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ORIGINAL ARTICLE

Convexity and the Hele–Shaw Equation

Thomas Alazard[1]

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Abstract
Walter Craig's seminal works on the water-wave problem established the importance of several exact identities: Zakharov's hamiltonian formulation, shape derivative formula for the Dirichlet-to-Neumann operator, and normal forms transformations. In this paper, we introduce several identities for the Hele–Shaw equation which are inspired by his nonlinear approach. First, we study convex changes of unknowns and obtain a large class of strong Lyapunov functions; in addition to be non-increasing, these Lyapunov functions are convex functions of time. The analysis relies on a new, simple compact elliptic formulation of the Hele–Shaw equation, which is of independent interest. Then, we study the role of convexity to control the spatial derivatives of the solutions. We consider the evolution equation for the Rayleigh–Taylor coefficient a (this is a positive function proportional to the opposite of the normal derivative of the pressure at the free surface). Inspired by the study of entropies for elliptic or parabolic equations, we consider the special function $\varphi(x) = x \log x$ and find that $\varphi(1/\sqrt{a})$ is a sub-solution of a well-posed equation.

Keywords Dirichlet-to-Neumann operator · Shape derivative · Convex functions · Lyapunov functionals

1 Introduction

1.1 The Hele–Shaw Equation

Consider an incompressible liquid having a free surface given as a graph, so that, at time $t \geq 0$, the fluid domain is of the form:

$$\Omega(t) = \{(x, y) \in \mathbf{T}^n \times \mathbf{R} \,;\; y < h(t, x)\},$$

✉ Thomas Alazard
thomas.alazard@ens-paris-saclay.fr

[1] CNRS and Université Paris-Saclay, ENS Paris-Saclay, CNRS, Centre de mathématiques et de leurs applications, 94235 Cachan, France

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where $\mathbf{T}^n$ denotes a n-dimensional torus, x (resp. y) is the horizontal (resp. vertical) space variable. In the Eulerian coordinate system, in addition to the free surface elevation h, the unknowns are the velocity field $v\colon \Omega \rightarrow \mathbf{R}^{n+1}$ and the scalar pressure $P\colon \Omega \rightarrow \mathbf{R}$. We assume that they satisfy the Darcy's equations:

$$\operatorname{div}_{x,y} v = 0 \quad \text{and} \quad v = -\nabla_{x,y}(P + gy) \quad \text{in } \Omega. \tag{1}$$

A timescale may be chosen, so that the acceleration of gravity is $g = 1$.

These equations are supplemented by two boundary conditions. First, one assumes that the pressure vanishes on the free surface:

$$P = 0 \quad \text{on} \quad \partial\Omega.$$

The second boundary condition states that the normal velocity of the free surface is equal to the normal component of the fluid velocity on the free surface. It follows that:

$$\partial_t h = \sqrt{1 + |\nabla h|^2}\, v \cdot n, \tag{2}$$

where $\nabla = \nabla_x$ and n is the outward unit normal to $\partial\Omega$, given by:

$$n = \frac{1}{\sqrt{1 + |\nabla h|^2}} \begin{pmatrix} -\nabla h \\ 1 \end{pmatrix}.$$

Notice that the velocity field v is a gradient, that is $v = -\nabla_{x,y}\phi$, where $\phi = P + y$ (recall that we set $g = 1$). Since $\operatorname{div}_{x,y} v = 0$, the potential ϕ is harmonic, and hence, it is fully determined by its trace on the boundary, which is h, since P vanishes on the boundary. We have:

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \qquad \phi|_{y=h} = h. \tag{3}$$

Consequently, v is fully determined by h which implies that the Hele–Shaw problem simplifies to an evolution equation for h only; namely, Eq. (2). Once h is determined, one obtains ϕ by solving (3), and then, one sets $v = -\nabla_{x,y}\phi$ and $P = -\phi - y$.

The previous reduction to an evolution equation for h is better formulated by introducing the Dirichlet-to-Neumann operator (this operator plays a key role in the analysis by Walter Craig and Catherine Sulem of the water-wave equations). For a given time t, that is omitted here, and a function $\psi = \psi(x)$, $G(h)\psi$ is defined by (see Sect. 2.1 for details):

$$G(h)\psi(x) = \sqrt{1 + |\nabla h|^2}\partial_n \varphi|_{y=h(x)} = \partial_y \varphi(x, h(x)) - \nabla h(x) \cdot \nabla\varphi(x, h(x)),$$

where φ is the harmonic extension of ψ, given by:

$$\Delta_{x,y}\varphi = 0 \quad \text{in } \Omega, \qquad \varphi|_{y=h} = \psi. \tag{4}$$

Then, with this notation, it follows from Eq. (2) that (see Sect. 2.2):

$$\partial_t h + G(h)h = 0. \tag{5}$$

This equation is analogous to the Craig–Sulem–Zakharov formulation of the water-wave equations (following Zakharov [30] and Craig–Sulem [17]).

There are many other possible approaches to study the Cauchy problem for the Hele–Shaw equation. One can study the existence of weak solutions, viscosity solutions, or classical solutions; we refer the reader to [5–7,13,18–21,23,24,28]. These papers consider different formulations of the Hele–Shaw problem and we notice that, for rough solutions, it is not obvious to check that these formulations are equivalent. In this article, we are interested in proving some qualitative properties of the flow. To do so, we consider classical solutions (in the sense of Definition 1 below). The parabolic smoothing effect implies that, for positive times, these solutions are C^∞ in space and time, so that it is elementary to rigorously justify the computations.

1.2 Main Results

In this paper, we study some properties of the Hele–Shaw equation which are related to convexity. First, we study the existence of Lyapunov functions of the form:

$$I_\Phi(t) = \int_{\mathbf{T}^n} \Phi(h(t,x))\,\mathrm{d}x.$$

We show that if both Φ and Φ' are convex, then $I_\Phi(t)$ is a strong Lyapunov function; by this, we mean that $t \mapsto I_\Phi(t)$ is a non-increasing *convex* function. To study these strong Lyapunov functions, we will introduce a new elliptic formulation of the Hele–Shaw equation. Namely, we observe that the linearized Hele–Shaw equation can be written as $\Delta_{t,x} h = 0$ and find an analogous elliptic formulation equation for the nonlinear Hele–Shaw equation. Eventually, we study the role of convexity by seeking entropy-type inequalities.

Lyapunov functions. Consider a convex function $\Phi\colon \mathbf{R} \to \mathbf{R}^+$. With Nicolas Meunier and Didier Smets, we proved in [3] that:

$$I_\Phi\colon [0,T] \to \mathbf{R}^+, \quad t \mapsto \int \Phi(h(t,x))\,\mathrm{d}x$$

is a Lyapunov function (which means that the latter quantity is a non-increasing positive function). The first main result of this paper is that, if one further assumes that the derivative Φ' is also convex, then the latter quantity is a *strong Lyapunov function*; by this, we mean that it is a non-increasing convex function.

Theorem 1 *Consider a smooth solution h to the Hele–Shaw equation.*

(i) *If $\Phi\colon \mathbf{R} \to \mathbf{R}^+$ is a C^2 convex function, then:*

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\Phi \le 0 \quad \textit{where} \quad I_\Phi(t) = \int_{\mathbf{T}^n} \Phi(h(t,x))\,\mathrm{d}x. \tag{6}$$

(ii) *Assume that* $\Phi : \mathbf{R} \to \mathbf{R}^+$ *is a* C^3 *convex function whose derivative is also convex. Then:*

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\Phi \leq 0 \quad \textit{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} I_\Phi \geq 0. \tag{7}$$

Remark 1 (i) In [3], the inequality (6) is proved only for $\Phi(h) = h^{2p}$ for all p in $\{1\} \cup 2\mathbf{N}$; but the generalization to an arbitrary convex function is straightforward.

(ii) To the author's knowledge, the study of the existence of strong convex Lyapunov function is new.

(iii) It follows from Stokes' theorem that $\int_{\mathbf{T}^n} hG(h)h \,\mathrm{d}x \geq 0$ (see 11). Therefore, by multiplying the equation $\partial_t h + G(h)h = 0$ by h and integrating over $\mathbf{T}^n$, one obtains the classical result that the L^2-norm is a Lyapunov function:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{T}^n} h(t,x)^2 \,\mathrm{d}x \leq 0.$$

This is the special case for (6) with $\Phi(h) = h^2$. On the other hand, the fact that (7) holds for $\Phi(h) = h^2$ is already highly non trivial. Indeed, this follows from the following identity (first proved in [3]):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\mathbf{T}^n} h^2 \,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{T}^n} hG(h)h \,\mathrm{d}x = \int_{\mathbf{T}^n} a \left|\nabla_{t,x} h\right|^2 \mathrm{d}x \geq 0,$$

where a is a positive coefficient (this is the so-called Taylor coefficient).

An elliptic formulation. To prove Theorem 1, we will introduce an elliptic formulation of the Hele–Shaw problem. To explain this, we begin by considering the linearized equation $\partial_t h + G(0)h = 0$. Recall that the Dirichlet-to-Neumann operator $G(0)$ associated with a flat half-space is given explicitly by $G(0) = |D_x|$, that is the Fourier multiplier defined by $|D_x| \mathrm{e}^{ix\cdot\xi} = |\xi| \mathrm{e}^{ix\cdot\xi}$. Then, the linearized Hele–Shaw equation reads:

$$\partial_t h + |D_x|\, h = 0.$$

Now, observe that the previous equation is elliptic. Indeed, its symbol $i\tau + |\xi|$ is obviously an elliptic symbol or order 1. Another way to see this is to make act $\partial_t - |D_x|$ on the equation. Since $-|D_x|^2 = \Delta_x$, we find:

$$\Delta_{t,x} h = \partial_t^2 h + \Delta_x h = 0.$$

The next result generalizes this observation to the Hele–Shaw equation.

Theorem 2 *If h is a smooth solution to $\partial_t h + G(h)h = 0$, then:*

$$\Delta_{t,x} h + B(h)^* \big(\left|\nabla_{t,x} h\right|^2 \big) = 0,$$

where $B(h)^*$ is the adjoint (for the $L^2(\mathbf{T}^n)$-scalar product) of the operator defined by:

$$B(h)\psi = \partial_y \varphi|_{y=h},$$

where φ is the harmonic extension of ψ (given by (4)).

An entropy inequality. Then, we study the role of convexity to control the spatial derivatives of the solutions. We consider the Rayleigh–Taylor coefficient a, which is a positive function. Inspired by the study of entropies for elliptic or parabolic equations, we consider the convex function $\varphi(x) = x \log x$ and find that $\varphi(1/\sqrt{a})$ is a sub-solution of a well-posed equation.

The Rayleigh–Taylor coefficient a is defined by:

$$a = -(\partial_y P)|_{y=h}.$$

It is known that this coefficient is always positive when the free surface is at least $C^{1,\alpha}$ for some $\alpha > 0$ (see [3, Prop. 4.3]). As a consequence, we may consider $\sqrt{a}$ and $\log(a)$.

Proposition 1 *Introduce the operator $L(h)$ defined by:*

$$L(h)f = -V \cdot \nabla f - \frac{1}{2}(\operatorname{div} V)f + \sqrt{a}G(h)\big(\sqrt{a}f\big).$$

The function

$$v := \frac{1}{\sqrt{a}} \log\left(\frac{1}{\sqrt{a}}\right)$$

satisfies:

$$\partial_t v + L(h)v + cv = f, \tag{8}$$

where $f(t,x) \leq 0$ and $c = c(t,x) \geq 0$.

Remark 2 Observe that $L(h)$ is a non-negative operator. For any function f, it follows from the inequality (11) below that:

$$\int_{\mathbf{T}^n} f L(h) f \,\mathrm{d}x = \int (\sqrt{a}f) G(h)(\sqrt{a}f)\,\mathrm{d}x \geq 0.$$

The main interest of the previous result lies in the fact that it was surprising to find an equation involving derivatives of the unknown where both c and f have favorable signs (for other candidates, one obtains equations of the form (8) where either f has no sign or $c \leq 0$). As an application of the previous entropy inequality, we will give an alternate proof of the following result first proved in [3].

Corollary 1 *Let $n \geq 1$ and consider a regular solution h to the Hele–Shaw equation defined on $[0, T]$. Then, for all time t in $[0, T]$:*

$$\inf_{x \in \mathbf{T}^n} a(t, x) \geq \inf_{x \in \mathbf{T}^n} a(0, x).$$

To Walter With Guy Métivier [2], we started working on the water-wave equations and the Dirichlet-to-Neumann operator by reading a very well-written paper, in French, by Walter Craig and Ana-Maria Matei [15]. Over the years, I met Walter frequently during conferences, in Canada or during his visits in France. He was always generous with his ideas. His original points of view, his enthusiasm, and his questions deeply influenced me. I wish that I could thank him one more time for all he did to help me.

2 Preliminaries

In this section, we review several results about the Dirichlet-to-Neumann operator as well as some identities proved in [3] about the Hele–Shaw equation.

2.1 The Dirichlet-to-Neumann Operator

In this paragraph, the time variable is seen as a parameter and we skip it. We denote by $H^s(\mathbf{T}^n)$ the Sobolev space of periodic functions u, such that $(I - \Delta)^{s/2} u$ belongs to $L^2(\mathbf{T}^n)$, where $(I - \Delta)^{s/2}$ is the Fourier multiplier with symbol $(1 + |\xi|^2)^{s/2}$.

Now, consider a smooth function $h \in C^\infty(\mathbf{T}^n)$ and a function ψ in the Sobolev space $H^{\frac{1}{2}}(\mathbf{T}^n)$. Then, it follows from classical arguments that there is a unique variational solution φ to the problem:

$$\Delta_{x,y} \varphi = 0 \quad \text{in } \Omega = \{y < h(x)\}, \quad \varphi|_{y=h} = \psi. \tag{9}$$

Notice that $\nabla_{x,y}\varphi$ belongs only to $L^2(\Omega)$, so it is not obvious that one can consider the trace $\partial_n \varphi|_{\partial\Omega}$. However, since $\Delta_{x,y}\varphi = 0$, one can express the normal derivative in terms of the tangential derivatives and $\sqrt{1 + |\nabla h|^2}\partial_n \varphi|_{\partial\Omega}$ is well defined and belongs to $H^{-\frac{1}{2}}(\mathbf{T}^n)$. As a result, one can define the Dirichlet-to-Neumann operator $G(h)$ by:

$$G(h)\psi(x) = \sqrt{1 + |\nabla h|^2}\partial_n \varphi|_{y=h(x)} = \partial_y \varphi(x, h(x)) - \nabla h(x) \cdot \nabla \varphi(x, h(x)).$$

Let us recall two results. First, it follows from classical elliptic regularity results that, for any $s \geq 1/2$, $G(h)$ is bounded from $H^s(\mathbf{T}^n)$ into $H^{s-1}(\mathbf{T}^n)$. This property still holds in the case where h has limited regularity. Many results have been obtained since the pioneering works of Craig and Nicholls ([16]; see also [20,25,29]). It is known that (see [1,26]), for any $s > n/2 + 1$:

$$\|G(h)\psi\|_{H^{s-1}} \leq C\big(\|h\|_{H^s}\big)\|\psi\|_{H^s}. \tag{10}$$

Second, we will frequently use the fact that $G(h)$ is a positive operator. Namely, consider a function $\psi = \psi(x)$ and its harmonic extension $\varphi = \varphi(x, y)$, solution to (9). It follows from Stokes theorem that:

$$\int_{\mathbf{T}^n} \psi G(h)\psi \,\mathrm{d}x = \int_{\partial\Omega} \varphi \partial_n \varphi \,\mathrm{d}\sigma = \iint_{\Omega} \left|\nabla_{x,y}\varphi\right|^2 \mathrm{d}y\,\mathrm{d}x \geq 0. \tag{11}$$

In addition to the Dirichlet-to-Neumann operator, we will use the operators $B(h)$, $V(h)$ defined by:

$$B(h)\psi = \partial_y \varphi|_{y=h},$$
$$V(h)\psi = (\nabla_x \varphi)|_{y=h},$$

where, again, φ is the harmonic extension of ψ given by (9).

We recall the following identities.

Lemma 1 *We have:*

$$B(h)\psi = \frac{G(h)\psi + \nabla h \cdot \nabla\psi}{1 + |\nabla h|^2}, \qquad V(h)\psi = \nabla\psi - (B(h)\psi)\nabla h, \tag{12}$$

and

$$G(h)B(h)\psi = -\operatorname{div} V(h)\psi. \tag{13}$$

Proof By definition of the operator $G(h)$:

$$G(h)h = \big(\partial_y \varphi - \nabla h \cdot \nabla\varphi\big)|_{y=h} = B(h)\psi - \nabla h \cdot V(h)\psi. \tag{14}$$

On the other hand, it follows from the chain rule that:

$$\nabla\psi = \nabla_x(\varphi|_{y=h}) = (\nabla_x \varphi|_{y=h}) + (\partial_y \varphi)|_{y=h} \nabla h = V(h)\psi + (B(h)\psi)\nabla h.$$

Consequently, we obtain the wanted identity for $V(h)\psi$:

$$V(h)\psi = \nabla\psi - (B(h)\psi)\nabla h.$$

Now, by reporting this formula in (14), we get:

$$G(h)h = (1 + |\nabla h|^2)B(h)\psi - \nabla\psi \cdot \nabla h,$$

which immediately implies the desired result for $B(h)\psi$.

The identity (13) is proved in [1,4,25], see also Proposition 5.1 in [3]. □

2.2 A Reformulation

In this paragraph, we give more details about the formulation of the Hele–Shaw equation in terms of the Dirichlet-to-Neumann operator given in the introduction.

The Dirichlet-to-Neumann operator plays a key role in the study of the water-wave problem since the seminal works of Zakharov [30] and Craig and Sulem [17]. It enters also in a very natural way in the study of the Hele–Shaw equation. Recall from the introduction that:

$$v = -\nabla_{x,y}\phi \quad \text{with} \quad \phi = P + y.$$

Since $\operatorname{div}_{x,y} v = 0$ and since $P|_{y=h} = 0$, the potential ϕ satisfies:

$$\Delta_{x,y}\phi = 0, \qquad \phi|_{y=h} = h.$$

We conclude that ϕ is the harmonic extension of gh, which implies that:

$$\sqrt{1+|\nabla h|^2}\, v \cdot n = -G(h)h.$$

Consequently, the evolution equation for h simplifies to:

$$\partial_t h + G(h)h = 0. \tag{15}$$

Recall from (10) that $G(h)h$ is well defined whenever h takes values in $H^s(\mathbf{T}^n)$ for some $s > n/2 + 1$. The following result allows to solve the Cauchy problem in this general setting.

Theorem 3 (From [3,27]) *Let $n \geq 1$ and consider a real number $s > n/2+1$. For any initial data h_0 in $H^s(\mathbf{T}^n)$, there exists a time $T > 0$, such that the Cauchy problem:*

$$\partial_t h + G(h)h = 0, \quad h|_{t=0} = h_0, \tag{16}$$

has a unique solution satisfying:

$$h \in C^0([0,T]; H^s(\mathbf{T}^n)) \cap C^1([0,T]; H^{s-1}(\mathbf{T}^n)) \cap L^2([0,T]; H^{s+\frac{1}{2}}(\mathbf{T}^n)).$$

Moreover, h belongs to $C^\infty((0,T] \times \mathbf{T}^n)$.

Definition 1 We say that h is a regular solution to (16) if h satisfies the conclusions of the above result on some time interval $[0,T]$.

2.3 Equations for the Derivatives

As shown in [3], it is very convenient to work with some special derivatives of the solutions. Guided by the analysis in [1], we introduce the horizontal and vertical traces of the velocity at the free surface:

$$B=(\partial_y\phi)|_{y=h},\quad V=(\nabla_x\phi)|_{y=h}. \tag{17}$$

They are given in terms of h by the following formulas (see Lemma 1):

$$B=\frac{G(h)h+|\nabla h|^2}{1+|\nabla h|^2},\qquad V=(1-B)\nabla h. \tag{18}$$

We also introduce the Rayleigh–Taylor coefficient a defined by:

$$a=-(\partial_y P)|_{y=h}=1-B. \tag{19}$$

There are two important positivity results which follow from the maximum principle (or the Hopf–Zaremba's principle). The first one is the well-known positivity of the Taylor coefficient (see [3, Prop. 4.3]).

Proposition 2 *For any regular solution h to the Hele–Shaw equation, there holds:*

$$a=1-B>0.$$

The next results gives an evolution equation for B and contains a positivity results for a coefficient γ.

Proposition 3 (See Prop. 5.2 in [3]) *Assume that h is a regular solution to the Hele–Shaw equation. Then, B satisfies:*

$$\partial_t B-V\cdot\nabla B+aG(h)B=\gamma, \tag{20}$$

where

$$\gamma=\frac{1}{1+|\nabla h|^2}\Big(G(h)\big(B^2+|V|^2\big)-2BG(h)B-2V\cdot G(h)V\Big). \tag{21}$$

Moreover, the coefficient γ satisfies:

$$\gamma\leq 0. \tag{22}$$

2.4 Shape Derivatives

Notice that $G(h)\psi$ is linear in ψ, but depends nonlinearly on h. This is one of the main difficulties to study the Hele–Shaw equation. The same problem appears for the water-wave problem. One tool to study the dependence in h is to consider the shape derivative formula, as given by the following

Proposition 4 (From Lannes [25,26]) *Consider a real number s, such that $s>1+n/2$. Let $\psi\in H^s(\mathbf{T}^n)$ and $h\in H^s(\mathbf{T}^n)$. Then, there is a neighborhood $\mathcal{U}_h\subset H^s(\mathbf{T}^n)$ of h, such that the mapping:*

$$h\in\mathcal{U}_h\mapsto G(h)\psi\in H^{s-1}(\mathbf{T}^n)$$

is differentiable. Moreover, for all $\zeta \in H^s(\mathbf{T}^n)$, we have:

$$dG(h)\psi \cdot \zeta := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big\{G(h+\varepsilon\zeta)\psi - G(h)\psi\big\} = -G(h)(\mathfrak{B}\zeta) - \operatorname{div}(\mathfrak{V}\zeta), \quad (23)$$

where

$$\mathfrak{B} = \frac{G(h)\psi + \nabla h \cdot \nabla \psi}{1+|\nabla h|^2}, \quad \mathfrak{V} = \nabla\psi - \mathfrak{B}\nabla h.$$

This result is proved for smoother function by Lannes in [25]. We refer to his monograph [26] for the proof in the general case. However, in this paper, to justify the computations, we only need this result for smooth functions.

3 Elliptic Formulation

In this section, we prove Theorem 2.

Let us recall (see Sect. 2.1) that the operators $B(h)$ and $V(h)$ are given by:

$$B(h)\psi = \partial_y \phi|_{y=h}, \qquad V(h)\psi = (\nabla_x \phi)|_{y=h},$$

where ϕ is the harmonic extension of ψ given by (9). Denote by $B(h)^*$ the adjoint for the $L^2(\mathbf{T}^n)$-scalar product. In light of the identity (12), one has:

$$B(h)^*\psi = G(h)\left(\frac{\psi}{1+|\nabla h|^2}\right) - \operatorname{div}\left(\frac{\psi}{1+|\nabla h|^2}\nabla h\right). \quad (24)$$

We are now ready to prove Theorem 2 whose statement is recall here.

Theorem 4 *If h is a smooth solution to $\partial_t h + G(h)h = 0$, then:*

$$\Delta_{t,x} h + B(h)^*\big(\left|\nabla_{t,x} h\right|^2\big) = 0. \quad (25)$$

Proof The proof is in two steps. We begin by differentiating in time the Hele–Shaw equation:

$$\partial_t h + G(h)h = 0.$$

It follows from the shape derivative formula (23) that:

$$\partial_t^2 h = -\partial_t G(h)h = -G(h)\big((1-B)\partial_t h\big) + \operatorname{div}\big((\partial_t h)V\big),$$

where

$$B = B(h)h = \frac{G(h)h + \nabla h \cdot \nabla h}{1+|\nabla h|^2}, \qquad V = V(h)h = (1-B)\nabla h. \quad (26)$$

We next compute $(1-B)\partial_t h$. To do so, we replace $\partial_t h$ by $-G(h)h$ and observe that, by definition of the operator $G(h)$:

$$G(h)h = \big(\partial_y \phi - \nabla h \cdot \nabla \phi\big)|_{y=h} = B - V \cdot \nabla h.$$

Recalling that $V = (1-B)\nabla h$, it follows that:

$$\begin{aligned}(1-B)\partial_t h &= -(1-B)G(h)h \\ &= -(1-B)(B - V \cdot \nabla h) \\ &= -B + B^2 + V \cdot \big((1-B)\nabla h\big) \\ &= -B + B^2 + |V|^2 .\end{aligned}$$

The previous results yield:

$$\partial_t^2 h - G(h)B + G(h)\big(B^2 + |V|^2\big) + \operatorname{div}\big((B - V \cdot \nabla h)V\big) = 0.$$

We then use the identity $G(h)B = -\operatorname{div} V$ (see 13) to infer that

$$\partial_t^2 h + \operatorname{div} V + G(h)\big(B^2 + |V|^2\big) + \operatorname{div}\big((B - V \cdot \nabla h)V\big) = 0.$$

Therefore, replacing V by $\nabla h - B\nabla h$ in $\operatorname{div} V$, we have:

$$\partial_t^2 h + \Delta_x h - \operatorname{div}(B\nabla h) + G(h)\big(B^2 + |V|^2\big) + \operatorname{div}\big((B - V \cdot \nabla h)V\big) = 0. \tag{27}$$

To simplify this expression, we begin by observing that:

$$-\operatorname{div}(B\nabla h) + \operatorname{div}\big((B - V \cdot \nabla h)V\big) = \operatorname{div}\Big(B(V - \nabla h) - (V \cdot \nabla h)V\Big). \tag{28}$$

Now:

$$V - \nabla h = -B\nabla h,$$

so the first term in the right-hand side of (28) can be written as:

$$\operatorname{div}(B(V - \nabla h)) = -\operatorname{div}(B^2 \nabla h).$$

Moving to the second term in the right-hand side of (28), using again $V = (1-B)\nabla h$, we verify that:

$$\begin{aligned}(V \cdot \nabla h)V &= ((1-B)\nabla h \cdot \nabla h)(1-B)\nabla h \\ &= (1-B)^2 |\nabla h|^2 \nabla h \\ &= |V|^2 \nabla h.\end{aligned}$$

Consequently, the identity (27) simplifies to:

$$\Delta_{t,x} h + G(h)(B^2 + |V|^2) - \operatorname{div}\big((B^2 + |V|^2)\nabla h\big) = 0. \tag{29}$$

The next step consists in expressing $B^2 + |V|^2$ in terms of $\nabla_{t,x} h$. To do so, using the identities in (26), we verify that:

$$\begin{aligned} B^2 + |V|^2 &= B^2 + (1 - B)^2 |\nabla h|^2 \\ &= \left(\frac{G(h)h + |\nabla h|^2}{1 + |\nabla h|^2}\right)^2 + \left(\frac{1 - G(h)h}{1 + |\nabla h|^2}\right)^2 |\nabla h|^2 \\ &= \frac{(G(h)h)^2 + |\nabla h|^2}{1 + |\nabla h|^2}. \end{aligned}$$

Since $\partial_t h = -G(h)h$, we conclude that:

$$B^2 + |V|^2 = \frac{(\partial_t h)^2 + |\nabla h|^2}{1 + |\nabla h|^2} = \frac{\left|\nabla_{t,x} h\right|^2}{1 + |\nabla h|^2}. \tag{30}$$

Therefore, the wanted identity (25) follows from (29), (30) and the definition (24) of $B(h)^*$. □

4 Lyapunov Functionals

In this section, we prove Theorem 1.

Lemma 2 *Consider a smooth solution h to the Hele–Shaw equation. If $\Phi\colon \mathbf{R} \to \mathbf{R}$ is a C^2 convex function, then:*

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\Phi \le 0 \quad \textit{where} \quad I_\Phi(t) = \int_{\mathbf{T}^n} \Phi(h(t,x))\,\mathrm{d}x. \tag{31}$$

Proof We follow the analysis in [3]. In [11,12], Córdoba and Córdoba proved that, for any exponent α in $[0,1]$ and any C^2 function f decaying sufficiently fast at infinity, one has the pointwise inequality:

$$2f(-\Delta)^\alpha f \ge (-\Delta)^\alpha (f^2).$$

This inequality has been generalized and applied to many different problems (see [8–10,22] and the numerous references there in). Recently, Córdoba and Martínez [14] proved that:

$$\Phi'(f)G(h)f \ge G(h)\big(\Phi(f)\big), \tag{32}$$

when h is a C^2 function and $\Phi(f) = f^{2m}$ for some positive integer m. In [3], this result is generalized to the case where $\Phi\colon \mathbf{R} \to \mathbf{R}$ is an arbitrary C^2 convex function and f, h belong to some Hölder space $C^{1,\alpha}(\mathbf{T}^n)$ with $\alpha > 0$. Using the latter result, we immediately obtain (31). Indeed, by multiplying the Hele–Shaw equation by $\Phi'(h)$ and integrating over $\mathbf{T}^n$, we get that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \Phi(h)\,\mathrm{d}x + \int \Phi'(h)G(h)h\,\mathrm{d}x = 0.$$

Now, we use the fact that $\int G(h)\psi\,\mathrm{d}x = 0$ for any function ψ to deduce from (32) that:

$$\int \Phi'(h)G(h)h\,\mathrm{d}x \geq \int G(h)\Phi(h)\,\mathrm{d}x = 0. \tag{33}$$

This completes the proof of (6). □

We now prove the main result.

Lemma 3 *Consider a smooth solution h to the Hele–Shaw equation. If $\Phi : \mathbf{R} \to \mathbf{R}$ is a C^3 convex function whose derivative Φ' is also convex, then:*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_\Phi \geq 0. \tag{34}$$

Proof We have seen in the previous proof that:

$$\frac{\mathrm{d}}{\mathrm{d}t} I_\Phi + \int \Phi'(h)G(h)h\,\mathrm{d}x = 0.$$

Therefore, it is sufficient to show that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \Phi'(h)G(h)h\,\mathrm{d}x \leq 0. \tag{35}$$

Notice that the latter result is interesting in itself, since it asserts that:

$$\int \Phi'(h)G(h)h\,\mathrm{d}x$$

is a Lyapunov functional (this is, indeed, a coercive quantity, see 33).

To prove (35), we use the elliptic formulation of the Hele–Shaw equation given by Theorem 2. Recall that:

$$-\Delta_{t,x}h - B(h)^*\big(\left|\nabla_{t,x}h\right|^2\big) = 0.$$

We multiply this equation by $\Phi'(h)$ and integrate first in space. This gives that:

$$-\int \Phi'(h)\partial_t^2 h\,\mathrm{d}x + \int \Phi''(h)\left|\nabla_x h\right|^2\mathrm{d}x - \int \big(B(h)\Phi'(h)\big)\left|\nabla_{t,x}h\right|^2\mathrm{d}x = 0.$$

It follows from the identity (12) for the operator $B(h)$ that:

$$B(h)\Phi'(h)=\frac{G(h)\Phi'(h)+\nabla h\cdot\nabla\Phi'(h)}{1+|\nabla h|^2}.$$

Since $\Phi'(h)$ is convex, the inequality (32) implies that:

$$G(h)\Phi'(h)\le\Phi''(h)G(h)h.$$

It follows that:

$$B(h)\Phi'(h)\le\Phi''(h)\frac{G(h)h+|\nabla h|^2}{1+|\nabla h|^2}=\Phi''(h)B\quad\text{where } B=\frac{G(h)h+|\nabla h|^2}{1+|\nabla h|^2}.$$

Consequently:

$$-\int\Phi'(h)\partial_t^2h\,\mathrm{d}x+\int\Phi''(h)\,|\nabla_x h|^2\,\mathrm{d}x-\int\Phi''(h)B\left|\nabla_{t,x}h\right|^2\mathrm{d}x\le 0.\tag{36}$$

Now, consider a time $T>0$ and integrate by parts in time on $[0,T]$ to obtain:

$$\int_0^T\int\Phi'(h)\partial_t^2h\,\mathrm{d}x\,\mathrm{d}t=\int\Phi'(h)\partial_t h\,\mathrm{d}x\bigg|_{t=0}^{t=T}-\int_0^T\int\Phi''(h)(\partial_t h)^2\,\mathrm{d}x\,\mathrm{d}t.$$

By combining this with (36), we find that:

$$-\int\Phi'(h)\partial_t h\,\mathrm{d}x\bigg|_{t=0}^{t=T}+\int_0^T\int\Phi''(h)(1-B)\left|\nabla_{t,x}h\right|^2\mathrm{d}x\,\mathrm{d}t\le 0.$$

Remembering that $a=1-B$ and $\partial_t h=-G(h)h$, the preceding inequality implies that:

$$\begin{aligned}&\int\Phi'(h)G(h)h\,\mathrm{d}x\bigg|_{t=T}+\int_0^T\int\Phi''(h)a\left|\nabla_{t,x}h\right|^2\mathrm{d}x\,\mathrm{d}t\\&\quad\le\int\Phi'(h)G(h)h\,\mathrm{d}x\bigg|_{t=0}.\end{aligned}$$

We now use the fact that the Taylor coefficient a is positive (see Proposition 2) and the fact that Φ is convex to deduce that $a\Phi''(h)\ge 0$. This concludes the proof of (35) and hence the proof of the lemma. □

5 Convexity and Entropy

Here, we prove Proposition 1 and its corollary. Recall the notation:

$$L(h)f = -V\cdot\nabla f - \frac{1}{2}(\operatorname{div} V)f + \sqrt{a}\,G(h)\big(\sqrt{a}f\big).$$

Recall also that $a(t,x)>0$ for all t,x, so that one may consider $\sqrt{a}$ and $\log(a)$.

Proposition 5 *For any positive constant $m>0$, the function:*

$$u := \frac{\log(ma)}{\sqrt{a}},$$

satisfies:

$$\partial_t u + L(h)u - \frac{\gamma}{2a}u \geq 0. \tag{37}$$

Remark 3 (i) With $m=1$, we have:

$$u=-2v \quad \text{where} \quad v=\frac{1}{\sqrt{a}}\log\left(\frac{1}{\sqrt{a}}\right).$$

Therefore, the preceding proposition implies the result of Proposition 1 with:

$$c=-\frac{\gamma}{2a}.$$

Since $\gamma\leq 0$, the latter function is non-negative.

(ii) Notice that the right-hand side in (37) does not depend on m.

(iii) We use the parameter m below to control $\inf_x a(t,x)$.

Proof Assume that h is a regular solution to the Hele–Shaw equation. As recalled in Proposition 3, the function B satisfies:

$$\partial_t B - V\cdot\nabla B + aG(h)B = \gamma,$$

where $\gamma\leq 0$ is given by:

$$\gamma = \frac{1}{1+|\nabla h|^2}\Big(G(h)\big(B^2+|V|^2\big) - 2BG(h)B - 2V\cdot G(h)V\Big).$$

Since $a=1-B$, using the fact that $G(h)1=0$, we deduce that:

$$\partial_t a - V\cdot\nabla a + aG(h)a + \gamma = 0, \tag{38}$$

together with

$$\gamma = \frac{1}{1+|\nabla h|^2}\Big(G(h)\big(a^2+|V|^2\big) - 2aG(h)a - 2V\cdot G(h)V\Big).$$

Since a is a positive function, we may multiply the equation (38) by $1/a$, to obtain at once:

$$(\partial_t - V \cdot \nabla) \log a + G(h)a + \frac{\gamma}{a} = 0. \tag{39}$$

We now claim that:

$$G(h)a \leq a\, G(h) \log a. \tag{40}$$

To do so, we use the fact that log is a concave function and the fact that a is bounded from below by a positive constant $c_0 > 0$ on $[0, T] \times \mathbf{T}^n$. This allows us to consider a smooth concave function $\theta \colon \mathbf{R} \to \mathbf{R}$ which coincides with log on $[c_0/2, +\infty)$. As a result, the inequality (32) implies that:

$$G(h) \log(a) = G(h)\theta(a) \geq \theta'(a) G(h)a = \frac{1}{a} G(h)a,$$

which in turn implies (40). We next apply (40) to deduce from (39) that:

$$(\partial_t - V \cdot \nabla) \log a + a\, G(h) \log a + \frac{\gamma}{a} \geq 0.$$

Since $G(h)C$ vanishes for any constant C, the preceding inequality implies that, for any positive constant $m > 0$:

$$(\partial_t - V \cdot \nabla) \log(ma) + a\, G(h) \log(ma) + \frac{\gamma}{a} \geq 0. \tag{41}$$

Now, we observe that:

$$\begin{aligned}
(\partial_t - V \cdot \nabla) \frac{1}{\sqrt{a}} &= -\frac{1}{2} \frac{(\partial_t - V \cdot \nabla)a}{a\sqrt{a}} \\
&= \frac{1}{2} \frac{a\, G(h)a + \gamma}{a\sqrt{a}} \qquad \text{(see (18))} \\
&= \frac{1}{2} \frac{a \operatorname{div} V + \gamma}{a\sqrt{a}},
\end{aligned}$$

where we used the identity $G(h)a = -G(h)B = \operatorname{div} V$ (see 13) in the last line. Consequently:

$$\begin{aligned}
&(\partial_t - V \cdot \nabla) \frac{\log(ma)}{\sqrt{a}} \\
&= \frac{1}{\sqrt{a}} (\partial_t - V \cdot \nabla) \log(ma) + \log(ma) (\partial_t - V \cdot \nabla) \frac{1}{\sqrt{a}} \\
&\geq \frac{1}{\sqrt{a}} \Big(- a\, G(h) \log(ma) - \frac{\gamma}{a} \Big) + \frac{1}{2} \frac{a \operatorname{div} V + \gamma}{a\sqrt{a}} \log(ma).
\end{aligned}$$

Then, one easily verifies that $u = \log(ma)/\sqrt{a}$ satisfies:

$$\partial_t u + L(h)u - \frac{1}{2}\frac{\gamma}{a}u \geq -\frac{\gamma}{a\sqrt{a}}.$$

Since $\gamma \leq 0$, this implies the wanted inequality (37). □

We now prove Corollary 1 whose statement is recalled here.

Corollary 2 *Let $n \geq 1$ and consider a regular solution h to the Hele–Shaw equation defined on $[0,T]$. Then, for all time t in $[0,T]$:*

$$\inf_{x\in\mathbf{T}^n} a(t,x) \geq \inf_{x\in\mathbf{T}^n} a(0,x). \tag{42}$$

Proof This result can be proved by exploiting only the fact that $\gamma \leq 0$. Here, we just want to explain how to recover this from the previous proposition.

Set

$$c_0 = \inf_{x\in\mathbf{T}^n} a(0,x), \qquad m = \frac{1}{c_0}.$$

Then $ma(0,x) \geq 1$ for all $x \in \mathbf{T}^n$. Set

$$u = \frac{\log(ma)}{\sqrt{a}}, \quad u_- = \min\{u, 0\}.$$

We claim that $u_- = 0$. This will at once imply that $\log(ma) \geq 0$ so $ma(t,x) \geq 1$ for all $(t,x) \in [0,T] \times \mathbf{T}^n$, which in turn implies $a(t,\cdot) \geq 1/m = c_0$, which is the asserted inequality (42).

To prove that $u_- = 0$, we use Stampacchia's method. By multiplying Eq. (37) by $u_- \leq 0$, one obtains:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int u_-^2 \,\mathrm{d}x + \int u_- L(h)u \,\mathrm{d}x - \frac{1}{2}\int \frac{\gamma}{a}u_-^2 \,\mathrm{d}x \leq 0.$$

Now, using that $\gamma \leq 0$ and $a > 0$, we have:

$$\int \frac{\gamma}{a}u_-^2 \,\mathrm{d}x \leq 0,$$

so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int u_-^2 \,\mathrm{d}x + \int u_- L(h)u \,\mathrm{d}x \leq 0. \tag{43}$$

On the other hand, proceeding as above, the convexity inequality (32) applied with the function $x \mapsto x^2 \mathbf{1}_{\mathbf{R}_-}(x)$ implies that:

$$\int u_- L(h)u \,\mathrm{d}x = \int \sqrt{a}u_- G(h)(\sqrt{a}u) \,\mathrm{d}x \geq \int G(h)\left(\frac{1}{2}au_-^2\right) \mathrm{d}x = 0.$$

As a result, the preceding inequality (43) simplifies to:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int u_-^2 \,\mathrm{d}x \leq 0.$$

Since $u_-(0,\cdot) = 0$ at initial time (by construction), we obtain $u_-(t,\cdot) = 0$ for all time t, which terminates the proof. □

Compliance with Ethical Standards

Conflict of interest The author declares that he has no conflict of interest.

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