - D_\alpha f,$$

we deduce that $|\partial_\alpha \mathcal{E}(\alpha,\cdot)| \lesssim |\partial_\alpha S_\alpha f| + |\partial_\alpha D_\alpha f|$. Then the second estimate (4.11) follows from the values for $\partial_\alpha S_\alpha f$ and $\partial_\alpha D_\alpha f$ given by Lemma 4.3. □

It follows directly from (4.8) and (4.9) that

$$\begin{aligned}\mathcal{T}_e(f)g = \frac{1}{2\pi}\frac{f_x^2}{1+f_x^2}\int \frac{s_\alpha g}{\alpha^2}\,\mathrm{d}\alpha \\ + \frac{1}{2\pi}\int \frac{s_\alpha g}{\alpha}\left(\frac{Q(\alpha,\cdot)}{\alpha} - \partial_\alpha \mathcal{E}(\alpha,\cdot)\right)\mathrm{d}\alpha.\end{aligned} \tag{4.15}$$

Observe that

$$\begin{aligned}\int \frac{s_\alpha g}{\alpha^2}\,\mathrm{d}\alpha &= -\int s_\alpha g \partial_\alpha\left(\frac{1}{\alpha}\right)\mathrm{d}\alpha\\ &= \int \frac{\partial_\alpha s_\alpha g}{\alpha}\,\mathrm{d}\alpha = \int \frac{g_x(x-\alpha)-g_x(x+\alpha)}{\alpha}\,\mathrm{d}\alpha\\ &= -2\int \Delta_\alpha g_x\,\mathrm{d}\alpha = 2\pi\Lambda g,\end{aligned}$$

where we used (1.3). So, the first term in the right-hand side of (4.15) is the wanted elliptic component

$$\frac{f_x^2}{1+f_x^2}\Lambda g.$$

To conclude the proof of the first statement in (4.6), it remains only to prove that the second term in the right-hand side of (4.15) is a remainder term. Putting for shortness

$$I = \left\lVert \int \frac{s_\alpha g}{\alpha}\left(\frac{Q(\alpha,\cdot)}{\alpha} - \partial_\alpha \mathcal{E}(\alpha,\cdot)\right)\mathrm{d}\alpha\right\rVert_{L^2},$$

we will prove that

$$I \lesssim \lVert f\rVert_{\dot H^{\frac{3}{2}+\epsilon}} \lVert g\rVert_{\dot B^{1-\epsilon}_{2,1}}. \tag{4.16}$$

The L^∞-norm of $\frac{Q(\alpha,\cdot)}{\alpha} - \partial_\alpha\mathcal{E}(\alpha,\cdot)$ is controlled from (4.10) and (4.11). We have

$$I \lesssim I_1 + I_2 + I_3 + I_4$$

with

$$\begin{aligned}I_1 &= \int \frac{\lVert s_\alpha g\rVert_{L^2}}{|\alpha|^{1-\epsilon}}\frac{\lVert \bar\delta_\alpha f_x\rVert_{L^\infty}}{|\alpha|^{\epsilon}}\frac{\mathrm{d}\alpha}{|\alpha|},\\ I_2 &= \int \frac{\lVert s_\alpha g\rVert_{L^2}}{|\alpha|^{1-\epsilon}}\frac{\lVert \delta_\alpha f_x\rVert_{L^\infty}}{|\alpha|^{\epsilon}}\frac{\mathrm{d}\alpha}{|\alpha|},\\ I_3 &= \int \frac{\lVert s_\alpha g\rVert_{L^2}}{|\alpha|^{1-\epsilon}}\frac{\lVert s_\alpha f\rVert_{L^\infty}}{|\alpha|^{1+\epsilon}}\frac{\mathrm{d}\alpha}{|\alpha|},\\ I_4 &= \int \frac{\lVert s_\alpha g\rVert_{L^2}}{|\alpha|^{1-\epsilon}}\frac{1}{|\alpha|^{1+\epsilon}}\left|\int_0^\alpha \lVert s_\eta f_x\rVert_{L^\infty}\,\mathrm{d}\eta\right|\frac{\mathrm{d}\alpha}{|\alpha|},\end{aligned} \tag{4.17}$$

where, as above, we have distributed the powers of $|\alpha|$ in a balanced way. Using the Cauchy-Schwarz inequality and the definition (2.1) of Besov semi-norms, it follows that

$$I_1 + I_2 \leq \lVert g\rVert_{\dot B^{1-\epsilon}_{2,2}} \lVert f_x\rVert_{\dot B^{\epsilon}_{\infty,2}}.$$

and, similarly, it results from (2.2) that

$$I_3 \leq \lVert g\rVert_{\dot B^{1-\epsilon}_{2,2}} \lVert f\rVert_{\dot B^{1+\epsilon}_{\infty,2}}.$$

Consequently, the Sobolev embeddings

$$\dot B^{1-\epsilon}_{2,1} \hookrightarrow \dot B^{1-\epsilon}_{2,2}, \qquad \dot H^{\frac{3}{2}+\epsilon}(\mathbb{R}) \hookrightarrow \dot B^{\epsilon}_{\infty,2}(\mathbb{R}),$$

imply that $I_1 + I_2 + I_3 \lesssim \lVert g\rVert_{\dot B^{1-\epsilon}_{2,1}} \lVert f\rVert_{\dot H^{\frac{3}{2}+\epsilon}}$.

To estimate I_4, the key point consists in using the Cauchy-Schwarz inequality to verify that

$$\frac{1}{|\alpha|^{1+\epsilon}}\left|\int_0^\alpha \lVert s_\eta f_x\rVert_{L^\infty}\,\mathrm{d}\eta\right| \lesssim \left(\int_0^\infty \frac{\lVert s_\mu f_x\rVert_{L^\infty}^2}{\mu^{2\epsilon}}\frac{\mathrm{d}\mu}{\mu}\right)^{\frac{1}{2}}$$

(notice that the variable η above could be negative, while μ here is always positive). It follows from (2.1) that

$$\left(\int_0^\infty \frac{\|s_\mu f_x\|_{L^\infty}^2}{\mu^{2\epsilon}}\frac{d\mu}{\mu}\right)^{\frac{1}{2}} \lesssim \|f_x\|_{\dot{B}^\epsilon_{\infty,2}} \lesssim \|f\|_{\dot{H}^{\frac{3}{2}+\epsilon}},$$

we obtain, using again (2.1) with $(p,q,s)=(2,1,1-\epsilon)$,

$$I_4 \lesssim \|f\|_{\dot{H}^{\frac{3}{2}+\epsilon}} \int \frac{\|s_\alpha g\|_{L^2}}{|\alpha|^{1-\epsilon}}\frac{d\alpha}{|\alpha|} \lesssim \|f\|_{\dot{H}^{\frac{3}{2}+\epsilon}} \|g\|_{\dot{B}^{1-\epsilon}_{2,1}}. \tag{4.18}$$

This completes the proof of (4.16) and hence the proof of the desired result (4.6).

ii) It remains to study $\mathcal{T}_o(f)g$. Recall that

$$\mathcal{T}_o(f)g = -\frac{1}{\pi}\int (\Delta_\alpha g_x)\mathcal{O}(\alpha,\cdot)\, d\alpha,$$

where $\mathcal{O}(\alpha,\cdot)$ is given by (4.5). By splitting the factor $\Delta_\alpha g_x$ into two parts

$$\Delta_\alpha g_x(x) = \frac{g_x(x)}{\alpha} - \frac{g_x(x-\alpha)}{\alpha},$$

we obtain at once that

$$\mathcal{T}_o(f)g = V\partial_x g + B$$

where V is given by (4.7) and where the remainder B is given by

$$B(x) = \frac{1}{\pi}\int \frac{1}{\alpha} g_x(x-\alpha)\mathcal{O}(\alpha,x)\, d\alpha.$$

The analysis of B is based on the observation that

$$g_x(x-\alpha) = \partial_\alpha(g(x)-g(x-\alpha)) = \partial_\alpha(\delta_\alpha g),$$

which allows to integrate by parts in α, to obtain

$$B = \frac{1}{\pi}\int \frac{\delta_\alpha g}{\alpha}\left(\frac{1}{\alpha}\mathcal{O}(\alpha,\cdot) - \partial_\alpha \mathcal{O}(\alpha,\cdot)\right) d\alpha.$$

Consequently, by writing

$$\|B\|_{L^2} \lesssim \int \frac{\|\delta_\alpha g\|_{L^2}}{|\alpha|}\left\|\frac{1}{\alpha}\mathcal{O}(\alpha,\cdot) - \partial_\alpha\mathcal{O}(\alpha,\cdot)\right\|_{L^\infty}\frac{d\alpha}{|\alpha|},$$

we are back to the situation already treated in the first step. The estimate for $\partial_\alpha\mathcal{O}$ is proved by repeating the arguments used to prove the estimate (4.11) for $\partial_\alpha\mathcal{E}$. To bound $\alpha^{-1}\mathcal{O}(\alpha,x)$, remembering the expression of $\mathcal{O}(\alpha,x)$ given by (4.5), it is sufficient to notice that

$$\begin{aligned} \left|\frac{\mathcal{O}(\alpha,\cdot)}{\alpha}\right| &= \frac{1}{2|\alpha|}\left|\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2} - \frac{(\bar{\Delta}_\alpha f)^2}{1+(\bar{\Delta}_\alpha f)^2}\right| \\ &\le \frac{1}{2|\alpha|}\left|\frac{\Delta_\alpha f - \bar{\Delta}_\alpha f}{(1+(\Delta_\alpha f)^2)(1+(\bar{\Delta}_\alpha f)^2}\right| \left|\Delta_\alpha f + \bar{\Delta}_\alpha f\right| \\ &\le \frac{|S_\alpha f|}{|\alpha|}. \end{aligned} \tag{4.19}$$

This gives that $|\mathcal{O}(\alpha,x)| \lesssim |s_\alpha f(x)| / |\alpha|^2$. Therefore, we obtain that the L^∞-norm of $\frac{\mathcal{O}(\alpha,\cdot)}{\alpha} - \partial_\alpha\mathcal{O}(\alpha,\cdot)$ is estimated by the right-hand side of (4.11). Then we may repeat the arguments used in the proof of the first step to estimate I. We call the attention to the fact that, previously, in (4.17), the expressions involved the more favorable symmetric differences $s_\alpha g$ instead of $\delta_\alpha g$. However, this is not important for our purpose since, to estimate $I_1,\dots,I_4$, we used only the characterization of

Besov norms valid for $0 < s < 1$, which involves only the finite differences $\delta_\alpha f$. This proves that $\|B\|_{L^2}$ is controlled by the right-hand side of (4.16), which implies that $B \sim 0$. □

Lemma 4.6. *Let $0 < \nu < \epsilon < 1/2$. There exists a positive constant $C > 0$ such that*

$$\|V\|_{C^{0,\nu}} = \|V\|_{L^\infty} + \sup_{y \in \mathbb{R}} \left(\frac{|V(x+y) - V(x)|}{|y|^\nu} \right) \le C \|f\|^2_{\dot{H}^1 \cap \dot{H}^{\frac{3}{2}+\epsilon}}.$$

Proof. As already seen, we have

$$V(x) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathcal{O}(\alpha, x)}{\alpha} \, \mathrm{d}\alpha,$$

where

$$\mathcal{O}(\alpha, \cdot) = M_\alpha(f) S_\alpha f \quad \text{with} \quad M_\alpha(f) = \frac{1}{2} \frac{\Delta_\alpha f - \bar{\Delta}_\alpha f}{(1 + (\bar{\Delta}_\alpha f)^2)(1 + (\bar{\Delta}_\alpha f)^2}.$$

Since $|M_\alpha(f)| \le |\Delta_\alpha f| + |\bar{\Delta}_\alpha f|$, we obtain that

$$\begin{aligned}
\|V\|_{L^\infty} &\le 2 \int \left(\|\Delta_\alpha f\|_{L^\infty} + \|\bar{\Delta}_\alpha f\|_{L^\infty} \right) \|S_\alpha f\|_{L^\infty} \frac{\mathrm{d}\alpha}{|\alpha|} \\
&\le \int \frac{\|\delta_\alpha f\|_{L^\infty} + \|\bar{\delta}_\alpha f\|_{L^\infty}}{|\alpha|^{1-\epsilon}} \frac{\|s_\alpha f\|_{L^\infty}}{|\alpha|^{1+\epsilon}} \frac{\mathrm{d}\alpha}{|\alpha|} \\
&\le \|f\|_{\dot{B}^{1-\epsilon}_{\infty,2}} \|f\|_{\dot{B}^{1+\epsilon}_{\infty,2}} \lesssim \|f\|^2_{\dot{H}^1 \cap \dot{H}^{\frac{3}{2}+\epsilon}},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality, the definitions (2.1) and (2.2) of the Besov semi-norms, and the Sobolev embedding.

We now have to estimate the Hölder-modulus of continuity of V. Given $y \in \mathbb{R}$ and a function $u = u(x)$, we introduce the function $[u]_y$ defined by

$$[u]_y(x) = \frac{u(x+y) - u(x)}{|y|^\nu}.$$

We want to estimate the L^∞-norm of $[V]_y$ uniformly in $y \in \mathbb{R}$. Notice that

$$[\mathcal{O}(\alpha, \cdot)]_y = M_\alpha(f) S_\alpha([f]_y) + [M_\alpha(f)]_y \tau_y (S_\alpha f)$$

where $\tau_y u(x) = u(x+y)$. The contribution of the first term is estimated as above: by setting $\delta = \epsilon - \nu > 0$, we have

$$\begin{aligned}
\int \|M_\alpha(f) S_\alpha([f]_y)\|_{L^\infty} \, \mathrm{d}\alpha &\le \int \left(\|\Delta_\alpha f\|_{L^\infty} + \|\bar{\Delta}_\alpha f\|_{L^\infty} \right) \|S_\alpha [f]_y\|_{L^\infty} \frac{\mathrm{d}\alpha}{|\alpha|} \\
&\le \int \frac{\|\delta_\alpha f\|_{L^\infty} + \|\bar{\delta}_\alpha f\|_{L^\infty}}{|\alpha|^{1-\delta}} \frac{\|s_\alpha [f]_y\|_{L^\infty}}{|\alpha|^{1+\delta}} \frac{\mathrm{d}\alpha}{|\alpha|} \\
&\le \|f\|_{\dot{B}^{1-\delta}_{\infty,2}} \|[f]_y\|_{\dot{B}^{1+\delta}_{\infty,2}}.
\end{aligned}$$

Now, using Plancherel theorem and the inequality $|e^{iy\xi} - 1| \le |y\xi|^\nu$, we have

$$\|[f]_y\|_{\dot{B}^{1+\delta}_{\infty,2}} \lesssim \|[f]_y\|_{\dot{H}^{\frac{3}{2}+\delta}} \lesssim \|f\|_{\dot{H}^{\frac{3}{2}+\delta+\nu}} = \|f\|_{\dot{H}^{\frac{3}{2}+\epsilon}},$$

since $\delta + \nu = \epsilon$. On the other hand, since

$$|[M_\alpha(f)]_y| \lesssim |\Delta_\alpha [f]_y| + |\bar{\Delta}_\alpha [f]_y|,$$

by repeating the previous arguments, we get

$$\int \|[M_\alpha(f)]_y \tau_y(S_\alpha f)\|_{L^\infty} \,\mathrm{d}\alpha \le \int \frac{\|\delta_\alpha[f]_y\|_{L^\infty} + \|\bar\delta_\alpha[f]_y\|_{L^\infty}}{|\alpha|^{1-\nu}} \frac{\|s_\alpha f\|_{L^\infty}}{|\alpha|^{1+\nu}} \frac{\mathrm{d}\alpha}{|\alpha|}$$
$$\le \|[f]_y\|_{\dot B^{1-\nu}_{\infty,2}} \|f\|_{\dot B^{1+\nu}_{\infty,2}} \lesssim \|f\|_{\dot H^{\frac{3}{2}}} \|f\|_{\dot H^{\frac{3}{2}+\nu}}.$$

This concludes the proof of Lemma 4.6 □

This completes the proof of Theorem 1.2.

5. Cauchy problem

In this section we prove Theorem 1.4 about the Cauchy problem.

We prove the uniqueness by estimating the difference of two solutions. With regards to the existence, we construct solutions to the Muskat equation as limits of solutions to a sequence of approximate nonlinear systems, following here [1, 2, 32, 33]. We split the analysis in three parts.

(1) Firstly, we prove that the Cauchy problem for these approximate systems are well-posed locally in time by means of an ODE argument.
(2) Secondly, we use Theorem 1.2 and an elementary L^2-estimate for the paralinearized equation to prove that the solutions of the later approximate systems are bounded in $C^0([0,T];\dot H^1(\mathbb{R})\cap \dot H^s(\mathbb{R}))$ on a uniform time interval.
(3) The third task consists in showing that these approximate solutions converge to a limit which is a solution of the Muskat equation. To do this, one cannot apply standard compactness results since the equation is non-local. Instead, we prove that the solutions form a Cauchy sequence in an appropriate space, by estimating the difference of two solutions.

5.1. **Approximate systems.** To define approximate systems, we use a version of Galerkin's method based on Friedrichs mollifiers. We find convenient to use smoothing operators which are projections and consider, for $n\in\mathbb{N}\setminus\{0\}$, the operators J_n defined by

$$\widehat{J_n u}(\xi) = \hat u(\xi) \quad \text{for } |\xi|\le n,$$
$$\widehat{J_n u}(\xi) = 0 \quad \text{for } |\xi| > n.$$

Notice that J_n is a projection, $J_n^2 = J_n$. This will allow us to simplify some technical arguments.

Now we consider the following approximate Cauchy problems:

$$\begin{cases} \partial_t f + \Lambda f = J_n\big(\mathcal{T}(f)f\big),\\ f|_{t=0} = J_n f_0. \end{cases} \tag{5.1}$$

The following lemma states that this system has smooth local in time solutions.

Lemma 5.1. *For all $f_0\in \dot H^1(\mathbb{R})$, and any $n\in\mathbb{N}\setminus\{0\}$, the initial value problem (5.1) has a unique maximal solution, for some time $T_n>0$, of the form $f_n = J_n f_0 + u_n$ where $u_n\in C^1([0,T_n[;H^\infty(\mathbb{R}))$ is such that $u_n(0)=0$. Moreover, either*

$$T_n = +\infty \quad \textit{or} \quad \limsup_{t\to T_n} \|u_n(t)\|_{L^2} = +\infty. \tag{5.2}$$

Proof. We begin by studying an auxiliary system. Consider the following Cauchy problem

$$\begin{cases} \partial_t f + J_n\Lambda f = J_n\big(\mathcal{T}(J_n f)J_n f\big),\\ f|_{t=0} = J_n f_0. \end{cases} \tag{5.3}$$

Set $u = f - J_n f_0$. Then the Cauchy problem (5.3) has the form

$$\partial_t u = F_n(u), \quad u|_{t=0} = 0, \tag{5.4}$$

where

$$F_n(u) = -\Lambda J_n u - \Lambda J_n f_0 + J_n\big(\mathcal{T}(J_n(f_0+u))J_n(f_0+u)\big)$$

(we have used $J_n^2 = J_n$ to simplify the expression of F). The operator J_n is a smoothing operator: it is bounded from $\dot{H}^1(\mathbb{R})$ into $\dot{H}^\mu(\mathbb{R})$ for any $\mu \geq 1$, and from $L^2(\mathbb{R})$ into $H^\mu(\mathbb{R})$ for any $\mu \geq 0$. Consequently, if u belongs to $L^2(\mathbb{R})$, then $J_n(f_0+u)$ belongs to $\dot{H}^\mu(\mathbb{R})$ for any $\mu \geq 1$. Thus, it follows from statement *i*) in Proposition 2.3 and the assumption $f_0 \in \dot{H}^1(\mathbb{R})$ that F_n maps $L^2(\mathbb{R})$ into itself. This shows that (5.4) is in fact an ODE with values in a Banach space for any $n \in \mathbb{N}\setminus\{0\}$. The key point is that statement *iii*) in Proposition 2.3 implies that the function F_n is locally Lipschitz from $L^2(\mathbb{R})$ to itself. Consequently, the Cauchy-Lipschitz theorem gives the existence of a unique maximal solution u_n in $C^1([0,T_n[;L^2(\mathbb{R}))$. Then the function $f_n = J_n f_0 + u_n$ is a solution to (5.3). Since $J_n^2 = J_n$, we check that the function $(I - J_n)f_n$ solves

$$\partial_t(I-J_n)f_n = 0, \quad (I-J_n)f_n|_{t=0} = 0.$$

This shows that $(I-J_n)f_n = 0$, so $J_n f_n = f_n$. Consequently, the fact that f_n solves (5.3) implies that f_n is also a solution to (5.1).

The alternative (5.2) is a consequence of the usual continuation principle for ordinary differential equations. Eventually, integrating (5.4) in time and using the fact that J_n is a smoothing operator, we obtain that u_n belongs to $C^0([0,T_n[;H^\infty(\mathbb{R}))$. Using again (5.4), we conclude that $\partial_t u_n$ belong to $C^0([0,T_n[;H^\infty(\mathbb{R}))$. □

5.2. **A priori estimate for the approximate systems.** In this paragraph we prove two *a priori* estimates which will play a key role to prove uniform estimates for the solutions (f_n) and also to estimate the differences between two such solutions. We begin with the following estimate in $L^2(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$.

Proposition 5.2. *For all real number $s \in (3/2, 2)$, there exists a positive constant $C > 0$ and a non-decreasing function $\mathcal{F} \colon \mathbb{R} \to \mathbb{R}$ such that, any $n \in \mathbb{N}\setminus\{0\}$, for any $T \in (0, T_n)$, the norm*

$$M_n(T) = \sup_{t\in[0,T]} \|f_n(t) - f_0\|^2_{L^2\cap\dot{H}^s}$$

satisfies

$$\begin{aligned} M_n(T) + \frac{C}{1+K^2}\int_0^T \|f_n(t)\|^2_{\dot{H}^{s+\frac{1}{2}}}\,\mathrm{d}t \\ \leq (2+T)^2\|f_0\|^2_{\dot{H}^1\cap\dot{H}^s} + T\mathcal{F}\big(\sup_{t\in[0,T]}\|f_n(t)\|^2_{\dot{H}^1\cap\dot{H}^s}\big), \end{aligned}\tag{5.5}$$

where

$$K := \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\partial_x f_n(t,x)|\,. \tag{5.6}$$

Proof. Set $\mathcal{T}_n(f) = J_n\big(\mathcal{T}(f)f\big)$. We estimate the $\|\cdot\|_{L^2}$-norm and $\|\cdot\|_{\dot{H}^s}$-norm by different methods.

First step : low-frequency estimate. Since

$$\partial_t f_n + \Lambda f_n = J_n\mathcal{T}(f_n)f_n, \quad f_n|_{t=0} = J_n f_0, \tag{5.7}$$

we have

$$f_n(t) = \exp(-t\Lambda) J_n f_0 + \int_0^t \exp\left(-(t-t')\Lambda\right) \mathcal{T}_n(f_n)(t')\, \mathrm{d}t',$$

so

$$f_n(t) - J_n f_0 = \left(\exp(-t\Lambda) - I\right) J_n f_0 + \int_0^t \exp\left(-(t-t')\Lambda\right) \mathcal{T}_n(f_n)(t')\, \mathrm{d}t',$$

where I denotes the identity operator. Using the Fourier transform and Plancherel identity, one obtains immediately that

$$\left\| \left(\exp(-t\Lambda) - I\right) J_n f_0 \right\|_{L^2} \le \left\| t\Lambda J_n f_0 \right\|_{L^2} \le T \left\| f_0 \right\|_{\dot{H}^1}.$$

On the other hand,

$$\left\| \exp\left(-(t-t')\Lambda\right) \mathcal{T}_n(f_n)(t') \right\|_{L^2} \le \left\| \mathcal{T}_n(f_n)(t') \right\|_{L^2}.$$

Consequently,

$$\left\| f_n(t) - J_n f_0 \right\|_{L^2} \le T \left\| f_0 \right\|_{\dot{H}^1} + T \sup_{t' \in [0,T]} \left\| \mathcal{T}_n(f_n)(t') \right\|_{L^2}.$$

Now we want to replace the left hand side of the above inequality by $\left\| f_n(t) - f_0 \right\|_{L^2}$. To do so, notice that, since the spectrum of $J_n f_0 - f_0$ is contained in $\{|\xi| \ge 1\}$, we have

$$\left\| J_n f_0 - f_0 \right\|_{L^2} \le \left\| f_0 \right\|_{\dot{H}^1}.$$

By combining the above estimates, we deduce that

$$\left\| f_n(t) - f_0 \right\|_{L^2} \le (1+T) \left\| f_0 \right\|_{\dot{H}^1} + T \sup_{t' \in [0,T]} \left\| \mathcal{T}_n(f_n)(t') \right\|_{L^2}.$$

Now, we estimate the L^2-norm of the nonlinearity $\mathcal{T}_n(f_n)$ by means of the first statement in Theorem 1.2. We conclude that, for $T < 1$,

$$\left\| f_n(t) - f_0 \right\|_{L^2}^2 \le 2(1+T)^2 \left\| f_0 \right\|_{\dot{H}^1}^2 + CT^2 \sup_{[0,T]} \left\| f_n \right\|_{\dot{H}^1}^2 \sup_{[0,T]} \left\| f_n \right\|_{\dot{H}^{\frac{3}{2}}}^2.$$

This is in turn estimated by the right side of (5.5). This concludes the first step.

Second step : High frequency estimate. Denote by $(\cdot,\cdot)$ the scalar product in $L^2(\mathbb{R})$. To estimate the $\dot{H}^s$-norm of f_n, we make act Λ^s on the equation, and then take its scalar product with $\Lambda^s f_n$. We get

$$(\partial_t \Lambda^s f_n, \Lambda^s f_n) + (\Lambda^{s+1} f_n, \Lambda^s f_n) = \left(\Lambda^s \mathcal{T}_n(f_n), \Lambda^s f_n\right).$$

Since the Muskat equation is parabolic of order one, we will be able to gain one half-derivative. We exploit this parabolic regularity by writing that

$$(\Lambda^{s+1} f_n, \Lambda^s f_n) = \left\| f_n \right\|_{\dot{H}^{s+\frac{1}{2}}}^2,$$

and

$$\begin{aligned}
\left(\Lambda^s \mathcal{T}_n(f_n), \Lambda^s f_n\right) &= \left(\Lambda^s J_n\left(\mathcal{T}(f_n) f_n\right), \Lambda^s f_n\right) \\
&= \left(\Lambda^s\left(\mathcal{T}(f_n) f_n\right), J_n \Lambda^s f_n\right) \\
&= \left(\Lambda^s\left(\mathcal{T}(f_n) f_n\right), \Lambda^s f_n\right) \quad \text{since } J_n f_n = f_n, \\
&= \left(\Lambda^{s-\frac{1}{2}} \mathcal{T}(f_n) f_n, \Lambda^{s+\frac{1}{2}} f_n\right).
\end{aligned}$$

Consequently, we find

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| f_n \right\|_{\dot{H}^s}^2 + \left\| f_n \right\|_{\dot{H}^{s+\frac{1}{2}}}^2 = \left(\Lambda^{s-\frac{1}{2}} \mathcal{T}(f_n) f_n, \Lambda^{s+\frac{1}{2}} f_n\right).$$

We next use a variant of the paralinearization formula given by Corollary 1.3. Set

$$\epsilon = s - \frac{3}{2}.$$

We claim that, for any function g,

$$\Lambda^{1+\epsilon}(T(g)g) = V(g)\partial_x \Lambda^{1+\epsilon} g + \frac{g_x^2}{1+g_x^2}\Lambda^{2+\epsilon} g + \Lambda^{1+\epsilon} R_\epsilon(g)$$

where $V(g)$ and $R_\epsilon(g)$ are two functions satisfying, for any fixed $\nu < \epsilon$,

$$\|R_\epsilon(g)\|_{\dot{H}^{1+\epsilon}} \le \mathcal{F}(\|g\|_{\dot{H}^1\cap \dot{H}^s}) \|g\|_{\dot{H}^{s+\frac{1}{2}-\frac{\epsilon}{2}}}, \tag{5.8}$$

$$\|V(g)\|_{C^{0,\nu}} \le \mathcal{F}(\|g\|_{\dot{H}^1\cap \dot{H}^s}) \|g\|_{\dot{H}^{s+\frac{1}{2}-\frac{\epsilon}{2}}}. \tag{5.9}$$

where $\mathcal{F}$ depends only on ϵ (that is s) and ν (which will be specified later). The proof of this claim is similar to the one of (1.9).

With notations as above, set

$$V_n = V(f_n), \quad R_n = \Lambda^{1+\epsilon} R_\epsilon(f_n), \quad \gamma_n = \frac{f_{nx}^2}{1+f_{nx}^2} \quad \text{where } f_{nx} = \partial_x f_n.$$

Then,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_n\|_{\dot{H}^s}^2 + \|f_n\|_{\dot{H}^{s+\frac{1}{2}}}^2 &= \big(\gamma_n \Lambda^{s+\frac{1}{2}} f_n, \Lambda^{s+\frac{1}{2}} f_n,\big) \\ &\quad + \Big(\big(V_n \partial_x \Lambda^{s-\frac{1}{2}} f_n + R_n\big), \Lambda^{s+\frac{1}{2}} f_n\Big). \end{aligned} \tag{5.10}$$

Now the key point is that

$$\|f_n\|_{\dot{H}^{s+\frac{1}{2}}}^2 - \big(\gamma_n \Lambda^{s+\frac{1}{2}} f_n, \Lambda^{s+\frac{1}{2}} f_n\big) = \int \frac{\big(\Lambda^{s+\frac{1}{2}} f_n\big)^2}{1+f_{nx}^2}\,\mathrm{d}x.$$

On the other hand, the Cauchy-Schwarz inequality and the estimate (5.8) imply that

$$\begin{aligned} \big|\big(R_n, \Lambda^{s+\frac{1}{2}} f_n\big)\big| &\le \|R_n\|_{L^2} \big\|\Lambda^{s+\frac{1}{2}} f_n\big\|_{L^2} \\ &\le \mathcal{F}(\|f_n\|_{\dot{H}^1\cap\dot{H}^s}) \|f_n\|_{\dot{H}^{s+\frac{1}{2}-\frac{\epsilon}{2}}} \|f_n\|_{\dot{H}^{s+\frac{1}{2}}}. \end{aligned}$$

It remains to estimate the contribution of V_n to the second term in the right-hand side of (5.10). Here we use the commutator estimate given by Lemma 2.6. To do so, one uses the identity $\mathcal{H}\Lambda = -\partial_x$ where $\mathcal{H}$ is the Hilbert transform, to write

$$V_n \partial_x \Lambda^{s-\frac{1}{2}} f_n = -V_n \Lambda^{s+\frac{1}{2}} \mathcal{H} f_n.$$

Since $\mathcal{H}$ is skew-symmetric, we deduce that

$$\Big(V_n \partial_x \Lambda^{s-\frac{1}{2}} f_n, \Lambda^{s+\frac{1}{2}} f_n\Big) = \frac{1}{2}\Big(\big[\mathcal{H}, V_n\big] \Lambda^{s+\frac{1}{2}} f_n, \Lambda^{s+\frac{1}{2}} f_n\Big).$$

Now we exploit the regularity result for V_n given by (5.9). Fix $\nu = 2\epsilon/3$ and $\theta = \epsilon/2$. By applying the commutator estimate in Lemma 2.6, we obtain

$$\begin{aligned} \big\|\big[\mathcal{H}, V_n\big]\Lambda^{s+\frac{1}{2}} f_n\big\|_{L^2} &\lesssim \|V_n\|_{C^{0,\nu}} \big\|\Lambda^{s+\frac{1}{2}} f_n\big\|_{H^{-\theta}} \\ &\lesssim \|V_n\|_{C^{0,\nu}} \|f_n\|_{\dot{H}^{s+\frac{1}{2}-\frac{\epsilon}{2}}} \\ &\le \mathcal{F}(\|f_n\|_{\dot{H}^1\cap\dot{H}^s}) \|f_n\|_{\dot{H}^{s+\frac{1}{2}-\frac{\epsilon}{2}}}. \end{aligned}$$

So, by combining the above estimates,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_n\|_{\dot{H}^s}^2 + \int \frac{\big(\Lambda^{s+\frac{1}{2}} f_n\big)^2}{1+f_{nx}^2}\,\mathrm{d}x \le \mathcal{F}(\|f_n\|_{\dot{H}^1\cap\dot{H}^s}) \|f_n\|_{\dot{H}^{s+\frac{1}{2}-\frac{\epsilon}{2}}} \|f_n\|_{\dot{H}^{s+\frac{1}{2}}}.$$

The end of the proof will consist in exploiting the parabolic regularity and a variant of Gronwall's lemma to absorb the right-hand side. Set $K(t) = \sup_{x\in\mathbb{R}} |\partial_x f_n(t,x)|$. Then

$$\frac{1}{1+f_{nx}^2} \geq \frac{1}{1+K^2},$$

so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_n\|_{\dot{H}^s}^2 + \frac{C}{1+K^2}\|f_n\|_{\dot{H}^{s+\frac12}}^2 \leq \mathcal{F}(\|f_n\|_{\dot{H}^1\cap\dot{H}^s})\|f_n\|_{\dot{H}^{s+\frac12-\frac{\epsilon}{2}}}\|f_n\|_{\dot{H}^{s+\frac12}}. \tag{5.11}$$

Then we observe that

$$\begin{aligned}
&\mathcal{F}(\|f_n\|_{\dot{H}^1\cap\dot{H}^s})\|f_n\|_{\dot{H}^{s+\frac12-\frac{\epsilon}{2}}}\|f_n\|_{\dot{H}^{s+\frac12}} \\
&\qquad\leq \frac{C}{2(1+K^2)}\|f_n\|_{\dot{H}^{s+\frac12}}^2 + \frac{1+K^2}{2C}\mathcal{F}(\|f_n\|_{\dot{H}^1\cap\dot{H}^s})^2\|f_n\|_{\dot{H}^{s+\frac12-\frac{\epsilon}{2}}}^2.
\end{aligned}$$

Since $K \lesssim \|f_n\|_{\dot{H}^1\cap\dot{H}^s}$ by Sobolev embedding, up to modifying the value of the function $\mathcal{F}$, by inserting the above inequality in (5.11), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_n\|_{\dot{H}^s}^2 + \frac{C}{2(1+K^2)}\|f_n\|_{\dot{H}^{s+\frac12}}^2 \leq \mathcal{F}(\|f_n\|_{\dot{H}^1\cap\dot{H}^s})\|f_n\|_{\dot{H}^{s+\frac12-\frac{\epsilon}{2}}}^2. \tag{5.12}$$

To conclude, it will suffice to replace in the right side the norm $\|f_n\|_{\dot{H}^{s+\frac12-\frac{\epsilon}{2}}}^2$ by $\|f_n\|_{\dot{H}^s}^2$. To do this, we begin by using the interpolation inequality:

$$\|f_n\|_{\dot{H}^{s+\frac12-\frac{\epsilon}{2}}}^2 \leq \|f_n\|_{\dot{H}^s}^{2\theta}\|f_n\|_{\dot{H}^{s+\frac12}}^{2-2\theta}, \tag{5.13}$$

for some $\theta \in (0,1)$. Next, because of the Young's inequality

$$xy \leq \frac{1}{p}x^p + \frac{1}{p'}y^{p'} \quad \text{with} \quad \frac{1}{p}+\frac{1}{p'}=1, \tag{5.14}$$

applied with $p = 2/(2-2\theta)$, we infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f_n\|_{\dot{H}^s}^2 + \frac{C}{4(1+K^2)}\|f_n\|_{\dot{H}^{s+\frac12}}^2 \leq \mathcal{F}(\|f_n\|_{\dot{H}^1\cap\dot{H}^s})\|f_n\|_{\dot{H}^s}^2, \tag{5.15}$$

where as above we modified the value of the function $\mathcal{F}$. From this, it is now an easy matter to obtain the conclusion of the proposition. Firstly, integration of the above estimate gives

$$\begin{aligned}
&\frac{1}{2}\|f_n(t)\|_{\dot{H}^s}^2 + \frac{C}{4(1+K^2)}\int_0^t \left\|f_n(t')\right\|_{\dot{H}^{s+\frac12}}^2 \mathrm{d}t' \\
&\qquad\qquad\leq \frac{1}{2}\|f_n(0)\|_{\dot{H}^s}^2 + t\sup_{t'\in[0,t]}\mathcal{F}(\left\|f_n(t')\right\|_{\dot{H}^1\cap\dot{H}^s})\left\|f_n(t')\right\|_{\dot{H}^s}^2.
\end{aligned}$$

Modifying $\mathcal{F}(\cdot)$ and C, we deduce that

$$\|f_n(t)\|_{\dot{H}^s}^2 + \frac{C}{1+K^2}\int_0^t \left\|f_n(t')\right\|_{\dot{H}^{s+\frac12}}^2 \mathrm{d}t' \leq \|f_n(0)\|_{\dot{H}^s}^2 + T\mathcal{F}\Big(\sup_{t\in[0,T]}\|f_n(t)\|_{\dot{H}^1\cap\dot{H}^s}^2\Big),$$

for any $t \in [0,T]$. By taking the supremum over $t \in [0,T]$, we deduce an estimate for $\sup_{t\in[0,T]}\|f_n(t)\|_{\dot{H}^s}^2$. Now, the desired estimate for $\sup_{t\in[0,T]}\|f_n(t) - f_0\|_{\dot{H}^s}^2$ follows from the triangle inequality and the fact that $\|f_n(0)\|_{\dot{H}^s} \leq \|f_0\|_{\dot{H}^s}$. □

We will also need another energy estimate to compare two different solutions f_1 and f_2. The main difficulty here will be to find the optimal space in which one can perform an energy estimate. The most simpler way to do so would be to estimate their difference $f_1 - f_2$ in the biggest possible space and to use an interpolation inequality to control the latter in a space of smoother function. This suggests to estimate $f_1 - f_2$ in $C^0([0,T];L^2(\mathbb{R}))$. On the other hand, by thinking of the fluid problem, we might think that it is compulsory to control the difference between the two functions parametrizing the two free surfaces in a space of smooth functions. We will see later that, somewhat unexpectedly, that it is enough to estimate $f_1 - f_2$ in $C^0([0,T];\dot{H}^{1/2}(\mathbb{R}))$. In this direction, we will use the following proposition.

Proposition 5.3. *i) For all s in $(3/2,2)$, there exists a non-decreasing function $\mathcal{F}\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $n \in \mathbb{N}\setminus\{0\}$, any $T>0$, and any functions*

$$\begin{aligned}&f \in C^0([0,T];\dot{H}^1(\mathbb{R})\cap \dot{H}^s(\mathbb{R})),\\ &g \in C^1([0,T];H^{\frac{1}{2}}(\mathbb{R})) \quad \text{with } J_n g = g,\\ &F \in C^0([0,T];L^2(\mathbb{R})),\end{aligned}$$

satisfying the equation

$$\partial_t g - J_n\big(V(f)\partial_x g\big) + J_n\Big(\frac{1}{1+f_x^2}\Lambda g\Big) = F, \tag{5.16}$$

where $V(f)$ is as above, we have the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\|^2_{\dot{H}^{\frac{1}{2}}} + \int \frac{(\Lambda g)^2}{1+f_x^2}\,\mathrm{d}x \le \mathcal{F}(\|f\|_{\dot{H}^1\cap\dot{H}^s})\,\|g\|_{\dot{H}^{1-\frac{\epsilon}{2}}}\,\|g\|_{\dot{H}^1} + (F,\Lambda g)_{L^2}. \tag{5.17}$$

where $\epsilon = s-3/2$ and $C = \mathcal{F}\big(\|f\|_{L^\infty([0,T];\dot{H}^1\cap\dot{H}^s)}\big)$.

ii) Moreover, the same result is true when one replaces J_n by the identity.

Proof. To prove (5.17) we take the L^2-scalar product of the equation (5.16) with Λg. Since

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\|^2_{\dot{H}^{\frac{1}{2}}} = (\partial_t g,\Lambda g), \qquad \Big(J_n\Big(\frac{1}{1+f_x^2}\Lambda g\Big),\Lambda g\Big) = \int \frac{(\Lambda g)^2}{1+f_x^2}\,\mathrm{d}x,$$

(where we used $J_n g = g$), we only have to estimate $\big(J_n\big(V(f)\partial_x g\big),\Lambda g\big)$. As above, writing $\partial_x = -\mathcal{H}\Lambda$, where $\mathcal{H}$ is the Hilbert transform satisfying $\mathcal{H}^* = -\mathcal{H}$, we obtain

$$\big|\big(J_n\big(V(f)\partial_x g\big),\Lambda g\big)\big| = \frac{1}{2}\left|\big(\big[\mathcal{H},V(f)\big]\Lambda g,\Lambda g\big)\right|.$$

Set $\epsilon = s-3/2$, $\nu = 2\epsilon/3$ and $\theta = \epsilon/2$. We use Lemma 2.6 to obtain

$$\begin{aligned}\big|\big(J_n\big(V(f)\partial_x g\big),\Lambda g\big)\big| &\le \frac{1}{2}\left\|\big[\mathcal{H},V(f)\big]\Lambda g\right\|_{L^2}\|\Lambda g\|_{L^2}\\ &\le \|V(f)\|_{C^{0,\nu}}\|\Lambda g\|_{H^{-\theta}}\|\Lambda g\|_{L^2}\\ &\le \mathcal{F}(\|f\|_{\dot{H}^1\cap\dot{H}^s})\|\Lambda g\|_{H^{-\frac{\epsilon}{2}}}\|\Lambda g\|_{L^2}\\ &\le \mathcal{F}(\|f\|_{\dot{H}^1\cap\dot{H}^s})\|g\|_{\dot{H}^{1-\frac{\epsilon}{2}}}\|g\|_{\dot{H}^1}.\end{aligned}$$

This completes the proof of $i)$ and the same arguments can be used to prove $ii)$. □

5.3. **End of the proof.** In this paragraph we complete the analysis of the Cauchy problem. We begin by proving the uniqueness part in Theorem 1.2.

Lemma 5.4. *Assume that f and f' are two solutions of the Muskat equation with the same initial data and satisfying the assumptions of Theorem 1.4. Then $f = f'$.*

Proof. Set

$$g = f - f', \quad M = \|f\|_{L^\infty([0,T];\dot{H}^1\cap\dot{H}^s)} + \|f'\|_{L^\infty([0,T];\dot{H}^1\cap\dot{H}^s)}.$$

We denote by $C(M)$ various constants depending only on M.

We want to prove that $g = 0$. To do so, we use the energy estimate in $\dot{H}^{1/2}(\mathbb{R})$. The key point is to write that g is a smooth function, in $C^1([0,T]; H^{\frac{1}{2}}(\mathbb{R}))$, satisfying

$$\partial_t g + \Lambda g = \mathcal{T}(f)g + F_1 \quad \text{with} \quad F_1 = \mathcal{T}(f)f' - \mathcal{T}(f')f'.$$

This term is estimated by means of point *ii*) in Proposition 2.3 with $\delta = \epsilon = s - 3/2$,

$$\|F_1\|_{L^2} = \left\|(\mathcal{T}(f) - \mathcal{T}(f'))f'\right\|_{L^2} \le C \left\|f - f'\right\|_{\dot{H}^{1-\epsilon}} \left\|f'\right\|_{\dot{H}^{\frac{3}{2}+\epsilon}} = C(M) \left\|g\right\|_{\dot{H}^{1-\epsilon}} . \tag{5.18}$$

Recall from (1.5) that

$$\mathcal{T}(f)g = \frac{f_x^2}{1+f_x^2}\Lambda g + V(f)\partial_x g + R(f,g)$$

where $R(f,g)$ satisfies (setting $\epsilon = s - 3/2$),

$$\|R(f,g)\|_{L^2} \le C \left\|f\right\|_{\dot{H}^{\frac{3}{2}+\epsilon}} \left\|g\right\|_{\dot{B}^{1-\epsilon}_{2,1}} \le C(M) \left\|g\right\|_{\dot{B}^{1-\epsilon}_{2,1}} . \tag{5.19}$$

Therefore, g satisfies

$$\partial_t g - V\partial_x g + \frac{1}{1+f_x^2}\Lambda g = F$$

where $F = F_1 + R(f)g$. In view of the estimates (5.18), (5.19) and the embedding $\dot{H}^{1-3\epsilon/2}(\mathbb{R}) \cap \dot{H}^{1-\epsilon/2}(\mathbb{R}) \hookrightarrow \dot{B}^{1-\epsilon}_{2,1}$ (see Lemma 2.2), we have

$$\left|(F, \Lambda g)\right| \le \left\|F\right\|_{L^2} \left\|g\right\|_{\dot{H}^1} \le C(M) \left\|g\right\|_{\dot{H}^{1-\frac{\epsilon}{2}} \cap \dot{H}^{1-\frac{3\epsilon}{2}}} \left\|g\right\|_{\dot{H}^1} .$$

Hence, it follows from Proposition 5.3 (see point *ii*)) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|g\right\|^2_{\dot{H}^{\frac{1}{2}}} + \int \frac{(\Lambda g)^2}{1+f_x^2}\,\mathrm{d}x \le C(M) \left\|g\right\|_{\dot{H}^{1-\frac{\epsilon}{2}} \cap \dot{H}^{1-\frac{3\epsilon}{2}}} \left\|g\right\|_{\dot{H}^1} .$$

Next, we use interpolation inequalities as in the proof of Proposition 5.2. More precisely, by using arguments parallel to those used to deduce (5.15) from (5.13)-(5.14), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|g\right\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{1}{4(1+\left\|f_x\right\|_{L^\infty_{t,x}})} \int (\Lambda g)^2\,\mathrm{d}x \le C(M) \left\|g\right\|^2_{\dot{H}^{\frac{1}{2}}} .$$

This obviously implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|g\right\|^2_{\dot{H}^{\frac{1}{2}}} \le C(M) \left\|g\right\|^2_{\dot{H}^{\frac{1}{2}}} .$$

Since $g(0) = 0$, the Gronwall's inequality implies that $\left\|g\right\|_{\dot{H}^{\frac{1}{2}}} = 0$ so $g = 0$, which completes the proof. □

Having proved the uniqueness of solutions, we now study their existence. The key step will be here to apply the *a priori* estimates proved in Proposition 5.2. This will give us uniform bounds for the solutions f_n defined in §5.1.

Lemma 5.5. *There exists $T_0 > 0$ such that $T_n \ge T_0$ for all $n \in \mathbb{N} \setminus \{0\}$ and such that $(f_n - f_0)_{n\in\mathbb{N}}$ is bounded in $C^0([0,T_0]; H^s(\mathbb{R}))$.*

Proof. We use the notations of §5.1 and Proposition 5.2. Given $T < T_n$, we define

$$M_n(T) = \sup_{t\in[0,T]} \left\|f_n(t) - f_0\right\|^2_{L^2\cap\dot{H}^s} , \qquad N_n(T) = M_n(T) + \left\|f_0\right\|^2_{\dot{H}^1\cap\dot{H}^s} .$$

Denote by $\mathcal{F}$ the function whose existence is the conclusion of Proposition 5.2 and set

$$A = 10 \left\|f_0\right\|^2_{\dot{H}^1\cap\dot{H}^s} .$$

We next pick $0 < T_0 \leq 1$ small enough such that

$$3(1+T_0)^2 \|f_0\|^2_{\dot{H}^1\cap\dot{H}^s} + T_0\mathcal{F}(A) < A,$$

We claim the uniform bound

$$\forall n \in \mathbb{N}\setminus\{0\},\ \forall T \in I_n := [0, \min\{T_0, T_n\}), \quad N_n(T) < A.$$

Let us prove this claim by contradiction. Assume that for some n there exists $\tau_n \in I_n$ such that $N_n(\tau_n) \geq A$ and consider the smallest of such times (then $\tau_n > 0$ since $T \mapsto N_n(T)$ is continuous and $N_n(0) < A$ by construction). Then, by definition, for all $0 < T \leq \tau_n$, one has $N_n(T) \leq A$ and $N_n(\tau_n) = A$. Since $\|\partial_x f_n(t)\|_{L^\infty(\mathbb{R})} \lesssim \|f_n(t)\|_{\dot{H}^1\cap\dot{H}^s}$ (see (3.4)), we have a uniform control of the L^∞_x-norm of $\partial_x f_n$ on $[0,T]$ in terms of A, hence we are in position to apply the *a priori* estimate (5.5). Now, if we add $\|f_0\|^2_{\dot{H}^1\cap\dot{H}^s}$ to both sides of (5.5) we deduce that

$$N_n(\tau_n) + \frac{C}{1+C(A)^2}\int_0^{\tau_n} \|f_n(t)\|^2_{\dot{H}^{s+\frac{1}{2}}}\,\mathrm{d}t \leq 3(1+\tau_n)^2\|f_0\|^2_{\dot{H}^1\cap\dot{H}^s} + \tau_n\mathcal{F}(N_n(\tau_n)).$$

We infer that

$$\begin{aligned} A = N_n(\tau_n) &\leq 3(1+\tau_n)^2\|f_0\|^2_{\dot{H}^1\cap\dot{H}^s} + \tau_n\mathcal{F}(N_n(\tau_n)) \\ &\leq 3(1+T_0)^2\|f_0\|^2_{\dot{H}^1\cap\dot{H}^s} + T_0\mathcal{F}(A) \\ &< A, \end{aligned}$$

hence the contradiction. We thus have proved that, for all $n \in \mathbb{N}$ and all $T \leq \min\{T_0, T_n\}$, we have

$$\sup_{t\in[0,T]} \|f_n(t) - f_0\|^2_{L^2\cap\dot{H}^s} \leq A.$$

This obviously implies that

$$\sup_{t\in[0,T]} \|f_n(t) - f_0\|_{L^2} \leq \sqrt{A}.$$

Since

$$u_n = f_n - J_n f_0 = f_n - f_0 + (I - J_n)f_0,$$

and since $\|(I-J_n)f_0\|_{L^2} \leq \|f_0\|_{\dot{H}^1}$, the previous bound implies that the norm $\|u_n(t)\|_{L^2}$ is bounded for all $t \leq \min\{T_0, T_n\}$. The alternative (5.2) then implies that the lifespan of f_n is bounded from below by T_0. And the previous inequality shows that $(f_n - f_0)$ is bounded in $C^0([0,T_0]; H^s(\mathbb{R}))$. This completes the proof. □

At that point, we have defined a sequence (f_n) of solutions to well-chosen approximate systems. The next task is to prove that this sequence converges. Here a word of caution is in order: $\dot{H}^s(\mathbb{R})$ is not a Banach space when $s > 1/2$. To overcome this difficulty, we use the fact that $u_n = f_n - f_0$ is bounded in $C^0([0,T_0]; H^s(\mathbb{R}))$, where $H^s(\mathbb{R})$ is the nonhomogeneous space $L^2(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$, which is a Banach space. We claim that, in addition, (u_n) is a Cauchy sequence in $C^0([0,T_0]; H^{s'}(\mathbb{R}))$ for any $s' < s$. Let us assume this claim for the moment. This will imply that (u_n) converges in the latter to some limit u. Now, setting $f = f_0 + u$ and using the continuity result for $\mathcal{T}(f)f$ given by *iii*) in Proposition 2.3, we verify immediately that f is a solution to the Cauchy problem for the Muskat equation. It would remain to prove that u is continuous in time with values in $H^s(\mathbb{R})$ (instead of $H^{s'}(\mathbb{R})$ for any $s' < s$). For the sake of shortness, this is the only point that we do not prove in details in this paper (referring to [2] for the proof of a similar result in a case with similar difficulties).

To conclude the proof of Theorem 1.4, it remains only to establish the following

Lemma 5.6. *For any real number s' in $[0,s)$, the sequence (u_n) is a Cauchy sequence in $C^0([0,T_0];H^{s'}(\mathbb{R}))$.*

Proof. The proof is in two steps. We begin by proving that (f_n) is a Cauchy sequence in $C^0([0,T_0];\dot{H}^{s'}(\mathbb{R}))$ for $1/2\le s'<s$. Then, we use this result and an elementary L^2-estimate to infer that (u_n) is a Cauchy sequence in $C^0([0,T_0];L^2(\mathbb{R}))$.

By using estimates parallel to those used to prove Lemma 5.4, one obtains that (f_n) is a Cauchy sequence in $C^0([0,T_0];\dot{H}^{\frac12}(\mathbb{R}))$. Now consider $1/2<s'<s$. By interpolation, there exists α in $(0,1)$ such that

$$\|u\|_{\dot{H}^{s'}}\lesssim \|u\|_{\dot{H}^{\frac12}}^{\alpha}\,\|u\|_{\dot{H}^{s}}^{1-\alpha}.$$

Consequently, since (f_n) is bounded in $C^0([0,T_0];\dot{H}^s(\mathbb{R}))$, we deduce that (f_n) is a Cauchy sequence in $C^0([0,T_0];\dot{H}^{s'}(\mathbb{R}))$ for any $s'<s$.

It remains only to prove that (u_n) is a Cauchy sequence in $C^0([0,T_0];L^2(\mathbb{R}))$. To do so, we proceed differently. Starting from (see (5.7)),

$$\partial_t f_n+\Lambda f_n=J_n\mathcal{T}(f_n)f_n,$$

we obtain that $u_n-u_p=f_n-f_p-(J_n-J_p)f_0$ satisfies

$$\partial_t(u_n-u_p)+\Lambda(u_n-u_p)=F_{np}+G_{np} \tag{5.20}$$

where

$$F_{np}=J_n(\mathcal{T}(f_n)f_n-\mathcal{T}(f_p)f_p),\quad G_{np}=(J_n-J_p)\big(-\Lambda f_0+\mathcal{T}(f_p)f_p\big).$$

We now use an elementary L^2-estimate. We take the L^2-scalar product of the equation (5.20) with u_n-u_p, to obtain, since $u_n(0)-u_p(0)=0$,

$$\sup_{t\in[0,T]}\|u_n(t)-u_p(t)\|_{L^2}\le \|F_{np}\|_{L^1([0,T];L^2)}+\|G_{np}\|_{L^2([0,T];\dot{H}^{-\frac12})}.$$

So it remains only to prove that $\|F_{np}\|_{L^1([0,T];L^2)}$ and $\|G_{np}\|_{L^2([0,T];\dot{H}^{-\frac12})}$ are arbitrarily small for n,p large enough. Here we use the result proved in the first part of the proof. Namely, since (f_n) is a Cauchy sequence in $C^0([0,T];\dot{H}^1(\mathbb{R})\cap\dot{H}^{\frac32}(\mathbb{R}))$, we deduce from point *iii*) in Proposition 2.3 that

$$\mathcal{T}(f_n)f_n-\mathcal{T}(f_p)f_p$$

is small in $C^0([0,T];L^2(\mathbb{R}))$ for n,p large enough. On the other hand, using the estimate

$$\|(J_n-J_p)u\|_{\dot{H}^{-\frac12}}\le\frac{1}{\sqrt{\min(n,p)}}\|u\|_{L^2},$$

we verify that $\|G_{np}\|_{L^2([0,T];\dot{H}^{-\frac12})}$ is arbitrarily small for n,p large enough. This completes the proof. □

Acknowledgments

T. A. acknowledges the support of the SingFlows project, grant ANR-18-CE40-0027 of the French National Research Agency (ANR). O. L. has been partially supported by the National Grant MTM2014-59488-P from the Spanish government and the ERC through the Starting Grant project H2020-EU.1.1.-63922

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