Free surface flows in fluid dynamics

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Description of the course

A classical subject in the mathematical theory of hydrodynamics consists in studying the evolution of the free surface separating the air from a perfect incompressible fluid. We will examine this question for two important sets of equations: the water-wave equations and the Hele-Shaw equations, including the Muskat problem. They are of a different nature, dispersive or parabolic, but we will see that they can be studied by related tools.

This course is intended for graduate students with a general interest in analysis and no prerequisites for advanced theory are required. A second and major part of the courses will consist of short self-contained introductions to the following topics: paradifferential calculus, study of the fractional Laplacian and Morawetz and Lions multiplier methods. I will give a detailed analysis of the Cauchy problem for the water-wave, Hele-Shaw and Muskat equations, and discuss some qualitative properties of their solutions.

Grading: 2-3 homeworks.

No required text. Lecture notes will be posted at the url

`http://talazard.perso.math.cnrs.fr/CL.pdf`

These lecture notes will be included in a book in preparation with Quoc-Hung Nguyen. It is based on several recent results with him as well as with Pietro Baldi, Didier Bresch, Nicolas Burq, Jean-Marc Delort, Daniel Han-Kwan, Mihaela Ifrim, Omar Lazar, Guy Métivier, Nicolas Meunier, Didier Smets, Daniel Tataru and Claude Zuily.

Part I

The equations

Chapter 1

The water-wave equations

The purpose of this chapter is to introduce the free surface Euler equation. We will begin by briefly explaining the meaning of these equations. I will also introduce the Dirichlet-to-Neumann operator and state some results about the Cauchy theory.

1.1 The equations

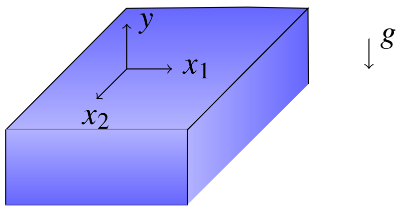
> *A heavy fluid mass, originally at rest, and of indefinite depth, has been set in motion by the effect of a given cause. One asks, after a given time, the shape of the external surface of the fluid and the velocity of each of the molecules located on the same surface.*

Une masse fluide pesante, primitivement en repos, et d'une profondeur indéfinie, a été mise en mouvement par l'effet d'une cause donnée. On demande, au bout d'un temps déterminé, la forme de la surface extérieure du fluide et la vitesse de chacune des molécules situées à cette même surface.

Figure 1.1: Question by the French Academy of Sciences in 1813.

Let us consider an ocean of infinite depth primarily at rest. For the sake of simplicity, let us suppose that this ocean is unbounded laterally as well as in depth. To visualize this, let us assimilate the space domain to $\mathbb{R}^3$, which is assumed to be divided into two distinct parts: the upper half-space occupied by air and the lower half-space occupied by water. Let us imagine that the wind is blowing over this ocean. Under certain conditions, this wind will generate waves, called wind waves. It is observed that they can propagate over immense distances until they reach the shore. We are going to be interested in the propagation of these waves far from the wind zone and far from the shore. We then speak of swell or gravity waves, because the waves propagate thanks to the restoring force of gravity.


To describe a wave, we need to introduce some notations. In the following, the fluid will always be referred to the rectangular coordinates of x_1, x_2, y, the plane (Ox_1x_2) being horizontal, and coinciding with the surface of the fluid when in equilibrium, the axis (Oy) being directed upwards. We write $x = (x_1, x_2)$ and assume that the gravity is oriented along the axis (Oy).



We introduce also the elevation of the surface of the sea, denoted by η and the velocity field of the fluid, denoted by u. The unknown η depends on the time t and the spatial variable $x = (x_1, x_2) \in \mathbb{R}^2$. At a given time t, the graph of the function η, denoted by $\Sigma(t)$, is what we call the free surface. The domain occupied by water at

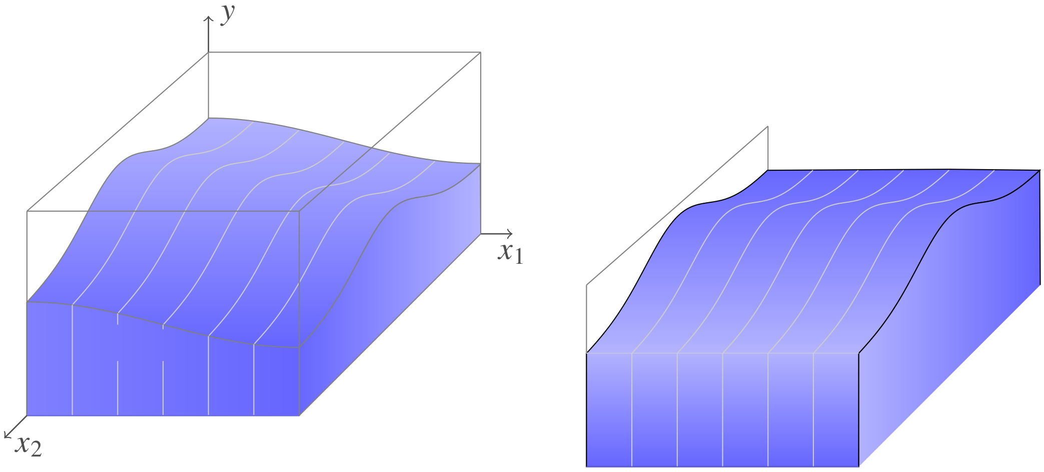


Figure 1.2: 3D and 2D waves

time t is the half-space, denoted by $\Omega(t)$, located underneath the free surface[1]:

$$\Sigma(t) = \left\{ (x,y) \in \mathbb{R}^2 \times \mathbb{R} \, : \, y = \eta(t,x) \right\},$$
$$\Omega(t) = \left\{ (x,y) \in \mathbb{R}^2 \times \mathbb{R} \, : \, y < \eta(t,x) \right\}.$$

In the following we will use the following notations.

Notation 1.1.1. *1. Given a function $f = f(x,y)$ and a function $h = h(x)$, we use $f|_{y=h}$ as a short notation for the function $x \mapsto f(x,h(x))$.*

2. We set

$$\nabla = (\partial_{x_1}, \partial_{x_2}), \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2, \quad \Delta_{x,y} = \Delta + \partial_y^2.$$

Let us insist on the fact that the operators ∇ and Δ contain only derivatives with respect to the horizontal variable x.

3. Given a function $f = f(t,x,y)$ and a time t, we have

$$\iint_{\Omega(t)} f(t,x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}^2} \left(\int_{-\infty}^{\eta(t,x)} f(t,x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

[1] For the sake of simplicity, we assumed that the free surface $\Sigma(t)$ is a graph, the domain $\Omega(t)$ is bottomless and that Ω has no lateral boundary ($x \in \mathbb{R}^2$). We refer to the review paper by Lannes [299] for the general case.

1.1.1 A word of caution

In this chapter we perform only formal computations and consider smooth enough functions. This is intended to mean that the functions are sufficiently regular, and decay sufficiently fast at spatial infinity, so that all the computations are justified.

In this direction, introduce the space

$$H^\infty(\mathbb{R}^d) = \left\{u \in C^\infty(\mathbb{R}^d)\,;\, \partial_x^\alpha u \in L^2(\mathbb{R}^d) \;\text{ for any } \alpha \in \mathbb{N}^d\right\}.$$

1.1.2 The incompressible Euler equations

The fluid we will study will be assumed to be subjected to the force of gravity and/or surface tension. Moreover, we assume that the eulerian velocity field $u = (u_1, u_2, u_3)\colon \Omega \to \mathbb{R}^3$ is solution to the incompressible Euler equations of fluid mechanics:

$$(1.1.1)\qquad \begin{cases} \dfrac{\partial u_1}{\partial t} + u_1\dfrac{\partial u_1}{\partial x_1} + u_2\dfrac{\partial u_1}{\partial x_2} + u_3\dfrac{\partial u_1}{\partial y} = -\dfrac{\partial P}{\partial x_1},\\ \dfrac{\partial u_2}{\partial t} + u_1\dfrac{\partial u_2}{\partial x_1} + u_2\dfrac{\partial u_2}{\partial x_2} + u_3\dfrac{\partial u_2}{\partial y} = -\dfrac{\partial P}{\partial x_2},\\ \dfrac{\partial u_3}{\partial t} + u_1\dfrac{\partial u_3}{\partial x_1} + u_2\dfrac{\partial u_3}{\partial x_2} + u_3\dfrac{\partial u_3}{\partial y} = -g - \dfrac{\partial P}{\partial y},\\ \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial y} = 0, \end{cases}$$

where $g > 0$ is the acceleration of gravity and $P\colon \Omega \to \mathbb{R}$ is the pressure. This is better formulated under the form

$$(1.1.2)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla_{x,y})u + \nabla_{x,y}(P + gy) = 0,\\ \operatorname{div}_{x,y} u = 0, \end{cases}$$

where $\cdot$ is the scalar product in $\mathbb{R}^3$, so that $u \cdot \nabla_{x,y} = u_1\partial_{x_1} + u_2\partial_{x_2} + u_3\partial_y$.

1.1.3 Boundary conditions on the free surface

We need two boundary conditions on the free surface. The first one corresponds to a balance of forces: it states that the pressure jump across the surface which separates the fluid from the air is proportional to the mean curvature of the interface. If in addition we assume that the air pressure above the fluid is constant (and then this constant can be chosen to be 0 without loss of generality) this results in the condition

$$P|_{y=\eta} = \lambda\kappa(\eta) \quad ; \quad \kappa(\eta) = -\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right), \tag{1.1.3}$$

where $\lambda \in [0,1]$ is a coefficient that measures the importance of surface tension.

The second boundary condition has to do with the normal velocity of the free surface. To introduce it, we need to introduce the outward pointing unit normal n to Ω. At a point $X = (x, \eta(t,x))$ of $\Sigma(t)$, it is given by

$$n(X) = \frac{1}{\sqrt{1+|\nabla\eta(t,x)|^2}}\begin{pmatrix}-\nabla\eta(t,x)\\ 1\end{pmatrix}.$$

The dynamics of the free surface $\Sigma(t) = \{y = \eta(t,x)\}$ is coupled to that of the fluid through the following kinematic boundary condition:

$$\frac{\partial\eta}{\partial t} = \sqrt{1+|\nabla\eta|^2}\,U\cdot n \qquad \text{where} \quad U = u|_{y=\eta}. \tag{1.1.4}$$

This simplifies to

$$\frac{\partial\eta}{\partial t} + U_1\partial_{x_1}\eta + U_2\partial_{x_2}\eta = U_3.$$

Let us make two elementary observations about this equation.

Definition 1.1.2. *A fluid particle is a curve* $\mathbb{R} \ni t \mapsto m(t) \in \mathbb{R}^3$ *solution to* $\dot{m}(t) = \frac{\mathrm{d}}{\mathrm{d}t}m(t) = u(t, m(t))$.

Proposition 1.1.3. *Any fluid particle which is on the free surface of the fluid at the initial time will remain on the free surface for any further time.*

Proof. Consider a fluid particle $m(t) = (x_1(t), x_2(t), y(t))$ and introduce the function $\theta(t) = y(t) - \eta(t, x(t))$. Then θ vanishes at $t = 0$ by assumption. Moreover, using (1.1.4),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\theta &= \dot{y}(t) - (\partial_t\eta)(t, m(t)) - \dot{x}_1(t)(\partial_{x_1}\eta)(t, m(t)) - \dot{x}_2(t)(\partial_{x_2}\eta)(t, m(t))\\ &= \left(U_3 - U_1\partial_{x_1}\eta - U_2\partial_{x_2}\eta - \partial_t\eta\right)\big|_{(x,y)=m(t)} = 0,\end{aligned}$$

and the result follows. □

The following lemma allows in many situations to compute the evolution of global quantities.

Lemma 1.1.4. *Consider two regular functions*[2] (η, u) *satisfying* (1.1.4)*. Then for any regular function* $f = f(t, x, y)$*, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega(t)} f(t, x, y) \,\mathrm{d}y \,\mathrm{d}x = \iint_{\Omega(t)} (\partial_t + u \cdot \nabla_{x,y}) f \,\mathrm{d}y \,\mathrm{d}x.$$

Proof. Write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega(t)} f(t, x, y) \,\mathrm{d}y \,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \left(\int_{-\infty}^{\eta(t,x)} f(t, x, y) \,\mathrm{d}y \right) \mathrm{d}x \\
&= \iint_{\Omega(t)} \partial_t f(t, x, y) \,\mathrm{d}y \,\mathrm{d}x + \int_{\mathbb{R}^2} (\partial_t \eta) f(t, x, \eta) \,\mathrm{d}x
\end{aligned}$$

and then use (1.1.4) and Stokes' theorem[3] to verify that

$$\begin{aligned}
\int_{\mathbb{R}^2} (\partial_t \eta) f(t, x, \eta) \,\mathrm{d}x &= \int_{\mathbb{R}^2} (U \cdot n) f(t, x, \eta) \sqrt{1 + |\nabla \eta|^2} \,\mathrm{d}x \\
&= \int_{\partial\Omega(t)} n \cdot (f u) \,\mathrm{d}\sigma = \iint_{\Omega(t)} \mathrm{div}_{x,y}(f u) \,\mathrm{d}y \,\mathrm{d}x,
\end{aligned}$$

where $\mathrm{d}\sigma = \sqrt{1 + |\nabla \eta|^2} \,\mathrm{d}x$. Since $\mathrm{div}_{x,y}\, u = 0$, this implies that

$$\int_{\mathbb{R}^2} (\partial_t \eta) f(t, x, \eta) \,\mathrm{d}x = \iint_{\Omega(t)} u \cdot \nabla_{x,y} f \,\mathrm{d}y \,\mathrm{d}x,$$

which concludes the proof. □

[2] See the warning in §1.1.1.

[3] Let us recall a fomulation of Stokes' theorem.

Definition 1.1.5. *We say that a domain* $D \subset \mathbb{R}^d$ *is* C^1 *if for each point* m *belonging to the boundary* ∂D *there is a cartesian coordinate system, a radius* $r > 0$ *and a* C^1 *function* $\zeta \colon \mathbb{R}^d \to \mathbb{R}$ *with compact support such that*

$$D \cap B(m, r) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}\,;\, \zeta(x) < y\} \cap B(m, r).$$

Theorem 1.1.6. *Suppose that* D *is a bounded* C^1 *domain and consider a function* $u \in C^1(\overline{D})$*. Then*

$$\int_D \mathrm{div}\, u \,\mathrm{d}x = \int_{\partial D} n \cdot u \,\mathrm{d}\sigma,$$

where n *is the outward pointing unit normal.*

We can apply this result in the unbounded domain $\Omega(t)$ provided that the functions at stake decay sufficiently at spatial infinity to justify a standard truncation argument.

1.1.4 Conserved quantities

Consider a solution (η, u) of the free surface Euler equations. At a given time t, we define its kinetic energy $E_k(t)$ and its potential energy $E_p(t)$ by

$$
\begin{aligned}
E_k(t) &= \frac{1}{2}\iint_{\Omega(t)} |u(t,x,y)|^2 \,\mathrm{d}y\,\mathrm{d}x \qquad (|\cdot| \text{ denotes the Euclidean norm on } \mathbb{R}^3)\\
E_p(t) &= \frac{g}{2}\int_{\mathbb{R}^2} \eta(t,x)^2 \,\mathrm{d}x + \lambda \int_{\mathbb{R}^d} \Big(\sqrt{1+|\nabla\eta|^2} - 1\Big)\,\mathrm{d}x.
\end{aligned}
$$

The total energy is then by definition the function $E = E_k + E_p$.

Proposition 1.1.7. *For any regular solution, there holds* $\dfrac{\mathrm{d}}{\mathrm{d}t}E = 0$.

Proof. It follows from Lemma 1.1.4 that

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_k = \iint_{\Omega(t)} \frac{1}{2}(\partial_t + u\cdot\nabla_{x,y})\,|u|^2\,\mathrm{d}y\,\mathrm{d}x.
$$

Now observe that

$$
\begin{aligned}
\frac{1}{2}(\partial_t + u\cdot\nabla_{x,y})\,|u|^2 &= u\cdot(\partial_t u + (u\cdot\nabla_{x,y})u) &&\text{(Leibniz rule)}\\
&= -u\cdot\nabla_{x,y}(P+gy) &&\text{(by (1.1.2))}\\
&= -\operatorname{div}_{x,y}((P+gy)u) &&(\text{since } \operatorname{div}_{x,y} u = 0).
\end{aligned}
$$

Consequently, Stokes' theorem implies that

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_k = -\iint_{\Omega(t)} \operatorname{div}_{x,y}((P+gy)u)\,\mathrm{d}y\,\mathrm{d}x = -\int_{\partial\Omega(t)} (P+gy)u\cdot n\,\mathrm{d}\sigma.
$$

Now, by assumption on the pressure (see (1.1.3)), on the free surface $\partial\Omega(t)$ we have $P + gy = g\eta + \lambda\kappa$. Also the kinematic boundary condition (1.1.4) implies that $u\cdot n\,\mathrm{d}\sigma = \partial_t\eta\,\mathrm{d}x$ (since $\mathrm{d}\sigma = \sqrt{1+|\nabla\eta|^2}\,\mathrm{d}x$). It follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_k = -\int_{\mathbb{R}^2} (g\eta + \lambda\kappa(\eta))\partial_t\eta\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}E_p.
$$

The identity is proven. □

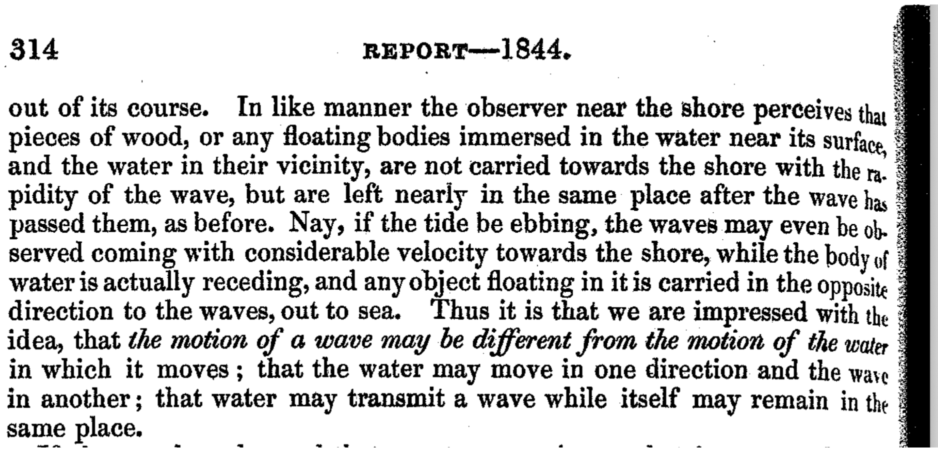
Exercise 1.1.8. *Write* $u = (u_1, u_2, u_3)$, *and introduce for* $j \in \{1,2\}$ *the momentum*

$$
\mathcal{M}_j = \iint_{\Omega(t)} u_j\,\mathrm{d}y\,\mathrm{d}x
$$

Prove that they are also conserved quantity (by formal computations).

1.1.5 Wave propagation

Another observation is in order. In the introduction of an article very famous [379] where he describes his discovery of the solitary wave, J. Scott Russell explains why it is fundamental to understand that this is a wave propagation problem. One of his arguments is



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out of its course. In like manner the observer near the shore perceives that pieces of wood, or any floating bodies immersed in the water near its surface, and the water in their vicinity, are not carried towards the shore with the rapidity of the wave, but are left nearly in the same place after the wave has passed them, as before. Nay, if the tide be ebbing, the waves may even be observed coming with considerable velocity towards the shore, while the body of water is actually receding, and any object floating in it is carried in the opposite direction to the waves, out to sea. Thus it is that we are impressed with the idea, that *the motion of a wave may be different from the motion of the water* in which it moves; that the water may move in one direction and the wave in another; that water may transmit a wave while itself may remain in the same place.

Figure 1.3: From Russell [379]

In other words: we can see that there is energy transfer without mass transfer, which is a circumstance of all wave phenomena.

Let us illustrate Russell's argument with a figure which represents the movement of a floating object (red dot) at successive times at the passage of a wave on the surface of the water moving to the right (naively represented by the graph of the function $y = \cos(x - t)$):

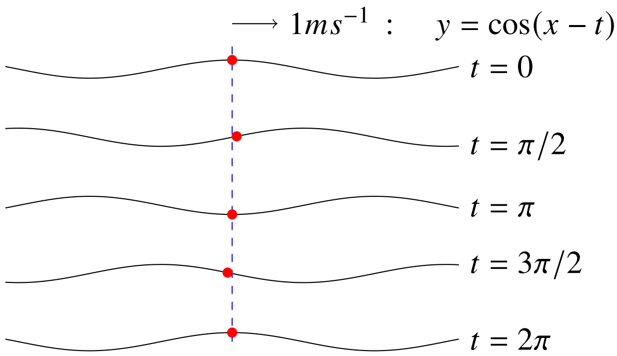


Figure 1.4: Russell's argument

1.2 Zakharov equations

1.2.1 Velocity potential

ABSTRACT: We study the stability of steady nonlinear waves on the surface of an infinitely deep fluid [1, 2]. In section 1, the equations of hydrodynamics for an ideal fluid with a free surface are transformed to canonical variables: the shape of the surface $\eta(\mathbf{r}, t)$ and the hydrodynamic potential $\Psi(\mathbf{r}, t)$ at the surface are expressed in terms of these variables. By introducing canonical variables, we can consider the problem of the stability of surface waves as part of the more general problem of nonlinear waves in media with dispersion [3, 4]. The results of the rest of the paper are also easily applicable to the general case.

Figure 1.5: From Zakharov' first paper [453]

In the sequel, we will always assume that the fluid was initially irrotational and furthermore has been set in motion by the action of conservative forces. This implies that the flow will remain irrotational for all time, that is

$$\mathrm{curl}_{x,y}\, u = \nabla_{x,y} \wedge u = 0.$$

Since the domain is simply connected and that the fluid is incompressible, there is a function ϕ defined on Ω with real values such that

$$u = \nabla_{x,y}\phi \quad ; \quad \Delta_{x,y}\phi = 0 \quad \text{in} \quad \Omega. \tag{1.2.1}$$

The function ϕ is called the velocity potential. The equation $\Delta_{x,y}\phi = 0$ means that ϕ is an harmonic function, which is a crucial property.

In view of (1.1.2), we see that up to translating ϕ by a factor that depends only on time, ϕ satisfies the following equation in Ω,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0. \tag{1.2.2}$$

This equation is called the Bernoulli's equation. When $u = \nabla_{x,y}\phi$, the kinematic boundary condition (1.1.4) can be written under the form

$$\frac{\partial \eta}{\partial t} = \left[\frac{\partial \phi}{\partial y} - \nabla\eta \cdot \nabla\phi\right]\Big|_{y=\eta}.$$

1.2.2 The Dirichlet-to-Neumann operator

Rather than studying the system in (η, ϕ), Zakharov ([453, 454]) suggests working with the unknown (η, ψ) where

$$\psi(t,x) = \phi(t,x,\eta(t,x))$$

is the trace of ϕ at the free surface Σ. The observation is that one can then subsequently deduct the properties of ϕ and u given that the function ϕ is harmonic in Ω (cf. (1.2.1)). An interest of this notation is that now the problem only depends on two unknowns which are functions of time t and $x \in \mathbb{R}^2$.

To state the equations that govern the propagation of (η, ψ), we need to introduce the Dirichlet–Neumann operator. This operator intervenes in many problems in analysis (harmonic analysis, inverse problem, spectral theory...). It plays a central role in the study of the water-waves problem since the work of Craig[4], C. Sulem [171]. By definition, this is the operator that associates to a function f, defined on the boundary of an open set Ω, the normal derivative of its harmonic extension. Here, it is more convenient to introduce a coefficient in the definition.

Definition 1.2.1. *i) Consider two smooth functions η, ψ defined on $\mathbb{R}^2$ with real values. Introduce the domain $\Omega := \{ (x,y) \in \mathbb{R}^2 \times \mathbb{R} \,;\, y < \eta(x) \}$ and denote by ϕ the harmonic extension of ψ in Ω, so that $\phi = \phi(x,y)$ satisfies*

$$\begin{aligned} &\Delta_{x,y}\phi = 0 \quad \text{in} \quad \Omega, \\ &\phi(x,\eta(x)) = \psi(x), \quad \nabla_{x,y}\phi(x,y) \longrightarrow 0 \text{ when } y \longrightarrow -\infty. \end{aligned} \tag{1.2.3}$$

Then the Dirichlet-to-Neumann operator, denoted by $G(\eta)$, is defined by

$$G(\eta)\psi(x) = \sqrt{1 + |\nabla\eta|^2}\, \partial_n \phi|_{y=\eta(x)},$$

where ∂_n is the normal derivative: $\partial_n = n \cdot \nabla_{x,y}$.

ii) We also define the operators

$$\begin{aligned} B(\eta)\psi &= (\partial_y \phi)|_{y=\eta(x)}, \\ V(\eta)\psi &= (\nabla_x \phi)|_{y=\eta(x)}. \end{aligned}$$

Remark 1.2.2. *i) We will always use B and V as compact notations for $B(\eta)\psi$ and $V(\eta)\psi$.*

[4] See the memorial tribute here.

ii) *To clarify the notations, note that*

$$\sqrt{1+|\nabla\eta|^2}\,\partial_n\phi\,\big|_{y=\eta(x)} = \partial_y\phi - \nabla\eta\cdot\nabla\phi\,\Big|_{y=\eta(x)} \tag{1.2.4}$$
$$= (\partial_y\phi)(x,\eta(x)) - \nabla\eta(x)\cdot(\nabla\phi)(x,\eta(x)).$$

iii)*If η and ψ belong to $H^\infty(\mathbb{R}^d)$, then $G(\eta)\psi$ is well-defined and belongs to $H^\infty(\mathbb{R}^d)$. We shall see later on that one can consider functions with limited regularity.*

A important observation is that the traces of the derivatives of an harmonic function satisfy several identities. Recall that B and V as compact notations for $B(\eta)\psi$ and $V(\eta)\psi$. We will make extensive use of the following

Lemma 1.2.3. *For any smooth functions, there holds*

$$B = \frac{G(\eta)\psi + \nabla\eta\cdot\nabla\psi}{1+|\nabla\eta|^2}, \tag{1.2.5}$$

$$V = \nabla\psi - B\nabla\eta, \tag{1.2.6}$$

$$G(\eta)B = -\operatorname{div} V. \tag{1.2.7}$$

Proof. *i*) The chain rule implies that

$$\nabla\psi = \nabla(\phi(x,\eta(x))) = (\nabla\phi + (\partial_y\phi)\nabla\eta)|_{y=\eta} = V + B\nabla\eta$$

which implies that $V = \nabla\psi - B\nabla\eta$. On the other hand, by definition of the operator $G(\eta)$, one has

$$G(\eta)\psi = \big(\partial_y\phi - \nabla\eta\cdot\nabla\phi\big)|_{y=\eta} = B - V\cdot\nabla\eta,$$

so the identity for B in (1.2.5) follows from $V = \nabla\psi - B\nabla\eta$.

ii) By definition, one has $B = (\partial_y\phi)|_{y=\eta}$. Therefore $\Phi(x,y) = \partial_y\phi(x,y)$ satisfies

$$\Delta_{x,y}\Phi = 0, \quad \Phi|_{y=h} = B.$$

Directly from the definition of $G(\eta)$, we have

$$G(\eta)B = \partial_y\Phi - \nabla\eta\cdot\nabla\Phi\,\big|_{y=\eta}.$$

So it suffices to show that $\partial_y\Phi - \nabla\eta\cdot\nabla\Phi\,\big|_{y=\eta} = -\operatorname{div} V$. To do that we first write that $\partial_y\Phi = \partial_y^2\phi = -\Delta\phi$ to obtain

$$\big(\partial_y\Phi - \nabla\eta\cdot\nabla\Phi\big)\,\big|_{y=\eta} = -\big(\Delta\phi + \nabla\eta\cdot\nabla\partial_y\phi\big)\,\big|_{y=\eta} = -\operatorname{div}(\nabla\phi\,\big|_{y=\eta}),$$

which proves statement *ii*). □

Exercise 1.2.4. *Assume that $d = 1$ and consider two functions η and ψ in $H^\infty(\mathbb{R})$ and set $B = B(\eta)\psi$ and $V = V(\eta)\psi$.*

i) Prove that $\partial_x B = G(\eta)V$.

ii) Consider an harmonic function $\varphi = \varphi(x, y)$. Verify that $(\partial_x\varphi)^2 - (\partial_y\varphi)^2$ is also harmonic.

iii) Denote by ϕ the harmonic extension of ψ. The result of the previous question implies that $(\partial_x\phi)^2 - (\partial_y\phi)^2$ is the harmonic extension of $V^2 - B^2$. Use this information to obtain that

$$G(\eta)(V^2 - B^2) = 2\partial_x(BV).$$

Craig and Sulem [172, 171] wrote the equations on η and ψ in an explicit form.

Proposition 1.2.5. *The water-wave equations can be written under the form*

$$\begin{cases} \dfrac{\partial\eta}{\partial t} - G(\eta)\psi = 0, \\ \dfrac{\partial\psi}{\partial t} + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{(\nabla\psi\cdot\nabla\eta + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} + g\eta + \lambda\kappa(\eta) = 0. \end{cases}$$

Proof. By definition of $G(\eta)\psi$, we get the first equation from (1.1.4) and (1.2.4).

Now, by evaluating the equation

$$\partial_t\phi + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0,$$

on the free surface, we obtain

$$(\partial_t\psi - B\partial_t\eta) + \frac{1}{2}|V|^2 + \frac{1}{2}B^2 + g\eta + \lambda\kappa(\eta) = 0.$$

Since $\partial_t\eta = G(\eta)\psi = B - V\cdot\nabla\eta$, it follows that

$$\partial_t\psi + g\eta + \frac{1}{2}|V|^2 - \frac{1}{2}B^2 + BV\cdot\nabla\eta = 0,$$

which gives the second equation. $\square$

1.2.3 Arbitrary space dimension

It is immediate to generalize the problem in case x belongs to $\mathbb{R}^d$ instead of $\mathbb{R}^2$. The case $d = 1$ is particularly useful to describe waves that do not depend, say, on x_2. We then speak of two-dimensional waves because, at a given instant t, the domain $\Omega(t)$ can be described by two variables only (here x_1 and y). The equations being invariant by rotation, we can reduce to this case the general case where all phenomena are identical in planes parallel to a fixed vertical plane. In all the following we will use the following notations:

$$\nabla = (\partial_{x_1}, \dots, \partial_{x_d}), \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \sum_{1 \le j \le d} \partial_{x_j}^2, \quad \Delta_{x,y} = \Delta_{x,y} + \partial_y^2.$$

1.2.4 The linearized equation around the rest state

It is important to determine the Dirichlet-to-Neumann operator in the simplest case where Ω is the half-space

$$\Omega = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y < 0\},$$

that is for the case $\eta = 0$.

Proposition 1.2.6. *There holds*

$$G(0)\psi = |D|\,\psi,$$

where $|D|$ is the square root of the opposite of the Laplacian, defined by

$$|D|\, e^{i\xi\cdot x} = |\xi|\, e^{i\xi\cdot x} \quad \textit{where } |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_d^2}.$$

(Notice that $|D|^2\, e^{i\xi\cdot x} = -\Delta e^{i\xi\cdot x}$.)

Remark 1.2.7. *More generally we define $|D|^s$ for $s \ge 0$ as the Fourier multiplier with symbol $|\xi|^s$. This means that if u is in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, then*

$$(|D|^s u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\, |\xi|^s\, \hat{u}(\xi)\, \mathrm{d}\xi.$$

Proof. Let us introduce the Fourier transform of ϕ with respect to x:

$$u(\xi, y) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \phi(x, y)\, \mathrm{d}x.$$

Then

$$\phi(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} u(\xi, y)\, \mathrm{d}\xi.$$

Since $\Delta_{x,y}\phi = 0$, we have $\partial_y^2 u - |\xi|^2 u = 0$. The solution $e^{-y|\xi|}$ being to be excluded because $y < 0$, we find that $u = e^{y|\xi|} u(\xi, 0)$ which leads to

$$(\partial_y u)(\xi, 0) = |\xi|\, u(\xi, 0),$$

which in turn implies that $(\partial_y \phi)(x, 0) = (|D|\, \phi)(x, 0)$ and thus the desired result. □

We can now derive the linearized equation around the trivial solution. Neglecting terms that are at least quadratic with respect to the unknowns, we obtain the following system

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi + g\eta - \lambda\Delta\eta = 0. \end{cases}$$

Using that

$$G(0) = |D| \quad , \quad \Delta = -|D|^2,$$

and deriving in time we get the equation we are looking for:

$$\partial_t^2 \eta + g\,|D|\,\eta + \lambda\,|D|^3\,\eta = 0. \tag{1.2.8}$$

Notice that the Cauchy problem for this equation is well posed on Sobolev's spaces in two cases : *i*) as soon as λ is strictly positive, for any real value g; *ii*) in the case of $\lambda = 0$, the Cauchy Problem remains well posed under the assumption that g is positive or zero. To verify this it is sufficient to note that the equation (1.2.8) is solved using the Fourier transform. For example, if $\lambda = 0$ and if $\partial_t \eta|_{t=0} = 0$, we find that

$$\begin{aligned} \widehat{\eta}(t,\xi) &= e^{i\sqrt{g}t|\xi|^{1/2}} \widehat{\eta}(0,\xi) \quad \text{if } g > 0, \\ \widehat{\eta}(t,\xi) &= e^{t\sqrt{-g}|\xi|^{1/2}} \widehat{\eta}(0,\xi) \quad \text{if } g < 0. \end{aligned}$$

Note **the stabilizing role of gravity** : the case $g > 0$ is stable while the case $g < 0$ is unstable; there is an exponential amplification of high frequencies. (The case $g < 0$ corresponds to the very unstable situation where a heavy fluid is placed over a light fluid).

1.2.5 Dispersive estimates

Assume that $d = 1$, $g = 0$ and $\lambda = 1$ so that the water-wave equations read

$$\begin{cases} \partial_t \eta - |D|\,\psi = 0, \\ \partial_t \psi + |D|^2\,\eta = 0. \end{cases}$$

We symmetrize this system by introducing $\varphi = |D|^{\frac{1}{2}}\,\eta + i\psi$, so that

$$\partial_t \varphi + A\varphi = 0 \quad \text{where} \quad A := i\,|D|^{\frac{3}{2}}\,. \tag{1.2.9}$$

Notice that $A^* = -A$, which means that

$$\langle u, Av\rangle = \int_{\mathbb{R}} u(x)\overline{Av(x)}\,\mathrm{d}x = -\int_{\mathbb{R}} Au(x)\overline{v(x)}\,\mathrm{d}x = -\langle Au, v\rangle.$$

This is another way to obtain the conservation of energy: take the L^2-scalar product with φ and then take the real part to obtain

$$\int_{\mathbb{R}} |\varphi(t,x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} |\varphi(0,x)|^2\,\mathrm{d}x.$$

A first dispersive estimate is given by the following

Proposition 1.2.8. *Let $T > 0$. For any $\delta > 0$, there is $C > 0$ such that, for any solution φ of (1.2.9) with initial data $\varphi_0 \in L^2(\mathbb{R})^2$ we have,*

$$\int_{-T}^{+T}\int_{\mathbb{R}} \langle x\rangle^{-(1+\delta)}\big||D_x|^{\frac{1}{4}}\varphi(t,x)\big|^2\,\mathrm{d}x\,\mathrm{d}t \le C\int_{\mathbb{R}} |\varphi_0(x)|^2\,\mathrm{d}x,$$

where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$.

Sketch of the proof. Denote by $\langle\cdot,\cdot\rangle$ the scalar product on $L^2(\mathbb{R})$. Consider an operator $C \in \mathcal{L}(L^2(\mathbb{R}))$, independent of time. We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle C\varphi,\varphi\rangle &= \langle C\partial_t\varphi,\varphi\rangle + \langle C\varphi,\partial_t\varphi\rangle = -\langle CA\varphi,\varphi\rangle - \langle C\varphi, A\varphi\rangle \\ &= \langle [A,C]\varphi,\varphi\rangle \qquad \text{where} \quad [A,C] = A\circ C - C\circ A. \end{aligned}$$

The operator $[A,C]$ is called the commutator. It vanishes if A and C commute. The idea consists in constructing an operator C (bounded on $L^2(\mathbb{R})$) such that $[A,C]$

is positive (the so-called *positive commutator method*). Then we will conclude the proof by writing

$$\left|\int_0^T \langle [A,C]\varphi,\varphi\rangle \,\mathrm{d}t\right| = \left|\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle C\varphi,\varphi\rangle \,\mathrm{d}t\right| \lesssim \|\varphi(T)\|_{L^2}^2 + \|\varphi(0)\|_{L^2}^2 .$$

We seek C under the form of a pseudo-differential operator, that is an operator of the form

$$Cu(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} c(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi,$$

for some function $c = c(x,\xi)$ to be chosen. We say that C is a pseudo-differential operator associated to the symbol c and we write $C = \mathrm{Op}(c)$. There is a general theory which allows to handle such operators and we will study it in a forthcoming chapter. It follows from the later that the commutator $[A,C]$ is given, up to some remainder term, by a pseudo-differential operator associated with a symbol γ given by

$$\gamma = \frac{1}{i}\big(\partial_\xi(i|\xi|^{\frac{3}{2}})(\partial_x c) - \partial_x(i|\xi|^{\frac{3}{2}})(\partial_\xi c)\big) = \frac{3}{2}(\partial_x c(x,\xi))\frac{\xi}{|\xi|}|\xi|^{\frac{1}{2}}.$$

Let us choose

$$c(x,\xi) = \frac{\xi}{|\xi|}\int_0^x \frac{\mathrm{d}y}{\langle y\rangle^{1+\delta}}.$$

Then

$$\gamma = (\partial_x c(x,\xi))\frac{\xi}{|\xi|}|\xi|^{\frac{1}{2}} = \frac{|\xi|^{\frac{1}{2}}}{\langle x\rangle^{1+\delta}}.$$

Then, the general theory alluded to above allows to show that the pseudo-differential operator $\mathrm{Op}(\gamma)$ with symbol γ is equal, modulo an admissible error, to the operator

$$E = |D|^{\frac{1}{4}}\left(\frac{1}{\langle x\rangle^{1+\delta}}\,|D|^{\frac{1}{4}}\,\cdot\right).$$

Now, notice that

$$\langle E\varphi,\varphi\rangle = \int_{\mathbb{R}} \langle x\rangle^{-(1+\delta)}\big||D|^{\frac{1}{4}}\varphi\big|^2\,\mathrm{d}x,$$

whence the result. $\square$

1.3 The Cauchy problem

The water-wave equations read as follows:

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + g\eta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\dfrac{\left(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\right)^2}{1+|\nabla\eta|^2} + \lambda\kappa(\eta) = 0, \end{cases} \tag{1.3.1}$$

where the unknowns are $\eta = \eta(t,x)$, $\psi = \psi(t,x)$ ($x \in \mathbb{R}^d$, $d \in \{1,2\}$), $G(\eta)$ is the Dirichlet–Neumann operator we introduced in the previous paragraph and $g > 0$ is the acceleration of gravity, $\lambda \geq 0$ is a coefficient and $\kappa(\eta)$ is the mean curvature given by

$$\kappa(\eta) = -\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right).$$

There is an abundant literature on Cauchy problem for water wave equations. We refer to the references in §1.5.1 and §1.5.2. We will examine this issue in detail in a next chapter. In this introduction, we simply cite two results concerning the existence and uniqueness of smooth solutions. The first result states that the Cauchy problem is well-posed locally in time for any smooth initial data. The second gives a global well-posedness result for initial data which are small and localized.

Theorem 1.3.1 (local existence)**.** *Let $d \geq 1$ and assume that either $(g,\lambda) \in \mathbb{R}\times(0,+\infty)$ or $g > 0$ and $\lambda = 0$. For any $(\eta_0,\psi_0) \in H^\infty(\mathbb{R}^d)$, the Cauchy problem for (1.3.1) with initial data (η_0,ψ_0) has a unique maximal solution $(\eta,\psi) \in C^\infty(I; H^\infty(\mathbb{R}^d))$, $I \subset \mathbb{R}$.*

Theorem 1.3.2 (global existence)**.** *Let $d \geq 1$ and assume that either $(g,\lambda) = (1,0)$ or $(g,\lambda) = (0,1)$. Let us consider $(\eta_0,\psi_0) \in C_0^\infty(\mathbb{R}^2)^2$. There is $\varepsilon_0 > 0$ such that for every $0 \leq \varepsilon < \varepsilon_0$, the Cauchy problem for (1.3.1) with initial data $(\varepsilon\eta_0, \varepsilon\psi_0)$ has a unique solution $(\eta,\psi) \in C^\infty(\mathbb{R}; H^\infty(\mathbb{R}^d))$.*

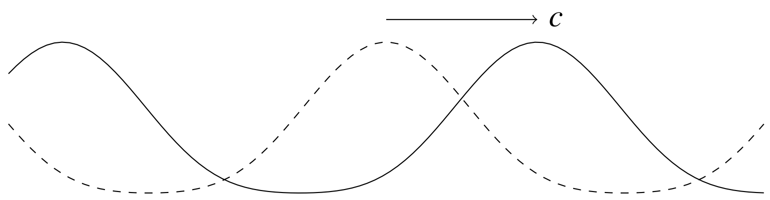
We will prove Theorem 1.3.1 later in this book. For Theorem 1.3.2 we refer to the original papers (see §1.5.1 and §1.5.2).

1.4 Stokes waves

In this section we discuss the existence of travelling waves on the surface of a heavy liquid, neglecting surface tension. This is a very old problem, going back to Stokes for periodic waves and to Boussinesq and Rayleigh for waves that decrease to 0 to infinity, called solitary waves. The problem consists in looking for waves propagating without alteration of shape, such that

$$\eta(t,x) = \tilde{\eta}(x - ct),$$

for some function $\tilde{\eta}$ and some constant vector $c \in \mathbb{R}^2$. People used to talk about permanent waves, today we talk about progressive waves.



We begin by studying this problem neglecting all the nonlinearities. The linear theory of gravity waves was much studied in the eighteenth and nineteenth centuries, notably by Laplace, Lagrange, Cauchy and Poisson. Recall that in this linear theory η solves

$$\frac{\partial^2 \eta}{\partial t^2} + g\,|D|\,\eta = 0, \tag{1.4.1}$$

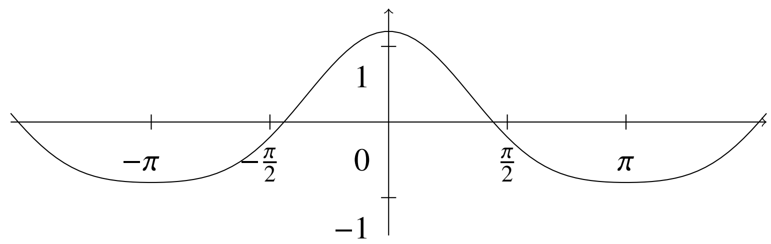
where $g > 0$ (recall that we neglect surface tension). One easily calculates that if $\varepsilon \cos(k \cdot (x - ct))$ is solution of the equation (1.4.1), then

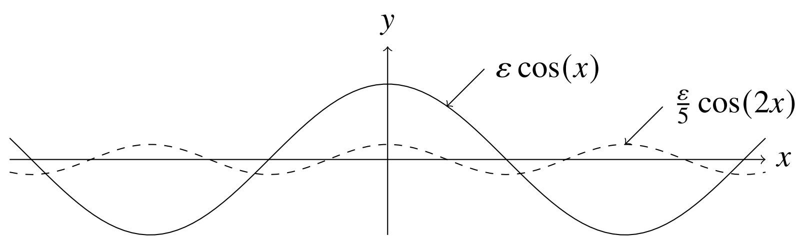
$$|c| = \sqrt{\frac{g}{|k|}}. \tag{1.4.2}$$

For a periodic progressive wave of the form $\varepsilon \cos(k \cdot (x - ct))$, we have a double periodicity: at a given instant, the quantity considered is spatially periodic, and at a given location, it oscillates periodically over time. The relation (1.4.2) gives a relationship between the spatial period and the temporal period.

As $|c|$ corresponds to the speed of the wave, this means that harmonics of different wavelengths propagate at different speeds and so they tend to disperse. Roughly summarizing what is found in oceanography books, this explains that at a certain place above the ocean, far from the wind zone, the ocean surface is very regular because it is essentially only possible to represent it with a single harmonic[5].

Nevertheless, the oceanographer observes that if the ocean surface has a sinuous shape, it is not the graph of a sinusoid. There are slightly marked edges and curves that suggest the presence of several harmonics. This corresponds to what we observe on the graph of the function $\cos(x) + 0.2\cos(2x)$ drawn below :





However, we have just seen that the linear theory predicts that two harmonics with different wavelengths will travel at different speeds. As a result their sum cannot be written as a function of $x - ct$. Stokes solved this problem. To explain the shape of the waves, he took into account terms neglected in linear theory.

Stokes' basic idea is that the relation of dispersion $\omega^2 = g\,|k|$ cannot be exact and that it should depend on the amplitude of the solution. By so doing, he discovered approximate solutions of the form:

$$\eta(t,x) = \varepsilon\cos\theta + \frac{1}{2}\,|k|\,\varepsilon^2\cos(2\theta) + \frac{3}{8}\,|k|^2\,\varepsilon^3\cos(3\theta) + O(\varepsilon^4), \tag{1.4.3}$$

[5]This is one of the great consequences of the memoir ([**?**]) by Cauchy: "As to the state of the fluid after a given time, it will be itself very irregular in the different points of the mass fluid originally subject to the immediate influence of causes that produced the movement. But, if we move away from these same points at increasingly greater distances, we will see the movement become more and more regular." We refer to page 83 of his memoir for a precise description of this result.

where the phase θ and the angular velocity ω are given by

$$\theta = kx - \omega t, \quad \omega = \sqrt{g\,|k|}(1 + \frac{1}{2}\varepsilon^2\,|k|^2 + O(\varepsilon^4)), \quad x \in \mathbb{R}.$$

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On this account I was induced to investigate the motion of oscillatory waves of the above form to a second approximation, that is, supposing the height of the waves finite, though small. I find that the expression for the velocity of propagation is independent of the height of the waves to a second approximation. With respect to the form of the waves, the elevations are no longer similar to the depressions, as is the case to a first approximation, but the elevations are narrower than the hollows, and the height of the former exceeds the depth of the latter. This is in accordance with Mr Russell's remarks at page 448 of his first Report*. I have proceeded to a third approximation in the particular case in which the depth of the fluid is very great, so as to find in this case the most important term, depending on the height of the waves, in the expression for the velocity of propagation. This term gives an increase in the velocity of propagation depending on the square of the ratio of the height of the waves to their length.

Figure 1.6: From Stokes' celebrated paper [405]

Stokes' work is a formal work (his result is that if a solution exists, then it admits the previous development). It was Levi-Civita and Nekrasov who much later managed to show the existence of exact solutions in the vicinity of these approximate solutions, having this development (see the original articles [312, 351] as well as the texts of Strauss [406] and Toland [424] for many extensions).

To observe experimentally the nonlinear dependence of ω on ε, let us go back to the figure 1.4. In fact, at the end of a cycle, the object has not returned exactly to its initial position, but is slightly shifted to the right, even in the absence of current; we speak of drift, which is related to the fact that the equations are non-linear. This phenomenon, discovered by Stokes, has been studied by Constantin [137] and Constantin and Escher [137, 138].

Let us conclude with two remarks on the original article by Stokes [405].

Bifurcation theory explains Stokes' fundamental observation that the dispersion relation depends on the amplitude. It is interesting to understand Stokes' calculations.

His method can be compared to the one used by Lindstedt (then Poincaré) to find know a good approximation for large times of periodic solutions of differential equations (a classical and relevant problem in astronomy, which arises to predict the position of the planets). The most classical example of an equation to implement this method is the Duffing equation. Let us consider the Cauchy problem

$$\ddot{u} + u^3 = 0, \qquad u(0) = \varepsilon,\ \dot{u}(0) = 0.$$

We look for u in the form $u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots$, then

$$\begin{aligned} &\ddot{u}_1 + u_1 = 0, && u_1(0) = 1, \quad \dot{u}_1(0) = 0,\\ &\ddot{u}_2 + u_2 = 0, && u_2(0) = 0, \quad \dot{u}_2(0) = 0,\\ &\ddot{u}_3 + u_3 + (u_1)^3 = 0, && u_3(0) = 0, \quad \dot{u}_3(0) = 0. \end{aligned}$$

We calculate successively that $u_1 = \cos(t)$, $u_2 = 0$, $u_1^3 = \frac{1}{4}\cos(3t) + \frac{3}{4}\cos(t)$ and

$$u_3 = -\frac{3}{8} t \sin(t) + \frac{1}{32}\cos(3t),$$

where to calculate u_3 we used that the solution of $\ddot{u} + \omega_0^2 u = a\cos(\omega t)$ is

$$\frac{a}{\omega_0^2 - \omega^2}\cos(\omega t) \text{ if } \omega \neq \omega_0, \qquad \frac{a}{2\omega_0} t \sin(\omega_0 t) \text{ if } \omega = \omega_0.$$

The disturbance u_3 contains an unbounded term, $t\sin(t)$, called the secular term. The idea of Stokes and Lindstedt is that in using a Taylor expansion of the cosine, one obtains an expression without secular term, of the form

$$u = \varepsilon \cos\Big((1 + \frac{3}{8}\varepsilon^2)t\Big) + O(\varepsilon^3).$$

We can in fact give an explicit formula for the solution that involves the amplitude function of Jacobi and the elliptic cosine of Jacobi, see the article by Baldi [60].

Still about Stokes' calculations, there is a remark to be made about the formula (1.4.3). This is, with one sign difference, the formula given by Stokes [405], page 211. In the original article, the equation for the free surface is written in the form

$$y = a\cos(mx) - \frac{1}{2} m a^2 \cos(2mx) + \frac{3}{8} m^2 a^3 \cos(3mx). \tag{1.4.4}$$

This contradiction in the signs is explained by the fact that for Stokes, the (Oy) axis is oriented downwards, whereas we have chosen it oriented upwards. We find the formula (1.4.3) by changing y to $-y$ and a to $-a$ (or x in $x + \pi$ if we want to give a physical meaning to the value of a).

1.5 References (work in progress)

1.5.1 References about the equations

For an introduction to linear gravity wave theory in physics, we refer the reader to paragraph §12 of Landau and Lifshitz [298], Chapters 3,4,5 of the book of Holthuijsen [242] and Chapter 16 of Stewart's book [404]. An aspect often highlighted is that the equations are Hamiltonian (see Zakharov [453]). A natural question is whether other conservation laws exist. Benjamin and Olver [67] have found 8 conservation laws for two-dimensional waves; Olver showed that under certain conditions in the definition of a conservation law, there are no other [355]. This contrasts with the discovery of Lax [306] for Korteweg de Vries' equation.

The equations can be written in much more general forms. One can thus consider an initial velocity that does not derive from a potential (see among others [130, 314, 456]), which is important if viscosity is to be taken into account. The case of a viscous fluid is of course also very interesting, but we will not talk about it because viscosity is a simplifying hypothesis for the problems we will consider. For the study of wave propagation, one can also *see* that the viscosity is negligible. Indeed, it can travel immense distances without being attenuated (for example 10000 kms) between the place where it is generated and the shore. There is therefore no energy dissipation by viscous friction (the argument can be faulty in other situations). Let us just quote the work of Solonnikov [399] and Beale [64] and the recent works of Guo and Tice [236], Bresch and Noble [95]. The reader will find in the article by Longuet-Higgins [319], on page 537, arguments that explain why it is sometimes important to consider data that are not irotational. A very large number of adaptations of the results on the Stokes waves in the so-called rotation are described in the review article by Strauss [406].

There are many possible formulations of the equations. One can model the water wave equations in holomorphic coordinates: this is described in detail in the papers [250] for the infinite depth case, respectively [241] for the finite depth case (see also [202]). Another idea that has been very fruitful in the field has been to introduce tools from geometric analysis, like the systematic use of the lagrangian formulation of the equations by Ambrose and Masmoudi [45], Coutand-Shkoller [159, 158], Kukavica and Tuffaha [295, 294, 296], or the tools of Riemannian geometry (parallel transport, covariant derivative) by Shatah and Zeng [388, 389, 390]. Geometrical methods have also been used by Christodoulou and Lindblad [130]. This approach was further developped by de Poyferré to initiate the study of the water-wave equation with emerging bottom [185].

Let us also mention that many papers focused on the numerical analysis of the Dirichlet-to-Neumann operator, we refer to [59, 171, 444, 366] and the references there in.

1.5.2 References about the Cauchy problem

The first results on the Cauchy problem go back to the pioneering works of Nalimov [348], Shinbrot [391], Reeder and Shinbrot [368, 369], [452] and Craig [162] (see also Tani [411], Hou, Teng and Zhang [248] and Beale, Hou and Lowengrub [65]). The first results without any hypothesis of smallness or analyticity are due to to Beyer and Günther [80] (for the case with surface tension, in any dimension) as well as Wu (for the case without surface tension, in dimension $d = 1$ first in [445], then in any dimension in [446]).

Numerous extensions of their results have been obtained by different methods: see Ogawa-Tani [354], Ambrose-Masmoudi [45, 46, 47], Schneider-Wayne [385, 386], Zhang-Zhang [456], Schweizer [387], Iguchi [257, 255, 256], Shatah-Zeng [388, 389, 390], Ming-Zhang [344], Coutand-Shkoller [159], Guo-Tice [237, 238], Masmoudi-Rousset [326], Rousset and Tzvetkov [378], Christianson, Hur and Staffilani [129], Alazard-Burq-Zuily [19], Lannes [302] or Berti and Delort [74] among others. The regularity of the flow map was studied by Chen, Marzuola, Spirn and Wright [124] and by Rimah-Said [381].

The main difference between the case $\lambda = 1$ (with surface tension) and the case $\lambda = 0$ (without surface tension) is that in the latter case we must make an assumption of positivity on the Taylor coefficient (named after Geoffrey Taylor [416]) which is defined by

$$a(t,x) = -(\partial_y P)(t,x,\eta(t,x)).$$

One of the main results in the articles [445, 446] by Wu states that this assumption is automatically satisfied for a domain of infinite depth. Lannes then showed that this result remains true for a regular bottom which is a small disturbance of a flat bottom ([300]); see also [203, 130].

There have been many recent results about rough solutions. One direction of research is the study of angled crested type water waves: See the recent papers by Kinsey-

Wu [285] and Wu [449]. In the latter reference, Sijue Wu proved the well-posedness of the 2D water-wave equations in a regime that allows for non-C^1 interfaces. In this regime, only a degenerate Taylor inequality $a \geq 0$ holds. Agrawal [4] proved that initial interface with angled crests remains angled crested, the Euler equations hold pointwise even on the boundary, the particle at the tip remains at the tip, and the acceleration at the tip is the one due to gravity. Also, Agrawal extended the study of angled crested water waves to the case with surface tension (see [3, 2]). On the other hand, several recent papers are devoted to the study of the Cauchy problem with rough initial data: starting from [22] and continued in [250, 23, 187, 186, 6, 7]. The best results at the time these notes are written are due to Ai, Ifrim and Tataru ([8, 9]) and Wu ([450]).

Let us briefly discuss some additional points concerning the regularity thresholds for the water-wave equations. As explained in §1.1.4, a well-known property of smooth solutions is that their energy is conserved

$$\frac{d}{dt}\left\{\frac{1}{2}\iint_{\Omega(t)}\left|\nabla_{x,y}\phi(t,x,y)\right|^2 \mathrm{d}y\,\mathrm{d}x + \frac{g}{2}\int_{\mathbb{R}^d}\eta(t,x)^2\,\mathrm{d}x\right\} = 0.$$

However, it is not known if weak solutions exist at this level of regularity (even the meaning of the equations is not clear). This is the only known coercive quantity (see [68]). Another regularity threshold is given by the scaling invariance, which is the following property: If ϕ and η are solutions of the gravity water waves equations, then ϕ_λ and η_λ defined by

$$\phi_\lambda(t,x,y) = \lambda^{-3/2}\phi(\sqrt{\lambda}t,\lambda x,\lambda y), \quad \eta_\lambda(t,x) = \lambda^{-1}\eta(\sqrt{\lambda}t,\lambda x),$$

solve the same system of equations. The (homogeneous) Hölder spaces invariant by this scaling (the scaling critical spaces) correspond to η_0 Lipschitz and ϕ_0 in $\dot{W}^{3/2,\infty}$ (one can replace the Hölder spaces by other spaces having the same invariance by scalings). One could expect that the problem exhibits some kind of "ill-posedness" for initial data such that the free surface is not Lipschitz. See e.g. [110, 128] for such ill-posedness results for semi-linear equations. However, the water waves equations are not semi-linear (see [381]) and it is not clear whether the scaling argument is the only relevant regularity threshold to determine the optimal regularity in the analysis of the Cauchy problem (One key result in this direction is the resolution of the bounded L^2 curvature conjecture by Klainerman, Rodnianski and Szeftel [290]). In particular, it remains an open problem to prove an ill-posedness result for the gravity water waves equations. We refer to the recent paper by Chen, Marzuola, Spirn and Wright [124] for a related result in the presence of surface tension.

There are very few results about the possible blow-up of water-waves. One reference here is the paper by Castro, Córdoba, Fefferman, Gancedo and Gómez Serrano [115] (see also [117, 160]). The authors consider the more general case where the surface is not a graph, but a curve drawn in $\mathbb{R}^2$, which does not self-intersect at initial time. The authors show the existence of an initial configuration such that in finite time the free surface will self-intersect; forming what is called in the literature a splash singularity.

On the contrary, in the case without surface tension, it has been recently demonstrated by Wu [448] and also by Germain, Masmoudi and Shatah [217] that we have global existence with small data for spatially localized three-dimensional waves ($d = 2$). This result was later extended by Alazard-Delort [24, 25], Ionescu-Pusateri [258], Hunter-Ifrim-Tataru [250, 251] to the critical problem, that is for $d = 1$, following earlier work by Sijue Wu [447] about almost global well-posedness. For the case with surface tension, the first result for $d = 2$ was obtained by Germain, Masmoudi and Shatah [218]. In the critical case ($d = 1$), the result was first obtained by Ifrim and Tataru [252] and then by Ionescu and Pusateri [259]. See also the paper by Ifrim-Tataru [253] for the case with vorticity and Harrop-Griffiths, Ifrim and Tataru [241] and Wang [440] for the finite depth case. See also the review article [190].

The well-posedness of the Cauchy problem is also true for the compressible equations, whether it is the isentropic system, or the complete entropic system. We refer the reader to articles by Tanaka and Tani [410], Lindblad [313], Coutand–Lindblad–Skholler [158], Coutand–Shkoller [160], Jang–Masmoudi [271, 270]. One can also couple the fluid mechanics equations to other equations; such systems have been studied by Trakhinin [425, 426]. Another very interesting extension consists in considering the case where several fluids are superimposed. This profoundly changes the nature of the equations and raises some very difficult questions; see the review text of Saut [384], recent articles by Shatah-Zeng [390] and Lannes [302] for the study of the Cauchy problem, and that of Iooss, Lombardi and Sun [261] for the study of travelling waves. In this direction, let us also mention the article by Bresch and Renardy [97]. Let us also mention the links with the Hele-Shaw problem studied by Günther and Prokert [234, 233].

In the study of the Cauchy problem, another hypothesis, hidden and much more problematic for applications, is that we consider a fixed initial data. A whole part of the theory consists precisely in deriving asymptotic equations in regimes where the solutions (and thus the initial data) depend on a small parameter that tends towards 0. One can think of estimates which are uniform with respect to the surface tension coefficient (see [45, 388, 344]). Interesting asymptotics are those that make appear new equations. In this the subject is magnificent because many classical equations (KdV,NLS,KP,BBM,...) are thus obtained from the free surface Euler equation (see [182, 407, 453]). Let us mention only the pioneering work of Craig [162], Kano and Nishida [278] as well as the work of David Lannes and his collaborators [42, 85, 84, 166, 199, 198, 197, 301, 324]. Several new models were found through mathematical analysis, I am thinking in particular of the article by Bona, Lannes and Saut [85].

As far as the Cauchy theory is concerned, we can show that the bottom has very little importance and that very irregular funds are appropriate. Other studies are interested in the case where the topography changes the dynamics (what happens in coastal oceanography, when near the shore the depth is shallow; which is fundamental in the study of refraction). This is the case, for example, of the work of Craig, Guyenne and Sulem [164], de Bouard, Craig, Díaz-Espinosa, Guyenne and Sulem [184], Alvarez-Samaniego and Lannes [41], Lannes [302], Chazel [121] and Israwi [269]. This raises a host of new questions and requires a wide variety of techniques. For example, in [166], Craig, Lannes and Sulem are interested in the so-called "shallow water" limit in case the bottom is strongly oscillating. The authors derive a new system, which corrects previous formal work, thanks to tools resulting from the analysis of homogenization problems.

1.5.3 Dispersive estimates

There is a vast literature concerning dispersive estimates. Here we mention a few results related to the smoothing effect mentioned previously. Let us quote the original article by Kato [280] for the Korteweg-de-Vries equation (KdV), the articles by Constantin and Saut [145], Sjölin [397] and Vega [436] for a general dispersive equation with constant coefficients, and for the Schrödinger equation with variable coefficients include the work of Craig, Kappeler and Strauss [165], Doi [193, 194, 195], Kenig-Ponce-Vega [282, 283], Robbiano and Zuily [375], Burq and Planchon [103]. Note that the regularizing effect of Kato for the KdV equation, intervenes in the demonstration by Rosier [377] of inequality observability for this equation (see also Coron's book [155, section 2.2]). There are links between the free surface Euler equation with surface tension and the Euler-Korteweg system (see Benzoni [69]); Audiard [54] has recently shown an analog of the theorem 1.2.8 for this system. Sharp versions of the smoothing effect were obtained recently by

Zhu [459] and Alazard-Ifrim-Tataru [26]. Dispersive properties can also be used to prove controllability properties of the waves [17, 458].

In addition to the smoothing effect, the water-wave equations enjoy Strichartz-type inequalities. For the water-wave equations with surface tension, in the special case of dimension $d = 1$, Strichartz estimates were proved by Christianson, Hur and Staffilani in [129] for smooth enough data and in [20] for the low regularity solutions constructed in [19].

One natural question is to combine these Strichartz estimates with the standard energy estimates to improve the threshold for the well-posedness theory. The main difficulty is that this requires to establish Strichartz inequalities at a lower level of regularity than the threshold where one is able to prove the existence of the solutions. This was first achieved in [22] for the gravity water-wave equations. This allowed to improve the Cauchy theory and in particular, for $d = 1$, to consider solutions such that the curvature of the initial free surface does not belong to L^2. Such Strichartz estimates are obtained by constructing parametrices on small time intervals tailored to the size of the frequencies considered (in the spirit of the works by Lebeau [308], Bahouri-Chemin [56], Tataru [415], Staffilani-Tataru [400], and Burq-Gérard-Tzvetkov [102]). These semi-classical Strichartz estimates have been extended in several directions: *i*) to the case with surface tension by de Poyferré and Nguyen [186], *ii*) Ai obtained ([5]) optimal lossless Strichartz estimates. This provides the sharp regularity threshold with respect to the approach of combining Strichartz estimates with energy estimates.

1.5.4 References about progressive waves

In the rest of these lectures, we will focus on the study of the Cauchy problem and we will not study the Stokes waves. Since the study of those special solutions lies at the heart of the water-wave theory, we discuss in this section many recent research directions about them.

We know today that the existence of 2D waves simply comes from a bifurcation from a simple eigenvalue. This corresponds to a local theory of bifurcations which studies all the solutions for ε close to 0. There is also a global theory that studies the possibility of find a branch of solutions parameterized by $\varepsilon \in \mathbb{R}$. Using this global theory we can prove the existence of what we call extreme Stokes waves (see [424, 101] for a complete introduction to the study of extremal waves and also the original articles of Toland [423], Amick, Fraenkel and Toland [49], Plotnikov [361]). For the problems we have studied, the difficulty is that the inversion of the linearized operator around a non-trivial state (in the orthogonal of its kernel) is possible, but at the price of a loss of derivatives (which comes from the presence of small divisors). The theory of bifurcations no longer applies because the implicit function theorem no longer applies. However, as has been observed in another context by Rabinowitz [367] (see also [161]), one can use the implicit function theorem of Nash-Moser.

The study of Stokes waves is a classical subject in physics: see the reference books of Whitham [442] and Wehausen and Laitone [441]. The question of reflection of a Stokes wave on a wall is illustrated in the book by Barber [62]. This is a classic subject for experimenters, and we refer here to the very complete review articles of Dias and Kharif [192] and Hammack, Henderson and Segur [240]. The more general question of the Stokes wave interaction is discussed in the books of Holthuijsen [242] and Kartashova [279]. The numerical study of surface waves is of course a huge subject and we refer the reader to the article by Clamond and Grue [132] for recent developments.

References on the problem of standing gravity waves (periodic in time and space) are given in the introduction of the article by Iooss, Plotnikov and Toland [267]. See also the note by Iooss [260] who cites Boussinesq as the first to consider this question in 1877. The question of the existence of these waves has remained open for a very long time. The first result is due to Plotnikov and Toland [362]. This work was extended by Iooss, Plotnikov and Toland ([267] then supplemented by the articles of Iooss and Plotnikov [262, 263] which show the existence of solutions unimodal ([267]) and multimodal ([262, 263]) in the completely degenerate case where the kernel is of infinite dimension (which comes from the fact that in [267, 262, 263] Ê the domain is of infinite depth, a more difficult case for this problem than the case of a finite depth domain [362]). The stability of standing waves with respect to harmonic perturbations are studied by Wilkening in [443]. A new phenomenon concerning a directional drift for 3D waves has been demonstrated by Iooss and Plotnikov [264] for small amplitude 3D waves non-symmetrical; whose existence was demonstrated by Iooss and Plotnikov [266] which was a very difficult question. The small divisor problems for 3D water waves was further studied by Alazard and Métivier [29]. The first result about the existence of gravity capillary standing water waves was obtained in [16]. By using the transformations introduced by Iooss-Plotnikov [265] and Alazard and Baldi [16] with methods to study quasi-periodic solutions, KAM results were then obtained, first for gravity-capillary water waves by Berti and Montalto [75] and then for the gravity water waves equations (see [61]).

The study of periodic travelling waves 3D can be traced back to the work of Reeder and Shinbrot [370]. It has been completed by Groves and Haragus [228] and Craig and Nicholls [169]. In the case where the surface tension is very high, the existence of localized capillary waves has been demonstrated by Groves and Sun [231] and Buffoni, Groves, Sun and Wahlén [100]. Let us also mention the previous papers by Groves and Mielke [230] (which we refer to for a discussion of Kirchgässner's spatial dynamics approach [286]). The linear instability of capillary waves is studied by Groves and Haragus [229]. The case non-linear, very difficult, was solved by Rousset and Tzvetkov [378]. There are very many results in the case where the rotational is non-zero, see for example [439] and the references of this article.

Solitary water waves have been proved to exist in many interesting situations. Their existence relies on a subtle balance of different factors, such as gravity, surface tension or the fluid bottom (see [212, 63, 50]). For this question, we refer to the article by Ifrim and Tataru [254] who prove that there are no solitary waves in 2D for purely gravity or capillary waves in infinite depth.

We have discussed the interaction of two progressive periodic waves. The interaction of two solitary waves is studied numerically and experimentally by Craig, Guyenne, Hammack, Henderson and C. Sulem in the article [163]. The interaction of solitons for the Korteweg de Vries equation has been studied, including in non-integrable situations, by Martel and Merle (see the survey [325]).

1.5.5 Historical References

The linear theory of gravity waves was much studied in the eighteenth and nineteenth centuries, notably by Laplace, Lagrange, Cauchy and Poisson. We refer to the articles of Craik [174, 175], to his book [173] as well as to that of Darrigol [179]. And note especially that the statues of Laplace, Lagrange, Cauchy and Poisson are located in the courtyard of the École normale supérieure rue d'Ulm in Paris, in the corner which is closest to the Pantheon.

Lagrange

Laplace

Poisson

Cauchy

The question by the French Academy question we cited can be found in Cauchy's memoir [**?**]. Darrigol explains[6] in his book [179, page 37] that it is due to Laplace.

Many microlocal analysis tools have their origin in the study of surface waves or in connection with questions that arise in this field. One can think of the introduction of the stationary phase method (in order to study the pattern of waves in the wake of a boat). Note that the notions of stationary phase, phase velocity and speed group appear in some calculations of Cauchy [**?**]. He had seen that harmonics of different wavelengths propagate at different speeds and thus they tend to disperse; they are said to disperse.

Let us also note that Cauchy and Poisson would have rediscovered Fourier's integral formula (which appears in Fourier's memoir, published in 1822, for which he won the prize for his answer (given in 1812) to a question asked by the Academy in 1811). This is explained in the article by Annaratone [51].

Part of the techniques we will use comes from the study of shock waves and rarefaction waves. A shock is a weak solution of a system of conservation laws,

[6]See the minutes of the sessions of the Academy, which can also be consulted at `http://gallica.bnf.fr/ark:/12148/cb32746437k/`; more generally, the original articles of Cauchy, Stokes, Russell and also those of Boussinesq, that we will mention later, can be easily found on the internet, in free access.

discontinuous through a regular hypersurface Σ, regular up to the boundary on either side of Σ. One can read in Tartar's book on Sobolev's spaces ([414, page 183]) that Stokes was the first to be interested in discontinuous solutions of the acoustic wave equation, in an article published one year after the publication of his 1847 article on the theory of oscillating waves [405].

LES

FONTAINES PUBLIQUES

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EXPOSITION ET APPLICATION

DES PRINCIPES A SUIVRE ET DES FORMULES A EMPLOYER

DANS LES QUESTIONS

DE

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PAR UN APPENDICE RELATIF AUX FOURNITURES D'EAU DE PLUSIEURS VILLES

AU FILTRAGE DES EAUX

ET

A LA FABRICATION DES TUYAUX DE FONTE, DE PLOMB, DE TOLE ET DE BITUME

PAR

HENRY DARCY

INSPECTEUR GÉNÉRAL DES PONTS ET CHAUSSÉES.

> La bonne qualité des eaux étant une des choses qui contribuent le plus à la santé des citoyens d'une ville, il n'y a rien à quoi les magistrats aient plus d'intérêt qu'à entretenir la salubrité de celles qui servent à la boisson commune des hommes et des animaux, et à remédier aux accidents par lesquels ces eaux pourraient être altérées, soit dans le lit des fontaines, des rivières, des ruisseaux où elles coulent, soit dans les lieux où sont conservées celles qu'on en dérive, soit enfin dans les puits d'où naissent des sources.
>
> (De Jussieu, *Hist. de l'Académie royale des sciences*, 1733, p. 351.

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1856

THE

Public Fountains

of the City of Dijon

EXPLANATION AND APPLICATION OF

Principles to be Followed and Formulas to be used

IN QUESTIONS ABOUT

Water Distribution Issues

COMPLETED WITH

an appendix relating to water supplies,

water filtering

AND TO

THE MANUFACTURE OF CAST IRON, SHEET METAL AND BITUMEN PIPES

BY

Henry Darcy

GENERAL MANAGER OF BRIDGES AND ROADWAYS

> *Since the good quality of water is one of the things that contributes most to the health of the citizens of a town, there is nothing in which magistrates have more interest than to maintain the wholesomeness of the water used for the common drink of men and animals, and to remedy the accidents by which this water could be altered, either in the beds of the fountains of rivers and streams where it flows, or in the places where the water derived from them is kept, or finally in the wells from which springs are born.*

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Chapter 2

Darcy's law and the Hele-Shaw equations

Consider a time-dependent surface Σ of the form

$$\Sigma(t) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}\,;\; y = h(t, x)\} \qquad (t \in \mathbb{R}_+).$$

In this chapter, we consider several free boundary problems that are described by evolution equations that express the velocity of Σ at each point in terms of some nonlinear expressions depending on h.

The most popular example is the **mean-curvature** equation, which stipulates that the normal component of the velocity of Σ is equal to the mean curvature at each point. It follows that:

$$\partial_t h + \sqrt{1 + |\nabla h|^2}\,\kappa = 0 \quad \text{where} \quad \kappa = -\operatorname{div}\left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right). \tag{2.0.1}$$

The previous equation plays a fundamental role in differential geometry. Many other free boundary problems appear in fluid dynamics. Among these, we are chiefly concerned by the equations modeling the dynamics of a free surface transported by the flow of an incompressible fluid evolving according to Darcy's law.

2.1 The Hele-Shaw equations

2.1.1 Darcy's law

Consider the fluid domain

$$\Omega(t) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}\,;\, y < h(t,x)\}.$$

One assumes that the velocity $v\colon \Omega \to \mathbb{R}^{d+1}$ and the pressure $P\colon \Omega \to \mathbb{R}$ solve the Darcy's equations. This means that

$$\operatorname{div}_{x,y} v = 0, \quad \text{and} \quad v = -\nabla_{x,y}(P + gy) \quad \text{in } \Omega, \tag{2.1.1}$$

where $g > 0$ is the acceleration of gravity.

2.1.2 Boundary conditions

One imposes that

$$\lim_{y\to-\infty} v = 0.$$

As for water waves, on the free surface, it is assumed that the normal component of v coincides with the normal component of the velocity of the free surface:

$$\partial_t h = \sqrt{1+|\nabla h|^2}\, v \cdot n \quad \text{on} \quad y = h, \tag{2.1.2}$$

where recall that $\nabla = \nabla_x$ and n is the outward unit normal to Σ, given by

$$n = \frac{1}{\sqrt{1+|\nabla h|^2}} \begin{pmatrix} -\nabla h \\ 1 \end{pmatrix}.$$

The final equation states that the restriction of the pressure to the free surface is proportional to the mean curvature:

$$P = \lambda\kappa \quad \text{on} \quad \Sigma,$$

where the parameter λ belongs to $[0,1]$ and κ is given by (2.0.1).

2.1.3 Formulation with the DN

Recall that the Dirichlet-to-Neumann operator $G(h)$ is defined by

$$G(h)\psi(x) = \sqrt{1+|\nabla h|^2}\partial_n\varphi|_{y=h(x)} = \partial_y\varphi(x,h(x)) - \nabla h(x)\cdot\nabla\varphi(x,h(x)),$$

where

$$\Delta_{x,y}\varphi = 0 \quad \text{in } \Omega, \qquad \varphi|_{y=h} = \psi.$$

Since

$$\Delta_{x,y}(P+gy) = \operatorname{div}_{x,y} v = 0 \quad ; \quad P+gy|_{y=h} = gh+\lambda\kappa,$$

it follows that the Hele-Shaw problem is equivalent to

$$\partial_t h + G(h)(gh+\lambda\kappa) = 0. \tag{2.1.3}$$

When $g = 1$ and $\lambda = 0$, the equation (2.1.3) is called the Hele-Shaw equation without surface tension. Hereafter, we will refer to this equation simply as the **Hele-Shaw** equation. If $g = 0$ and $\lambda = 1$, the equation is known as the Hele-Shaw equation with surface tension, also known as the **Mullins-Sekerka** equation. Let us record the terminology:

$$\partial_t h + G(h)h = 0 \qquad \text{(Hele-Shaw)}, \tag{2.1.4}$$

$$\partial_t h + G(h)\kappa = 0 \qquad \text{(Mullins-Sekerka)}. \tag{2.1.5}$$

Recalling that $G(0) = |D|$, we see that the linearized equations read

$$\partial_t h + |D|\, h = 0 \qquad \text{(Linearized Hele-Shaw)}, \tag{2.1.6}$$

$$\partial_t h + |D|^3\, h = 0 \qquad \text{(Linearized Mullins-Sekerka)}. \tag{2.1.7}$$

They are parabolic evolution equations.

2.1.4 Lubrication approximation

Other parabolic equations appear naturally to describe asymptotic regime in the **thin-film** approximation. They are

$$\partial_t h - \operatorname{div}(h\nabla h) = 0 \qquad \text{(Boussinesq)}, \tag{2.1.8}$$

$$\partial_t h + \operatorname{div}(h\nabla\Delta h) = 0 \qquad \text{(thin-film)}. \tag{2.1.9}$$

2.1.5 The Cauchy problem

The Cauchy problems for the Hele-Shaw and Mullins-Sekerka equations have been studied by different techniques, for weak solutions, viscosity solutions or also classical solutions (see the numerous references in §2.3.3 at the end of this chapter). In this book, we will restrict ourselves to the study of classical solutions. We consider initial data in Sobolev spaces.

Definition 2.1.1. *Given a real number $s \geq 0$, the Sobolev space $H^s(\mathbb{R}^d)$ consists of those functions $u \in L^2(\mathbb{R}^d)$ such that the following norm is finite:*

$$\|u\|_{H^s}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s \left|\hat{u}(\xi)\right|^2 \mathrm{d}\xi,$$

where $\widehat{u}$ is the Fourier transform of u:

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x.$$

The homogeneous Sobolev norms are defined by

$$\|u\|_{\dot{H}^s}^2 := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2s} \, |\hat{u}(\xi)|^2 \,\mathrm{d}\xi.$$

Theorem 2.1.2. *Let $d \geq 1$ and consider a real number $s > d/2 + 1$. For any initial data h_0 in $H^s(\mathbb{R}^d)$, there exists a time $T > 0$ such that the Cauchy problem*

$$\partial_t h + G(h)h = 0, \quad h|_{t=0} = h_0, \tag{2.1.10}$$

has a unique solution satisfying

$$h \in C^0([0,T]; H^s(\mathbb{R}^d)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^d)) \cap L^2([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d)).$$

Morevoer, h belongs to $C^\infty((0,T] \times \mathbb{R}^d)$.

Proof. See Chapter **??**. □

Definition 2.1.3. *i) Recall that*

$$\sigma > \frac{d}{2} \quad \Rightarrow H^\sigma(\mathbb{R}^d) \hookrightarrow C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).$$

So $s > 1 + d/2$ implies that $H^s(\mathbb{R}^d) \hookrightarrow C^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$.

ii) We say that h is a regular solution to (2.1.10) defined on $[0,T]$ if h satisfies the conclusions of the above result.

2.1.6 Maximum principles and Lyapunov functionals

The Hele-Shaw equation is a nonlinear parabolic equation, so a natural question is to find maximum principles. We will consider maximum principle for h itself or for its derivatives.

Proposition 2.1.4. *$i)$ Let $d \geq 1$. For any regular solution h of the Hele-Shaw equation $\partial_t h + G(h)h = 0$, there holds*

$$\sup_{x\in\mathbb{R}^d} |h(t,x)| \leq \sup_{x\in\mathbb{R}^d} |h(0,x)|.$$

$ii)$ Let h_1, h_2 be two regular solutions of the Hele-Shaw equation defined on the same time interval $[0,T]$, such that, initially,

$$h_1(0,\cdot) \leq h_2(0,\cdot).$$

Then

$$h_1(t,\cdot) \leq h_2(t,\cdot)$$

for all $t \in [0,T]$.

Proof. See Chapter **??**. □

Proposition 2.1.5. *Let $d \geq 1$ and consider a regular solution h to the Hele-Shaw equation $\partial_t h + G(h)h = 0$ defined on $[0,T]$. Then, whenever ω is a modulus of continuity for $h(0,\cdot)$, ω is also a modulus of continuity for $h(t,\cdot)$, for any $t \in [0,T]$. In particular, we have*

$$\sup_{x\in\mathbb{R}^d} |\nabla h(t,x)| \leq \sup_{x\in\mathbb{R}^d} |\nabla h(0,x)|.$$

Proof. See Chapter **??**. □

We will also study the monotonicity of some coercive quantities. Consider the L^2-norm and the area functional

$$\left(\int_{\mathbb{R}^d} h^2 \,\mathrm{d}x\right)^{\frac{1}{2}}, \qquad \mathcal{A}(\Sigma) = \int_{\mathbb{R}^d} \left(\sqrt{1+|\nabla h|^2} - 1\right) \mathrm{d}x.$$

Recall that $G(h)$ is a non-negative operator. Indeed, consider a function $\psi = \psi(x)$ and denote by $\varphi = \varphi(x, y)$ its harmonic extension given by

$$\Delta_{x,y}\varphi = 0 \quad \text{in } \Omega = \{y < h(t, x)\}, \qquad \varphi|_{y=h} = \psi.$$

It follows from Stokes' theorem that

$$\int_{\mathbb{R}^d} \psi G(h)\psi \, \mathrm{d}x = \int_{\partial\Omega} \varphi \partial_n \varphi \, \mathrm{d}\sigma = \iint_{\Omega} \left|\nabla_{x,y}\varphi\right|^2 \mathrm{d}y \, \mathrm{d}x \geq 0. \tag{2.1.11}$$

i) This immediately implies that, for any regular solution to $\partial_t h + G(h)h = 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} h^2 \, \mathrm{d}x = -\int_{\mathbb{R}^d} hG(h)h \, \mathrm{d}x \leq 0.$$

We say that $\int_{\mathbb{R}^d} h^2 \, \mathrm{d}x$ is a Lyapunov functional.

ii) On the other hand, if h is a regular solution to $\partial_t h + G(h)\kappa = 0$, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(\Sigma) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \left(\sqrt{1 + |\nabla h|^2} - 1\right) \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \frac{\nabla h \cdot \nabla \partial_t h}{\sqrt{1 + |\nabla h|^2}} \, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} (\partial_t h)\kappa \, \mathrm{d}x && \text{(integration by parts)} \\
&= -\int_{\mathbb{R}^d} \kappa G(h)\kappa \, \mathrm{d}x \leq 0 && \text{(using (2.1.11))}.
\end{aligned}$$

This proves that $\mathcal{A}(\Sigma)$ is a Lyapunov functional for the Mullins-Sekerka equation.

We refer to Section 2.3.5 for references and additional results related to these two examples. The next result generalizes the previous observations: the L^2-norm and the area functional are Lyapunov functionals for the Hele-Shaw and Mullins-Sekerka equations, in any dimension, uniformly in g and λ.

Proposition 2.1.6. *Let $d \geq 1$, $(g, \lambda) \in [0, +\infty)^2$ and assume that h is a smooth solution to*

$$\partial_t h + G(h)(gh + \lambda\kappa) = 0.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} h^2 \, \mathrm{d}x \leq 0 \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(\Sigma) \leq 0.$$

Proof. See Chapter **??**. □

In addition, for the Hele-Shaw equation, the square of the L^2-norm decays in a convex manner.

Proposition 2.1.7. *Let $d \geq 1$. For any regular solution h of the Hele-Shaw equation $\partial_t h + G(h)h = 0$, there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} h(t,x)^2 \,\mathrm{d}x \leq 0 \quad \textit{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{R}^d} h(t,x)^2 \,\mathrm{d}x \geq 0.$$

Proof. See Chapter **??**. □

2.2 The Muskat equation

2.2.1 The Muskat problem

The Muskat equation is a two-fluids in porous media, analogue of the Hele-Shaw equation. Here we consider a time-dependent free surface $\Sigma(t)$ separating two fluid domains $\Omega_1(t)$ and $\Omega_2(t)$. To simplify, we consider only two-dimensional fluids so that

$$\begin{aligned}
\Omega_1(t) &= \{(x,y) \in \mathbb{R}\times\mathbb{R}\,;\, y > f(t,x)\}\,,\\
\Omega_2(t) &= \{(x,y) \in \mathbb{R}\times\mathbb{R}\,;\, y < f(t,x)\}\,,\\
\Sigma(t) &= \partial\Omega_1(t) = \partial\Omega_2(t) = \{y = f(t,x)\}.
\end{aligned}$$

Introduce the density ρ_i, the velocity v_i and the pressure P_i in the domain Ω_i ($i = 1,2$). One assumes that the velocities v_1 and v_2 obey Darcy's law. Then, the equations by which the motion is to be determined are

$$\begin{aligned}
v_i &= -\nabla(P_i + \rho_i g y) && \text{in } \Omega_i,\\
\operatorname{div} v_i &= 0 && \text{in } \Omega_i,\\
P_1 &= P_2 && \text{on } \Sigma,\\
v_1\cdot n &= v_2\cdot n && \text{on } \Sigma,\\
\partial_t f &= \sqrt{1+(\partial_x f)^2}\, v_2\cdot n
\end{aligned}$$

where g is the gravity and n is the outward unit normal to Ω_2 on Σ,

$$n = \frac{1}{\sqrt{1 + (\partial_x f)^2}} \begin{pmatrix} -\partial_x f \\ 1 \end{pmatrix}.$$

2.2.2 The Córdoba-Gancedo formulation

Changes of unknowns, reducing the Muskat problem to an evolution equation for the free surface parametrization, have been known for quite a time (see the references at the end of this chapter). This approach was further developed by Córdoba and Gancedo who obtained the following beautiful compact formulation of the Muskat equation:

$$\partial_t f = \frac{\rho}{2\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \, \mathrm{d}\alpha, \tag{2.2.1}$$

where $\rho = \rho_2 - \rho_1$ is the difference of the densities of the two fluids and $\Delta_\alpha f$ is the slope

$$\Delta_\alpha f(t,x) = \frac{f(t,x) - f(t,x-\alpha)}{\alpha}. \tag{2.2.2}$$

We assume that $\rho_2 > \rho_1$ (heavier fluid below the lighter one) and then we may set $\rho = 2$ without loss of generality. It is not obvious that the right-hand side is well defined, and we will address this question later in this section.

Besides its esthetic aspect, this formulation allows to apply tools at interface of harmonic analysis and nonlinear partial differential equations. One can think of the circle of methods centering around the study of the Hilbert transform $\mathcal{H}$ and Riesz potentials, or Besov and Triebel-Lizorkin spaces.

Recall that the Hilbert transform $\mathcal{H}$ is defined by

$$\widehat{\mathcal{H}u}(\xi) = -i \frac{\xi}{|\xi|} \hat{u}(\xi).$$

It follows that the fractional Laplacian $|D| = (-\partial_{xx})^{\frac{1}{2}}$ satisfies

$$|D| = \partial_x \mathcal{H}.$$

Alternatively, it can be defined by a singular integral:

$$\mathcal{H}f(x) = \frac{1}{\pi}\mathrm{pv}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y, \tag{2.2.3}$$

where the integral is understood in the sense of principal values:

$$\begin{aligned}\mathcal{H}f(x) &= \frac{1}{\pi}\mathrm{pv}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y = \frac{1}{\pi}\mathrm{pv}\int_{\mathbb{R}} \frac{f(x-y)}{y}\,\mathrm{d}y\\ &= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<\frac{1}{\varepsilon}} \frac{f(x-y)}{y}\,\mathrm{d}y.\end{aligned}$$

Now, observe that

$$\mathcal{H}f(x) = -\frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \frac{f(x)-f(x-\alpha)}{\alpha}\,\mathrm{d}\alpha = -\frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \Delta_\alpha f\,\mathrm{d}\alpha.$$

It follows that the fractional Laplacian $|D| = \partial_x\mathcal{H}$ satisfies

$$|D|\,f = -\frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \partial_x\Delta_\alpha f\,\mathrm{d}\alpha.$$

Consequently, by writing

$$\frac{\partial_x\Delta_\alpha f}{1+(\Delta_\alpha f)^2} = \partial_x\Delta_\alpha f - (\partial_x\Delta_\alpha f)\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2},$$

we see that the Muskat equation can be written under the form

$$\partial_t f + |D|\,f = \mathcal{T}(f)f \quad \text{where} \quad \mathcal{T}(f)f = -\frac{1}{\pi}\int_{\mathbb{R}}(\partial_x\Delta_\alpha f)\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}\,\mathrm{d}\alpha.$$

To give a rigorous meaning to the Muskat equation, we will show in a moment that the map $f\mapsto \mathcal{T}(f)f$ is locally Lipschitz from $H^1(\mathbb{R})\cap H^{\frac{3}{2}}(\mathbb{R})$ to $L^2(\mathbb{R})$.

To do so, it is convenient to recall a characterization of Sobolev spaces in terms of finite differences. Given a function $f\colon\mathbb{R}\to\mathbb{R}$, an integer $m\in\mathbb{N}\setminus\{0\}$ and a real number $h\in\mathbb{R}$, we define the finite differences $\delta_h^m f$ as follows:

$$\delta_h f(x) = f(x) - f(x-h), \qquad \delta_h^{m+1} f = \delta_h(\delta_h^m f).$$

Notice that, by notations, we have

$$(\Delta_\alpha f)(x) = \frac{\delta_\alpha f}{\alpha}\cdot$$

Remark 2.2.1. *The operators ∂_x, $\mathcal{H}$, $|D|^s$ and δ_α are Fourier multipliers:*

$$\widehat{\partial_x u}(\xi) = i\xi\hat{u}(\xi), \qquad \widehat{\delta_\alpha u}(\xi) = (1 - e^{-i\alpha\xi})\hat{u}(\xi),$$
$$\widehat{\mathcal{H}u}(\xi) = -i\frac{\xi}{|\xi|}\hat{u}(\xi), \qquad \widehat{|D|^s u}(\xi) = |\xi|^s\hat{u}(\xi).$$

Consequently, they commute with each other.

2.2.3 Gagliardo semi-norms

Given $s \in (0,1)$, we define the following Gagliardo semi-norm

$$\|u\|^2_{\dot{F}^s_{2,2}} := \iint_{\mathbb{R}\times\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|^{2s}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|}. \tag{2.2.4}$$

Then, for $s \in (0,1)$,

$$\|u\|^2_{\dot{H}^s} = \frac{1}{4\pi c(s)} \|u\|^2_{\dot{F}^s_{2,2}} \quad \text{with} \quad c(s) = \int_{\mathbb{R}} \frac{1-\cos(h)}{|h|^{1+2s}}\,\mathrm{d}h. \tag{2.2.5}$$

Notice that this is equivalent to

$$\begin{aligned} \|u\|^2_{\dot{H}^s} &= \frac{1}{4\pi c(s)} \iint_{\mathbb{R}\times\mathbb{R}} \frac{|u(x)-u(x-\alpha)|^2}{|\alpha|^{2s}}\,\mathrm{d}x \frac{\mathrm{d}\alpha}{|\alpha|} \\ &= \frac{1}{4\pi c(s)} \int_{\mathbb{R}} \frac{\|\delta_\alpha u\|^2_{L^2}}{|\alpha|^{2s}} \frac{\mathrm{d}\alpha}{|\alpha|}. \end{aligned} \tag{2.2.6}$$

This immediately implies the following

Lemma 2.2.2. *For all $a \in [0,+\infty)$ and $b \in (0,1)$, there exists a positive constant $C > 0$ such that*

$$\int_{\mathbb{R}} \|\delta_\alpha f\|^2_{\dot{H}^a} \frac{\mathrm{d}\alpha}{|\alpha|^{1+2b}} = C\,\|f\|^2_{\dot{H}^{a+b}}. \tag{2.2.7}$$

Proof. Since $\|\delta_\alpha f\|_{\dot{H}^a} = \|\delta_\alpha(|D|^a f)\|_{L^2}$, (2.2.7) follows at once from (2.2.6)Ê applied with $u = |D|^a f$. □

As an example of properties that is very simple to prove using the definition of Sobolev spaces in terms of finite differences, let us prove the following proposition (from Alazard and Lazar [28]).

Proposition 2.2.3. *Consider the operator*

$$\mathcal{T}(f)g = -\frac{1}{\pi}\int_{\mathbb{R}} \Delta_\alpha g_x \, \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2} \, \mathrm{d}\alpha,$$

where $g_x := \partial_x g$.

i) For all f *in* $H^1(\mathbb{R})$ *and all* g *in* $H^{\frac{3}{2}}(\mathbb{R})$*, the function*

$$\alpha \mapsto \Delta_\alpha g_x \, \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}$$

belongs to $L^1_\alpha(\mathbb{R}; L^2_x(\mathbb{R}))$*. Consequently,* $\mathcal{T}(f)g$ *belongs to* $L^2(\mathbb{R})$*. Moreover, there is a constant* C *such that*

$$\|\mathcal{T}(f)g\|_{L^2} \leq C \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^{\frac{3}{2}}} . \tag{2.2.8}$$

ii) There exists a constant $C > 0$ *such that, for all functions* f_1, f_2 *in* $H^1(\mathbb{R})$ *and for all* g *in* $H^{\frac{3}{2}}(\mathbb{R})$*,*

$$\|(\mathcal{T}(f_1) - \mathcal{T}(f_2))g\|_{L^2} \leq C \|f_1 - f_2\|_{\dot{H}^1} \|g\|_{\dot{H}^{\frac{3}{2}}} .$$

iii) The map $f \mapsto \mathcal{T}(f)f$ *is locally Lipschitz from* $H^{\frac{3}{2}}(\mathbb{R})$ *to* $L^2(\mathbb{R})$*.*

Proof. *i*) The proof will use the following Sobolev embedding (see Theorem 5.3.1 as well as the remark which follows the latter)

$$\dot{H}^t(\mathbb{R}) \hookrightarrow L^{\frac{2}{1-2t}}(\mathbb{R}) \quad \text{for} \quad 0 \leq t < \frac{1}{2}. \tag{2.2.9}$$

In particular, for $t = 1/4$, this gives that $\dot{H}^{1/4}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$.

Recall that, by notation, we have

$$\Delta_\alpha f = \frac{\delta_\alpha f}{\alpha}.$$

By combining the previous Sobolev embedding with Hölder's inequality, we obtain

$$\begin{aligned}\left\|\Delta_\alpha g_x \frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}\right\|_{L^2} &\leq \|(\Delta_\alpha g_x)(\Delta_\alpha f)\|_{L^2} \leq \|\Delta_\alpha g_x\|_{L^4} \|\Delta_\alpha f\|_{L^4} \\ &\lesssim \frac{\|\delta_\alpha g_x\|_{\dot{H}^{\frac14}}}{|\alpha|} \frac{\|\delta_\alpha f\|_{\dot{H}^{\frac14}}}{|\alpha|}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathcal{T}(f)g\|_{L^2} &\lesssim \int \frac{\|\delta_\alpha g_x\|_{\dot{H}^{\frac14}}}{|\alpha|} \frac{\|\delta_\alpha f\|_{\dot{H}^{\frac14}}}{|\alpha|} \, d\alpha \\ &\lesssim \int \frac{\|\delta_\alpha g_x\|_{\dot{H}^{\frac14}}}{|\alpha|^{\frac14}} \frac{\|\delta_\alpha f\|_{\dot{H}^{\frac14}}}{|\alpha|^{\frac34}} \frac{d\alpha}{|\alpha|} \\ &\lesssim \left(\int \frac{\|\delta_\alpha g_x\|^2_{\dot{H}^{\frac14}}}{|\alpha|^{2\frac14}} \frac{d\alpha}{|\alpha|}\right)^{\frac12} \left(\int \frac{\|\delta_\alpha f\|^2_{\dot{H}^{\frac14}}}{|\alpha|^{2\frac34}} \frac{d\alpha}{|\alpha|}\right)^{\frac12} \\ &\lesssim \|g_x\|_{\dot{H}^{\frac12}} \|f\|_{\dot{H}^1} = \|g\|_{\dot{H}^{\frac32}} \|f\|_{\dot{H}^1},\end{aligned}$$

hence the wanted inequality (2.2.8) .

ii) Write that

$$(\mathcal{T}(f_1) - \mathcal{T}(f_2))g = -\frac{1}{\pi} \int \Delta_\alpha g_x \Delta_\alpha (f_1 - f_2) M(\alpha, x) \, d\alpha$$

where

$$M(\alpha, x) = \frac{(\Delta_\alpha f_1) + \Delta_\alpha f_2}{(1 + (\Delta_\alpha f_1)^2)(1 + (\Delta_\alpha f_2)^2)}.$$

Since $|M(\alpha, x)| \leq 1$, by repeating similar arguments to those used in the first part, we get

$$\begin{aligned}\|(\mathcal{T}(f_1) - \mathcal{T}(f_2))g\|_{L^2} &\leq \frac{1}{\pi} \int \frac{\|\delta_\alpha g_x\|_{L^4}}{|\alpha|} \frac{\|\delta_\alpha (f_1 - f_2)\|_{L^4}}{|\alpha|} \, d\alpha \\ &\leq \frac{1}{\pi} \int \frac{\|\delta_\alpha g_x\|_{\dot{H}^{\frac14}}}{|\alpha|^{1/4}} \frac{\|\delta_\alpha (f_1 - f_2)\|_{\dot{H}^{\frac14}}}{|\alpha|^{3/4}} \frac{d\alpha}{|\alpha|} \\ &\lesssim \|g_x\|_{\dot{H}^{\frac14+\frac14}} \|f_1 - f_2\|_{\dot{H}^{\frac14+\frac34}}.\end{aligned}$$

which implies

$$\|(\mathcal{T}(f_1)-\mathcal{T}(f_2))g\|_{L^2} \lesssim \|f_1-f_2\|_{\dot H^1}\|g\|_{\dot H^{\frac32}}.$$

iii) Consider f_1 and f_2 in $H^{\frac32}(\mathbb{R})$. Then

$$\mathcal{T}(f_1)f_1-\mathcal{T}(f_2)f_2=\mathcal{T}(f_1)(f_1-f_2)+(\mathcal{T}(f_1)-\mathcal{T}(f_2))f_2.$$

Then (2.2.8) implies that the L^2-norm of the first term is bounded by

$$C\|f_1\|_{\dot H^1}\|f_1-f_2\|_{\dot H^{\frac32}}.$$

We estimate the second term by using statement *ii*). It follows that

$$\|\mathcal{T}(f_1)f_1-\mathcal{T}(f_2)f_2\|_{L^2} \lesssim \left(\|f_1\|_{H^{\frac32}}+\|f_2\|_{H^{\frac32}}\right)\|f_1-f_2\|_{H^{\frac32}},$$

which completes the proof. □

2.2.4 The critical Cauchy problem

In section, I discuss the main result from a series of papers with Quoc-Hung Nguyen (see [33, 32, 31]) devoted to the study of solutions with critical regularity for the two-dimensional Muskat equation. We prove that the Cauchy problem is well-posed on the endpoint Sobolev space $H^{3/2}(\mathbb{R})$ of L^2 functions with three-half derivative in L^2. This result is optimal with respect to the scaling of the equation.

A key feature of the Muskat equation is that (2.2.1) is preserved by the change of unknowns:

$$f(t,x)\mapsto f_\lambda(t,x):=\frac{1}{\lambda}f(\lambda t,\lambda x).$$

Now, by a direct calculation, one verifies that

$$\left\|f_\lambda\,\big|_{t=0}\right\|_{\dot H^{\frac32}}=\|f_0\|_{\dot H^{\frac32}}.$$

This means that the space $\dot H^{\frac32}(\mathbb{R})$ is a critical space for the study of the Cauchy problem.

Exercise 2.2.4. *Check the previous claims.*

Theorem 2.2.5 (from [31]). *i) For any $f_0 \in H^{\frac{3}{2}}(\mathbb{R})$, there is $T > 0$ such that the Cauchy problem has a unique solution f in*

$$X^{\frac{3}{2}}(T) = \left\{ f \in C^0\big([0,T]; H^{\frac{3}{2}}(\mathbb{R})\big) \; ; \; \int_0^T \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial_x f)^2} \,\mathrm{d}x \,\mathrm{d}t < \infty \right\}.$$

ii) There exists ε_0 such that if $\|f_0\|_{\dot{H}^{\frac{3}{2}}} \le \varepsilon_0$ then the solution exists for all time.

The proof requires to

- uncover a certain null-type structure because of a degenerate parabolic behavior;
- estimate the solutions for a norm which depends on the initial data themselves.

2.2.5 Weighted fractional Laplacians

To prove Theorem 2.2.5, one well-known difficulty is that one cannot define a flow map such that the lifespan is bounded from below on bounded subsets of this critical Sobolev space. To overcome this, in [31] we estimate the solutions for a norm which depends on the initial data themselves, using the weighted fractional Laplacians introduced in our previous works [33, 32].

As the name indicates, a weighted fractional Laplacian attempts to be a slight modulation of the usual fractional Laplacian $|D|^s = (-\partial_{xx})^{s/2}$.

Notation 2.2.6. *Consider $s \in [0, +\infty)$ and a function $\phi\colon [0,+\infty) \to [1,\infty)$. The weighted fractional Laplacian $|D|^{s,\phi}$ denotes the Fourier multiplier with symbol $|\xi|^s \phi(|\xi|)$, such that*

$$\mathcal{F}(|D|^{s,\phi} f)(\xi) = |\xi|^s \phi(|\xi|) \mathcal{F}(f)(\xi).$$

Moreover, we define the space

$$\mathcal{H}^{s,\phi}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) \; : \; |D|^{s,\phi} f \in L^2(\mathbb{R}) \}. \tag{2.2.10}$$

We will consider special weight functions ϕ depending on some extra functions $\kappa\colon [0,\infty) \to [1,\infty)$, of the form

$$\phi(\lambda) = \int_0^\infty \frac{1 - \cos(h)}{h^2} \kappa\left(\frac{\lambda}{h}\right) \mathrm{d}h, \quad \text{for } \lambda \ge 0. \tag{2.2.11}$$

In addition we will always assume that κ is an admissible weight, in the sense of the following definition.

Definition 2.2.7. *An admissible weight is a function* $\kappa\colon [0,\infty) \to [1,\infty)$ *satisfying the following three conditions:*

(H1) *κ is increasing;*

(H2) *there exists a positive constant c_0 such that $\kappa(2r) \le c_0\kappa(r)$ for any $r \ge 0$;*

(H3) *the function $r \mapsto \kappa(r)/\log(1+r)$ is decreasing on $[r_0,\infty)$, for some $r_0 > 0$ large enough.*

The next results contain the main results about these operators.

Lemma 2.2.8. *For all $\sigma > 0$, there exists $C_\sigma > 0$ such that, for all $0 < r \le \mu$,*

$$r^\sigma \kappa\left(\frac{1}{r}\right) \le C_\sigma\, \mu^\sigma \kappa\left(\frac{1}{\mu}\right), \tag{2.2.12}$$

$$r^\sigma \kappa^2\left(\frac{1}{r}\right) \le C_\sigma\, \mu^\sigma \kappa^2\left(\frac{1}{\mu}\right). \tag{2.2.13}$$

Proof. *i*) Use the decomposition:

$$r^\sigma \kappa\left(\frac{1}{r}\right) = \frac{\kappa\left(\frac{1}{r}\right)}{\log\left(4+\frac{1}{r}\right)} \times \left[r^\sigma \log\left(e^{\frac{1}{\sigma}} + \frac{1}{r}\right)\right] \times \frac{\log\left(4+\frac{1}{r}\right)}{\log\left(e^{\frac{1}{\sigma}}+\frac{1}{r}\right)}.$$

By assumption, the first factor is an increasing function of r for r large enough. By computing the derivative, it is easily verified that the second factor is also an increasing function of r. Eventually, the third factor is a bounded function on $(0,+\infty)$.

ii) We proceed as above, using this time the decomposition

$$r^\sigma \kappa^2\left(\frac{1}{r}\right) = \frac{\kappa^2\left(\frac{1}{r}\right)}{\log^2\left(4+\frac{1}{r}\right)} \times \left[r^\sigma \log^2\left(e^{\frac{2}{\sigma}} + \frac{1}{r}\right)\right] \times \frac{\log^2\left(4+\frac{1}{r}\right)}{\log^2\left(e^{\frac{2}{\sigma}}+\frac{1}{r}\right)}.$$

This completes the proof. □

Remark 2.2.9. *These inequalities have the following interpretation: even if the function $r \to \kappa(1/r)$ and $r \to \kappa^2(1/r)$ are decreasing, since the function $\kappa(r)/\log(2+r)$ is decreasing for r large enough, one expects that $r^\sigma \kappa(1/r)$ and $r^\sigma \kappa^2(1/r)$ behave as increasing functions of r.*

We will see that κ and ϕ are equivalent. The reason for introducing two different functions to code a single operator is that we will use them for different purposes. We use ϕ when we prefer to work with the frequency variable, whereas we will use κ when the physical variable is more practical. The next proposition will be used later to switch calculations between frequency and physical variables.

Proposition 2.2.10. *Assume that ϕ is as defined in (2.2.11) for some function κ satisfying Assumption 2.2.7. Then, for all $g \in \mathcal{S}(\mathbb{R})$, there holds*

$$|D|^{1,\phi} g(x) = \frac{1}{4} \int_{\mathbb{R}} \frac{2g(x) - g(x+h) - g(x-h)}{h^2} \kappa\left(\frac{1}{|h|}\right) \mathrm{d}h.$$

Proof. Notice that the Fourier transform of the function

$$\int_{\mathbb{R}} \frac{2g(x) - g(x+h) - g(x-h)}{h^2} \kappa\left(\frac{1}{|h|}\right) \mathrm{d}h,$$

is given by

$$\left(\int_{\mathbb{R}} \frac{2 - 2\cos(h\xi)}{h^2} \kappa\left(\frac{1}{|h|}\right) \mathrm{d}h\right) \hat{g}(\xi).$$

Therefore

$$\left(4|\xi| \int_0^\infty \frac{1-\cos(h)}{h^2} \kappa\left(\frac{|\xi|}{|h|}\right) \mathrm{d}h\right) \hat{g}(\xi) = 4\phi(\xi)\,|\xi|\,\hat{g}(\xi) = 4\widehat{|D|^{1,\phi} g}(\xi),$$

equivalent to the wanted result. □

Eventually, we will need the following link between $|D|^{s,\phi}$ and the function κ.

Proposition 2.2.11. *i) There exist $c, C > 0$ such that, for all $\lambda \geq 0$,*

$$c\kappa(\lambda) \leq \phi(\lambda) \leq C\kappa(\lambda). \tag{2.2.14}$$

ii) Given $g \in \mathcal{S}(\mathbb{R})$, define the semi-norm

$$\|g\|_{s,\kappa} = \left(\iint_{\mathbb{R}^2} |2g(x) - g(x+h) - g(x-h)|^2 \left(\frac{1}{|h|^s} \kappa\left(\frac{1}{|h|}\right)\right)^2 \frac{\mathrm{d}x\,\mathrm{d}h}{|h|}\right)^{\frac{1}{2}}.$$

Then, for all $0 < s < 2$*, there exist* $c, C > 0$ *such that, for all* $g \in \mathcal{S}(\mathbb{R})$*,*

$$c \int_{\mathbb{R}} \left| |D|^{s,\phi} g(x) \right|^2 \mathrm{d}x \le \|g\|_{s,\kappa}^2 \le C \int_{\mathbb{R}} \left| |D|^{s,\phi} g(x) \right|^2 \mathrm{d}x.$$

Proof. We prove statement $ii)$ only, the proof of statement $i)$ is similar.

$ii)$ For $h \in \mathbb{R}$, the Fourier transform of $x \mapsto 2g(x) - g(x+h) - g(x-h)$ is given by $(2 - 2\cos(\xi h))\hat{g}(\xi)$. So, Plancherel's identity implies that

$$\begin{aligned}\|g\|_{s,\kappa}^2 &= \iint_{\mathbb{R}^2} |2g(x) - g(x+h) - g(x-h)|^2 \kappa^2 \left(\frac{1}{|h|}\right) \frac{1}{|h|^{1+2s}} \,\mathrm{d}x\,\mathrm{d}h \\ &= \int_{\mathbb{R}} I(\xi) \left|\hat{g}(\xi)\right|^2 \mathrm{d}\xi,\end{aligned}$$

where

$$I(\xi) = \frac{2}{\pi} \int_{\mathbb{R}^2} (1 - \cos(\xi h))^2 \kappa^2 \left(\frac{1}{|h|}\right) \frac{1}{|h|^{1+2s}} \,\mathrm{d}h.$$

We claim that $I(\xi) \sim |\xi|^{2s} \phi(|\xi|)^2$.

Since $|1 - \cos(\theta)| \le \min\{2, \theta^2\}$ for all $\theta \in \mathbb{R}$, we have

$$I(\xi) \le \frac{4}{\pi^2} \int_{|h\xi| \ge 1} \kappa^2 \left(\frac{1}{|h|}\right) \frac{1}{|h|^{1+2s}} \,\mathrm{d}h + \frac{1}{\pi^2} \int_{|h\xi| \le 1} |\xi|^4 \, |h|^4 \, \kappa^2 \left(\frac{1}{|h|}\right) \frac{1}{|h|^{1+2s}} \,\mathrm{d}h.$$

Since κ is increasing (by assumption), the first integral is estimated by

$$\int_{|h\xi| \ge 1} \kappa^2 \left(\frac{1}{|h|}\right) \frac{1}{|h|^{1+2s}} \,\mathrm{d}h \le (\kappa(|\xi|))^2 \int_{|h\xi| \ge 1} \frac{\mathrm{d}h}{|h|^{1+2s}} \lesssim \kappa^2(|\xi|) \, |\xi|^{2s}.$$

To estimate the second integral, we use Lemma 2.2.8 to infer

$$|h| \le \frac{1}{|\xi|} \quad \Rightarrow \quad |h|^{2-s} \kappa^2 \left(\frac{1}{|h|}\right) \lesssim \frac{1}{|\xi|^{2-s}} \kappa^2 \left(|\xi|\right).$$

It follows that

$$\begin{aligned}\int_{|h\xi| \le 1} |\xi|^4 \, |h|^4 \, \kappa^2 \left(\frac{1}{|h|}\right) \frac{1}{|h|^{1+2s}} \,\mathrm{d}h &= \int_{|h\xi| \le 1} |\xi|^4 \, |h|^4 \, \kappa^2 \left(\frac{1}{|h|}\right) \frac{1}{|h|^{1+2s}} \,\mathrm{d}h \\ &= |\xi|^4 \int_{|h\xi| \le 1} |h|^{2-s} \, \kappa^2 \left(\frac{1}{|h|}\right) \frac{1}{|h|^{s-1}} \,\mathrm{d}h \\ &\lesssim \kappa(|\xi|)^2 |\xi|^{2+s} \int_{|h| \le 1/|\xi|} \frac{\mathrm{d}h}{|h|^{s-1}} \lesssim \kappa(|\xi|)^2 |\xi|^{2s},\end{aligned}$$

where we have used the assumption $s < 2$ to obtain the last inequality.

Now, by combining the previous estimates, we deduce that $I(\xi) \lesssim |\xi|^{2s}\kappa(|\xi|)^2$. Since $\phi \sim \kappa$, this proves that $I(\xi) \lesssim |\xi|^{2s}\phi(|\xi|)^2$.

The proof of the lower bound is straightforward:

$$\begin{aligned} I(\xi) &\gtrsim \int_{\frac{2\pi}{5} \le \xi h \le \frac{3\pi}{5}} \kappa^2\left(\frac{1}{|h|}\right)\frac{1}{|h|^{1+2s}}\,\mathrm{d}h \gtrsim \left(\int_{\frac{2\pi}{5} \le \xi h \le \frac{3\pi}{5}} \mathrm{d}h\right)\kappa^2(|\xi|)|\xi|^{1+2s} \\ &\gtrsim (\kappa(|\xi|))^2|\xi|^{2s}. \end{aligned}$$

Using again the equivalence $\phi \sim \kappa$, this concludes the proof of the claim $I(\xi) \sim |\xi|^{2s}\phi(|\xi|)^2$, which in turn concludes the proof of statement *ii*). □

For large data, the time of existence must depend on the initial data themselves and not only on its critical Sobolev norm. To overcome this problem, our strategy is to estimate the solution for a stronger norm whose definition involves the initial data themselves. To do so, we use the following elementary result.

Proposition 2.2.12. *For all f_0 in $H^{\frac{3}{2}}(\mathbb{R})$, there exists an admissible weight κ satisfying $\lim_{r\to+\infty}\kappa(r) = \infty$ and such that f_0 belongs to $\mathcal{H}^{\frac{3}{2},\phi}(\mathbb{R})$ where ϕ is given by* (2.2.11).

Lemma 2.2.13. *For any nonnegative integrable function $\omega \in L^1(\mathbb{R})$, there exists a function $\eta\colon [0,\infty) \to [1,\infty)$ satisfying the following properties:*

1. *η is increasing and $\lim\limits_{r\to\infty}\eta(r) = \infty$,*
2. *$\eta(2r) \le 2\eta(r)$ for any $r \ge 0$,*
3. *ω satisfies the enhanced integrability condition:*

$$\int_{\mathbb{R}} \eta(|r|)\omega(r)\,\mathrm{d}r < \infty, \tag{2.2.15}$$

4. *moreover, the function $r \mapsto \eta(r)/\log(4+r)$ is decreasing on $[0,\infty)$.*

Proof. Consider a sequence $(\alpha_k)_{k\ge1}$ such that $\alpha_1 \ge e^5$ and $\alpha_k \ge \alpha_{k-1}^{10}$ and in addition

$$\forall k \ge 1, \qquad \int_{|r|\ge\alpha_k} \omega(r)\,\mathrm{d}r \le 2^{-k}. \tag{2.2.16}$$

We set

$$\eta(r)=\begin{cases}2 & \text{if } \ 0\le r<\alpha_1,\\[2ex] k+1+\dfrac{\log(\frac{4+r}{4+\alpha_k})}{\log(\frac{4+\alpha_{k+1}}{4+\alpha_k})} & \text{if } \ \alpha_k\le r<\alpha_{k+1}.\end{cases}\tag{2.2.17}$$

It is easy to check that $\eta\colon [0,\infty)\to[1,\infty)$ is an increasing function converging to $+\infty$ when r goes to $+\infty$. Moreover, η satisfies $\eta(2r)\le 2\eta(r)$ for any $r\ge 0$.

In addition,

$$\begin{aligned}\int \eta(|r|)\omega(r)\,\mathrm{d}r &\le \int_{|r|\le\alpha_1} 2\omega(r)\,\mathrm{d}r+\sum_{k=1}^{\infty}(k+2)\int_{\alpha_k\le|r|\le\alpha_{k+1}}\omega(r)\,\mathrm{d}r\\ &\le 2||\omega||_{L^1}+\sum_{k=1}^{\infty}(k+2)2^{-k}\\ &\le 2||\omega||_{L^1}+C.\end{aligned}$$

It remains to prove that $r\mapsto \eta(r)/\log(4+r)$ is decreasing. To do so, write

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\eta(r)}{\log(4+r)}\right)=\frac{1}{\log(4+r)}\left(\eta'(r)-\frac{1}{4+r}\frac{\eta(r)}{\log(4+r)}\right).\tag{2.2.18}$$

So, for $0\le r<\alpha_1$,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\eta(r)}{\log(4+r)}\right)<0,\tag{2.2.19}$$

while for $\alpha_k\le r<\alpha_{k+1}$ with $k\ge 1$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\eta(r)}{\log(4+r)}\right) &\le \frac{1}{(4+r)\log(4+r)^2}\left(\frac{\log(4+r)}{\log(\frac{4+\alpha_{k+1}}{4+\alpha_k})}-k-1\right)\\ &\le \frac{1}{(4+r)\log(4+r)^2}\left(\frac{\log(4+\alpha_{k+1})}{\log(\frac{4+\alpha_{k+1}}{4+\alpha_{k+1}^{1/10}})}-2\right)<0,\end{aligned}$$

where we have used $\alpha_{k+1}\ge e^{5\times 10^k}$.

This proves that $r\mapsto\eta(r)/\log(4+r)$ is decreasing on $[0,\infty)$. The proof is complete. $\square$

Then, we are in position to state an improvement of statement i) in Theorem 2.2.5 which asserts that, whenever one controls a bigger norm than the critical one, the time of existence depends only on the norm. Recall that $\mathcal{H}^{\frac{3}{2},\phi}(\mathbb{R})$ is defined by (2.2.10).

Theorem 2.2.14. *Consider a real number $M_0 > 0$ and a function ϕ given by* (2.2.11) *for some admissible weight κ satisfying $\lim_{r\to+\infty}\kappa(r) = \infty$. Then there exists a time $T_0 > 0$ depending on M_0, κ such that, for any initial data f_0 in $\mathcal{H}^{\frac{3}{2},\phi}(\mathbb{R})$ satisfying $\|f_0\|_{\mathcal{H}^{\frac{3}{2},\phi}} \le M_0$, the Cauchy problem for* (2.2.1) *has a unique solution in the space*

$$\mathcal{X}^{\frac{3}{2},\phi}(T_0) := \left\{ f \in C^0\big([0,T_0];\mathcal{H}^{\frac{3}{2},\phi}(\mathbb{R})\big)\,;\ \int_0^{T_0}\int_{\mathbb{R}} \frac{\big||D|^{2,\phi} f\big|^2}{1+(\partial_x f)^2}\,\mathrm{d}x\,\mathrm{d}t < \infty \right\}.$$

2.3 References (work in progress)

2.3.1 References about the equations

The first rigorous derivation of DarcyÕs law was done by Tartar [413] in the appendix of the Book [383]. Further extensions can be found in [39, 317, 341]. A review of the literature in terms of both mathematical results and physical motivation can be found in the book by Hornung [247]. The latter reference discuss many extensions of Darcy law, including Darcy law with memory, nonlinear Darcy law, or models of one-phase Newtonian flows as well as of non-Newtonian fluids, to name a few. The analysis of filtration in porous media plays a key role in many situations, see for instance [342, 434, 435] and the references there in.

The Hele-Shaw equation plays a key role in Mathematical Biology (see [331, 358, 360, 359, 460, 305]). Muskat introduced the equations that bear his name in [345], to model problems in petroleum engineering (see [346, 349] for many historical comments).

It is often convenient to observe that there is a gradient flow structure. The fact Mullins-Sekerka equation is a gradient flow for the area functional $\mathcal{H}^d(\Sigma)$ was first observed by Almgren [40] and Giacomelli and Otto [219].

The Boussinesq equation (2.1.8) was derived from the Hele-Shaw equation (2.1.6) by Boussinesq [92] to study groundwater infiltration; it also models the flow of gas in porous media (see the monograph of Vázquez [432]). The thin-film equation (2.1.9) was derived from the Mullins-Sekerka equation (2.1.7) by Constantin, Dupont, Goldstein, Kadanoff, Shelley and Zhou in [140] as a lubrication approximation model of the interface between two immiscible fluids in a Hele-Shaw cell.

It has long been understood that the Muskat problem can be reduced to a parabolic evolution equation for the unknown function f (see [105, 207, 364, 395]). The formulation (2.2.1) that we are using arises from the work by Córdoba and Gancedo [153] who have studied this problem using contour integrals, and obtained this beautiful formulation of the Muskat equation in terms of finite differences.

2.3.2 The Cauchy problem for the Hele-Shaw equation

The Cauchy problems for the Hele-Shaw and Mullins-Sekerka equations have been studied by different techniques, for weak solutions, viscosity solutions or also classical solutions (see [27, 30, 120, 125, 126, 150, 208, 213, 235, 239, 284, 291, 352, 365]). In this book, we will restrict ourselves to the study of classical solutions. There are many possible ways to study this problem: to mention a few approaches we quote various PDE methods based on L^2-energy estimates (see the works of Chen [125], Córdoba, Córdoba and Gancedo [150], Knüpfer and Masmoudi [291], Günther and Prokert [235], Cheng, Granero-Belinchón and Shkoller [126]), there are also methods based on functional analysis tools and maximal estimates (see Escher and Simonett [208], the results reviewed in the book by Prüss and Simonett [365]).

2.3.3 The Cauchy problem for the Muskat equation

The study of the Cauchy problem for the Muskat equation begun two decades ago. Inspired by the analysis of free boundary flows, several different approaches succeeded to establish local well-posedness results for smooth enough initial data starting with the works of Yi [451], Ambrose [43, 44], Caflisch, Howison and Siegel [395]. In the last several years, this problem was extensively studied. There are now many different proofs that the Cauchy problem is well-posed, locally in time. The well-posedness of the Cauchy problem was proved in [153] by Córdoba and Gancedo for initial data in $H^3(\mathbb{R})$ in the stable regime $\rho_2 > \rho_1$ (they also proved that the problem is ill-posed in Sobolev spaces when $\rho_2 < \rho_1$). In [126], Cheng, Granero-Belinchón, Shkoller proved the well-posedness of the Cauchy problem in $H^2(\mathbb{R})$ (introducing a Lagrangian point of view which can be used in a broad setting, see [224]). The Cauchy problem was then studied in various sub-critical spaces. Firstly, by Constantin, Gancedo, Shvydkoy and Vicol [141] in the Sobolev space $W^{2,p}(\mathbb{R})$ for some $p > 1$, by Deng, Lei and Lin in Hölder spaces [191], and by Matioc [327, 328] for initial data in $H^s(\mathbb{R})$ with $s > 3/2$ (see also Alazard-Lazar [28] and Nguyen and Pausader [352]). Regularity criteria were obtained in [141, 222] in terms of a control of some critical quantities.

Since the Muskat equation is parabolic, the proof of the local well-posedness results also gives global well-posedness results under a smallness assumption, see Yi [451]. The first global well-posedness results under mild smallness assumptions, namely assuming that the Lipschitz semi-norm is smaller than 1, was obtained by Constantin, Córdoba, Gancedo, Rodríguez-Piazza and Strain [139] (see also [141, 357]).

On the other hand, there are blow-up results for some large enough data by Castro, Córdoba, Fefferman, Gancedo and López-Fernández ([113, 114, 116]). They prove the existence of solutions such that at time $t = 0$ the interface is a graph, at a later time $t_1 > 0$ the interface is no longer a graph and then at a subsequent time $t_2 > t_1$, the interface is C^3 but not C^4.

The previous discussion raises a question about the possible existence of a criteria on the slopes of the solutions which would force/prevent them to enter the unstable regime where the slope is infinite. Surprisingly, it was shown that it is possible to solve the Cauchy problem for initial data whose slope can be arbitrarily large. Deng, Lei and Lin in [191] obtained the first result in this direction, under the assumption that the initial data are monotone. Cameron [108] proved the existence of a modulus of continuity for the derivative, and hence a global existence result assuming only that the product of the maximal and minimal slopes is bounded by 1; thereby allowing arbitrarily large slopes too (recently, Abedin and Schwab also obtained the existence of a modulus of continuity in [1] via Krylov-Safonov estimates). Then, by using a new formulation of the Muskat equation involving oscillatory integrals, Córdoba and Lazar established in [154] that the Muskat equation is globally well-posed in time, assuming only that the initial data is sufficiently smooth and that the $\dot{H}^{3/2}(\mathbb{R})$-norm is small enough. This result was extended to the 3D case by Gancedo and Lazar [214]. Let us also quote papers by Vazquez [433], Granero-Belinchón and Scrobogna [223] for related global existence results for different equations.

Eventually, let us mention that many recent results focus on the existence and possible non-uniqueness of weak-solutions (we refer the reader to [94, 152, 409, 112, 211, 353]). These problems arise for instance in the unstable regime $\rho_1 > \rho_2$, to study the existence mixing zones or the dynamic between the two different regimes.

2.3.4 References about critical Cauchy problems

The study of the well-posedness of the Cauchy problem for various partial differential equations in critical spaces has attracted a lot of attention in the last decades.

The Muskat equation is parabolic, but it is interesting to also discuss other type of equations. We begin with the Schrödinger equation, which is the prototypical example of a semi-linear dispersive equation. For this equation, the study of Cauchy problem in the energy critical case goes back to the works of Cazenave and Weissler [118, 119] and culminates with the global existence results of Bourgain [91], Grillakis [226] and Colliander, Keel, Staffilani, Takaoka and Tao [136]. In sharp contrast with sub-critical problems, the time of existence given by the local theory in [118, 119] depends on the profile of the data and not only on its norm. As a result, the conservation of the energy is unsufficient to obtain a global existence result. Detailed historical accounts of the subject can be found in the book by Tao [412]. We also refer to the recent paper by Merle, Raphäel, Rodnianski and Szeftel [332] which establishes an unexpected blow-up result for supercritical defocusing nonlinear Schrödinger equations. If instead of a semi-linear equation, one considers a quasi-linear problem, then the scaling is not necessarily the only relevant criteria. One key result in this direction is about a hyperbolic equation in general relativity, namely the resolution of the bounded L^2 curvature conjecture by Klainerman, Rodnianski and Szeftel [290].

Many papers have been devoted to the study of critical problems for parabolic equations. Consider for instance the equation

$$\partial_t \theta + u \cdot \nabla \theta + (-\Delta)^{\frac{\alpha}{2}} \theta = 0 \quad \text{with} \quad u = \nabla^{\perp} (-\Delta)^{-\frac{1}{2}} \theta.$$

This equation arises as a dissipative version of the surface quasi-geostrophic equation introduced by Constantin-Majda-Tabak [144]. Here the critical case corresponds to $\alpha = 1$. In this case, the global in time well-posedness has been proved by Kiselev-Nazarov-Volberg [288], Caffarelli-Vasseur [104] and Constantin-Vicol [147] (see also [287, 396, 431]).

2.3.5 Entropies and maximum principles

In the literature, there are many definitions of entropies for various evolution equations. The common idea is that entropy dissipation methods allow to study the large time behavior or to prove functional inequalities (see [76, 111, 53, 209, 438, 83, 196, 82, 462, 276]). For a thorough discussion of the role of entropy methods in information theory, we refer to Villani's lecture notes [437] and his book [438, Chapters 20, 21, 22].

The study of entropies plays a key role in the study of the thin-film equation (and its variant) since the works of Bernis and Friedman [73] and Bertozzi and Pugh [78]. The simplest observation is that, if h is a non-negative solution to $\partial_t h+\partial_x(h\partial_x^3 h)=0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} h^2\,\mathrm{d}x \leq 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}(\partial_x h)^2\,\mathrm{d}x \leq 0.$$

(This can be verified by elementary integrations by parts.) To give an example of hidden Lyapunov functionals, consider, for $p\geq 0$ and a function $h>0$, the functionals

$$H_p(h)=\int_{\mathbb{T}}\frac{h_x^2}{h^p}\,\mathrm{d}x.$$

Laugesen discovered ([303]) that, for $0\leq p\leq 1/2$, $H_p(h)$ is a Lyapunov functional. This result was complemented by Carlen and Ulusoy ([109]) who showed that $H_p(f)$ is an entropy when $0<p<(9+4\sqrt{15})/53$. We also refer to [71, 177, 79, 277] for the study of entropies of the form $\int h^p\,\mathrm{d}x$ with $1/2\leq p\leq 2$.

Still for the thin-film equation, the study of the decay of Lebesgue norms was initiated by Bernis and Friedman [73] and continued by Beretta-Bertsch-Dal Passo [71], Dal Passo–Garcke–Grün [177] and more recently by Jüngel and Matthes [277], who performed a systematic study of entropies for the thin-film equation, by means of a computer assisted proof. The study of these decay estimates is related to the study of functional inequalities: we refer to Bernis [72], Dal Passo–Garcke–Grün [177] and [18].

In addition to Lyapunov functionals, maximum principles also play a key role in the study of these parabolic equations. One can think of the maximum principles for the mean-curvature equation obtained by Huisken [249] and Ecker and Huisken (see [205, 204]), used to obtain a very sharp global existence result of smooth solutions. Many maximum principles exist also for the Hele-Shaw equations (see [284, 120]). In particular, we will use the maximum principle for space-time derivatives proved in [30]. For the thin-film equations of the form $\partial_t h+\partial_x(f(h)\partial_x^3 h)=0$ with $f(h)=h^m$ with $m\geq 3.5$, in one space dimension, if the initial data h_0 is positive, then the solution $h(x,t)$ is guaranteed to stay positive (see [73, 77] and [177, 79, 457, 96]).

Part II

Tools

Chapter 3

Introduction to paradifferential calculus

This chapter is a short self-contained introduction to the application of paradifferential calculus to the study of the Dirichlet-to-Neumann operator.

3.1 Introduction

3.1.1 Bony's paradifferential operators

Bony's theory of paradiferential operators allows to study the regularity of the solutions of nonlinear partial differential equations. This theory lies at the interface between harmonic analysis and microlocal analysis. It has a long history that owes a lot to Calderón and Zygmund, Coifman and Meyer, Kohn and Nirenberg, as well as Hörmander.

Since the work of Kohn–Nirenberg and Hörmander it is said that that T is a pseudo-differential operator if we can define it from a function $a = a(x,\xi)$ by the relation

$$T\big(e^{ix\cdot\xi}\big) = a(x,\xi)e^{ix\cdot\xi}. \tag{3.1.1}$$

We then say that a is the symbol for T and we denote $T = \operatorname{Op}(a)$. For instance, the operator associated with the symbol $a = \sum_\alpha a_\alpha(x)(i\xi)^\alpha$ is simply the differential operator $T = \sum_\alpha a_\alpha(x)\partial_x^\alpha$ (with classical notations). Another fundamental example is the case of the operator $|D_x|$ that we have already defined by

$$|D_x|\, e^{ix\cdot\xi} = |\xi|\, e^{ix\cdot\xi},$$

so that $|D_x| = \mathrm{Op}(|\xi|)$.

The pseudo-differential calculus is a process that associates to a symbol $a = a(x,\xi)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ an operator $\mathrm{Op}(a)$ such that one can understand the properties of these operators (product, adjoint, boundedness on the usual spaces of functions...) simply by looking at the properties of the symbols.

The application that associates an operator $\mathrm{Op}(a)$ to the symbol a is called a quantification. There are very many quantizations that are known to be useful, which are variants of (3.1.1). Bony's quantization is perfectly suited for non-linear problems. Its specificity is to quantize symbols that have a limited regularity in x. It will allow us to quantize, among others, the symbol

$$\sqrt{(1+|\nabla\eta(x)|^2)\,|\xi|^2 - (\nabla\eta(x)\cdot\xi)^2},$$

in situations where η is not very regular (for our subject matter, we are interested in the case where η is lipschitzian, so that this symbol is barely bounded in x).

On the other hand, Calderón and his school favoured the point of view of the study of an operator by the study of its kernel. This distinction is also found in the study of free boundary problems. This is how the operator $|D_x|$ appears much more often written in the form $\partial_x\mathcal{H}$ where $\mathcal{H}$ is the Hilbert transform, which is defined on the functions of a single variable by

$$\mathcal{H}f(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-t|>\varepsilon}\frac{f(t)}{x-t}\,\mathrm{d}t.$$

A central result in the theory of singular integrals concerns the commutator $A\circ|D_x| - |D_x|\circ A$ where A is the operator of multiplication by one function $a(x)$. Calderón proved that this commutator is bounded to L^2 if and only if the function $a(x)$ is lipschitzian (see the introduction of Meyer's book [338]). This estimate will play an important role in the problems we will consider.

3.1.2 The Dirichlet-to-Neumann operator

A notable part of the analysis of free boundary problems consists in studying the Dirichlet-to-Neumann operator. We present here some of the results on this topic which are proved later in Chapter 4.

Let $\eta\colon \mathbb{R}^d \to \mathbb{R}$ be a smooth enough function and consider the open set

$$\Omega := \{\, (x,y) \in \mathbb{R}^d \times \mathbb{R}\, ;\; y < \eta(x)\, \}.$$

If $\psi\colon \mathbb{R}^d \to \mathbb{R}$ is another function, and if we call $\phi\colon \Omega \to \mathbb{R}$ the unique solution of $\Delta_{x,y}\phi = 0$ in Ω satisfying $\phi|_{y=\eta(x)} = \psi$ and a convenient vanishing condition at $y \to -\infty$, one defines the Dirichlet-to-Neumann operator $G(\eta)$ by

$$G(\eta)\psi = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta},$$

where ∂_n is the outward normal derivative on $\partial\Omega$. In Chapter 4 we make precise the above definition and study the action of $G(\eta)$ on different spaces. By standard variational arguments, one can prove that $G(\eta)\psi$ is well-defined for any function ψ that belongs to the Sobolev space $H^{1/2}(\mathbb{R}^d)$.

Proposition 3.1.1. *Let η be a Lipschitz function in $W^{1,\infty}(\mathbb{R}^d)$. Then $G(\eta)$ is well-defined and bounded from $H^{1/2}(\mathbb{R}^d)$ to $H^{-1/2}(\mathbb{R}^d)$ and satisfies an estimate*

$$\|G(\eta)\psi\|_{H^{-\frac{1}{2}}} \le C\big(\|\nabla\eta\|_{L^\infty}\big)\big\|\,|D|^{\frac{1}{2}}\,\psi\big\|_{L^2},$$

where $C(\cdot)$ is a non decreasing continuous function of its argument.

The previous proposition will be proven using classical variational arguments. On the other hand, when the domain is smoother, this problem can be studied by means of of microlocal analysis. In particular, if $\eta \in C^\infty_b(\mathbb{R}^d)$, it is known since Calderón that $G(\eta)$ is a pseudo-differential operator of order 1. This is true in any dimension $d \ge 1$. If $d = 1$, one can give a very simple rigorous meaning to the previous statement. Indeed, the latter simplifies to the following

Proposition 3.1.2. *Assume that $d = 1$ and $\eta \in C^\infty_b(\mathbb{R})$. Then $G(\eta)$ can be written under the form*

$$G(\eta)\psi = |D_x|\,\psi \;+\; R(\eta)\psi, \tag{3.1.2}$$

where $R(\eta)f$ is a smoothing operator, bounded from $H^\mu(\mathbb{R})$ to $H^{\mu+m}(\mathbb{R})$ for any integer m. Namely, for any $m \in \mathbb{N}$, there exists a constant $K \ge 1$ such that

$$\forall \mu \ge \frac{1}{2},\quad \|R_0(\eta)\psi\|_{H^{\mu+m}} \le C\left(\|\eta\|_{H^{\mu+K}}\right)\|\psi\|_{H^\mu}\,, \tag{3.1.3}$$

where $C(\cdot)$ is a non decreasing continuous function of its argument.

This result is not satisfactory for the analysis of the free boundary problems we want to study. For our subject matters, η and ψ are expected to have essentially the same regularity so that the constant K that appears in (3.1.3) corresponds to a loss of derivatives. We need estimates without loss of derivatives. In this direction, let us state the following result about the boundedness of $G(\eta)$ on Sobolev spaces.

Proposition 3.1.3. *Let $d \geq 1$, $s > 1 + \frac{d}{2}$ and $\frac{1}{2} \leq \sigma \leq s$. Then, for all $\eta \in H^s(\mathbb{R}^d)$ and all $f \in H^\sigma(\mathbb{R}^d)$, $G(\eta)f$ belongs to $H^{\sigma-1}(\mathbb{R}^d)$, together with the estimate*

$$\|G(\eta)f\|_{H^{\sigma-1}} \leq C\big(\|\eta\|_{H^s}\big)\,\|f\|_{H^\sigma}\,, \tag{3.1.4}$$

where $C(\cdot)$ is a non decreasing continuous function of its argument.

Ideally, we would like a result which includes the two previous propositions. To do so, we need to pause to introduce paraproducts.

3.1.3 Paraproducts

To make this introductory chapter self-contained, we recall here the definition of a paraproduct, which is the simplest example of a paradifferal operator.

One can define a paraproduct very simply, using the Fourier inversion formula:

$$a(x)b(x) = \frac{1}{(2\pi)^{2d}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} e^{ix\cdot(\xi_1+\xi_2)}\,\widehat{a}(\xi_1)\,\widehat{b}(\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

Let us decompose the integral in three terms

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} = \iint_{|\xi_1+\xi_2|\sim|\xi_2|} + \iint_{|\xi_1+\xi_2|\sim|\xi_1|} + \iint_{|\xi_1|\sim|\xi_2|}$$

to obtain the following decomposition of the product

$$ab = T_a b + T_b a + R(a,b).$$

To define these operators more precisely, let us consider a cut-off function θ in $C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ such that

$$\theta(\xi_1,\xi_2) = 1 \quad \text{if} \quad |\xi_1| \leq \varepsilon_1\,|\xi_2|\,, \qquad \theta(\xi_1,\xi_2) = 0 \quad \text{if} \quad |\xi_1| \geq \varepsilon_2\,|\xi_2|\,,$$

with $0 < \varepsilon_1 < \varepsilon_2 < 1$. Given two functions $a = a(x)$ and $b = b(x)$ one defines

$$
\begin{aligned}
T_a b &= \frac{1}{(2\pi)^{2d}} \iint e^{ix\cdot(\xi_1+\xi_2)} \theta(\xi_1, \xi_2) \widehat{a}(\xi_1) \widehat{b}(\xi_2) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2, \\
T_b a &= \frac{1}{(2\pi)^{2d}} \iint e^{ix\cdot(\xi_1+\xi_2)} \theta(\xi_2, \xi_1) \widehat{a}(\xi_1) \widehat{b}(\xi_2) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2, \\
R_{\mathcal{B}}(a, b) &= \frac{1}{(2\pi)^{2d}} \iint e^{ix\cdot(\xi_1+\xi_2)} \big(1 - \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1)\big) \widehat{a}(\xi_1) \widehat{b}(\xi_2) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2.
\end{aligned}
$$

This is Bony's decomposition of the product of two functions. One says that $T_a b$ and $T_b a$ are paraproducts, while $R_{\mathcal{B}}(a, b)$ is a remainder. The key property is that a paraproduct by an L^∞ function acts on any Sobolev spaces H^s with s in $\mathbb{R}$. The remainder term $R_{\mathcal{B}}(a, b)$ is smoother than the paraproducts $T_a b$ and $T_b a$. We have the following results:

$$
\begin{aligned}
&\forall \sigma \in \mathbb{R}, && a \in L^\infty(\mathbb{R}^d), \quad b \in H^\sigma(\mathbb{R}^d) && \Rightarrow \quad T_a b \in H^\sigma(\mathbb{R}^d), \\
&\forall \sigma \in (0, +\infty), && a \in H^\sigma(\mathbb{R}^d), \quad b \in H^\sigma(\mathbb{R}^d) && \Rightarrow \quad R_{\mathcal{B}}(a, b) \in H^{2\sigma - d/2}(\mathbb{R}^d).
\end{aligned}
$$

3.1.4 Paralinearization of the Dirichlet–Neumann operator

Once paraproducts have been defined, we are in position to introduce a paralinearization formula for the Dirichlet-to-Neumann operator. The next result, proved in collaboration with Guy Métivier ([29]) allows to study the microlocal properties of the Dirichlet-to-Neumann operator $G(\eta)\psi$ in the case where η and ψ have the same regularity. Again, for the sake of simplicity, we state the result in the particular case where $d = 1$ and refer to Chapter 4 for the general case.

Proposition 3.1.4 (from [29]). *There exists $K > 0$ such that the following property holds: For all functions η and ψ belonging to the Sobolev space $H^s(\mathbb{R})$, with $s \geq K$, we have*

$$
G(\eta)\psi = |D_x| \left(\psi - T_{\partial_y \phi|_{y=\eta}} \eta\right) - \partial_x \left(T_{\partial_x \phi|_{y=\eta}} \eta\right) + F(\eta, \psi),
$$

where ϕ denotes the harmonic extension of ψ,

$$
\Delta_{x,y}\phi = 0 \quad \text{in } \Omega = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < \eta(x)\}, \qquad \phi|_{y=\eta} = \psi,
$$

and where

$$
\|F(\eta, \psi)\|_{H^{2s-K}} \leq C\big(\|\eta\|_{H^s} \big) \|\psi\|_{H^s},
$$

where C is a non-decreasing function of its argument.

For $\eta \in C^\infty$, this gives the Calderón result alluded to earlier. This result makes it easy to prove estimates that would otherwise only be obtained using (difficult) estimates of solutions to elliptic problems. For instance, the commutator properties between the Dirichlet-to-Neumann operator and a Fourier multiplier can be deduced directly from the properties of symbolic calculus.

The novelty is that we understand perfectly how the operator $G(\eta)$ depends on η, except for a regular remainder. The fact that the remainder R is twice as regular as η and ψ plays an important role for applications to the study of small divisors problems (see [29, 16]) or to the study of exact controllability (see [17, 458]) in the study of 3D travelling waves.

On the other hand, in the study of Cauchy's problem with irregular data, it is enough to have a remainder which has the same regularity as η. The difficulty is then to know how to deal with indices s which are small. In this direction, we have proven with Nicolas Burq and Claude Zuily various results. For instance, in the article [19] we have shown that the previous theorem is true with $s > 2+d/2$ and $K = (3+d)/2$. In addition we gave an estimate of the differential of $R(\eta, \psi)$ with respect to η. Another useful result is the following

Proposition 3.1.5 (from [22])**.** *Consider real numbers s, σ, ε such that*

$$s > 1 + \frac{1}{2}, \qquad \frac{1}{2} \le \sigma \le s - \frac{1}{2}, \qquad 0 < \varepsilon \le \frac{1}{2}, \qquad \varepsilon < s - 1 - \frac{1}{2}.$$

Then there exists a non-decreasing function $C \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\| G(\eta) f - |D_x| f \|_{H^{\sigma-1+\varepsilon}(\mathbb{R})} \le C\big(\|\eta\|_{H^s(\mathbb{R})}\big) \, \|f\|_{H^\sigma(\mathbb{R})} \, .$$

3.2 The Calderón-Vaillancourt theorem

Given a function $a = a(x, \xi)$ and a function $u = u(x)$, we want to study operators A defined by expression of the form

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x, \xi) \widehat{u}(\xi) \, \mathrm{d}\xi.$$

Notice that, if $a = 1$, then the Fourier inversion theorem implies that $A = \mathrm{Id}$. More generally, if $a(x, \xi) = \sum a_\alpha(x)(i\xi)^\alpha$, then A is the differential operator $\sum a_\alpha \partial_x^\alpha$.

A pseudo-differential operator is an operator of the previous form, but where the function $a(x,\xi)$ is not necessarily a polynomial function. In this chapter, we propose to study two elementary results about these operators. We will assume that a is a smooth bounded function and prove that A is bounded on $L^2(\mathbb{R}^d)$.

3.2.1 Continuity on the Schwartz class

Consider a function $a \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. By definition, this means that, for all multi-indices α and β in $\mathbb{N}^d$, we have

$$\sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| < +\infty.$$

Given any function $u \in \mathcal{S}(\mathbb{R}^d)$ in the Schwartz class and a fixed $x \in \mathbb{R}^d$, the function $\xi \mapsto a(x,\xi)\widehat{u}(\xi)$ belongs to $\mathcal{S}(\mathbb{R}^d_\xi)$. In particular, it is integrable and we may define the function $\mathrm{Op}(a)u$ by

$$\mathrm{Op}(a)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi.$$

We say that $\mathrm{Op}(a)$ is a pseudo-differential operator and we call a its symbol.

Proposition 3.2.1. *For any $a \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and any $u \in \mathcal{S}(\mathbb{R}^d)$, the function* $\mathrm{Op}(a)u$ *is well-defined and belongs to $\mathcal{S}(\mathbb{R}^d)$. Moreover* $\mathrm{Op}(a)$ *is continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$.*

Proof. Since $\widehat{u} \in \mathcal{S}(\mathbb{R}^d)$, we can apply Lebesgue's differentiation theorem to check easily that $\mathrm{Op}(a)u \in C^\infty(\mathbb{R}^d)$. So it will suffice to prove estimates.

Using $\|a\|_{L^\infty} < +\infty$ and $\left\|\langle\xi\rangle^{2d}\widehat{u}\right\|_{L^\infty} < +\infty$, we get the inequality

$$|\mathrm{Op}(a)u(x)| \leqslant (2\pi)^{-n} \int \|a\|_{L^\infty} \left\|\langle\xi\rangle^{2d}\widehat{u}\right\|_{L^\infty} \langle\xi\rangle^{-2d}\,\mathrm{d}\xi,$$

which implies that $\mathrm{Op}(a)u$ is bounded together with the estimate

$$\|\mathrm{Op}(a)u\|_{L^\infty} \leqslant C\mathcal{N}_{2d}(\widehat{u})$$

where we used the notation $\mathcal{N}_p(\varphi) = \sum_{|\alpha|\leqslant p,|\beta|\leqslant p} \left\|x^\alpha \partial_x^\beta \varphi\right\|_{L^\infty}$ to denote the canonical semi-norms on the Schwartz space; let us recall that the Fourier transform is continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$ and that

$$\mathcal{N}_{2d}(\widehat{u}) \leqslant C_{2d}\, \mathcal{N}_{3d+1}(u).$$

To estimate the other semi-norms in $\mathcal{S}(\mathbb{R}^d)$ of $\mathrm{Op}(a)u$, we use the following formulas (to be checked as an exercise)

$$\begin{aligned}\partial_{x_j}\,\mathrm{Op}(a)u &= \mathrm{Op}(a)(\partial_{x_j}u) + \mathrm{Op}(\partial_{x_j}a)u,\\ x_j\,\mathrm{Op}(a)u &= \mathrm{Op}(a)(x_j u) + i\,\mathrm{Op}(\partial_{\xi_j}a)u.\end{aligned}$$

Thus, $x^\alpha \partial_x^\beta \,\mathrm{Op}(a)u$ can written as a linear combination of terms of the form

$$\mathrm{Op}\big(\partial_x^\gamma \partial_\xi^\delta a\big)(x^{\alpha-\delta}\partial_x^{\beta-\gamma}u).$$

Since $\partial_x^\gamma \partial_\xi^\delta a$ belongs to $C_b^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ and since $x^{\alpha-\delta}\partial_x^{\beta-\gamma}u$ belongs to $\mathcal{S}(\mathbb{R}^d)$, we are back to the previous case. This shows that $\mathrm{Op}(a)u$ belongs to $\mathcal{S}(\mathbb{R}^d)$ and that we have estimates for the semi-norms of $\mathrm{Op}(a)u$ in terms of the semi-norms of u, which in turn proves that $\mathrm{Op}(a)$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to itself. □

3.2.2 Boundedness on L^2

We can now state the main result, which asserts that one can extend $\mathrm{Op}(a)$ as a bounded operator from $L^2(\mathbb{R}^d)$ into itself.

Theorem 3.2.2. *If $a \in C_b^\infty(\mathbb{R}^{2d})$, the operator $\mathrm{Op}(a)$ can be uniquely extended as a bounded linear operator in $\mathcal{L}(L^2(\mathbb{R}^d))$.*

We will demonstrate this result by assuming, to simplify the notations, that the space dimension d is less than or equal to 3 (otherwise just replace the polynomial $P(\zeta)$ below by $(1+|\zeta|^2)^k$ where k is an integer such that $4k > d$).

Let us introduce the polynomial

$$P(\zeta) = 1 + |\zeta|^2 \qquad (\zeta\in\mathbb{R}^d,\ d=1,2,3).$$

Lemma 3.2.3. *Given a function $u\in\mathcal{S}(\mathbb{R}^d)$, we introduce the function*

$$Wu(x,\xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\xi} P(x-y)^{-1}u(y)\,\mathrm{d}y \qquad ((x,\xi)\in\mathbb{R}^{2d}).$$

i) Then Wu is a function $C_b^\infty(\mathbb{R}^{2d})$ and moreover for any multi-indices α,β,γ,

$$\sup_{\mathbb{R}^{2d}} P(x)|\xi|^\gamma \left|(\partial_x^\alpha \partial_\xi^\beta Wu)(x,\xi)\right| < +\infty.$$

ii) *There is a constant A such that*

$$\|Wu\|_{L^2(\mathbb{R}^{2d})} = A \|u\|_{L^2(\mathbb{R}^d)} \tag{3.2.1}$$

for any u in $\mathcal{S}(\mathbb{R}^d)$.

iii) *For any* $\gamma \in \mathbb{N}^d$, *there are* A_γ *such as*

$$\left\|\partial_x^\gamma Wu\right\|_{L^2(\mathbb{R}^{2d})} \leqslant A_\gamma \|u\|_{L^2(\mathbb{R}^d)} .$$

Proof. *i*) We verify that

$$\xi^\gamma(\partial_x^\alpha \partial_\xi^\beta Wu)(x,\xi) = \int i^{|\gamma|}\partial_y^\gamma\big(e^{-iy\cdot\xi}\big)(-iy)^\beta \partial_x^\alpha(P(x-y)^{-1})u(y)\,\mathrm{d}y,$$

so, by integrating by parts

$$\begin{aligned}&\xi^\gamma(\partial_x^\alpha \partial_\xi^\beta Wu)(x,\xi)\\ &= \sum_{\gamma'+\gamma''=\gamma}\frac{\gamma!}{\gamma'!\gamma''!}\int (-i)^{|\gamma|}\partial_y^{\gamma'}\big(u(y)(-iy)^\beta\big)(-1)^{|\gamma''|}(\partial^{\gamma''+\alpha}1/P)(x-y)e^{-iy\cdot\xi}\,\mathrm{d}y.\end{aligned}$$

We next use the elementary estimates

$$\left|\partial_\zeta^\alpha\langle\zeta\rangle^{-2}\right| \leqslant C_\alpha\langle\zeta\rangle^{-2-|\alpha|} \leqslant C_\alpha\langle\zeta\rangle^{-2}$$

to deduce that

$$|\partial^\alpha(1/P)(x-y)| \leqslant C_\alpha(1+|x-y|^2)^{-1} \leqslant 2C_\alpha(1+|x|^2)^{-1}(1+|y|^2),$$

where the last inequality comes from the fact that

$$1+|x|^2 = 1+|x-y+y|^2 \leqslant 1+2|x-y|^2+2|y|^2 \leqslant 2(1+|x-y|^2)(1+|y|^2).$$

ii) For any $x \in \mathbb{R}^d$, $Wu(x,\cdot)$ is the Fourier transform of $y \mapsto u(y)P(x-y)^{-1}$. So

$$\int |W(x,\xi)|^2\,\mathrm{d}\xi = (2\pi)^d\int \left|u(y)P(x-y)^{-1}\right|^2\mathrm{d}y$$

according to Plancherel's formula. So

$$\iint |W(x,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}x = (2\pi)^d\iint \left|u(y)P(x-y)^{-1}\right|^2\mathrm{d}y dx = A^2\|u\|^2_{L^2(\mathbb{R}^d)} .$$

iii) By combining the above observations. □

Lemma 3.2.4. *We have*

$$\widehat{u}(\xi) = e^{-ix\cdot\xi}(I - \Delta_\xi)\big(e^{ix\cdot\xi} Wu(x,\xi)\big)$$

and

$$\overline{v}(x) = \frac{1}{(2\pi)^d} e^{-ix\cdot\xi}(I - \Delta_x)\big(e^{ix\cdot\xi} W\widehat{\overline{v}}(\xi, x)\big).$$

Proof. Like $(I - \Delta_\xi)e^{iX\cdot\xi} = P(X)$ we have

$$e^{ix\cdot\xi}\widehat{u}(\xi) = \int e^{i(x-y)\cdot\xi} u(y)\,\mathrm{d}y = (I - \Delta_\xi)\int e^{i(x-y)\cdot\xi} P(x-y)^{-1} u(y)\,\mathrm{d}y.$$

In a dual way, using the inverse Fourier transform, we have

$$\begin{aligned} e^{ix\cdot\xi}\overline{v}(x) &= \frac{1}{(2\pi)^n}\int e^{i(\xi-\eta)\cdot x}\widehat{\overline{v}}(\eta)\,\mathrm{d}\eta \\ &= \frac{1}{(2\pi)^n}(I - \Delta_x)\int e^{i(\xi-\eta)\cdot x} P(\xi-\eta)^{-1}\widehat{\overline{v}}(\eta)\,\mathrm{d}\eta, \end{aligned}$$

which implies the second identity. □

Proof of Theorem 3.2.2. Given the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, it is enough to demonstrate the inequality

$$\|\mathrm{Op}(a)u\|_{L^2} \leqslant C\,\|u\|_{L^2}$$

for any u in $\mathcal{S}(\mathbb{R}^d)$. Let us consider two functions u, v in $\mathcal{S}(\mathbb{R}^d)$ and let us set

$$I := \iint e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\overline{v}(x)\,\mathrm{d}\xi\,\mathrm{d}x.$$

We want to show that $|I| \leqslant C\,\|u\|_{L^2}\,\|v\|_{L^2}$. For this we will rewrite I as a scalar product in $L^2(\mathbb{R}^{2d})$ of functions involving Wu and $W\widehat{\overline{v}}$.

Let us start by writing I in the form

$$I = \iint a(x,\xi)\Big[(I - \Delta_\xi)\big(e^{ix\cdot\xi} Wu(x,\xi)\Big]\overline{v}(x)\,\mathrm{d}\xi\,\mathrm{d}x.$$

Since $(I - \Delta_\xi)\big(e^{ix\cdot\xi} Wu(x,\xi)\overline{v}(x)\big)$ belongs to $\mathcal{S}(\mathbb{R}^{2d})$, we can integrate by parts in ξ and deduce that

$$I = \iint \Big[(I - \Delta_\xi)a(x,\xi)\Big] Wu(x,\xi)e^{ix\cdot\xi}\overline{v}(x)\,\mathrm{d}\xi\,\mathrm{d}x.$$

Using the identity for v it comes

$$I = \iint \Big[(I - \Delta_\xi) a(x, \xi)\Big] Wu(x, \xi)(I - \Delta_x)\big(e^{ix\cdot\xi} W\overline{\hat{v}}(\xi, x)\big)\, \mathrm{d}x$$

and integrating by parts in x,

$$I = \iint (I - \Delta_x)\Big[\big((I - \Delta_\xi) a(x, \xi)\big) Wu(x, \xi)\Big] e^{ix\cdot\xi} W\overline{\hat{v}}(\xi, x)\, \mathrm{d}\xi\, \mathrm{d}x$$

so

$$I = \sum_{|\beta|\leqslant 2, |\alpha|+|\gamma|\leqslant 2} C_{\alpha\beta\gamma} \iint (\partial_x^\alpha \partial_\xi^\beta a(x, \xi)) \partial_x^\gamma Wu(x, \xi) W\overline{\hat{v}}(\xi, x) e^{ix\cdot\xi}\, \mathrm{d}x\, \mathrm{d}\xi.$$

We conclude the proof with the Cauchy-Schwarz inequality and the previous results:

$$\begin{aligned}
\left\|\partial_x^\gamma Wu\right\|_{L^2(\mathbb{R}^{2d})} &\leqslant A_\gamma \left\|u\right\|_{L^2(\mathbb{R}^d)}, \\
\left\|W\overline{\hat{v}}(\xi, x)\right\|_{L^2(\mathbb{R}^{2d})} &= A\left\|\overline{\hat{v}}\right\|_{L^2} = A(2\pi)^{\frac{d}{2}} \left\|v\right\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where the Plancherel formula was used in the last inequality. □

3.3 Littlewood-Paley Decomposition

We begin by introducing a dyadic decomposition of the unity. This decomposition allows to introduce a parameter (large or small) in a problem which does not have any. It is a simple and extremely fruitful idea.

Lemma 3.3.1. *Let $d \geq 1$. There exist $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that the following properties hold:*

(i) (Support conditions) We have $0 \leq \psi \leq 1$, $0 \leq \varphi \leq 1$ and

$$\operatorname{supp} \psi \subset \{|\xi| \leqslant 1\}, \qquad \operatorname{supp} \varphi \subset \left\{\frac{3}{4} \leqslant |\xi| \leqslant 2\right\}.$$

(ii) (Decomposition of the unity) For any $\xi \in \mathbb{R}^d$,

$$1 = \psi(\xi) + \sum_{p=0}^{\infty} \varphi(2^{-p}\xi). \tag{3.3.1}$$

(iii) (Almost orthogonality) For any $\xi \in \mathbb{R}^d$,

$$\frac{1}{3} \leq \psi^2(\xi) + \sum_{p=0}^{+\infty} \varphi^2(2^{-p}\xi) \leq 1. \tag{3.3.2}$$

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ be a radial function verifying $\psi(\xi) = 1$ for $|\xi| \leqslant 3/4$, and $\psi(\xi) = 0$ for $|\xi| \geq 1$, and decreasing (if $|\xi| \geq |\eta|$ then $\psi(\eta) \leq \psi(\eta)$). Then, we set $\varphi(\xi) = \psi(\xi/2) - \psi(\xi)$ and notice that φ is supported in the annulus $\{3/4 \leqslant |\xi| \leqslant 2\}$. For any integer N and any $\xi \in \mathbb{R}^d$, we have

$$\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi) = \psi(2^{-N-1}\xi),$$

which immediately implies (3.4.2) by letting N goes to $+\infty$.

It remains to prove (3.3.2). For any integer N we have

$$\psi^2(\xi) + \sum_{p=0}^{N} \varphi^2(2^{-p}\xi) \leq \Big(\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi)\Big)^2.$$

On the other hand, notice that, for all $\xi \in \mathbb{R}^d$, there are never more than three non-zero terms in the set $\{\psi(\xi),\ \varphi(\xi), \ldots, \varphi(2^{-p}\xi), \ldots\}$. Consequently, using the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we easily get

$$\Big(\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi)\Big)^2 \leq 3\Big(\psi^2(\xi) + \sum_{p=0}^{N} \varphi^2(2^{-p}\xi)\Big)^2.$$

Then we obtain (3.3.2) by letting N goes to $+\infty$ in the previous inequalities. $\square$

Let us define, for $p \geq -1$, the Fourier multipliers Δ_p as follows:

$$\Delta_{-1} := \psi(D_x) \quad \text{and} \quad \Delta_p := \varphi\big(2^{-p}D_x\big) \quad (p \geq 0).$$

Let us also introduce, for $p \geq 0$, the Fourier multipliers S_p:

$$S_p := \psi\big(2^{-p}D_x\big) = \sum_{k=-1}^{p-1} \Delta_k.$$

The partition of the unit also implies a partition of the identity.

Proposition 3.3.2. *We have*

$$I = \sum_{p \geq -1} \Delta_p,$$

in the sense of distributions: For any $u \in \mathcal{S}'(\mathbb{R}^d)$, the series $\sum u_p$ converges to u in $\mathcal{S}'(\mathbb{R}^d)$, which means that $\sum_p \langle \Delta_p u, \varphi \rangle_{\mathcal{S}' \times \mathcal{S}}$ converges to $\langle u, \varphi \rangle_{\mathcal{S}' \times \mathcal{S}}$ for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\theta \in \mathcal{S}(\mathbb{R}^d)$. The partial sums $S_N u = \sum_{p \geq 0}^{N-1} \Delta_p u$ are well defined and

$$\langle \mathcal{F}\left(S_N u\right), \theta \rangle = \langle \psi(2^{-N}\xi) \mathcal{F}(u), \theta \rangle = \langle \mathcal{F}(u), \psi(2^{-N}\xi)\theta \rangle.$$

Now $\lim_{N \to +\infty} \psi(2^{-N}\xi)\theta = \theta$ in $\mathcal{S}(\mathbb{R}^d)$, so

$$\mathcal{F}\left(S_N u\right) \underset{p \to +\infty}{\longrightarrow} \mathcal{F}(u) \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

By continuity of $\mathcal{F}^{-1} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ we have $u = \sum_{p \geq -1} \Delta_p u$. $\square$

Proposition 3.3.3. *(i) For all $u \in L^2(\mathbb{R}^d)$,*

$$\sum_{p \geq -1} \left\| \Delta_p u \right\|_{L^2}^2 \leq \| u \|_{L^2}^2 \leq 3 \sum_{p \geq -1} \left\| \Delta_p u \right\|_{L^2}^2 . \tag{3.3.3}$$

(ii) Consider $s \in \mathbb{R}$. A temperate distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the Sobolev space $H^s(\mathbb{R}^d)$ if and only if

(a) $\Delta_{-1} u \in L^2(\mathbb{R}^d)$ and for all $p \geq 0$, $\Delta_p u \in L^2(\mathbb{R}^d)$;

(b) the sequence $\delta_p = 2^{ps} \left\| \Delta_p u \right\|_{L^2}$ belongs to $\ell^2(\mathbb{N} \cup \{-1\})$.

Moreover, there exists a constant C such that

$$\frac{1}{C} \| u \|_{H^s} \leq \Big(\sum_{p=-1}^{+\infty} \delta_p^2 \Big)^{\frac{1}{2}} \leq C \| u \|_{H^s} . \tag{3.3.4}$$

Proof. The first point follows immediately from (3.3.2) and Plancherel's identity.

Since $\| u \|_{H^s} = \| \langle D_x \rangle^s u \|_{L^2}$, by applying (3.3.3) with u replaced by $\langle D_x \rangle^s u$, we obtain

$$\sum_{p \geq -1} \left\| \Delta_p \langle D_x \rangle^s u \right\|_{L^2}^2 \leq \| u \|_{H^s}^2 \leq 3 \sum_{p \geq -1} \left\| \Delta_p \langle D_x \rangle^s u \right\|_{L^2}^2 .$$

Consider $p \geq 0$ and write that

$$\left\|\Delta_p \langle D_x\rangle^s u\right\|_{L^2}^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} (1+|\xi|^2)^s \varphi(2^{-p}\xi) \left|\hat{u}(\xi)\right|^2 \mathrm{d}\xi.$$

Since $(1+|\xi|^2)^s \varphi(2^{-p}\xi) \sim 2^{2ps}$ on the support of $\varphi(2^{-p}\xi)$, we see that

$$\frac{1}{C} 2^{2ps} \left\|\Delta_p u\right\|_{L^2}^2 \leq \left\|\Delta_p \langle D_x\rangle^s u\right\|_{L^2}^2 \leq C 2^{2ps} \left\|\Delta_p u\right\|_{L^2}^2, \tag{3.3.5}$$

for some constant C depending only on s. We have a similar estimate for $\Delta_{-1} u$ and the wanted result easily follows. □

Proposition 3.3.4. *i) Consider $s \in \mathbb{R}$ and $R \geq 1$. Assume that $(u_j)_{j\geq -1}$ is a sequence of functions in $L^2(\mathbb{R}^d)$ such that*

$$\operatorname{supp} \widehat{u_{-1}} \subset \{|\xi| \leq R\}, \qquad \operatorname{supp} \widehat{u_j} \subset \left\{\frac{1}{R} 2^j \leq |\xi| \leq R 2^j\right\},$$

and, in addition,

$$\sum_{j\geq -1} 2^{2js} \left\|u_j\right\|_{L^2}^2 < +\infty. \tag{3.3.6}$$

Then the series $\sum u_j$ converges to a function $u \in H^s(\mathbb{R}^d)$ and moreover,

$$\|u\|_{H^s}^2 \leq C \sum_{j\geq -1} 2^{2js} \left\|u_j\right\|_{L^2}^2,$$

for some constant C depending only on s and R.

ii) If $s > 0$, then the previous result holds under the weaker assumption that $\operatorname{supp} \widehat{u_j}$ is included in the ball $B(0, R2^j)$.

Proof. *i)* We begin by proving that the series $\sum u_j$ is normally convergent in $H^r(\mathbb{R}^d)$ for any $0 < r < s$. Assuming that $\operatorname{supp} \widehat{u_j}$ is included in an annulus $\left\{\frac{1}{R} 2^j \leq |\xi| \leq R 2^j\right\}$, parallel to (3.3.5), we see that $2^{jr} \left\|u_j\right\|_{L^2} \sim \left\|u_j\right\|_{H^r}$. So, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}\sum_{j\geq -1} \left\|u_j\right\|_{H^r} &\leq \Big(\sum_{j\geq -1} 2^{2js} \left\|u_j\right\|_{L^2}^2\Big)^{\frac{1}{2}} \Big(\sum_{j\geq -1} 2^{2j(r-s)}\Big)^{\frac{1}{2}} \\ &\lesssim \Big(\sum 2^{2js} \left\|u_j\right\|_{L^2}^2\Big)^{\frac{1}{2}} < +\infty.\end{aligned}$$

This shows that the series $\sum u_j$ is normally convergent and hence convergent in $H^s(\mathbb{R}^d)$. Now we can set $u = \sum_{j\geq -1} u_j$. Our goal is then to prove that u belongs to $H^s(\mathbb{R}^d)$.

The assumption that $\operatorname{supp}\widehat{u_j}$ is included in an annulus $\left\{\frac{1}{R}2^j \leq |\xi| \leq R2^j\right\}$ implies that there exists some integer N depending only on R such that $\Delta_p u_j = 0$ if $|j-p| > N$. Therefore

$$\left\|\Delta_p u\right\|_{L^2} \leq \sum_{|j-p|\leq N} \left\|\Delta_p u_j\right\|_{L^2} \leq \sum_{|j-p|\leq N} \left\|u_j\right\|_{L^2},$$

whence the result.

ii) If one only assumes that $\operatorname{supp}\widehat{u_j}$ is included in a ball $\{|\xi| \leq R2^j\}$, then we just have for some integer N,

$$\Delta_p u = \sum_{j\geq p-N} \Delta_p u_j.$$

It follows from the triangle inequality that

$$2^{ps}\left\|\Delta_p u\right\|_{L^2} \leq \sum_{j\geq p-N} 2^{(p-j)s} 2^{js}\left\|u_j\right\|_{L^2}.$$

Now, since $s > 0$, the sequence $(2^{(p-j)s})_{j\geq p-N}$ belongs to ℓ^1 and the convolution inequality $\ell^1 * \ell^2 \hookrightarrow \ell^2$ gives the result. □

3.4 Operators of type $(1,1)$

We continue the study of pseudo-differential operators of the form

(3.4.1) $$\operatorname{Op}(a)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi.$$

In Section §3.2, we have introduced the study of these operators, assuming that $a \in C_b^\infty(\mathbb{R}^{2d};\mathbb{C})$. Here we consider more general symbols, belonging to the following spaces.

Definition 3.4.1. *For $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, the symbol class $S^m_{\rho,\delta}(\mathbb{R}^d)$ is the space of functions $a \in C^\infty(\mathbb{R}^{2d};\mathbb{C})$ such that, for all multi-indices $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^d$, there exists a constant C such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}.$$

We say that a is a symbol of order m and type (ρ, δ).

Remark 3.4.2. *Notice that* $C_b^\infty(\mathbb{R}^{2d}; \mathbb{C}) = S^0_{0,0}(\mathbb{R}^d)$.

For any real numbers $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, and for any symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^d)$, by using similar arguments to those used to prove Proposition 3.2.1, one can prove that the relation (3.4.1) defines $\mathrm{Op}(a)$ as a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.

Let us state a generalization of Theorem 3.2.2 to the case of general symbol.

Theorem 3.4.3 (Calderón-Vaillancourt)**.** *Let* $a \in S^0_{\rho,\delta}(\mathbb{R}^d)$ *with* $0 \leq \delta \leq \rho \leq 1$ *and* $\delta < 1$*. Then the operator* $\mathrm{Op}(a)$ *can be extended as a bounded operator from* $L^2(\mathbb{R}^d)$ *to itself. Moreover,*

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(L^2)} \leq C \sup_{|\alpha| \leq [\frac{d}{2}]+1} \sup_{|\beta| \leq [\frac{d}{2}]+1} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \left|(1+|\xi|)^{\delta|\alpha|-\rho|\beta|} \partial_x^\alpha \partial_\xi^\beta p(x,\xi)\right|,$$

for some absolute constant C depending only on d, ρ, δ.

Proof. We will not use this result and refer to [107] for the proof. The precise bound in terms of a the semi-norms of p is proved for instance by Coifman and Meyer [135]. □

It is proved in Exercise 3.9.3 that the statement of Theorem 3.4.3 does not hold for $(\rho, \delta) = (1, 1)$. This means that an operator of 0 and type $(1, 1)$ is not bounded in general from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. However, the following result, due to Stein, states that such an operator is bounded from $H^s(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for any $s > 0$.

Theorem 3.4.4 (Stein)**.** *Assume that* $a \in S^0_{1,1}(\mathbb{R}^d)$*. Then the operator* $\mathrm{Op}(a)$ *is bounded from* $H^s(\mathbb{R}^d)$ *to* $H^s(\mathbb{R}^d)$ *for all* $s > 0$.

Proof. We will not use this result and refer to [335] for the proof. □

Theorem 3.4.5. *Let* $\varepsilon \in [0, 1)$ *and consider a function* $a \in C^\infty(\mathbb{R}^{2d}; \mathbb{C})$ *such that*

$$M := \sup_{|\beta| \leq [\frac{d}{2}]+1} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \left|(1+|\xi|)^{|\beta|} \partial_\xi^\beta a(x,\xi)\right| < +\infty.$$

Assume in addition that, for all $\xi \in \mathbb{R}^d$*, the partial Fourier transform*

$$\hat{a}(\eta, \xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\eta} a(y, \xi) \, \mathrm{d}y$$

is supported in the ball $\{\eta \in \mathbb{R}^d\,;\, |\eta| \le \varepsilon\, |\xi|\}$*. Then* $\mathrm{Op}(a) \in \mathcal{L}(L^2(\mathbb{R}^d))$ *and*

$$\|\mathrm{Op}(a)\|_{L^2\to L^2} \le CM,$$

for some constant C *depending only on* ε*.*

Remark 3.4.6. *We will see in the proof that a belongs to* $S^0_{1,1}(\mathbb{R}^d)$*.*

Proof. Set $N = 1 + [d/2]$.

Step 1: Littlewood-Paley decomposition. We use the decomposition of the unity introduced in Lemma 3.3.1. Write

$$a(x,\xi) = a(x,\xi)\psi(\xi) + \sum_{p=0}^{\infty} a(x,\xi)\varphi(2^{-p}\xi), \tag{3.4.2}$$

and then set

$$a_{-1}(x,\xi) = a(x,\xi)\psi(\xi) \quad ; \quad a_p(x,\xi) = a(x,\xi)\varphi(2^{-p}\xi) \quad \text{for } p \ge 0.$$

Step 2: Bernstein Lemma. We claim that, for any multi-indices $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^d$ with $|\beta| \le N$, there exists a positive constant C such that,

$$\left|\partial_x^\alpha \partial_\xi^\beta a_{-1}(x,\xi)\right| \le CM, \tag{3.4.3}$$

and, for $p \ge 0$,

$$\left|\partial_x^\alpha \partial_\xi^\beta a_p(x,\xi)\right| \le CM2^{p(|\alpha|-|\beta|)}.$$

Since $|\xi| \sim 2^p$ on the support of $\varphi(2^{-p}\xi)$ (resp. $\psi(\xi)$ for $p = 0$), this follows from the assumption that the partial Fourier transform $\hat{a}(\eta,\xi)$ is supported in the ball $\{|\eta| \le \varepsilon\,|\xi|\}$, by using the following

Lemma 3.4.7. *Consider a function* $f \in L^\infty(\mathbb{R}^d)$ *whose Fourier transform is included in the ball* $\{|\xi| \le \lambda\}$*. Then* $f \in C^\infty(\mathbb{R}^d)$ *and, for all* $\alpha \in \mathbb{N}^d$*, there exists a constant* $C = C(d,\alpha)$ *such that*

$$\left\|\partial_x^\alpha f\right\|_{L^\infty} \le C\lambda^{|\alpha|}\,\|f\|_{L^\infty}.$$

Proof. Introduce $\theta \in C_0^\infty(\mathbb{R}^d)$ such that $\theta(\xi) = 1$ for $|\xi| \le 1$ and set $\theta_\lambda(\xi) = \theta(\xi/\lambda)$. Then $\theta_\lambda \hat{f} = \hat{f}$, which implies that

$$f = \kappa_\lambda * f, \quad \text{where} \quad \kappa_\lambda = \mathcal{F}^{-1}(\theta_\lambda).$$

We are now in position to estimate the derivatives of f by exploiting the relation

$$\partial_x^\alpha f = (\partial_x^\alpha \kappa_\lambda) * f.$$

Observing that $\kappa_\lambda(x) = \lambda^d \kappa(\lambda x)$ with $\kappa = \mathcal{F}^{-1}(\theta)$, we obtain that

$$\left\| \partial_x^\alpha \kappa_\lambda \right\|_{L^1(\mathbb{R}^d)} = \lambda^{|\alpha|} \left\| \partial_x^\alpha \kappa \right\|_{L^1(\mathbb{R}^d)},$$

and the result follows. □

Step 3: low frequency component. In view of (3.4.3), it follows directly from the Calderón-Vaillancourt theorem (see Theorem 3.2.2), implies that $\mathrm{Op}(a_{-1})$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, and satisfies the estimate

$$\|\mathrm{Op}(a_{-1})\|_{L^2 \to L^2} \leq CM.$$

Step 4: rescaling. We want to prove that the operators $\mathrm{Op}(a_p)$ are also bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. To do so, we use a rescaling argument. More precisely, given a positive real-number $\lambda > 0$, introduce the operator H_λ defined by

$$(H_\lambda u)(x) = \lambda^{\frac{d}{2}} u(\lambda x).$$

Then

$$\|H_\lambda u\|_{L^2} = \|u\|_{L^2}.$$

In addition, for any symbol $p = p(x, \xi)$, we have

$$\mathrm{Op}(p)(H_\lambda u) = H_\lambda(\mathrm{Op}(p_\lambda)u) \quad \text{where} \quad p_\lambda(x, \xi) = p\Big(\frac{x}{\lambda}, \lambda\xi\Big).$$

This implies that $\mathrm{Op}(a_p) \in \mathcal{L}(L^2(\mathbb{R}^d))$ if and only if $\mathrm{Op}(b_p) \in \mathcal{L}(L^2(\mathbb{R}^d))$ where

$$b_p(x, \xi) = a_p\big(2^{-p}x, 2^p\xi\big),$$

and then $\left\|\mathrm{Op}(a_p)\right\|_{L^2 \to L^2} = \left\|\mathrm{Op}(b_p)\right\|_{L^2 \to L^2}$.

Step 5: boundedness of the rescaled operators. Notice that, for any multi-indices $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^d$ with $|\beta| \leq N$, there holds

$$\left|\partial_x^\alpha \partial_\xi^\beta b_p(x, \xi)\right| \leq CM.$$

Then, as already explained above, it follows from the Calderón-Vaillancourt theorem (see Theorem 3.2.2) that

$$\left\|\mathrm{Op}(a_p)\right\|_{L^2\to L^2} = \left\|\mathrm{Op}(b_p)\right\|_{L^2\to L^2} \le CM.$$

Step 6: spectral localization. Notice that

$$\widehat{\mathrm{Op}(a_p)u}(\eta) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{a}(\eta-\xi,\xi)\varphi(2^{-p}\xi)\hat{u}(\xi)\,\mathrm{d}\xi.$$

Introduce the function u_p defined by

$$\hat{u_p}(\xi) = \hat{u}(\xi) \quad \text{if} \quad \xi \in \Gamma_p := \{3^{-1}\cdot 2^p \le |\xi| \le 3\cdot 2^p\},$$

and $\hat{u_p}(\xi) = 0$ whenever $\xi \notin \Gamma_p$. Then $\varphi(2^{-p}\xi)\hat{u}(\xi) = \varphi(2^{-p}\xi)\hat{u_p}(\xi)$, which in turn implies that

$$\mathrm{Op}(a_p)u = \mathrm{Op}(a_p)u_p.$$

Exploiting again that the partial Fourier transform $\hat{a}(\eta,\xi)$ is supported in the ball $\{|\eta| \le \varepsilon|\xi|\}$, we verify that the support of $\mathcal{F}(\mathrm{Op}(a)u_p)$ is included in the larger shell

$$\Gamma_p' = \left\{\xi\in\mathbb{R}^d\,;\, \frac{1-\epsilon}{3}2^p \le |\xi| \le 3\cdot(1+\epsilon)2^p\right\}.$$

Now, since any η is included in at most $2\log(3/(1-\varepsilon))/\log(2)$ dyadic shells Γ_p', we deduce from the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ that

$$\left|\sum_p \widehat{\mathrm{Op}(a_p)u}(\eta)\right|^2 \le C(\varepsilon)\sum_p \left|\widehat{\mathrm{Op}(a_p)u}(\eta)\right|^2. \tag{3.4.4}$$

It follows from Plancherel's theorem that

$$\left\|\mathrm{Op}(a)u\right\|_{L^2}^2 \sim \left\|\sum_p \widehat{\mathrm{Op}(a_p)u}\right\|_{L^2}^2 \lesssim \sum_p \left\|\widehat{\mathrm{Op}(a_p)u}\right\|_{L^2}^2 \sim \sum_p \left\|\mathrm{Op}(a_p)u\right\|_{L^2}^2.$$

Remembering that $\mathrm{Op}(a_p)u = \mathrm{Op}(a_p)u_p$ and using the fact that $\mathrm{Op}(a_p) \in \mathcal{L}(L^2(\mathbb{R}^d))$, we get from (3.4.4) that

$$\sum_p \left\|\mathrm{Op}(a_p)u\right\|_{L^2}^2 = \left\|\mathrm{Op}(a_p)u_p\right\|_{L^2}^2 \lesssim M^2\left\|u_p\right\|_{L^2}^2,$$

we conclude that

$$\left\|\mathrm{Op}(a_p)u\right\|_{L^2}^2 \lesssim M^2\sum_p \left\|u_p\right\|_{L^2}^2.$$

Eventually, since each ξ is contained in at most a fix number of dyadic shells Γ_p, we have

$$\sum_p \|u_p\|_{L^2}^2 \lesssim \|u\|_{L^2}^2 .$$

This concludes the proof. □

3.5 Symbolic calculus

In this paragraph we review notations and results about Bony's paradifferential calculus. We refer to [88, 244, 335, 336, 418] for the general theory. Here we follow the presentation by Métivier in [335]. We refer also to the recent book of Benzoni-Gavage and Serre [70] for applications of paradifferential calculus to hyperbolic systems.

3.5.1 notations

Given an integer $k \in \mathbb{N}$, we note $W^{k,\infty}(\mathbb{R}^d)$ the Sobolev space of distributions f such that $\partial_x^\alpha f \in L^\infty(\mathbb{R}^d)$ for $|\alpha| \leq k$. This space is equipped with the norm

$$\|u\|_{W^{k,\infty}} = \sum_{|\alpha|\leq k} \left\|\partial_x^\alpha u\right\|_{L^\infty} .$$

Given $\rho \in]0, +\infty[\backslash\mathbb{N}$, $W^{\rho,\infty}(\mathbb{R}^d)$ is the space of bounded functions whose derivatives of order $[\rho] \in \mathbb{N}$ are uniformly Hölder continuous with exponent $\rho - [\rho]$. This space is provided with the norm

$$\|u\|_{W^{\rho,\infty}} = \|u\|_{W^{[\rho],\infty}} + \sum_{|\alpha|=[\rho]} \frac{\left|\partial_x^\alpha u(x) - \partial_x^\alpha u(y)\right|}{|x-y|^{\rho-[\rho]}}.$$

The Fourier transform of a temperate distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ is denoted by $\widehat{u}$ or $\mathcal{F}u$. The spectrum of u is the support of $\mathcal{F}u$. In the following we call Fourier multipliers any operator defined by

$$p(D_x)u = \mathcal{F}^{-1}\left(p\mathcal{F}u\right),$$

which makes sense as soon as the multiplication by p is defined from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$; we say that $p(D_x)$ is the Fourier multiplier with symbol $p(\xi)$.

Definition 3.5.1. *Consider ρ in $[0,+\infty)$ and m in $\mathbb{R}$. One denotes by $\Gamma^m_\rho(\mathbb{R}^d)$ the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$, which are C^∞ functions of ξ outside the origin and such that, for any $\alpha\in\mathbb{N}^d$ and any $\xi\neq 0$, the function $x\mapsto\partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbb{R}^d)$ and there exists a constant C_α such that,*

$$\forall\,|\xi|\geq\frac{1}{2},\quad \left\|\partial_\xi^\alpha a(\cdot,\xi)\right\|_{W^{\rho,\infty}}\leq C_\alpha(1+|\xi|)^{m-|\alpha|}. \tag{3.5.1}$$

Remark 3.5.2. *Note that we consider symbols $a(x,\xi)$ that need not be smooth for $\xi=0$ (for instance $a(x,\xi)=|\xi|^m$ with $m\in\mathbb{R}^*$). The main motivation for considering such symbols comes from the principal symbol of the Dirichlet to Neumann operator. As we shall see this symbol is given by*

$$\sqrt{(1+|\nabla\eta(x)|^2)\,|\xi|^2-(\nabla\eta(x)\cdot\xi)^2}.$$

If $\eta\in W^{\rho,\infty}(\mathbb{R}^d)$ then this symbol belongs to $\Gamma^1_{\rho-1}(\mathbb{R}^d)$. Of course, this symbol is not C^∞ with respect to $\xi\in\mathbb{R}^d$.

Given a symbol a, to define the paradifferential operator T_a we need to introduce a cutoff function θ.

Definition 3.5.3. *Fix $\theta\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ satisfying the three following properties.*

(i) There exists $\varepsilon_1,\varepsilon_2$ satisfying $0<2\varepsilon_1<\varepsilon_2<1/2$ such that

$$\begin{aligned}\theta(\xi_1,\xi_2)&=1 \quad \text{if}\quad |\xi_1|\leq\varepsilon_1\,|\xi_2| \quad\text{and } |\xi_2|\geq 2,\\ \theta(\xi_1,\xi_2)&=0 \quad \text{if}\quad |\xi_1|\geq\varepsilon_2\,|\xi_2| \quad\text{or}\quad |\xi_2|\leq 1.\end{aligned}$$

(ii) For all $(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d$, there is $C_{\alpha,\beta}$ such that

$$\forall(\xi_1,\xi_2)\in\mathbb{R}^d\times\mathbb{R}^d,\quad \left|\partial_{\xi_1}^\alpha\partial_{\xi_2}^\beta\theta(\xi_1,\xi_2)\right|\leq C_{\alpha,\beta}(1+|\xi_2|)^{-|\alpha|-|\beta|}.$$

(iii) θ satisfies the following symmetry conditions:

$$\theta(\xi_1,\xi_2)=\theta(-\xi_1,-\xi_2)=\theta(-\xi_1,\xi_2). \tag{3.5.2}$$

Remark 3.5.4. *Notice that $\theta(\xi_1,\xi_2)=0$ for $|\xi_2|$ small enough. This choice is important in our analysis since we have to handle symbols which are homogeneous in ξ_2 and hence not regular for $\xi_2=0$.*

Example 3.5.5. *As an example, fix $d = 1$ and $\varepsilon_1, \varepsilon_2$ such that $0 < 2\varepsilon_1 < \varepsilon_2 < 1/2$ and a function f in $C_0^\infty(\mathbb{R})$ satisfying $f(t) = f(-t)$, $f(t) = 1$ for $|t| \leq 2\varepsilon_1$ and $f(t) = 0$ for $|t| \geq \varepsilon_2$. Then set*

$$\theta(\xi_1, \xi_2) = (1 - f(\xi_2)) f\left(\frac{\xi_1}{\xi_2}\right).$$

Properties (i), (ii) and (iii) are clearly satisfied.

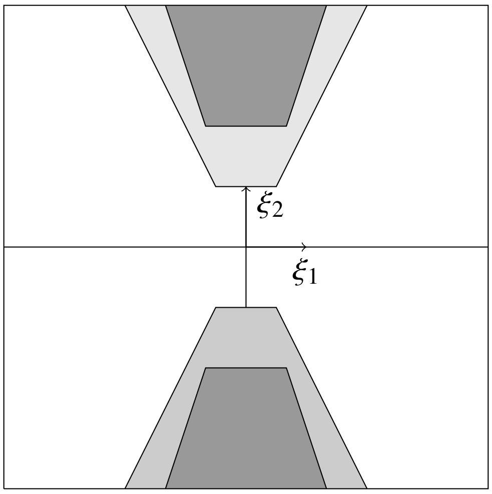


Figure 3.1: The support of the cut-off function $\theta(\xi_1, \xi_2)$ is in grey. The set of points (ξ_1, ξ_2) where $\theta(\xi_1, \xi_2) = 1$ is in darker grey.

The paradifferential operator T_a with symbol a is defined by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \theta(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \widehat{u}(\eta)\, \mathrm{d}\eta, \tag{3.5.3}$$

where $\widehat{a}(\theta, \xi) = \int e^{-ix\cdot\theta} a(x, \xi)\, \mathrm{d}x$ is the Fourier transform of a with respect to x.

Remark 3.5.6. *It follows from (3.5.2) that, if a and u are real-valued functions, so is $T_a u$.*

Remark 3.5.7. *One says that $\Theta = \Theta(\xi_1, \xi_2)$ is an admissible cut-off function if Θ satisfies the properties (i) and (ii) in Definition 3.5.3. All the results given in this appendix remain true for any admissible cut-off function (except Remark 3.5.6).*

Remark 3.5.8. *Given a symbol $a = a(x, \xi)$ which is homogeneous in ξ and smooth in x, we define the pseudo-differential operator* $\mathrm{Op}(a)$ *by*

$$\widehat{\mathrm{Op}(a) u}(\xi) = (2\pi)^{-d} \int \widehat{a}(\xi - \eta, \eta) \psi(\eta) \widehat{u}(\eta)\, d\eta. \tag{3.5.4}$$

where again $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta} a(x,\xi)\,\mathrm{d}x$ is the Fourier transform of a with respect to the first variable; and where ψ is as in (6.2.2). Note that the only difference betwenn (6.2.2) and (3.5.4) is the cut-off function χ; this cut-off allows to define operators for non smooth symbols by means of symbol smoothing.

We shall use quantitative results from [335]. To do so, introduce the following semi-norms.

Definition 3.5.9. *For $m \in \mathbb{R}$, $\rho \geq 0$ and $a \in \Gamma^m_\rho(\mathbb{R}^d)$, we set*

$$M^m_\rho(a) = \sup_{|\alpha| \leq \frac{d}{2}+1+\rho} \ \sup_{|\xi|\geq 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial^\alpha_\xi a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbb{R}^d)}. \tag{3.5.5}$$

We shall use quantitative results from [335] about operator norms estimates in symbolic calculus. To do so, introduce the following semi-norms.

Definition 3.5.10. *For $m \in \mathbb{R}$, $\rho \geq 0$ and $a \in \Gamma^m_\rho(\mathbb{R}^d)$, we set*

$$M^m_\rho(a) = \sup_{|\alpha| \leq \frac{d}{2}+1+\rho} \ \sup_{|\xi|\geq 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial^\alpha_\xi a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbb{R}^d)}. \tag{3.5.6}$$

The main features of symbolic calculus for paradifferential operators are given by the following theorems.

Definition 3.5.11. *Let $m \in \mathbb{R}$. An operator T is said of order m if, for all $\mu \in \mathbb{R}$, it is bounded from H^μ to $H^{\mu-m}$.*

Theorem 3.5.12. *Let $m \in \mathbb{R}$. If $a \in \Gamma^m_0(\mathbb{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that*

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \leq K M^m_0(a). \tag{3.5.7}$$

Theorem 3.5.13 (Composition)**.** *Let $m \in \mathbb{R}$ and $\rho > 0$. If $a \in \Gamma^m_\rho(\mathbb{R}^d)$, $b \in \Gamma^{m'}_\rho(\mathbb{R}^d)$ then $T_aT_b - T_{a\#b}$ is of order $m+m'-\rho$ where*

$$a\#b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial^\alpha_\xi a \partial^\alpha_x b.$$

Moreover, for all $\mu \in \mathbb{R}$ there exists a constant K such that

$$\|T_aT_b - T_{a\#b}\|_{H^\mu \to H^{\mu-m-m'+\rho}} \leq K M^m_\rho(a) M^{m'}_\rho(b). \tag{3.5.8}$$

Remark 3.5.14. *Note that the definition of the symbol $a\#b$ depends on the regularity of the symbols at stake. To clarify possible confusion, we will sometimes use a notation with an index ρ and write*

$$a\#_\rho b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b.$$

Theorem 3.5.15 (Adjoint)**.** *Let $m \in \mathbb{R}$, $\rho > 0$ and $a \in \Gamma_\rho^m(\mathbb{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of T_a and by $\overline{a}$ the complex-conjugated of a. Then $(T_a)^* - T_{a^*}$ is of order $m - \rho$ where*

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a}.$$

Moreover, for all μ there exists a constant K such that

$$\|(T_a)^* - T_{a^*}\|_{H^\mu \to H^{\mu-m+\rho}} \le K M_\rho^m(a). \tag{3.5.9}$$

3.6 Paraproducts

If $a = a(x)$ is a function of x only, then T_a is a called a paraproduct.

If $a \in L^\infty(\mathbb{R})$ then T_a is an operator of order 0, together with the estimate

$$\forall \sigma \in \mathbb{R}, \quad \|T_a u\|_{H^\sigma} \lesssim \|a\|_{L^\infty} \|u\|_{H^\sigma}. \tag{3.6.1}$$

A paraproduct with an L^∞-function acts on any Hölder space $W^{\rho,\infty}(\mathbb{R})$ with ρ in $\mathbb{R}_+^* \setminus \mathbb{N}$,

$$\forall \rho \in \mathbb{R}_+^* \setminus \mathbb{N}, \quad \|T_a u\|_{W^{\rho,\infty}} \lesssim \|a\|_{L^\infty} \|u\|_{W^{\rho,\infty}}. \tag{3.6.2}$$

If $a = a(x)$ and $b = b(x)$ are two functions then (**??**) simplifies to $a\sharp b = ab$ and hence (**??**) implies that, for any $\rho > 0$,

$$\|T_a T_b - T_{ab}\|_{H^\mu \to H^{\mu+\rho}} \le K \|a\|_{W^{\rho,\infty}} \|b\|_{W^{\rho,\infty}}, \tag{3.6.3}$$

provided that a and b belong to $W^{\rho,\infty}(\mathbb{R})$.

A key feature of paraproducts is that one can replace nonlinear expressions by paradifferential expressions, to the price of error terms which are smoother than the main terms. As an illustration, we give the following result of Bony [88].

Definition 3.6.1. *Given two functions a, b, we define the remainder*

$$R_{\mathcal{B}}(a,u) = au - T_a u - T_u a. \tag{3.6.4}$$

We record here two estimates about the remainder $R_{\mathcal{B}}(a,b)$ (see chapter 2 in [58]).

Theorem 3.6.2. *Let $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ be such that $\alpha + \beta > 0$. Then*

$$\|R_{\mathcal{B}}(a,u)\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbb{R})} \leq K \|a\|_{H^\alpha(\mathbb{R})} \|u\|_{H^\beta(\mathbb{R})}, \tag{3.6.5}$$

$$\|R_{\mathcal{B}}(a,u)\|_{H^{\alpha+\beta}(\mathbb{R})} \leq K \|a\|_{W^{\alpha,\infty}(\mathbb{R})} \|u\|_{H^\beta(\mathbb{R})}. \tag{3.6.6}$$

We next recall a well-known property of products of functions in Sobolev spaces (see chapter 8 in [244]) that can be obtained from (3.6.1) and (3.6.6): If $u_1, u_2 \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $s > 0$ then

$$\|u_1 u_2\|_{H^s} \leq K \|u_1\|_{L^\infty} \|u_2\|_{H^s} + K \|u_2\|_{L^\infty} \|u_1\|_{H^s}. \tag{3.6.7}$$

Similarly, recall that, for $s > 0$ and $F \in C^\infty(\mathbb{C}^N)$ such that $F(0) = 0$, there exists a non-decreasing function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|F(U)\|_{H^s} \leq C\big(\|U\|_{L^\infty}\big) \|U\|_{H^s}, \tag{3.6.8}$$

for any $U \in (H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$.

Theorem 3.6.3. *For all C^∞ function F, if $a \in H^\alpha(\mathbb{R}^d)$ then*

$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{d}{2}}(\mathbb{R}^d).$$

Moreover,

$$\left\| F(a) - F(0) - T_{F'(a)} a \right\|_{H^{2\alpha-\frac{d}{2}}(\mathbb{R}^d)} \leq C\left(\|a\|_{H^\alpha(\mathbb{R}^d)} \right),$$

for some non-decreasing function C depending only on F.

3.7 Sobolev energy estimates for hyperbolic equations

3.7.1 Introduction

Let $d \geq 1$ and $v \in \mathbb{R}^d$. The transport equation is the prototype of a hyperbolic equation of the first order. It is the equation

$$\partial_t u + v \cdot \nabla u = 0$$

where the unknown $u = u(t,x)$ is a real function of class C^1, defined on $\mathbb{R} \times \mathbb{R}^d$.

Proposition 3.7.1. *Let $u_0 \in C^1(\mathbb{R}^d)$. There exists a unique function $u \in C^1(\mathbb{R}\times\mathbb{R}^d)$ which is a solution of the Cauchy problem*

$$\begin{cases} \partial_t u + v \cdot \nabla u = 0, \\ u_{|t=0} = u_0. \end{cases}$$

This solution is given by the formula $u(t,x) = u_0(x - tv)$.

In the case where the constant vector v is replaced by a function with variable coefficients, we still have a formula for representing the solution based on the use of the characteristics curves. We will not study use this approach. Instead we will study energy estimates, which is a powerful tool to study PDE of different natures.

Let us now consider $V \in C_b^\infty(\mathbb{R} \times \mathbb{R}^d)$ with real values and a regular solution of the equation

$$\partial_t u + V(t,x) \cdot \nabla u = 0.$$

By multiplying the equation by u and integrating we obtain that we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u(t,x)^2 \,\mathrm{d}x = 2 \int u \partial_t u \,\mathrm{d}x = -2 \int u(V \cdot \nabla u) \,\mathrm{d}x$$

and by integrating by parts we deduce that

$$\int u(V \cdot \nabla u) \,\mathrm{d}x = -\frac{1}{2} \int (\operatorname{div} V) u^2 \,\mathrm{d}x,$$

from which

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |u(t,x)|^2 \,\mathrm{d}x \leqslant \|\operatorname{div} V\|_{L^\infty} \int |u|^2 \,\mathrm{d}x.$$

Gronwall's lemma then gives

$$\|u(t)\|_{L^2} \leqslant e^{Ct} \|u_0\|_{L^2}.$$

Note that if $\operatorname{div} V = 0$ then the $L^2(\mathbb{R}^d, \mathrm{d}x)$-norm is preserved.

3.7.2 Paradifferential hyperbolic equations

We consider complex valued symbols $a(x,\xi)$ depending on a the variables x, ξ in $\mathbb{R}^d$ where $d \geq 1$ is a fixed integer.

Definition 3.7.2. *Consider a symbol $a = a(x,\xi)$. We say that a is hyperbolic if a can be written as $a = a_1 + a_0$ where $a_1 \in \Gamma_1^1(\mathbb{R}^n)$ is purely imaginary and a_0 belongs to $\Gamma^0(\mathbb{R}^d)$.*

Example 3.7.3. *The simplest example of a hyperbolic symbol is the following: $a(x,\xi) = i\xi$. Then $\mathrm{Op}(a)u = \partial_x u$. If $a(x,\xi) = iV(x)\xi$, then $T_a u = T_V \cdot \nabla u$.*

We consider in addition:

- a time $T > 0$ and a real number s;
- an initial data $u_0 \in H^s(\mathbb{R}^n)$;
- a source term $f \in C^0([0,T]; H^s(\mathbb{R}^n))$.

We are interested in the following Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + T_a u = f, \\ u_{|t=0} = u_0, \end{cases} \tag{3.7.1}$$

where the unknown is the function $u = u(t,x)$, the variable $t \in \mathbb{R}_+$ corresponds to time and the variable $x \in \mathbb{R}^d$ ($d \geq 1$) corresponds to the space variable.

Theorem 3.7.4. *Let $T > 0$, $d \geq 1$ and $s \in \mathbb{R}$. For any initial $u_0 \in H^s(\mathbb{R}^d)$ and any $f \in C^0([0,T]; H^s(\mathbb{R}^d))$ there exists a unique function*

$$u \in C^0([0,T]; H^s(\mathbb{R}^n)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^n))$$

which verifies

$$\frac{\partial u}{\partial t} + T_a u = f$$

and which is such that $u(0) = u_0$.

3.7.3 Proof of Theorem 3.7.4

The first ingredient to prove this theorem is an estimate *a priori*.

Lemma 3.7.5. *Let $s \in \mathbb{R}$, $T > 0$. There exists a constant C such that for all $u \in C^1([0,T];H^s) \cap C^0([0,T];H^{s+1})$, any $f \in C^0([0,T];H^s)$, any $u_0 \in H^s$ and any $t \in [0,T]$, if u is a solution of* (3.7.1) *then*

$$\|u(t)\|_{H^s} \leqslant e^{Ct}\,\|u_0\|_{H^s} + 2\int_0^t e^{C(t-t')}\,\|f(t')\|_{H^s}\,\mathrm{d}t'. \tag{3.7.2}$$

Moreover there are two constants K and N which depend only on s such that

$$C \leqslant K \sum_{|\alpha|+|\beta|\leqslant N} \sup_{t\in[0,T]} \sup_{x,\xi} \left|\langle\xi\rangle^{-|\beta|}\partial_x^\alpha\partial_\xi^\beta \alpha_t(x,\xi)\right| \quad \textit{avec } \alpha_t := \left(a_t^* - \overline{a}_t\right) + 2\operatorname{Re} a_t.$$

where a_t^* is the symbol for the adjoint of $\operatorname{Op}(a_t)$.

Proof. We start with the case $s = 0$. Since u is C^1 with values in L^2 we can write that

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_{L^2}^2 &= \frac{d}{dt}(u(t),u(t)) \\
&= 2\operatorname{Re}\left(\partial_t u(t),u(t)\right) \\
&= -2\operatorname{Re}\left(T_a u(t),u(t)\right) + 2\operatorname{Re}\left(f(t),u(t)\right).
\end{aligned} \tag{3.7.3}$$

We have seen in the previous chapter that the adjoint $T_a^* - T_{\overline{a}}$ is of order 0. We can then write

$$(T_a u(t),u(t)) = (u(t),(T_a)^* u(t)) = (u(t), T_{\overline{a}}u(t) + \operatorname{Op}(b_t)u(t)).$$

The assumption that (a_t) is hyperbolic means that $\overline{a} = -a + 2\operatorname{Re} a$ with $\operatorname{Re} a \in \Gamma^0(\mathbb{R}^d)$. So $a^* = -a + \alpha$ where α belongs to $\Gamma^0(\mathbb{R}^d)$. We deduce that

$$(T_a u(t),u(t)) = (u(t), -T_a u(t) + T_\alpha u(t)),$$

from which

$$2\operatorname{Re}\left(T_a u(t),u(t)\right) = (u(t),T_a u(t)).$$

The Cauchy-Schwarz inequality and the continuity theorem of ΨDO of order 0 on L^2 imply that

$$|(T_\alpha u(t),u(t))| \leqslant K\sup_t \|T_\alpha\|_{\mathcal{L}(L^2)}\,\|u(t)\|_{L^2}^2 \leqslant C_0\,\|u(t)\|_{L^2}^2$$

where C_0 is a constant that does not depend on t. By transferring this inequality into (3.7.3) we conclude that

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \leqslant C_0\|u(t)\|_{L^2}^2 + 2\|f(t)\|_{L^2}\|u(t)\|_{L^2}. \tag{3.7.4}$$

We would like to write that $\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\|u(t)\|_{L^2}\frac{d}{dt}\|u(t)\|_{L^2}$ and simplify the inequality by dividing by $\|u(t)\|_{L^2}$. We proceed as follows: Given $\delta > 0$, we deduce from (3.7.4) that the function $y(t) = \sqrt{\|u(t)\|_{L^2}^2 + \delta}$ verifies

$$\frac{d}{dt}y(t)^2 \leqslant C_0 y(t)^2 + 2\|f(t)\|_{L^2}\, y(t),$$

and since $\|u(t)\|_{L^2}^2 + \delta > 0$, the function $y(t)$ is C^1 and we can simplify

$$2\frac{d}{dt}y(t) \leqslant C_0 y(t) + 2\|f(t)\|_{L^2}.$$

Gronwall's lemma implies that

$$\|u(t)\|_{L^2} \leqslant y(t) \leqslant y(0)e^{C_0 t/2} + \int_0^t \|f(t')\|_{L^2}\, e^{C_0(t-t')/2}\,\mathrm{d}t',$$

for all $\delta > 0$. By making δ tend to 0 we obtain that

$$\|u(t)\|_{L^2} \leqslant \|u(0)\|_{L^2}\, e^{C_0 t/2} + \int_0^t \|f(t')\|_{L^2}\, e^{C_0(t-t')/2}\,\mathrm{d}t',$$

which concludes the proof of the lemma in the case $s = 0$.

Now for any s we commute $L = \partial_t + \mathrm{Op}(a)$ to $\Lambda_s = \langle D_x\rangle^s$, which gives

$$\Lambda_s L u = \widetilde{L}\Lambda_s u, \quad \widetilde{L} = \partial_t + \widetilde{A}, \quad \widetilde{A} = \Lambda_s\,\mathrm{Op}(a)\Lambda_{-s}.$$

Note that $\widetilde{A}$ is a ΨDO operator of hyperbolic symbol. We conclude the proof by applying the estimate L^2 to $\widetilde{L}$. □

In this section we will show that the solution obtained in the previous section is the limit of solutions of approximate problems. is the limit of solutions of approximate problems. This result can be seen as another way of proving the existence of a solution once we have proved an estimate *a priori* (recall that the proof of uniqueness is direct).

Let $\varepsilon > 0$. Let us consider the Cauchy problem

$$\partial_t u + T_a J_\varepsilon u = f, \quad u(0) = u_0 \tag{3.7.5}$$

where J_ε, called Friedrichs mollifier, is defined by

$$\widehat{J_\varepsilon v}(\xi) = \chi(\varepsilon\xi)\widehat{v}(\xi),$$

where χ is a function C^∞ on $\mathbb{R}^n$, with support in the ball of center 0 and radius 2, and value 1 on the ball of center 0 and radius 1.

Theorem 3.7.6. *Let $T > 0$ and $s \in \mathbb{R}$. For all $u_0 \in H^s$ and all $f \in C^0([0,T]; H^s)$, there exists a unique solution u_ε belonging to $C^1([0,T]; H^s)$ of the Cauchy problem (3.7.5). Moreover, for all $\sigma < s$, this sequence is a Cauchy sequence in $C^0([0,T]; H^\sigma) \cap C^1([0,T]; H^{\sigma-1})$ and converges in this space to the unique solution $u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ of the Cauchy problem*

$$\partial_t u + T_a u = f, \quad u(0) = u_0.$$

Proof. We will note C several constants (whose value can vary from one expression to another) which depend only on T and s.

The main difference between the Cauchy problem (3.7.5) and the same problem without the operator J_ε is that it is very easy to show that the problem (3.7.5) has a solution.

Lemma 3.7.7. *For any $u_0 \in H^s$ and all $f \in C([0,T]; H^s)$, there exists a unique solution u_ε belonging to $C^1([0,T]; H^s)$ of the Cauchy problem (3.7.5).*

Proof. We have already seen that if $a = a(x,\xi)$ and $b = b(\xi)$ (symbol independent of x) then $T_a \circ b(D_x) = T_{ab}$. We deduce that, for all $\varepsilon > 0$,

$$T_a J_\varepsilon = T_{a^\varepsilon}$$

where

$$a^\varepsilon(x,\xi) = a(x,\xi)\chi(\varepsilon\xi).$$

For all $\varepsilon > 0$, the symbol a^ε is compactly supported in ξ and in particular it belongs to $\Gamma^0_0(\mathbb{R}^n)$. The continuity theorem for paradifferential operators of order 0 on the Sobolev implies that T_{a^ε} is a continuous operator on $H^s(\mathbb{R}^n)$. Then the equation $\partial_t u + T_{a^\varepsilon} u = f$ is an ordinary differential equation, for which the Cauchy-Lipschitz theorem applies. □

In the following we use the notation $a^\varepsilon(x,\xi) = a(x,\xi)\chi(\varepsilon\xi)$.

Lemma 3.7.8. *There exists a constant C such that for any $\varepsilon > 0$ and any $t \in [0,T]$, and any function $v \in C^1([0,T]; H^s(\mathbb{R}^n))$,*

$$\|v(t)\|_{H^s} \leqslant C\,\|v(0)\|_{H^s} + C\int_0^t \|(\partial_t v + T_{a^\varepsilon} v(\tau)\|_{H^s}\, d\tau.$$

Proof. Note that the symbol $a^\varepsilon(x,\xi) = a(x,\xi)\chi(\varepsilon\xi)$ is bounded in $\Gamma^1_1(\mathbb{R}^n)$ uniformly in ε, in the sense that $\{\, a_1(x,\xi)\chi(\varepsilon\xi) \,:\, \varepsilon \in]0,1],\ t\in[0,T]\}$ is a bounded part of $\Gamma^1_1(\mathbb{R}^d)$. Moreover $\operatorname{Re} a^\varepsilon$ is uniformly bounded in $\Gamma^0_0(\mathbb{R}^n)$. The desired inequality is therefore a consequence of (3.7.2). □

Applying the previous inequality to $v = u_\varepsilon$, we obtain that there is a constant C such that exists a constant C such that for all $\varepsilon > 0$ and all $tin[0,T]$,

$$\|u_\varepsilon\|_{C^0([0,T];H^s)} \leqslant C\,\|u_0\|_{H^s} + C\int_0^T \|f(t)\|_{H^s}\,\mathrm{d}t. \tag{3.7.6}$$

This implies that $(u_\varepsilon)_{\varepsilon\in]0,1]}$ is a bounded family in $C^0([0,T];H^s(\mathbb{R}^d))$. Using the equation, we verify that $(u_\varepsilon)_{\varepsilon\in]0,1]}$ is a bounded family in $C^1([0,T];H^{s-1}(\mathbb{R}^d))$. Our goal is to show that u_ε converges when ε tends to 0 to a solution of the Cauchy problem. For this we will show the following lemma.

Lemma 3.7.9. *The family $(u_\varepsilon)_{\varepsilon\in]0,1]}$ is a Cauchy sequence in $C^0([0,T];H^{s-2}(\mathbb{R}^d))$.*

Proof. Let ε and ε' in $]0,1]$. Starting from

$$\partial_t u_\varepsilon + \mathrm{Op}(a)J_\varepsilon u_\varepsilon = f,$$
$$\partial_t u_{\varepsilon'} + \mathrm{Op}(a)J_{\varepsilon'} u_{\varepsilon'} = f,$$

we deduce that $v = u_\varepsilon - u_{\varepsilon'}$ verifies

$$\partial_t v + T_{a^\varepsilon} v = f_\varepsilon \quad \text{avec} \quad f_\varepsilon = T_a\big(J_{\varepsilon'} - J_\varepsilon\big)u_{\varepsilon'}.$$

Since u_ε and $u_{\varepsilon'}$ coincide for $t = 0$, we have $v(0) = 0$ and we can then use the inequality of the previous lemma to obtain that

$$\|v\|_{C^0([0,T];H^{s-2})} \leqslant C\int_0^T \|f_\varepsilon(t)\|_{H^{s-2}}\, dt.$$

Now

$$\|f^{\varepsilon}(t)\|_{H^{s-2}} = \left\|\mathrm{Op}(a)\big(J_{\varepsilon'} - J_{\varepsilon}\big)u_{\varepsilon'}(t)\right\|_{H^{s-2}} \leqslant K \left\|\big(J_{\varepsilon'} - J_{\varepsilon}\big)u_{\varepsilon'}(t)\right\|_{H^{s-1}}.$$

By definition

$$\left\|\big(J_{\varepsilon'} - J_{\varepsilon}\big)u_{\varepsilon'}(t)\right\|_{H^{s-1}}^2 = (2\pi)^{-2d} \int \langle\xi\rangle^{2(s-2)} \left|\chi(\varepsilon\xi) - \chi(\varepsilon'\xi)\right|^2 \left|\widehat{u}_{\varepsilon'}(t,\xi)\right|^2 \,\mathrm{d}\xi.$$

We use the elementary inequality $|\chi(\varepsilon\xi) - \chi(\varepsilon'\xi)| \leqslant K\,|\varepsilon - \varepsilon'|\,|\xi|$ to conclude that

$$\int_0^T \|f_{\varepsilon}(t)\|_{H^{s-2}} \,\mathrm{d}t \leqslant K' \,|\varepsilon - \varepsilon'| \int_0^T \|u_{\varepsilon'}(t)\|_{H^s} \,\mathrm{d}t.$$

Since $\|u_{\varepsilon'}\|_{C^0([0,T];H^s)}$ is uniformly bounded according to (3.7.6), we obtain

$$\|v\|_{C^0([0,T];H^{s-2})} = O(|\varepsilon - \varepsilon'|),$$

which is the desired result. □

Lemma 3.7.10 (Interpolation in Sobolev spaces)**.** *Consider three real numbers $s_1 < s_2$ and $\sigma \in]s_1, s_2[$ with $\sigma = \alpha s_1 + (1-\alpha)s_2$ for some $\alpha \in [0,1]$, and for all $u \in H^{s_2}(\mathbb{R}^d)$,*

$$\|u\|_{H^{\sigma}} \leqslant \|u\|_{H^{s_1}}^{\alpha} \,\|u\|_{H^{s_2}}^{1-\alpha}. \tag{3.7.7}$$

Proof. Let us write that

$$\begin{aligned}\|u\|_{H^s}^2 &= (2\pi)^{-d} \int \langle\xi\rangle^{2s} \left|\widehat{u}(\xi)\right|^2 \,\mathrm{d}\xi\\ &= (2\pi)^{-d} \int \langle\xi\rangle^{2\alpha s_1} \left|\widehat{u}(\xi)\right|^{2\alpha} \langle\xi\rangle^{2(1-\alpha)s_2} \left|\widehat{u}(\xi)\right|^{2(1-\alpha)} \,\mathrm{d}\xi\end{aligned}$$

so that the desired inequality is a consequence of the Hölder inequality. □

We have seen that the family $(u_{\varepsilon})_{\varepsilon\in(0,1]}$ is bounded in $C^0([0,T];H^s)$ and that it is also Cauchy in $C^0([0,T];H^{s-2})$. Given $\sigma \in [s-2, s)$, the inequality $(s_1, s_2, \sigma) = (s-2, s, \sigma)$, leads to that $(u_{\varepsilon})_{\varepsilon\in(0,1]}$ is a Cauchy sequence in $C^0([0,T];H^{\sigma})$. Using the equation, we further obtain that $(\partial_t u_{\varepsilon})_{\varepsilon\in]0,1]}$ is Cauchy in $C^1([0,T];H^{\sigma-1})$. So u_{ε} converges in $C^0([0,T];H^{\sigma}) \cap C^1([0,T];H^{\sigma-1})$ to a limit denoted u.

By passing to the limit, we find that u is a solution of the Cauchy problem

$$\partial_t u + T_a u = f, \quad u(0) = u_0.$$

To conclude we just have to show that u belongs to $C^0([0,T];H^s)\cap C^1([0,T];H^{s-1})$. We have seen at the end of the previous section how to show that u belongs to $C^0([0,T];H^s)$. We then show that $\partial_t u$ belongs to $C^0([0,T];H^{s-1})$ using the equation.

By uniqueness of the solution of the Cauchy problem, we deduce that u is the solution whose existence we have shown in the previous section. □

3.8 References

3.8.1 Pseudo-differential operators

The study of pseudo-differential operators is a very vast subject of which we have barely touched the surface here. We refer for instance to Alinhac and Gérard [38], Grigis and Sjöstrand [225], Hörmander [245], Lerner [311], Saint-Raymond [382], Taylor [417] or Zworski [463].

We have studied the boundedness of pseudo-differential operators on $L^2(\mathbb{R}^d)$. Let us briefly discuss the boundedness on other functions spaces.

(i) Firstly, a pseudo-differential operator of order 0 and type (ρ,δ) is not bounded in general on Lebesgue spaces $L^p(\mathbb{R}^d)$ with $p\neq 2$. Nevertheless, Fefferman proved in [210] that, for any $0\leq\delta\leq\rho\leq 1$ with $\delta<1$, and any symbol $a\in S^m_{\rho,\delta}(\mathbb{R}^d)$, the operator $\mathrm{Op}(a)$ belong to $\mathcal{L}(L^p(\mathbb{R}^d))$ provided that

$$m\leq -d(1-\rho)\left|\frac{1}{2}-\frac{1}{p}\right|.$$

We also refer to David and Journé (see [183]) for the boundedness of pseudo-differential operators on $L^p(\mathbb{R}^d)$ when $\rho=1=\delta$.

(ii) Assume that $a\in S^0_{1,1}(\mathbb{R}^d)$. Then $\mathrm{Op}(a)$ is also bounded from $C^{k,\rho}(\mathbb{R}^d)$ to $C^{k,\rho}(\mathbb{R}^d)$ for all $k\in\mathbb{N}$ and all $\rho\in(0,1)$.

(iii) One can also consider the case where $\delta>\rho$, see Hörmander [243].

3.8.2 Littlewood-Paley decomposition

For an introduction to this topic, we refer the reader to Bahouri [55] or Danchin [178]. There are many books which develop a systematic study of this tool. In addition to the book by Coifman and Meyer [135, 339] and Métivier [335] that we have already mentioned, we refer to Alinhac and Gérard [38], Bahouri, Danchin and Chemin [58], Tao [412] or Taylor [419, 421].

The decomposition of a product into two paraproducts and a remainder has been introduced by Bony in [88]. However, the discrete version of it using Littlewood-Paley decomposition is due to Gérard and Rauch (see [215]).

3.8.3 Variants of paradifferential caculus

Bony's paradifferential calculus has been completed in particular by Chemin who introduced a bilinear symbolic computation and it has been enriched by the introduction of the operators of paracomposition by Alinhac and also by Fourier para-integral operators (the interested reader is referred to the original articles [35, 36, 123, 122] as well as to Taylor's book [419]). Let us mention that we can use versions much more simple of this calculus in particular situations (see the articles by Shnirelman [393, 394] for what is perhaps a first appearance of the ideas of decomposition underlying the paradiferential calculus). Paraproducts play a key role in the study of multi-linear Fourier multipliers. The reference works are the books by Stein [401], Coifman and Meyer [135, 339], Meyer [337, 338], Christ [127] and now that of Bahouri, Chemin and Danchin [58]. To return to the initial question, the application of paradifferential calculus to the non-linear interaction of singularities and to their propagation, we refer the reader to to the notes of Bony [89].

If $a \in \Gamma^m_\rho(\mathbb{R}^d)$ then the symbol σ defined by (**??**) belongs to the class $S^m_{1,1}(\mathbb{R}^d)$ and satisfy some additional properties (see [90, 89, 244, 335]). Other frequency regularization lead to symbols $S^m_{1,\delta}$ with $\delta < 1$. As noted Lebeau [309], the case $\delta < 1$ is very useful, for instance to construct parametrixes and prove Strichartz's estimates ([20, 23]).

Calderón's projectors have been studied in very general frameworks; see the books by Egorov, Komech and Shubin [206], Hörmander [246], Trèves [427], Grubb [232]. The ΨDO operators with irregular coefficients have been extensively studied, see Nagase [347], Kumano-go and Nagase [297], Beals and Reed [66] and the books of

Taylor [418, 420, 421, 419]. Lerner's book contains a lot of additional results [311].

3.8.4 About the Dirichlet-to-Neumann operator

The idea of studying the incompressible Euler equation at free surface using tools derived from the analysis of singular integrals goes back to an article by Craig–Schanz–Sulem [170]; this idea has been pursued by, among others, Craig–Schanz–Sulem [170], Lannes [300], Ming-Zhang [344] and Iooss–Plotnikov [265]. The paradifferential analysis of the Dirichlet-to-Neumann operator is introduced by Alazard-Métivier in [29].

The paradiferential approach is inspired by another context of free boundary problems: the study of shock waves. and rarefaction waves for conservation law systems by Majda [323, 322], Alinhac [37, 36] and Métivier [333, 334]. The good unknown of Alinhac plays a key role in the study of surface waves. For the water-wave problems, it was used by Lannes [300] and Trakhinin [425] to study the linearized equations.

There are close links between Alinhac's good unknown and the geometric analysis of Ambrose and Masmoudi [45, 46, 47], Lindblad [314, 313] and Shatah and Zeng [388]. There are also links with the work of Coulombel and Secchi [156] on the study of the problem of vortex pockets for the equation compressible Euler in dimension two (see also [157]).

We now discuss the boundedness of the Dirichlet-to-Neumann operator on Sobolev spaces. Expressing $G(\eta)$ as a singular integral operator, it was proved by Craig, Schanz and Sulem [170] that if η is in C^{k+1} and ψ is in H^{k+1} for some integer k, then $G(\eta)\psi$ belongs to H^k. Moreover, it was proved by Lannes [300] that when η is a function with limited smoothness, then $G(\eta)$ is a pseudo-differential operator with symbol of limited regularity. This implies that if η is in H^s and ψ is in H^s for some s large enough, then $G(\eta)\psi$ belongs to H^{s-1} (which was first established by Craig and Nicholls [169] and Wu [445, 446] by different methods). We refer to [22, 19, 388, 389] for results in rough domains.

Numerous works have been devoted to the study of elliptic equations in domains whose boundary is only Lipschitz. One could cite many works by Dahlberg, Dauge,Jerison, Kenig, Maz'ya and many others (see for instance [170, 181, 176, 274, 281] for many references). However, we are exclusively interested in to situations where the boundary has a Sobolev regularity, the threshold of lipschitzian regularity being seen only by the Sobolev injection; this allows to show results that are impossible to show for boundaries that are only C^1. One can nevertheless cite works closer to those of interest to us we refer, for instance, to chapter 14 of Maz'ya and Shaposhnikova [330] (see also [329]) and the article by Gérard-Varet and Hillairet [216].

An important extension of the estimates for Dirichlet-Neumann is given by Alvarez-Samniego and Lannes in [41] which prove estimates that are uniform with respect to the various physical parameters encountered in the study of surface waves. Let us also mention that these estimates are true for any bottom (as shown in [19]) under the assumption that the distance h between the free surface and the bottom is strictly positive. See also Lannes [302] for an analysis of the case where the constant h tends towards 0.

There are many extensions of the Theorem **??** for higher order derivatives (see [302]). That is, one can give the Taylor development of $G(\eta)$. This intervenes in the computation of formal solutions ([265]), to derive the equations in order to study the problem of Cauchy ([302, 80, 378]) and in the normal form methods ([217, 25]).

3.8.5 Propagation of singularities

Let us conclude this chapter by discussing the applications of paradifferential calculus to the study of singularities for nonlinear equations.

An important question in PDE is to determine the wavefront of the distribution solutions of the equation $Pf = 0$ where P is a differential operator of order m with coefficients C^∞,

$$P = \sum p_\alpha(x) D_x^\alpha, \quad (|\alpha| \le m,\ D_x = -i\partial_x).$$

An important geometrical object is constituted by the characteristic variety of P that we denote $\mathrm{Car}(P)$ and which is the closed (homogeneous in ξ) defined by

$$\mathrm{Car}(P) = \left\{ (x,\xi) \in T^*\mathbb{R}^d\,;\ p_m(x,\xi) = 0 \right\} \quad \text{where} \quad p_m(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha.$$

The first important result of the theory is the following: the singularities are contained in the characteristic variety. What means that in any point $(x_0, \xi_0) \in \mathbb{R}^d \setminus \{0\})$ such that $p_m(x_0,\xi_0) \neq 0$, there exists a symbol $\varphi = \varphi(x,\xi)$ homogeneous of order 0 in ξ, satisfying $\varphi(x_0,\xi_0) \neq 0$, such that $\mathrm{Op}(\varphi)u \in C^\infty$. See the text of Bony [87] and that of Lebeau [310] for two excellent introductions to these issues.

The paradiferential calculus has been introduced by Jean-Michel Bony to study the singularities of the equations to the non-linear partial derivatives of the form

$$F((\partial_x^\alpha f)_{|\alpha|\leq m}) = 0, \tag{3.8.1}$$

where $f \in H^s(\mathbb{R}^d)$ with $s > s_0 = d/2 + m$ so that $\partial_x^\alpha f$ is continuous and bounded for all multi-index α of length less than m.

We cannot describe for such a general equation the set of points where the function is not microlocally C^∞. On the other hand, thanks to the parodirectional calculus, we can say things if we replace C^∞ by a space of functions with limited regularity. Let us introduce

$$p_m(f; x, \xi) = \sum_{|\alpha|=m} \frac{\partial F}{\partial f_\alpha}((\partial_x^\alpha f(x))_{|\alpha|\leq m})(i\xi)^\alpha.$$

(The big difference between the linear and the non-linear case is that here the characteristic variety depends on the unknown of the problem). Bony then shows two results. The simplest one says that, in any point $(x_0, \xi_0) \in \mathbb{R}^d \setminus \{0\})$ such that $p_m(x_0, \xi_0) \neq 0$, f is microlocally twice as regular: it is microlocally of class H^t. for all $t < 2s - s_0$, which means that there exists a symbol $\varphi = \varphi(x, \xi)$ homogeneous of order 0 in ξ, satisfying $\varphi(x_0, \xi_0) \neq 0$, such that $\mathrm{Op}(\varphi)u \in H^t$.

The central point in the proof is to show that if f satisfies the equation (3.8.1) then $T_p f \in H^{t-m}(\mathbb{R}^d)$, where

$$p(x, \xi) = \sum_{|\beta|>2m-(s-d/2)} \frac{\partial F}{\partial f_\beta}((\partial_x^\alpha f(x))_{|\alpha|\leq m})(i\xi)^\beta.$$

It is said that we have parallelized (3.8.1) (we have replaced a non-linear equation by a linear paradiferential equation).

3.9 Exercises

Exercise 3.9.1 (Semi-classical operators)**.** *Consider a real number $h \in (0, 1]$ and a symbol $a = a(x, \xi)$ which belongs to $C_b^\infty(\mathbb{R}^{2n})$. We define*

$$\mathrm{Op}_h(a)u(x) = \frac{1}{(2\pi)^d} \int e^{ix\cdot\xi} a(x, h\xi)\widehat{u}(\xi)\, \mathrm{d}\xi.$$

We want to show that

$$\left\|\operatorname{Op}_h(a)\right\|_{\mathcal{L}(L^2)} \leqslant C \sup_{\mathbb{R}^{2d}} |a| + O(h^{\frac{1}{2}}).$$

1. *Show that*
$$\operatorname{Op}_h(a)u(x) = \big(\operatorname{Op}(a_h)u_h\big)(h^{-\frac{1}{2}}x)$$
where
$$a_h(x,\xi) = a(h^{\frac{1}{2}}x, h^{\frac{1}{2}}\xi),\ u_h(y) = u(h^{\frac{1}{2}}y).$$

2. *Deduce that there is a constant C and an integer M such that for all $a \in C_b^\infty(\mathbb{R}^{2d})$ and all $h \in (0,1]$,*
$$\left\|\operatorname{Op}_h(a)\right\|_{\mathcal{L}(L^2)} \leqslant C \sup_{(x,\xi)\in\mathbb{R}^{2n}} |a(x,\xi)| \\ + C \sup_{1\leqslant|\alpha|+|\beta|\leqslant M}\ \sup_{(x,\xi)\in\mathbb{R}^{2n}} h^{\frac{1}{2}(|\alpha|+|\beta|)} \left|\partial_x^\alpha \partial_\xi^\beta a\right|.$$

Exercise 3.9.2 (Wave packet transformation)**.** *Let $u\colon \mathbb{R} \to \mathbb{C}$ in the class of Schwartz $\mathcal{S}(\mathbb{R})$. The wave packet transform of u is the function $Wu\colon \mathbb{R} \to \mathbb{C}$ defined by*

$$Wu(x,\xi) = \int_{\mathbb{R}} e^{i(x-y)\xi - \frac{1}{2}(x-y)^2} u(y)\,\mathrm{d}y.$$

1. Show that $(x,\xi) \mapsto xWu(x,\xi)$ and $(x,\xi) \mapsto \xi Wu(x,\xi)$ are bounded on $\mathbb{R}^2$. Show more generally that Wu belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^2)$.

2. Show that, for any $x \in \mathbb{R}$,

$$\int |Wu(x,\xi)|^2\,\mathrm{d}\xi = 2\pi \int e^{-(x-y)^2/2}\,|u(y)|^2\,\mathrm{d}y.$$

Deduce that there is a constant $A > 0$ such that, for every u in $\mathcal{S}(\mathbb{R})$, we have

$$\iint |Wu(x,\xi)|^2\,\mathrm{d}x\,\mathrm{d}\xi = A \int |u(y)|^2\,\mathrm{d}y.$$

(It is not required to calculate A.)

3. Show that for any function u in the Schwartz class $\mathcal{S}(\mathbb{R})$,

$$Wu(x,\xi) = c\, e^{ix\xi}\,(W\widehat{u})(\xi,-x),$$

for a certain constant c (it is not required to calculate c).

4. *Let* $\varepsilon \in (0,1]$ *and u in Schwartz's class* $\mathcal{S}(\mathbb{R}^2)$. *We introduce*

$$W^\varepsilon u(x,\xi) = \varepsilon^{-3/4} \int_{\mathbb{R}} e^{i(x-y)\cdot\xi/\varepsilon - (x-y)^2/2\varepsilon} u(y)\,\mathrm{d}y.$$

Check that $A^{-1/2}W^\varepsilon$ *is an isometry and then show that there is K such that for all* $\varepsilon \in (0,1]$ *and all functions u and v in the Schwartz class* $\mathcal{S}(\mathbb{R})$,

$$\left\| vW^\varepsilon u - W^\varepsilon(vu) \right\|_{L^2(\mathbb{R}^2)} \le K\varepsilon^{1/2} \|\partial_x v\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}.$$

5. *Show that there is* K' *such that, for all* $\varepsilon \in (0,1]$ *and for all function u in the Schwartz class* $\mathcal{S}(\mathbb{R})$,

$$\left\| i\xi W^\varepsilon u - W^\varepsilon(\varepsilon\partial_x u) \right\|_{L^2(\mathbb{R}^2)} \leqslant K'\varepsilon^{1/2} \|u\|_{L^2(\mathbb{R})}.$$

Exercise 3.9.3 (An unbounded operator on L^2)**.** *Let* $\chi \in C_0^\infty(\mathbb{R})$ *be such that*

$$\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}\,,\, 2^{-1/2} \leqslant |\xi| \leqslant 2^{1/2}\}, \quad \chi(\xi) = 1 \quad si\ 2^{-1/4} \leqslant |\xi| \leqslant 2^{1/4}.$$

Set

$$a(x,\xi) = \sum_{j=1}^{+\infty} \exp(-i2^j x)\chi(2^{-j}\xi).$$

1. *Show that* $a \in C^\infty(\mathbb{R}^2)$ *verifies*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{|\alpha|-|\beta|} \quad \forall \alpha,\beta \in \mathbb{N}^2,\ \forall (x,\xi) \in \mathbb{R}^2.$$

2. *Let* f_0 *be a function of the Schwartz class whose Fourier transform* $\widehat{f_0}$ *is to support in the range* $[-1/2.1/2]$. *For* $N \in \mathbb{N}$ *we set*

$$f_N(x) = \sum_{j=2}^{N} \frac{1}{j} \exp(i2^j x) f_0(x).$$

Using Plancherel's formula, show that

$$\|f_N\|_{L^2}^2 = \Big(\sum_{j=2}^{N} j^{-2}\Big) \|f_0\|_{L^2}^2 \leqslant c.$$

3. *Show that*

$$\operatorname{Op}(a) f_N = \Big(\sum_{j=2}^{N} j^{-1}\Big) f_0.$$

4. *Conclude.*

Chapter 4

The Dirichlet-to-Neumann operator

In this chapter we gather various results about the Dirichlet-to-Neumann operator that are obtained by means of the multiplier method.

We begin by studying the definition of the Dirichlet–Neumann operator under general assumptions on the fluid domain. Allowing a general bottom and assuming only that $\eta \in W^{1,\infty}(\mathbb{R}^d)$ and $\psi \in H^{\frac{1}{2}}(\mathbb{R}^d)$, we show how to define an harmonic extension ϕ of ψ. Then we prove that $G(\eta)\psi$ is well-defined and that the map

$$\psi \in H^{\frac{1}{2}}(\mathbb{R}^d) \mapsto G(\eta)\psi \in H^{-\frac{1}{2}}(\mathbb{R}^d)$$

is continuous. Next, we obtain three inequalities for the Dirichlet-to-Neumann operator. The first one gives a control of the $H^{\frac{1}{2}}(\mathbb{R}^d)$ norm of ψ in terms of the $L^2(\Omega)$-norm of $\nabla_{x,y}\phi$ assuming only that $\nabla\eta$ is estimated in BMO. The last two inequalities will be obtained by means of the multiplier method, inspired by Rellich and Pohozaev inequalities.

4.1 Definition

All the results in this section come from works with Nicolas Burq and Claude Zuily ([19, 21, 22]).

4.1.1 General assumptions on the domain

We consider a general fluid domain Ω located underneath a free surface Σ and moving in a fixed container denoted by $\mathcal{O}$, so that

$$\Omega = \{(x, y) \in \mathcal{O} \,:\, y < \eta(x)\}\,,$$

where $\mathcal{O}$ is a given open domain which contains a fixed strip around the free surface

$$\Sigma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \,:\, y = \eta(x)\}.$$

This assumption implies that there exists $h > 0$ such that,

$$\Omega_h := \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R} \,:\, \eta(x) - h < y < \eta(x)\right\} \subset \Omega. \tag{4.1.1}$$

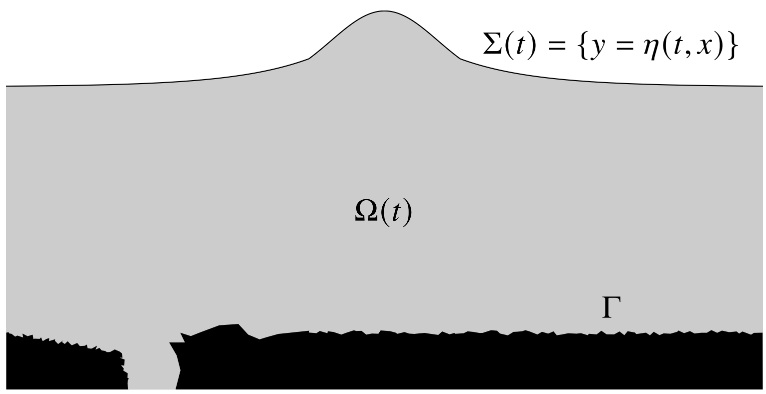
We also assume that the domain $\mathcal{O}$ (and hence the domain Ω) is connected.

We denote by Γ the bottom, which is given by

$$\Gamma = \partial\Omega \setminus \Sigma.$$

Examples.

1. $\mathcal{O} = \mathbb{R}^d \times \mathbb{R}$ corresponds to the infinite depth case ($\Gamma = \emptyset$);
2. The finite depth case corresponds to $\mathcal{O} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \,:\, y > b(x)\}$ for some continuous function b such that $\eta(t, x) - h > b(x)$ for any time t (then $\Gamma = \{y = b(x)\}$). Notice that no regularity assumption is required on b.
3. See the picture below.



4.1.2 Harmonic extension

The goal is to study the boundary value problem

$$\Delta_{x,y}\phi = 0 \text{ in } \Omega, \quad \phi|_{y=\eta} = f, \quad \partial_n\phi|_\Gamma = 0. \tag{4.1.2}$$

Since we make no assumption on Γ, the definition of ϕ requires some care. We introduce the space of functions that vanish on the free surface Σ and prove a variant of the Poincaré inequality in this setting.

Notation 4.1.1. *Denote by $\mathscr{D}$ the space of functions $u \in C^\infty(\Omega)$ such that $\nabla_{x,y}u \in L^2(\Omega)$. We then define $\mathscr{D}_0$ as the subspace of functions $u \in \mathscr{D}$ such that u is equal to 0 in a neighborhood of the top boundary Σ.*

Proposition 4.1.2. *There exists a positive weight $g \in L^\infty_{loc}(\Omega)$, equal to 1 on a neighborhood of Σ and a constant $C > 0$ such that for all $u \in \mathscr{D}_0$,*

$$\iint_\Omega g(x,y)u(x,y)^2 \,\mathrm{d}y\,\mathrm{d}x \le C \iint_\Omega |\nabla_{x,y}u(x,y)|^2 \,\mathrm{d}y\,\mathrm{d}x. \tag{4.1.3}$$

Proof. Let us set

$$\begin{aligned} \mathcal{O}_1 &= \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R} \,:\, \eta(x) - h < y < \eta(x)\right\}, \\ \mathcal{O}_2 &= \left\{(x,y) \in \Omega \,:\, y < \eta(x) - h\right\}. \end{aligned} \tag{4.1.4}$$

To prove Proposition 4.1.2, the starting point is the following Poincaré inequality on $\mathcal{O}_1$.

Lemma 4.1.3. *For all $u \in \mathscr{D}_0$ we have*

$$\iint_{\mathcal{O}_1} u^2 \,\mathrm{d}y\,\mathrm{d}x \le h^2 \iint_\Omega |\nabla_{x,y}u|^2 \,\mathrm{d}y\,\mathrm{d}x.$$

Proof. For $(x,y) \in \mathcal{O}_1$ we can write $u(x,y) = -\int_y^{\eta(x)} (\partial_y u)(x,z)\,\mathrm{d}z$, so using the Cauchy-Schwarz inequality we obtain

$$|u(x,y)|^2 \le h \int_{\eta(x)-h}^{\eta(x)} \left|(\partial_y u)(x,z)\right|^2 \mathrm{d}z.$$

Integrating on $\mathcal{O}_1$ we obtain the desired conclusion. □

Lemma 4.1.4. *Let $m_0 \in \Omega$ and $\delta > 0$ such that*

$$B(m_0, 2\delta) = \{m \in \mathbb{R}^d \times \mathbb{R} \, : \, |m - m_0| < 2\delta\} \subset \Omega.$$

Then for any $m_1 \in B(m_0, \delta)$ and any $u \in \mathscr{D}$,

$$\iint_{B(m_0,\delta)} u^2 \, \mathrm{d}y \, \mathrm{d}x \leq 2 \iint_{B(m_1,\delta)} u^2 \, \mathrm{d}y \, \mathrm{d}x + 2\delta^2 \iint_{B(m_0,2\delta)} \left|\nabla_{x,y} u\right|^2 \mathrm{d}y \, \mathrm{d}x. \tag{4.1.5}$$

Proof. Denote by $v = m_0 - m_1$ and write

$$u(m+v) = u(m) + \int_0^1 v \cdot \nabla_{x,y} u(m+tv) \, \mathrm{d}t.$$

As a consequence, we get

$$u(m+v)^2 \leq 2u(m)^2 + 2|v|^2 \int_0^1 \left|\nabla_{x,y} u(m+tv)\right|^2 \mathrm{d}t,$$

and integrating this last inequality on $B(m_1, \delta) \subset B(m_0, 2\delta) \subset \Omega$, we obtain (4.1.5). □

Lemma 4.1.5. *For any compact $K \subset \mathcal{O}_2$, there exists a constant $C(K) > 0$ such that, for all $u \in \mathscr{D}_0$, we have*

$$\iint_K u^2 \, \mathrm{d}y \, \mathrm{d}x \leq C(K) \iint_\Omega |\nabla_{x,y} u|^2 \, \mathrm{d}y \, \mathrm{d}x.$$

Proof. Consider now an arbitrary point $m_0 \in \mathcal{O}_2$. Since Ω is open and connected, there exists a continuous map $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = m_0$ and $\gamma(1) \in \mathcal{O}_1$. By compactness, there exists $\delta > 0$ such that for any $t \in [0, 1]$ $B(\gamma(t), 2\delta) \subset \Omega$. Taking smaller δ if necessary, we can also assume that $B(\gamma(1), \delta) \subset \mathcal{O}_1$ so that by Lemma 4.1.3

$$\iint_{B(\gamma(1),\delta)} u^2 \, \mathrm{d}y \, \mathrm{d}x \leq C \iint_\Omega |\nabla_{x,y} u|^2 \, \mathrm{d}y \, \mathrm{d}x.$$

We now can find a sequence $t_0 = 0, t_1, \cdots, t_N = 1$ such that the points $m_n = \gamma(t_n)$ satisfy $m_{n+1} \in B(m_n, \delta)$. Applying Lemma 4.1.4 successively, we obtain

$$\iint_{B(m_0,\delta)} u^2 \, \mathrm{d}y \, \mathrm{d}x \leq C' \iint_\Omega |\nabla_{x,y} u|^2 \, \mathrm{d}y \, \mathrm{d}x.$$

Then Lemma 4.1.5 follows by compactness. □

We are now in position to prove Proposition 4.1.2. Writing $\mathcal{O}_2 = \cup_{n=1}^{\infty} K_n$, and taking a partition of unity (χ_n) such that $0 \le \chi_n \le 1$ and $\operatorname{supp} \chi_n \subset K_n$, we can define the continuous function

$$\widetilde{g}(x,y) = \sum_{n=1}^{\infty} \frac{\chi_n(x,y)}{(1 + C(K_n))n^2},$$

which is clearly positive. Then by Corollary 4.1.5,

$$\begin{aligned}\iint_{\mathcal{O}_2} \widetilde{g}(x,y)\, |u|^2 \,\mathrm{d}y\,\mathrm{d}x &\le \sum_{n=1}^{\infty} \frac{1}{(1 + C(K_n))n^2} \iint_{K_n} |u|^2 \,\mathrm{d}y\,\mathrm{d}x \\ &\le 2 \iint_{\mathcal{O}_2} \left|\nabla_{x,y} u\right|^2 \mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{4.1.6}$$

Finally, let us set

$$g(x,y) = 1 \quad \text{for } (x,y) \in \mathcal{O}_1, \quad g(x,y) = \widetilde{g}(x,y) \quad \text{for } (x,y) \in \mathcal{O}_2.$$

Then Proposition 4.1.2 follows from Lemma 4.1.3 and (4.1.6). □

Definition 4.1.6. *Denote by $H^{1,0}(\Omega)$ the space of functions u on Ω such that there exists a sequence $(u_n) \in \mathscr{D}_0$ such that,*

$$\nabla_{x,y} u_n \to \nabla_{x,y} u \text{ in } L^2(\Omega, \mathrm{d}y\,\mathrm{d}x), \qquad u_n \to u \text{ in } L^2(\Omega, g(x,y)\,\mathrm{d}y\,\mathrm{d}x).$$

We endow the space $H^{1,0}(\Omega)$ with the norm $\|u\| = \left\|\nabla_{x,y} u\right\|_{L^2(\Omega)}$.

Proposition 4.1.7. *The space $H^{1,0}(\Omega)$ is a Hilbert space.*

Proof. Indeed, passing to the limit in (4.1.3), we obtain first that by definition, the norm on $H^{1,0}(\Omega)$ is equivalent to

$$\left\|\nabla_{x,y} u\right\|_{L^2(\Omega,\mathrm{d}y\,\mathrm{d}x)} + \|u\|_{L^2(\Omega, g(x,y)\,\mathrm{d}y\,\mathrm{d}x)}\,.$$

As a consequence, if (u_n) is a Cauchy sequence in $H^{1,0}(\Omega)$, we obtain easily from the completeness of L^2 spaces that there exists $u \in L^2(\Omega, g(x,y)\,\mathrm{d}y\,\mathrm{d}x)$ and $v \in L^2(\Omega, \mathrm{d}y\,\mathrm{d}x)$ such that

$$u_n \to u \text{ in } L^2(\Omega, g(x,y)\,\mathrm{d}y\,\mathrm{d}x), \qquad \nabla_{x,y} u_n \to v \text{ in } L^2(\Omega, \mathrm{d}y\,\mathrm{d}x).$$

But the convergence in $L^2(\Omega, g(x,y)\,\mathrm{d}y\,\mathrm{d}x)$ implies the convergence in $\mathcal{D}'(\Omega)$ and consequently $v = \nabla_{x,y} u$ in $\mathcal{D}'(\Omega)$ and it is easy to see that $u \in H^{1,0}(\Omega)$. □

We are able now to define a variational solution to the Dirichlet boundary value problem.

Let $f \in H^{\frac{1}{2}}(\mathbb{R}^d)$. We first define an H^1 lifting of f in Ω.

Lemma 4.1.8. *There exists a function $F \in H^1(\Omega)$ satisfying the three following properties: (i) $F|_{y=\eta} = f$, (ii) F vanishes on a neighborhood of the bottom Γ and (iii)*

$$\|F\|_{H^1(\Omega)} \leq K(1 + \|\eta\|_{W^{1,\infty}})\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}, \tag{4.1.7}$$

for some universal constant K.

Proof. We begin by constructing such a lifting in the case where Ω is replaced by a half space $\mathbb{R}^{d+1}_-$. Then we extend the construction to Ω by a simple change of variables.

Let $\chi_0 \in C^\infty(\mathbb{R})$ be such that $\chi_0(z) = 1$ if $z \geq -\frac{1}{2}$ and $\chi_0(z) = 0$ if $z \leq -1$. We set

$$\tilde{F}(x, z) = \chi_0(z) e^{z\langle D_x \rangle} f(x), \quad x \in \mathbb{R}^d, z \leq 0.$$

By using the Plancherel's identity, we verify (exercise) that

$$\left\|\nabla_{x,z}\tilde{F}\right\|_{L^2([-1,0]\times\mathbb{R}^d)} \leq C \|f\|_{H^{\frac{1}{2}}(\mathbb{R}^d)}.$$

Then we set

$$F(x, y) = \tilde{F}\Big(x, \frac{y - \eta(x)}{h}\Big), \quad (x, y) \in \Omega.$$

This is well defined since $\Omega \subset \{(x, y) : y < \eta(x)\}$. Moreover since the bottom Γ is contained in $\{(x, y) : y < \eta(x) - h\}$, we see that F vanishes identically near Γ.

Now we have obviously $F|_\Sigma = f$ and since $\nabla\eta \in L^\infty(\mathbb{R}^d)$, an easy computation shows that $F \in H^1(\Omega)$ together with (4.1.7). This completes the proof. □

Then the map

$$v \mapsto -\iint_\Omega \nabla_{x,y} F \cdot \nabla_{x,y} v \,\mathrm{d}y\,\mathrm{d}x$$

is a bounded linear form on $H^{1,0}(\Omega)$. It follows from the Riesz theorem that there exists a unique $u \in H^{1,0}(\Omega)$ such that

$$\forall v \in H^{1,0}(\Omega), \qquad \iint_\Omega \nabla_{x,y} u \cdot \nabla_{x,y} v \,\mathrm{d}y\,\mathrm{d}x = -\iint_\Omega \nabla_{x,y} F \cdot \nabla_{x,y} v \,\mathrm{d}y\,\mathrm{d}x. \tag{4.1.8}$$

Then u is the variational solution to the problem

$$-\Delta_{x,y}u = \Delta_{x,y}F \quad \text{in } \mathcal{D}'(\Omega), \qquad u\,|_{\Sigma}= 0, \qquad \partial_n u\,|_{\Gamma}= 0,$$

the latter condition being justified as soon as the bottom Γ is regular enough.

Lemma 4.1.9. *The function $\phi = u + F$ constructed by this procedure is independent on the choice of the lifting function F as long as it remains bounded in $H^1(\Omega)$ and vanishes near the bottom.*

Proof. Consider two functions constructed by this procedure, $\phi_k = u_k + F_k$, $k = 1, 2$. Then, by standard density arguments, since $F_1 - F_2$ vanishes at the top boundary Σ and in a neighborhood of the bottom Γ, there exists a sequence of functions $\psi_n \in C_0^\infty(\Omega)$ supported in a fixed Lipschitz domain $\widetilde{\Omega} \subset \Omega$ tending to $F_1 - F_2$ in $H_0^1(\widetilde{\Omega})$ and hence also in $H^{1,0}(\Omega)$. As a consequence, $F_1 - F_2 \in H^{1,0}(\Omega)$ and the function $\phi = \phi_1 - \phi_2$ is the unique (trivial) solution in $H^{1,0}(\Omega)$ of the equation $\Delta_{x,y}\phi = 0$ given by the Riesz Theorem. □

Definition 4.1.10. *We shall say that the function $\phi = u + \underline{\psi}$ constructed by the above procedure is the variational solution of* (4.1.2). *It satisfies*

(4.1.9)
$$\iint_{\Omega} \left|\nabla_{x,y}\phi\right|^2 \mathrm{d}y\, \mathrm{d}x \leq K \left\|f\right\|^2_{H^{\frac{1}{2}}(\mathbb{R}^d)},$$

for some constant K depending only on the Lipschitz norm of η.

4.1.3 About the behavior at infinite depth

When the fluid domain has infinite depth, it is often convenient to truncate the domain Ω to work with a domain with finite depth (to apply for instance the Stokes theorem or the maximum principe). To do so, one considers $\beta > 0$ such that the affine plane $\mathbb{R}^d \times \{-\beta\}$ is located underneath the free surface $\Sigma = \{y = h(x)\}$ and set

$$\Omega_\beta = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}\,;\, -\beta < y < h(x)\}.$$

Then one performs computations in Ω_β and then let β goes to $+\infty$. To justify that one can pass to the limit, it is important to justify the fact that ϕ and its derivative converges to 0 when y goes to $-\infty$. This is the content of the following lemma.

Proposition 4.1.11. *Assume that $\eta \in W^{1,\infty}(\mathbb{R}^d)$ and consider a real number $h > 0$ such that $\eta(x) > -h$ for all $x \in \mathbb{R}^d$. Set*

$$\Pi = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \, ; \, y < -h\}.$$

Then, for any multi-index $\alpha \in \mathbb{N}^d$ and any $\beta \in \mathbb{N}$ with $|\alpha| + \beta > 0$, one has

$$\partial_x^\alpha \partial_y^\beta \phi \in L^2(\Pi) \quad \text{and} \quad \lim_{y \to -\infty} \sup_{x \in \mathbb{R}^d} \left| \partial_x^\alpha \partial_y^\beta \phi(x, y) \right| = 0. \tag{4.1.10}$$

Proof. Consider another real number $0 < h' < h$ such that $\eta(x) > -h'$ for all $x \in \mathbb{R}^d$, and consider the strip

$$S = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \, ; \, -h < y < -h'\}.$$

Since $\nabla_{x,y}\phi$ belongs to $L^2(\Omega)$, we have $\nabla_{x,y}\phi \in L^2(S')$. Then, the trace $\zeta := \phi|_{y=-h}$ is well-defined and belongs to $H^{-1/2}(\mathbb{R}^d)$ (using $\partial_y^2 \phi = -\Delta_x \phi$ and Lemma **??**). Set $\zeta = \phi_{|y=-h}$.

Now, in the domain $\Pi := \{(x, y) \, ; \, y < -h\}$, ϕ coincides with the harmonic extension of ζ, by uniqueness of the harmonic extension. Since Π is invariant by translation in x, we can compute the latter function by using the Fourier transform in x. It results that,

$$\forall x \in \mathbb{R}^d, \quad \forall y < -h, \qquad \phi(x, y) = (e^{(y+h)|D|}\zeta)(x). \tag{4.1.11}$$

(Here, for $\tau < 0$, $e^{\tau|D|}$ denotes the Fourier multiplier with symbol $e^{\tau|\xi|}$.) Indeed, the function $(e^{(y+h)|D|}\zeta)(x)$ is clearly harmonic and is equal to ζ on $\{y = h\}$. Then, for $|\alpha| + \beta > 0$, it easily follows from (4.1.11) and the Plancherel theorem that $\partial_x^\alpha \partial_y^\beta \phi$ belongs to $L^2(\Pi)$.

To prove the second half of (4.1.10), we use again the formula (4.1.11) and the Plancherel theorem, to infer that $\partial_x^\alpha \partial_y^\beta \phi(\cdot, y)$ converges to 0 in any Sobolev space $H^\mu(\mathbb{R}^d)$ ($\mu \geq 0$) when y goes to $-\infty$. The desired decay result now follows from the Sobolev embedding $H^\mu(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ for $\mu > d/2$. □

4.2 Definition of the Dirichlet-to-Neumann operator

Consider two functions $\eta \in W^{1,\infty}(\mathbb{R}^d)$ and $\psi \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Denote by ϕ the harmonic extension of ψ as defined in the previous section, which is a variational solution to

$$\Delta_{x,y}\phi = 0 \text{ in } \Omega, \quad \phi|_{y=\eta} = f, \quad \partial_n \phi|_\Gamma = 0. \tag{4.2.1}$$

Formally the Dirichlet-to-Neumann operator is defined by

$$G(\eta)\psi = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi\,\Big|_{y=\eta(x)} = \big[\partial_y\phi - \nabla\eta\cdot\nabla\phi\big]\,\Big|_{y=\eta}. \tag{4.2.2}$$

Since we merely have $\nabla_{x,y}\phi \in L^2(\Omega)$, some care is needed to give a rigorous meaning to the trace of $\partial_n\phi\,\big|_{y=\eta}$. This is the subject matter of this paragraph.

A classical idea in free boundary problems is to use a change of variables to reduce the problem to a fixed domain. The simplest way to proceed is to map the domain $\Omega_h = \eta(x) - h < y < \eta(x)$ to the flat strip $\mathbb{R}^d \times (-h, 0)$ via the correspondence

$$(x, y) \mapsto (x, z) \quad \text{where} \quad z = y - \eta(x).$$

This change of variables takes $\Delta_{x,y}$ to a strictly elliptic operator and ∂_n to a vector field which is transverse to the boundary $\{z = 0\}$. Namely, introduce $v\colon \mathbb{R}^d\times[-h,0] \to \mathbb{R}$ defined by

$$v(x,z) = \phi(x, z+\eta(x)),$$

so that v satisfies

$$v|_{z=0} = \phi|_{y=\eta} = f.$$

Notice that

$$\begin{aligned}(\partial_z v)(x,z) &= (\partial_y\phi)(x, z+\eta(x)),\\ \nabla v(x,z) &= \nabla\phi(x, z+\eta(x)) + (\partial_y\phi)(x, z+\eta(x)).\end{aligned}$$

It follows that

$$(\nabla\phi)(x, z+\eta(x)) = (\nabla v)(x,z) - (\partial_z v)(x,z)\nabla\eta(x),$$

and v satisfies the equation

$$\Big(\partial_z^2 + (\nabla - (\nabla\eta)\partial_z)^2\Big)v = 0,$$

which gives

$$(1+|\nabla\eta|^2)\partial_z^2 v + \Delta v - 2\nabla\eta\cdot\nabla\partial_z v - (\partial_z v)\Delta\eta = 0, \tag{4.2.3}$$

in $\mathbb{R}^d \times (-\infty, 0)$. Then,

$$G(\eta)f = U\,\Big|_{z=0} \quad \text{with} \quad U := (1+|\nabla\eta|^2)\partial_z v - \nabla\eta\cdot\nabla v\,\Big|_{z=0}. \tag{4.2.4}$$

We will also use the fact that

$$\iint_{\Omega_h h} \big|\nabla_{x,y}\phi\big|^2 \,\mathrm{d}y\,\mathrm{d}x = \iint_{\mathbb{R}^d\times(-h,0)} \big(\,|\nabla v - (\partial_z v)\nabla\eta|^2 + (\partial_z v)^2\big)\,\mathrm{d}x\,\mathrm{d}z. \tag{4.2.5}$$

Proposition 4.2.1. *Let $d \geq 1$ and consider $\eta \in W^{1,\infty}(\mathbb{R}^d)$. Then for all $f \in H^{\frac{1}{2}}(\mathbb{R}^d)$, the formula (4.2.4) defines a function $G(\eta)f$ in $H^{-\frac{1}{2}}(\mathbb{R}^d)$.*

Now, the key point is that U satisfies

$$\begin{aligned} U &= \partial_z v - \nabla\eta \cdot (\nabla v - (\partial_z v)\nabla\eta), \\ \partial_z U &= -\operatorname{div}(\nabla v - (\partial_z v)\nabla\eta). \end{aligned}$$

Hence it follows from (4.2.5) that

$$U \in L^2_z((-h,0); L^2(\mathbb{R}^d)) \quad \text{and} \quad \partial_z U \in L^2_z((-h,0); H^{-1}(\mathbb{R}^d)).$$

Then the proposition follows from the following classical lemma (see [315, Theorem 3.1]).

Lemma 4.2.2. *Let $I = (-1,0)$ and $s \in \mathbb{R}$. Let $u \in L^2_z(I, H^{s+\frac{1}{2}}(\mathbb{R}^d))$ such that $\partial_z u \in L^2_z(I, H^{s-\frac{1}{2}}(\mathbb{R}^d))$. Then $u \in C^0([-1,0]; H^s(\mathbb{R}^d))$ and there exists an absolute constant $C > 0$ such that*

$$\sup_{z\in[-1,0]} \|u(z,\cdot)\|_{H^s(\mathbf{R}^d))} \leq C\Big(\|u\|_{L^2(I,H^{s+\frac{1}{2}}(\mathbb{R}^d))} + \|\partial_z u\|_{L^2(I,H^{s-\frac{1}{2}}(\mathbb{R}^d))}\Big).$$

4.3 A trace estimate

The next result is from a joint work with Quoc-Hung Nguyen [34]. It gives a trace estimate assuming only that the gradient of η is estimated in $\mathrm{BMO}(\mathbb{R}^{\mathrm{d}})$. This is a strictly smaller of $L^\infty(\mathbb{R}^d)$, introduced by John and Nirenberg, which is defined as follows.

Definition 4.3.1. *The space $\mathrm{BMO}(\mathbb{R}^{\mathrm{d}})$ (for Bounded Mean Oscillations) consists of those functions $f \in L^1(\mathbb{R}^d_{loc}(\mathbb{R}^d)$ such that $\|f\|_{\mathrm{BMO}} < +\infty$, with*

$$\|f\|_{\mathrm{BMO}} = \sup_{Q\subset\mathbb{R}^d} \frac{1}{|Q|}\int_Q \left|f - f_Q\right| \mathrm{d}x,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ and

$$f_Q = \frac{1}{|Q|}\int_Q f \,\mathrm{d}x.$$

Proposition 4.3.2. *There exists a positive constant c such that,*

$$\int_{\mathbb{R}^d} f G(h) f \,\mathrm{d}x \geq \frac{c}{1 + \|\nabla h\|_{\mathrm{BMO}}} \|f\|^2_{\dot{H}^{\frac{1}{2}}} .$$

Proof. We use the notations introduced in the previous section. Recall that

$$\int_{\mathbb{R}^d} f G(h) f \,\mathrm{d}x = Q(f), \tag{4.3.1}$$

where the quadratic form $Q(f)$ is defined by

$$Q(f) = \iint_{\mathbb{R}^{d+1}_-} |\nabla v - \partial_z v \nabla h|^2 \,\mathrm{d}x\,\mathrm{d}z + \iint_{\mathbb{R}^{d+1}_-} (\partial_z v)^2 \,\mathrm{d}x\,\mathrm{d}z. \tag{4.3.2}$$

On the other hand, by writing that $f = \partial_z v|_{z=0}$, we have

$$\int_{\mathbb{R}^d} f\, |D|\, f \,\mathrm{d}x = \iint_{\mathbb{R}^{d+1}_-} \partial_z (v\, |D|\, v) \,\mathrm{d}x\,\mathrm{d}z,$$

where $|D|\, v$ is the function defined by $(|D|\, v)(\cdot, z) = |D|\, (v(\cdot, z))$. Now, we use the Leibniz rule and the fact that $|D|$ is self-adjoint to write

$$\begin{aligned} \int_{\mathbb{R}^d} f\, |D|\, f \,\mathrm{d}x &= \iint_{\mathbb{R}^{d+1}_-} \big((\partial_z v)\, |D|\, v + v \partial_z\, |D|\, v\big) \,\mathrm{d}x\,\mathrm{d}z \\ &= 2 \iint_{\mathbb{R}^{d+1}_-} (\partial_z v)\, |D|\, v \,\mathrm{d}x\,\mathrm{d}z. \end{aligned}$$

Now introduce the Riesz transform $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_d)$ where $\mathcal{R}_j$ is the Fourier multiplier with symbol $-i\xi_j/|\xi|$, so that $|D| = \mathcal{R} \cdot \nabla$. With this operator, one can write

$$\begin{aligned} \int_{\mathbb{R}^d} f\, |D|\, f \,\mathrm{d}x = 2 &\iint_{\mathbb{R}^{d+1}_-} (\partial_z v) \mathcal{R} \cdot \big(\nabla v - (\partial_z v) \nabla h\big) \,\mathrm{d}x\,\mathrm{d}z \\ &+ 2 \iint_{\mathbb{R}^{d+1}_-} (\partial_z v) \mathcal{R} \cdot \big((\partial_z v) \nabla h\big) \,\mathrm{d}x\,\mathrm{d}z. \end{aligned} \tag{4.3.3}$$

The end of the proof is in two steps. We will compare the first term in the right-hand side to $Q(f)$. Secondly, we will estimate the second term by exploiting a commutator estimate.

Step 1.

Notice that $\mathcal{R}$ is a skew-symmetric operator. Therefore

$$\iint_{\mathbb{R}^{d+1}_-} (\partial_z v)\mathcal{R}\cdot\big(\nabla v-(\partial_z v)\nabla h\big)\,\mathrm{d}x\,\mathrm{d}z=-\iint_{\mathbb{R}^{d+1}_-}(\mathcal{R}(\partial_z v))\cdot\big(\nabla v-(\partial_z v)\nabla h\big)\,\mathrm{d}x\,\mathrm{d}z,$$

where $\mathcal{R}(\partial_z v)$ is the vector valued function with cooordinates $\mathcal{R}_j\partial_z v$.

Given two vectors a,b in $\mathbb{R}^d$, we use the inequality $-2a\cdot b\le |a|^2+|b|^2$ to deduce that

$$-2\mathcal{R}(\partial_z v)\cdot\big(\nabla v-(\partial_z v)\nabla h\big)\le|\mathcal{R}(\partial_z v)|^2+|\nabla v-(\partial_z v)\nabla h|^2 \tag{4.3.4}$$

On the other hand, by Plancherel's theorem, we have

$$\int_{\mathbb{R}^d}|\mathcal{R}(\partial_z v)|^2\,\mathrm{d}x=\int_{\mathbb{R}^d}(\partial_z v)^2\,\mathrm{d}x.$$

Consequently, it follows from (4.4.4) that

$$Q(f)\ge -2\iint_{\mathbb{R}^{d+1}_-}\mathcal{R}(\partial_z v)\cdot\big(\nabla v-(\partial_z v)\nabla h\big)\,\mathrm{d}x\,\mathrm{d}z$$

In light of (4.3.3), we conclude that

$$Q(f)\ge\int_{\mathbb{R}^d} f\,|D|\,f\,\mathrm{d}x+2\iint_{\mathbb{R}^{d+1}_-}(\partial_z v)\mathcal{R}\cdot\big((\partial_z v)\nabla h\big)\,\mathrm{d}x\,\mathrm{d}z. \tag{4.3.5}$$

Step 2.

In this step we estimate

$$2\iint_{\mathbb{R}^{d+1}_-}(\partial_z v)\mathcal{R}\cdot\big((\partial_z v)\nabla h\big)\,\mathrm{d}x\,\mathrm{d}z.$$

To do so, we exploit the fact that $\mathcal{R}$ is a skew-symmetric operator to make appear a commutator. More precisely, notice that, for any functions $\zeta=\zeta(x)$ and $g=g(x)$, and for any index $j\in\{1,\dots,d\}$, there holds

$$\begin{aligned}2\int_{\mathbb{R}^d}\zeta\mathcal{R}_j(g\zeta)\,\mathrm{d}x&=\int_{\mathbb{R}^d}\zeta\mathcal{R}_j(g\zeta)\,\mathrm{d}x+\int_{\mathbb{R}^d}(\mathcal{R}_j^*\zeta)(g\zeta)\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}\zeta\big[\mathcal{R}_j,g\big]\zeta\,\mathrm{d}x.\end{aligned}$$

By using this identity with $\zeta = v(\cdot, z)$ at fixed z and then integrating in z, we get that

$$2 \iint_{\mathbb{R}^{d+1}_-} (\partial_z v)\mathcal{R} \cdot \big((\partial_z v)\nabla h\big) \, \mathrm{d}x \, \mathrm{d}z = \iint_{\mathbb{R}^{d+1}_-} (\partial_z v) \sum_{j=1}^{d} \big[\mathcal{R}_j, \partial_{x_j} h\big] \partial_z v \, \mathrm{d}x \, \mathrm{d}z.$$

Now, we use the following commutator estimate

$$\big\| \big[\mathcal{R}_j, g\big] \zeta \big\|_{L^2(\mathbb{R}^d)} \lesssim \| g \|_{\mathrm{BMO}(\mathbb{R}^d)} \| \zeta \|_{L^2(\mathbb{R}^d)} \, .$$

By using the Cauchy-Schwarz inequality, this implies that

$$\begin{aligned} \left| 2 \iint_{\mathbb{R}^{d+1}_-} (\partial_z v)\mathcal{R} \cdot \big((\partial_z v)\nabla h\big) \, \mathrm{d}x \, \mathrm{d}z \right| &\lesssim \| \nabla h \|_{\mathrm{BMO}} \iint_{\mathbb{R}^{d+1}_-} (\partial_z v)^2 \, \mathrm{d}x \, \mathrm{d}z \\ &\leq \| \nabla h \|_{\mathrm{BMO}} \, Q(f). \end{aligned} \tag{4.3.6}$$

End of the proof.

By combining (4.3.5) and (4.3.6), we get

$$\int_{\mathbb{R}^d} f G(h) f \, \mathrm{d}x = Q(f) \geq \int_{\mathbb{R}^d} f \, |D| \, f \, \mathrm{d}x - \| \nabla h \|_{\mathrm{BMO}} \, Q(f).$$

Remembering (4.3.1), this concludes the proof. □

4.4 Rellich estimates

Proposition 4.4.1. *For any smooth functions h and ζ, there holds*

$$\int_{\mathbb{R}^d} (G(h)\zeta)^2 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} (1 + |\nabla h|^2) |\nabla \zeta - \mathcal{B} \nabla h|^2 \, \mathrm{d}x, \tag{4.4.1}$$

where

$$\mathcal{B} = \frac{G(h)\zeta + \nabla \zeta \cdot \nabla h}{1 + |\nabla h|^2}. \tag{4.4.2}$$

Moreover, when $d = 1$, this is an equality:

$$\int_{\mathbb{R}} (G(h)\zeta)^2 \, \mathrm{d}x = \int_{\mathbb{R}} (1 + (\partial_x h)^2)(\partial_x \zeta - \mathcal{B} \partial_x h)^2 \, \mathrm{d}x. \tag{4.4.3}$$

Proof. *i*) Set

$$\Omega = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}\,; y < h(x)\},$$

and denote by ϕ the harmonic function defined by

$$\begin{cases} \Delta_{x,y}\phi = 0 & \text{in } \Omega = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}\,;\ y < h(x)\}, \\ \phi(x,h(x)) = \zeta(x). \end{cases} \tag{4.4.4}$$

This is a classical elliptic boundary problem, which admits a unique smooth solution. Moreover, it satisfies

$$\lim_{y\to-\infty} \sup_{x\in\mathbb{R}^d} \left|\nabla_{x,y}\phi(x,y)\right| = 0. \tag{4.4.5}$$

Introduce the notations

$$\mathcal{V} = (\nabla\phi)_{|y=h}, \qquad \mathcal{B} = (\partial_y\phi)_{|y=h}.$$

(We parenthetically recall that ∇ denotes the gradient with respect to the horizontal variables $x = (x_1, \dots, x_d)$ only.) As we already seen in the first chapter, it follows from the chain rule that

$$\mathcal{V} = \nabla\zeta - \mathcal{B}\nabla h,$$

where $\mathcal{B}$ is given by (4.4.2). On the other hand, by definition of the Dirichlet-to-Neumann operator, one has

$$G(h)\zeta = \left(\partial_y\phi - \nabla h \cdot \nabla\phi\right)_{|y=h},$$

so

$$G(h)\zeta = \mathcal{B} - \nabla h \cdot \mathcal{V}.$$

Squaring this identity yields

$$(G(h)\zeta)^2 = \mathcal{B}^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V} + (\nabla h \cdot \mathcal{V})^2.$$

Since $(\nabla h \cdot \mathcal{V})^2 \le |\nabla h|^2 \cdot |\mathcal{V}|^2$, this implies the inequality:

$$(G(h)\zeta)^2 \le \mathcal{B}^2 - |\mathcal{V}|^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V} + (1 + |\nabla h|^2)\mathcal{V}^2. \tag{4.4.6}$$

Integrating this gives

$$\int_{\mathbb{R}^d} (G(h)\zeta)^2 \,\mathrm{d}x \le \int_{\mathbb{R}^d} (1 + |\nabla h|^2)\, |\mathcal{V}|^2 \,\mathrm{d}x + R,$$

where

$$R = \int_{\mathbb{R}^d} \left(\mathcal{B}^2 - |\mathcal{V}|^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V}\right) \mathrm{d}x.$$

Since $|\mathcal{V}| = |\nabla\zeta - \mathcal{B}\nabla h|$, we immediately see that, to obtain the wanted estimate (4.4.1), it is sufficient to prove that $R = 0$. To do so, we begin by noticing that R is the flux associated to a vector field. Indeed,

$$R = \int_{\partial\Omega} X \cdot n \, \mathrm{d}\mathcal{H}^d$$

where $X\colon \Omega \longrightarrow \mathbb{R}^{d+1}$ is given by

$$X = (2(\partial_y\phi)\nabla\phi; (\partial_y\phi)^2 - |\nabla\phi|^2).$$

Then the key observation is that this vector field satisfies $\operatorname{div}_{x,y} X = 0$ since

$$\partial_y\left((\partial_y\phi)^2 - |\nabla\phi|^2\right) + 2\operatorname{div}\left((\partial_y\phi)\nabla\phi\right) = 2(\partial_y\phi)\Delta_{x,y}\phi = 0,$$

as can be verified by an elementary computation. Now, we see that the cancellation $R = 0$ comes from the Stokes' theorem. To rigorously justify this point, we truncate Ω in order to work in a smooth bounded domain. Given a parameter $\beta > 0$, set

$$\Omega_\beta = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}\,; -\beta < y < h(x)\}.$$

An application of the divergence theorem in Ω_β gives that

$$0 = \iint_{\Omega_\beta} \operatorname{div}_{x,y} X \,\mathrm{d}y\,\mathrm{d}x = R + \int_{\{y=-\beta\}} X \cdot (-e_y)\,\mathrm{d}x,$$

where e_y is the vector $(0, \dots, 0, 1)$ in $\mathbb{R}^{d+1}$. Sending β to $+\infty$ and remembering that X converges to 0 uniformly when y goes to $-\infty$ (see (4.4.5)), we obtain the expected result $R = 0$. This completes the proof of (4.4.1).

ii) Now assume that $d = 1$ and let us prove that we have an identity instead of an inequality. To do so, it is sufficient to notice that to derive (4.4.6) we use the inequality $(\nabla h \cdot \mathcal{V})^2 \leq |\nabla h|^2 \cdot |\mathcal{V}|^2$, which is clearly an equality when $d = 1$. □

4.5 A Pohozaev estimate

Consider a three-dimensional fluid in a rectangular tank, bounded by a flat bottom, vertical walls and a free surface evolving under the influence of gravity. The inequality that we will prove in this section is motivated by the following problem: Can we estimate the energy of the water-wave evolving in this tank by looking only at the motion of the points of contact between the free surface and the vertical walls?

Denote by $d \in \{1,2\}$ the dimension of the free surface. For a 3D (resp. 2D) fluid one has $d = 2$ (resp. $d = 1$) and we use the notation $\nabla = (\partial_{x_1}, \partial_{x_2})$ (resp. $\nabla = \partial_x$). We consider water waves over an incompressible liquid, evolving under the influence of gravity, in the case where the fluid is located inside a fixed rectangular tank $\mathscr{R}$ of the form $\mathscr{R} = Q \times [-h, +\infty)$ where $Q = [0, L_1] \times [0, L_2]$ for $d = 2$ (resp. $Q = [0, L_1]$ for $d = 1$) and h is a positive constant. At time t, the fluid domain $\Omega(t)$ is given by

$$\Omega(t) = \{ (x, y) \ : \ x \in Q, \ -h \le y \le \eta(t, x) \}, \tag{4.5.1}$$

where $x = (x_1, x_2)$ (resp. y) is the horizontal (resp. vertical) space variable.

We want to determine the energy by observing the motion of the curves of contact between the free surface and the walls $\{x_1 = L_1\}$ and $\{x_2 = L_2\}$:

$$\mathscr{C}_1(t) = \{(L_1, x_2, y), \ x_2 \in [0, L_2], \ y = \eta(t, L_1, x_2)\},$$
$$\mathscr{C}_2(t) = \{(x_1, L_2, y), \ x_1 \in [0, L_1], \ y = \eta(t, x_1, L_2)\}.$$

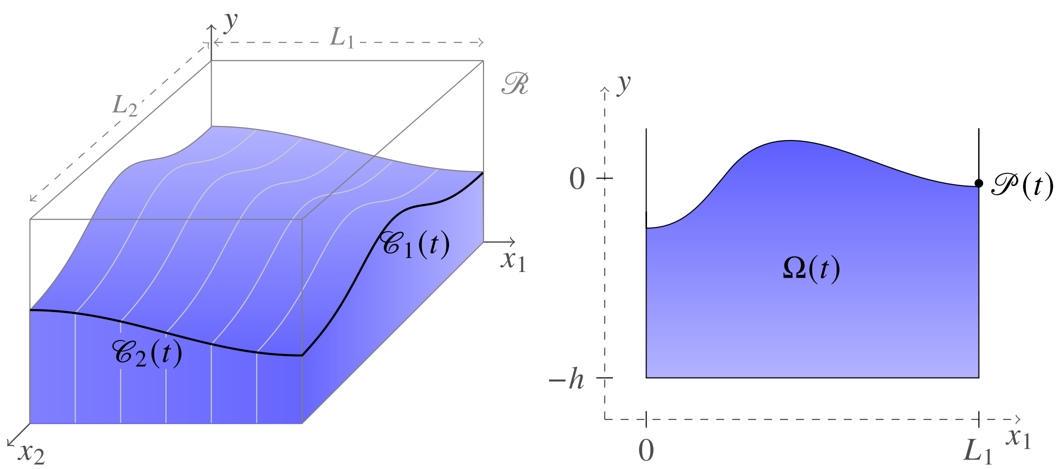


Figure 4.1: Three-dimensional and two-dimensional waves in a rectangular tank

To study the previous problem, the key point is to prove a Pohozaev type identity for $G(\eta)\psi$, that is we want to compute

$$\int_Q (G(\eta)\psi)(x \cdot \nabla\psi) \, \mathrm{d}x.$$

Since we are working in a bounded domain, we need to clarify the boundary conditions on the lateral walls. One denotes by ν the outward unit normal to Q ($\nu = (1,0)$ if $x_1 = L_1$, $\nu = (0,-1)$ if $x_2 = 0$,...). We assume that η and ψ are such that

$$\partial_\nu \eta = 0, \quad \partial_\nu \psi = 0 \text{ on } \partial Q.$$

In this case, one can prove that ψ has a unique harmonic extension satisfying

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \phi|_{y=\eta} = \psi, \tag{4.5.2}$$

together with $\partial_n \phi = 0$ on $\partial\mathcal{R} \cap \partial\Omega$, which implies that

$$\partial_{x_1}\phi = 0 \quad \text{for} \quad x_1 = 0 \text{ or } x_1 = L_1, \tag{4.5.3}$$

$$\partial_{x_2}\phi = 0 \quad \text{for} \quad x_2 = 0 \text{ or } x_2 = L_2, \tag{4.5.4}$$

$$\partial_y \phi = 0 \quad \text{for} \quad y = -h. \tag{4.5.5}$$

It is possible to prove by elementary arguments that this problem has a unique solution such that $\nabla_{x,y}\phi \in C^1(\overline{\Omega})$ and $\partial_n\phi = 0$ on $\partial\mathcal{R} \cap \partial\Omega$. (This is sufficient to justify all the computations done below.)

Proposition 4.5.1 (Pohozaev identity)**.** *Denote by R the solid part of $\partial\Omega$:*

$$R := \partial\mathcal{R} \cap \partial\Omega,$$

($\mathcal{R} = Q \times [-h, +\infty)$) and denote by n the unit outward normal to $\partial\Omega$. Then, the following identity holds

$$\begin{aligned}
&\int_Q (G(\eta)\psi)(x\cdot\nabla\psi)\,\mathrm{d}x \\
&\quad = \frac{1}{2}\int_R \left|\nabla_{x,y}\phi\right|^2 \begin{pmatrix} x \\ y \end{pmatrix} \cdot n \,\mathrm{d}\mathcal{H}^d - \frac{d-1}{2}\iint_\Omega \left|\nabla_{x,y}\phi\right|^2 \mathrm{d}y\,\mathrm{d}x \\
&\qquad + \frac{1}{2}\int_Q (\eta - x\cdot\nabla\eta)\left[V^2 + B^2 - 2BG(\eta)\psi\right]\mathrm{d}x,
\end{aligned} \tag{4.5.6}$$

where $B = (\partial_y\phi)|_{y=\eta(x)}$ and $V = (\nabla_x\phi)|_{y=\eta(x)}$.

Remark 4.5.2. *i) If $d = 1$ then the second term in the right-hand side of* (4.5.6) *vanishes and, since $n\cdot\nabla_{x,y}\phi = 0$ on R, the first one simplifies to*

$$\frac{1}{2}\int_R \left|\nabla_{x,y}\phi\right|^2 \begin{pmatrix} x \\ y \end{pmatrix} \cdot n \,\mathrm{d}\mathcal{H}^d = \frac{L_1}{2}\int_{-h}^{\eta(L_1)} (\partial_y\phi(L_1,y))^2\,\mathrm{d}y + \frac{h}{2}\int_0^{L_1} (\partial_x\phi(x,-h))^2\,\mathrm{d}x.$$

Consider now the case $d = 2$. Then one has also $(x, y) \cdot n \geq 0$. Indeed,

$$\begin{array}{lll} \text{on } \{x_1 = L_1\} & \text{one has } n = (1,0,0) & \text{and } (x,y)\cdot n = L_1, \\ \text{on } \{x_2 = L_2\} & \text{one has } n = (0,1,0) & \text{and } (x,y)\cdot n = L_2, \\ \text{on } \{y = -h\} & \text{one has } n = (0,0,-1) & \text{and } (x,y)\cdot n = h, \end{array}$$

and moreover, $(x, y) \cdot n \equiv 0$ on the two other faces $\{x_1 = 0\}$ and $\{x_2 = 0\}$.

Proof. The proof of this proposition relies on the divergence theorem applied to a well chosen vector field. Introduce the scalar function

$$\theta := x \cdot \nabla_x \phi + y \partial_y \phi$$

and the vector field

$$X = \theta \nabla_{x,y} \phi.$$

We are going to compute the integral of $\operatorname{div}_{x,y} X$ by two different ways. The wanted identity (4.5.6) will be deduced by comparing the two results.

First computation. We want to exploit the fact that, since $\partial_n \phi = 0$ on R, one has $X \cdot n = 0$ on R. To do so we begin by writing

$$\iint_\Omega \operatorname{div}_{x,y} X \, \mathrm{d}y \, \mathrm{d}x = \int_{\partial\Omega} X \cdot n \, \mathrm{d}\mathcal{H}^d = \int_{\partial\Omega \setminus R} X \cdot n \, \mathrm{d}\mathcal{H}^d.$$

Since $\partial\Omega \setminus R = \{(x, y), x \in Q, y = \eta(x)\}$, by definition of $G(\eta)\psi$, the previous identity simplifies to

$$\begin{aligned} \iint_\Omega \operatorname{div}_{x,y} X \, \mathrm{d}y \, \mathrm{d}x &= \int_{\partial\Omega \setminus R} \theta \partial_n \phi \, \mathrm{d}\mathcal{H}^d \\ &= \int_Q \theta(x, \eta) \sqrt{1 + |\nabla \eta|^2} \, \partial_n \phi|_{y=\eta} \, \mathrm{d}x \\ &= \int_Q \theta(x, \eta) G(\eta)\psi \, \mathrm{d}x. \end{aligned}$$

Now, write

$$\nabla_x \psi = \nabla_x(\phi(x, \eta(x))) = (\nabla_x \phi)(x, \eta(x)) + (\partial_y \phi)(x, \eta(x)) \nabla_x \eta.$$

Since $B = \partial_y \phi(x, \eta)$, we get that $(\nabla_x \phi)(x, \eta(x)) = \nabla_x \psi - B \nabla_x \eta$. By definition of θ, we deduce that

$$\theta(x, \eta) = x \cdot (\nabla_x \psi - B \nabla_x \eta) + \eta B = x \cdot \nabla_x \psi + (\eta - x \cdot \nabla_x \eta) B.$$

We thus end up with

$$\text{(4.5.7)} \quad \iint_{\Omega} \operatorname{div}_{x,y} X \,\mathrm{d}y\,\mathrm{d}x = \int_Q (G(\eta)\psi)(x\cdot\nabla\psi)\,\mathrm{d}x + \int_Q (\eta - x\cdot\nabla_x\eta)BG(\eta)\psi\,\mathrm{d}x.$$

Second computation. Set

$$\mathcal{W} = \left|\nabla_{x,y}\phi\right|^2.$$

As can be verified by a direct computation, one has

$$\operatorname{div}_{x,y} X = \mathcal{W} + \frac{1}{2}x\cdot\nabla_x\mathcal{W} + \frac{1}{2}y\partial_y\mathcal{W},$$

and hence

$$\operatorname{div}_{x,y} X = \operatorname{div}\left(\frac{\mathcal{W}}{2}x\right) + \partial_y\left(\frac{\mathcal{W}}{2}y\right) - \frac{d-1}{2}\mathcal{W}.$$

Introduce the vector field

$$Y = \frac{\mathcal{W}}{2}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the previous identity reads $\operatorname{div}_{x,y} X = \operatorname{div}_{x,y} Y - \frac{d-1}{2}\mathcal{W}$. Consequently

$$\iint_{\Omega} \operatorname{div}_{x,y} X \,\mathrm{d}y\,\mathrm{d}x = \int_{\partial\Omega} Y\cdot n\,\mathrm{d}\mathcal{H}^d - \frac{d-1}{2}\iint_{\Omega}\mathcal{W}\,\mathrm{d}y\,\mathrm{d}x.$$

Now observe that,

$$\int_R Y\cdot n\,\mathrm{d}\mathcal{H}^d = \frac{1}{2}\int_R \left|\nabla_{x,y}\phi\right|^2\begin{pmatrix} x \\ y \end{pmatrix}\cdot n\,\mathrm{d}\mathcal{H}^d,$$

$$\int_{\partial\Omega\setminus R} Y\cdot n\,\mathrm{d}\mathcal{H}^d = \frac{1}{2}\int_Q \mathcal{W}|_{y=\eta}\begin{pmatrix} x \\ \eta \end{pmatrix}\cdot\begin{pmatrix} -\nabla\eta \\ 1 \end{pmatrix}\mathrm{d}x = \frac{1}{2}\int_Q (\eta - x\cdot\nabla\eta)\mathcal{W}|_{y=\eta}\,\mathrm{d}x.$$

Therefore

$$\begin{aligned}\iint_{\Omega} \operatorname{div}_{x,y} X \,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}&\int_R \left|\nabla_{x,y}\phi\right|^2\begin{pmatrix} x \\ y \end{pmatrix}\cdot n\,\mathrm{d}\mathcal{H}^d \\ &+ \frac{1}{2}\int_Q (\eta - x\cdot\nabla\eta)\mathcal{W}|_{y=\eta}\,\mathrm{d}x - \frac{d-1}{2}\iint_{\Omega}\mathcal{W}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

By combining this identity with (4.5.7) we obtain the wanted result (4.5.6). □

4.6 Pointwise inequalities

In order to consider various different situations, in this paragraph we assume that the fluid domain is of the form

$$\Omega = \{(x, y) \in \mathbb{T}^d \times \mathbb{R} \, : \, y < h(x)\},$$

where h is a periodic function. As above, the free surface is denoted by Σ:

$$\Sigma = \{(x, h(x)) \, ; \, x \in \mathbb{T}^d\},$$

where we use $\mathbb{T}^d$ to denote a d-dimensional torus.

4.6.1 Applications of the maximum principle

We begin by recalling the following maximum principle.

Proposition 4.6.1. *Let $h \in W^{1,\infty}(\mathbb{T}^d)$. Consider a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $\nabla_{x,y} u \in L^2(\Omega)$ satisfying*

$$-\Delta u \geq 0 \quad \textit{in } \Omega, \qquad u|_{\Sigma} \geq 0.$$

Then

$$u \geq 0.$$

Proof. See Chapter 3 in [220]. □

We next recall the Zaremba principle (see [455]; also known as Hopf's maximum principle). It states that, if $\partial\Omega$ is C^2 and $x_0 \in \partial\Omega$, then

$$-\Delta u = f \geq 0 \quad \text{in } \Omega, \quad u(x) > u(x_0) \quad \text{in } \Omega \quad \Rightarrow \quad \partial_n u(x_0) < 0.$$

We shall use a version which holds in domain which are less regular.

Theorem 4.6.2. *Let $h \in C^{1,\alpha}(\mathbb{T}^d)$ with $\alpha \in (0, 1)$. Consider a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with $\nabla_{x,y} u \in L^2(\Omega)$ satisfying*

$$-\Delta u \geq 0 \quad \textit{in } \Omega, \qquad u|_{\Sigma} \geq 0.$$

If u attains its minimum at a point x_0 of the boundary Σ, then

$$\partial_n u(x_0) < 0.$$

Proof. See Safonov [380], Apushkinskaya-Nazarov [180] and Nazarov [350]. □

We deduce from this inequality the following pointwise bound for the Dirichlet-to-Neumann operator.

Proposition 4.6.3. *For any function $h \in C^{1,\alpha}(\mathbb{T}^d)$ with $\alpha \in (0,1)$, there holds*

$$G(h)h < 1.$$

Proof. We will prove that

$$\frac{G(h)h + |\nabla h|^2}{1 + |\nabla h|^2} < 1,$$

equivalent to the wanted result. It follows from Lemma 1.2.3 that

$$\frac{G(h)h + |\nabla h|^2}{1 + |\nabla h|^2} = B(h)h = (\partial_y \phi)|_{y=h},$$

where ϕ is the variational solution to

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \quad \phi(x, h(x)) = h(x).$$

Hence the proof reduces to showing that $(\partial_y \phi)|_{y=h} < 1$.

Given $\ell > 0$, set

$$\Omega_\ell = \left\{(x,y) \in \mathbb{T}^d \times \mathbb{R}\,;\, -\ell < y < h(x)\right\},$$

and introduce

$$P = \phi - y.$$

Then P is an harmonic function in Ω_ℓ vanishing on $\Sigma := \{y = h(x)\}$. Moreover, since $\partial_y \phi$ goes to 0 when y goes to $-\infty$ (see Proposition 4.1.11), one gets that, if ℓ is large enough, then

$$\partial_y P|_{y=-\ell} < 0.$$

Since $\partial_n P = -\partial_y P$ on $\{y = -\ell\}$, one infers from the Zaremba principle that P cannot reach its minimum on $\{y = -\ell\}$. So P reaches its minimum on Σ.

On the other hand, P is constant on Σ. This shows that P reaches its minimum on any point of Σ. Using again the Zaremba principle, one concludes that $\partial_n P < 0$ on any point of Σ. So, to conclude the proof, it remains only to relate $\partial_n P$ and $\partial_y P$ on Σ. To do so, we apply the chain rule to the equation $P(x, h(x)) = 0$. This gives

$$(\nabla P)|_{y=h} = -(\partial_y P)|_{y=h} \nabla h.$$

Recalling that $n = (1+|\nabla h|^2)^{-1/2}\begin{pmatrix}-\nabla h\\ 1\end{pmatrix}$, and using the previous identity, one has

$$(\partial_n P)|_{y=h} = \frac{1}{\sqrt{1+|\nabla h|^2}}(\partial_y P - \nabla h\cdot\nabla P)|_{y=h} = \sqrt{1+|\nabla h|^2}(\partial_y P)|_{y=h}.$$

This proves that $(\partial_y P)|_{y=h} < 0$ on Σ, which means that $(\partial_y \phi)|_{y=h} - 1 < 0$, which is the desired inequality. □

4.6.2 A convexity inequality

Assume that $d = 1$. Then

$$|D| = H\partial_x,$$

where H is the periodic Hilbert transform, defined by

$$\widehat{Hu}(0) = 0, \quad \widehat{Hu}(k) = -i\,\mathrm{sign}(k)\hat{u}(k) \qquad (k\in\mathbb{Z}\setminus\{0\}),$$

where we use $\hat{u}(k)$ to denote the Fourier coefficient of a function u in $L^1(\mathbb{T})$. We also have

$$Hu(x) = \frac{1}{2\pi}\,\mathrm{pv}\int_{-\pi}^{\pi} u(y)\cot\left(\frac{1}{2}(x-y)\right)\mathrm{d}y$$

(Compare with the formula (2.2.3) for the Hilbert transform for functions which are not periodic.)

Lemma 4.6.4 (Toland)**.** *Consider $u\in C^\infty(\mathbb{T})$. Then*

$$u(x)(|D|\,u)(x) - \frac{1}{2}\,|D|\,(u^2)(x) = \frac{1}{8\pi}\int_{-\pi}^{\pi}\left(\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right)^2\mathrm{d}y.$$

Proof. Since $|D| = H\partial_x$, we have

$$\begin{aligned}u(x)(|D|\,u)(x) - \frac{1}{2}\,|D|\,(u^2)(x) &= u(x)Hu'(x) - H(uu')(x)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(u(x)-u(y))u'(y)\cot\big(\frac{1}{2}(x-y)\big)\,\mathrm{d}y\\ &= -\frac{1}{4\pi}\int_{-\pi}^{\pi}\cot\big(\frac{1}{2}(x-y)\big)\partial_y(u(y)-u(x))^2\,\mathrm{d}y\\ &= \frac{1}{8\pi}\int_{-\pi}^{\pi}\left(\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)}\right)^2\mathrm{d}y,\end{aligned}$$

where we integrate by parts to deduce the final identity. □

It follows from Parseval's identity that and

$$\sum_{k\in\mathbb{Z}} |k| \left|\hat{u}(k)\right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\, |D|\, u \,\mathrm{d}x = \frac{1}{16\pi} \iint_{\mathbb{T}\times\mathbb{T}} \left(\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)} \right)^2 \mathrm{d}x\,\mathrm{d}y.$$

In particular, the $H^{\frac{1}{2}}(\mathbb{T})$-norm of u is equivalent to

$$\iint_{\mathbb{T}\times\mathbb{T}} \left(\frac{u(x)-u(y)}{\sin\frac{1}{2}(x-y)} \right)^2 \mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{T}} (u(x))^2 \,\mathrm{d}x.$$

(Compare with (2.2.6).)

Recall that $|D| = G(0)$. In particular, it follows from Lemma 4.6.4 that

$$2fG(0)f \geq G(0)(f^2).$$

Several extensions of this inequality are known (see §4.7.4 for many references). We will use later the following

Proposition 4.6.5. *Let $\alpha \in (0,1)$ and consider two functions f, h in $C^{1,\alpha}(\mathbb{T}^d)$. For any C^2 convex function $\Phi\colon \mathbb{R} \to \mathbb{R}$, it holds the pointwise inequality*

$$\Phi'(f)G(h)f \geq G(h)\big(\Phi(f)\big). \tag{4.6.1}$$

Remark 4.6.6. *We consider only periodic functions but the proof is extremely simple and easy to adapt to other settings.*

Proof. Denote by ζ (resp. ξ) the harmonic extension of f (resp. $\Phi(f)$), so that

$$\Delta_{x,y}\zeta = 0 \quad \text{in } \Omega, \qquad \zeta|_{y=h} = f,$$
$$\Delta_{x,y}\xi = 0 \quad \text{in } \Omega, \qquad \xi|_{y=h} = \Phi(f).$$

By assumption, ζ and ξ belong to $C^2(\Omega)\cap C^1(\overline{\Omega})$. By definition of the Dirichlet to Neumann operator and using the chain rule, one has

$$G(h)\big(\Phi(f)\big) - \Phi'(f)G(h)f = \sqrt{1+|\nabla h|^2}\,\partial_n\big(\xi - \Phi(\zeta)\big).$$

It suffices then to prove that the difference $u = \xi - \Phi(\zeta)$ satisfies $\partial_n u \leq 0$ on $\partial\Omega$. To do so, using that Φ is convex, we observe that

$$\Delta\Phi(\zeta) = \Phi'(\zeta)\Delta\zeta + \Phi''(\zeta)|\nabla\zeta|^2 = \Phi''(\zeta)|\nabla\zeta|^2 \geq 0.$$

Thus, we deduce that

$$-\Delta_{x,y} u \geq 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

It follows from the maximum principle that $u \geq 0$ in Ω. Since u vanishes on $\partial\Omega$, we infer that

$$\forall x \in \partial\Omega,\ \forall t > 0, \quad u(x - tn) \geq u(x),$$

where n is the outward unit normal to the boundary. Since u belongs to $C^1(\overline{\Omega})$, this immediately implies that $\partial_n u \leq 0$, which completes the proof. □

4.7 References

4.7.1 Campanato and BMO spaces

Consider a function $f \colon \mathbb{R}^d \to \mathbb{R}$. To study the regularity of f at a point x_0, a fruitful point of view is to study the integrability properties of the difference between f and its mean value on balls centered at x_0. To explain this, we begin by proving an elementary lemma.

Definition 4.7.1. *Let $x_0 \in \mathbb{R}^d$ and $r > 0$. If $f \colon \Omega \to \mathbb{R}$, one poses*

$$f_{x_0,r} = \fint_{B(x_0,r)} f \,\mathrm{d}x = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f \,\mathrm{d}x.$$

Lemma 4.7.2. *Suppose $f \in C^{0,\alpha}(\mathbb{R}^d)$. Then for all x_0 and all $r > 0$,*

$$\int_{B(x_0,r)} \left| f(x) - f_{x_0,r} \right|^p \mathrm{d}x \leqslant 2^{\alpha p} \left\| f \right\|_{C^{0,\alpha}}^p \omega_d r^{d+\alpha p},$$

where ω_d is the volume of the unit ball (note that the integral of the left-hand side is a usual integral).

Proof. Since

$$f(x) - f_{x_0,r} = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} \big(f(x) - f(y) \big) \,\mathrm{d}y$$

we have

$$\left| f(x) - f_{x_0,r} \right| \leqslant \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} \left\| f \right\|_{C^{0,\alpha}} |x-y|^\alpha \,\mathrm{d}y \leqslant \left\| f \right\|_{C^{0,\alpha}} (2r)^\alpha.$$

We conclude by raising the left-hand side to the power p and integrating on $B(x_0, r)$. $\square$

This idea lead to introduce the following spaces.

Definition 4.7.3 (Campanato Spaces)**.** *Let $\lambda > 0$, $p \in [1, \infty[$. A function $f \in L^1_{loc}(\mathbb{R}^d)$ belongs to the Campanato space $\mathcal{L}^{p,\lambda}(\mathbb{R}^d)$ if*

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \sup_{0<r} \sup_{x_0 \in \mathbb{R}^d} \left(r^{-\lambda} \int_{B(x_0,r)} \left| f(x) - f_{x_0,r} \right|^p \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

These spaces allow to study the Hölder regularity to solutions to elliptic equations, by means of the following

Theorem 4.7.4 (Campanato)**.** *Let $p \in [1, \infty[$ and $\lambda \in]d, d+p]$. Then*

$$\mathcal{L}^{p,\lambda}(\mathbb{R}^d) \subset C^{0,\alpha}(\mathbb{R}^d) \quad \textit{with } \alpha = \frac{\lambda - d}{p}.$$

Proof. See [15, 48]. $\square$

The endpoint space $p = 1$ and $\alpha = 0$ plays a central role in harmonic analysis. It was introduced by John and Nirenberg, and is named BMO (for functions with Bounded Mean Oscillations). Namely, we set

$$\mathrm{BMO}(\mathbb{R}^{\mathrm{d}}) = \mathcal{L}^{1,\mathrm{d}}(\mathbb{R}^{\mathrm{d}}).$$

We have clearly

$$L^\infty(\mathbb{R}^d) \hookrightarrow \mathrm{BMO}(\mathbb{R}^{\mathrm{d}}).$$

However there exist functions in $\mathrm{BMO}(\mathbb{R}^{\mathrm{d}})$ which are not bounded, for instance the function $x \mapsto \exp(-|x|^2)\log(|x|)$ belongs to $\mathrm{BMO}(\mathbb{R}^{\mathrm{d}})$. In the same direction, we also have

$$W^{1,d}(\mathbb{R}^d) \hookrightarrow \mathrm{BMO}(\mathbb{R}^{\mathrm{d}}).$$

Let us mention a couple of subtle estimates for paraproducts involving the BMO-semi-norm.

Theorem 4.7.5 (Coifman-Meyer)**.** *The paraproduct and remainder term satisfy*

$$\|T_a b\|_{L^2} \leq C \|a\|_{L^2} \|b\|_{\mathrm{BMO}},$$
$$\|R_{\mathcal{B}}(a, b)\|_{L^2} \leq C \|a\|_{L^2} \|b\|_{\mathrm{BMO}},$$

for some universal constant C.

4.7.2 Commutator estimates

Having introduced the BMO space, in this paragraph, we are going to state three different commutators estimates. The goal is to explain the different roles played by the spaces $L^\infty(\mathbb{R}^d)$, $\mathrm{BMO}(\mathbb{R}^\mathrm{d})$ and the Höder space $C^{0,\alpha}(\mathbb{R}^d)$ for some $\alpha > 0$.

We begin by a fundamental commutator estimate.

Theorem 4.7.6 (Calderón; Coifman-Meyer)**.** *Let $p \in (1, +\infty)$ and consider a symbol $a \in S^1_{1,0}(\mathbb{R}^d)$ of order 1. Then for any Lipschitz function $f \in W^{1,\infty}(\mathbb{R}^d)$, the commutator $[\mathrm{Op}(a), f]$ satisfies*

$$\|[\mathrm{Op}(a), f]u\|_{L^p} \le C \|f\|_{W^{1,\infty}} \|u\|_{L^p},$$

for some universal constant C.

Proof. See [134] and [418, Chapter 3]. □

This theorem was first proved by Calderón ([106]) when a is homogeneous in ξ. For instance, if $a = (1 - \chi(\xi))|\xi|$ where $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\chi(\xi) = 1$ on a neighborhood of 0, then $\mathrm{Op}(a) = (1 - \chi(D))\,|D|$. Since $\chi(D)\,|D|$ is a smoothing operator (continuous from $H^{-\infty}$ to H^∞), we infer that

$$\||D|\,(fu) - f\,|D|\,u\|_{L^p} \le C \|f\|_{W^{1,\infty}} \|u\|_{L^p}.$$

Our second commutator estimate has to do with BMO functions.

Theorem 4.7.7 (Coifman-Rochberg-Weiss)**.** *Let $p \in (1, +\infty)$ and consider a symbol $a \in S^0_{1,0}(\mathbb{R}^d)$ of order 0. Then, for any a function $f \in \mathrm{BMO}(\mathbb{R}^\mathrm{d})$, then the commutator $[\mathrm{Op}(a), f]$ satisfies*

$$\|[\mathrm{Op}(a), f]u\|_{L^p} \le C \|f\|_{\mathrm{BMO}} \|u\|_{L^p},$$

for some universal constant C.

Remark 4.7.8. *This theorem implies the following corollary for the Hilbert transform:*

$$\|[b, \mathcal{H}]u\|_{L^p} \le C \|b\|_{\mathrm{BMO}} \|u\|_{L^p}, \quad 1 < p < \infty,$$

where $[b, \mathcal{H}]u$ denotes the commutator

$$[b, \mathcal{H}]u = b(\mathcal{H}u) - \mathcal{H}(bu).$$

Eventually, we mention the following estimate for the commutator between the Hilbert transform and a function in some Hölder space $C^{0,\nu}(\mathbb{R}^d)$.

Proposition 4.7.9. *Let $0 < \theta < \nu < 1$. There exists a constant K such that for all $f \in C^{0,\nu}(\mathbb{R})$, and all u in $H^{-\theta}(\mathbb{R})$,*

$$\|\mathcal{H}(fu) - f\mathcal{H}u\|_{L^2} \leq K \|f\|_{C^{0,\nu}} \|u\|_{H^{-\theta}} . \tag{4.7.1}$$

Proof. See [**?**]. □

4.7.3 Pohozaev identity

The proof of (4.5.6) given above is guided by the study of the commutator $[x\partial_x, G(\eta)]$ in Alazard-Delort [25, Chapter 4]. However, one cannot apply the results of [25] because of the boundary conditions on R (and also because we consider the case $d \geq 1$ while the analysis in [25] is restricted to $d = 1$). Compared to [25, Chapter 4], the main new result here is the observation that the contribution of these boundary conditions is given by a positive term (namely the first term in the right-hand side of (4.5.6)).

Another Pohozaev identity for the fractional Laplacian has been proved by Ros-Oton and Serra [376]. Moreover, Biccari [81] deduced from the Pohozaev identity in [376] a Pohozaev identity for solutions to fractional Schrödinger equations. Namely, it is proved in [376] that, if Q is any $C^{1,1}$ domain of $\mathbb{R}^d$, $s \in (0,1)$ and $u \in H^s(\mathbb{R}^d)$ vanishes in $\mathbb{R}^d \setminus Q$, then

$$\begin{aligned}\int_Q (x \cdot \nabla u)(-\Delta)^s u \,\mathrm{d}x &= \frac{2s-d}{2}\int_Q u(-\Delta)^s u \,\mathrm{d}x \\ &\quad - \frac{\Gamma(1+s)^2}{2}\int_{\partial Q}\left(\frac{u}{\operatorname{dist}(x,\partial Q)^s}\right)^2 (x\cdot\nu)\,\mathrm{d}\mathcal{H}^d.\end{aligned} \tag{4.7.2}$$

4.7.4 About the pointwise inequalities

Proposition 4.6.3 is related to the Rayleigh–Taylor stability condition for the Hele-Shaw problem. In particular, the inequality $G(h)h < 1$ is classical and plays a central role in the study of the Cauchy problem for the latter equation (see [116, 113, 114, 126]). The proof of Proposition 4.6.3 is taken from [30] and follows from the proof by Lannes [300]Ê of a similar result for the gravity water-wave equations.

Lemma 4.6.4 is proved in [422] and applied to study Stokes waves. Similar computations appear in the study of parabolic free boundary problems. In particular, in [148, 149], Córdoba and Córdoba proved that, for any exponent α in $[0,1]$ and any C^2 function f decaying sufficiently fast at infinity, one has the pointwise inequality

$$2f(-\Delta)^\alpha f \geq (-\Delta)^\alpha (f^2).$$

This inequality has been generalized and applied to many different problems. To mention a few results, we quote the papers by Ju [275], Constantin and Ignatova ([142, 143]), Constantin, Tarfulea and Vicol ([146]), and we refer to the numerous references there in. Recently, Córdoba and Martínez ([151]) proved that

$$\Phi'(f)G(h)f \geq G(h)\big(\Phi(f)\big)$$

when h is a C^2 function and $\Phi(f) = f^{2m}$ for some positive integer m. The next proposition (from [30]) gives an extension to the general case where Φ is a convex function and h is C^s for some $s > 1$.

4.7.5 The multiplier method in control theory

The control theory of wave equations is well developed and many techniques have been introduced (microlocal analysis, Carleman estimates...). In this paper, we use the multiplier method. The key point is that this method allows us to work directly at the level of the nonlinear equations.

For the sake of readability, we begin by recalling some well-known results for the linear wave equation

$$\partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad u|_{\partial\Omega} = 0. \tag{4.7.3}$$

The multiplier method, introduced by Morawetz, consists in multiplying the equations by $m(x) \cdot \nabla u(t,x)$, for some well-chosen function m, and to integrate by parts in space and time. For instance, by considering a smooth extension $m\colon \Omega \to \mathbb{R}^n$ of the normal $\nu(x)$ to the boundary $\partial\Omega$, one obtains

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} (\partial_n u)^2 \,\mathrm{d}\sigma\,\mathrm{d}t \leq K(T)\mathcal{E}(u) \text{ where} \\ &\mathcal{E}(u) := \|u(0,\cdot)\|_{H_0^1(\Omega)}^2 + \|\partial_t u(0,\cdot)\|_{L^2(\Omega)}^2 . \end{aligned} \tag{4.7.4}$$

This is the so-called *hidden regularity* property. The name comes from the fact that, using energy estimates, one controls only the $C^0([0,T];L^2(\Omega))$-norm of $\nabla_x u$ by means of the right-hand side of (9.4.2), which is insufficient to control the left-hand side of (9.4.2) by means of classical trace theorems.

Another key estimate is the so-called *boundary observability inequality*, which is, compared to (9.4.2), a reverse inequality where one can bound the norms of the initial data by the integral of $\partial_n u$ restricted to a domain $\Gamma_0 \subset \partial\Omega$. Such an inequality can be obtained by the multiplier method applied in this way: fix $x_0 \in \mathbb{R}^n$ and set

$$\Gamma(x_0) = \big\{x \in \partial\Omega\,,\ (x - x_0)\cdot \nu(x) > 0\big\}, \quad T(x_0) = 2\max_{x\in\overline{\Omega}} |x - x_0|\,.$$

Then, multiplying the equation by $(x-x_0)\cdot\nabla u$ and integrating by parts, we get that, for $T > T(x_0)$,

$$(T - T(x_0))\mathcal{E}(u) \le \frac{T(x_0)}{2}\int_0^T\int_{\Gamma(x_0)} (\partial_n u)^2\,\mathrm{d}\sigma\,\mathrm{d}t. \tag{4.7.5}$$

For more details about the previous two inequalities, we refer the reader to the SIAM Review article by Lions [316] and the books by Komornik [293], Micu and Zuazua [340], Tucsnak and Weiss [429] and the lecture notes by Alabau-Boussouira in [10].

Now consider a domain ω surrounding $\Gamma(x_0)$. The proof of the hidden regularity property (9.4.2) allows us to bound the right-hand side in (4.7.5) by the sum of $C_1\mathcal{E}$ (where C_1 is independent of time) and the integral of $|\nabla u|^2$ on $(0,T)\times\omega$. Then, for T large enough, one can absorb the term $C_1\mathcal{E}$ in the left-hand side of (4.7.5) to deduce the following *internal observability inequality*:

$$\mathcal{E}(u) \le C(T)\int_0^T\int_\omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.7.6}$$

This inequality can be used to obtain directly a stabilization result for the following damped wave equation (see [461])

$$\partial_t^2 v - \Delta v + a(x)\partial_t v = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad v|_{\partial\Omega} = 0,$$

where $a \in C_0^\infty(\Omega)$ is a non-negative function satisfying $a(x) = 1$ for x in $\omega \subset\subset \Omega$. One can write v as $v = u + w$ where u and w are given by solving

$$\begin{aligned}
&\partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad u|_{\partial\Omega} = 0,\\
&\partial_t^2 w - \Delta w + a(x)\partial_t w = -a(x)\partial_t u \quad \text{in } \Omega \subset \mathbb{R}^n, \quad w|_{\partial\Omega} = 0,\\
&u(0,\cdot) = v(0,\cdot), \quad \partial_t u(t,0) = \partial_t v(0,\cdot); \quad w(0,\cdot) = 0, \quad \partial_t w(t,0) = 0.
\end{aligned} \tag{4.7.7}$$

Using the internal observability inequality for u and a straightforward estimate for w based on the Duhamel formula, one can deduce that $\mathcal{E}(v)(t) \leq ce^{-c't}$ for some positive constants c, c'.

Similar results are known for many other wave equations and we only mention the paper by Machtyngier [320] (see also [321]) for the Schrödinger equation $i\partial_t u + \Delta u = 0$. Biccari [81] introduced recently the use of the multiplier method to analyze the interior controllability problem for the fractional Schrödinger equation $i\partial_t u + (-\Delta)^s u = 0$ with $s \geq 1/2$ in a $C^{1,1}$ bounded domain with Dirichlet boundary condition. The key difference between the Schrödinger equation ($s = 1$) and the fractional equation (for $1/2 \leq s < 1$) is that the latter is nonlocal. This is a source of difficulty since one seeks an observability result involving integrals over small localized domains. The multiplier method is applied to the water-wave problem in the general case in [11, 14, 13].

Chapter 5

Sobolev estimates

5.1 Inequalities in Lebesgue spaces

Proposition 5.1.1 (Hölder)**.** *Consider three real numbers p, q, r in $[1, +\infty]$, satisfying*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}\cdot$$

Then, for any couple of functions $(f, g) \in L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$, the product fg belongs to $L^r(\mathbb{R}^d)$. Moreover, the following estimate holds

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} . \tag{5.1.1}$$

Proposition 5.1.2 (Minkowski)**.** *Suppose that (S_1, μ_1) and (S_2, μ_2) are two σ-finite measure spaces and $F\colon S_1 \times S_2 \to \mathbb{R}$ is measurable. Then for all $p \in [1, +\infty)$,*

$$\left(\int_{S_2}\left|\int_{S_1} F(x,y)\mu_1(\mathrm{d}x)\right|^p \mu_2(\mathrm{d}y)\right)^{\frac{1}{p}} \leq \left(\int_{S_1}\left(\int_{S_2} |F(x,y)|^p \mu_2(\mathrm{d}y)\right)^{\frac{1}{p}} \mu_1(\mathrm{d}x)\right). \tag{5.1.2}$$

Proposition 5.1.3 (Hardy)**.** *For any real number $p \in]1, +\infty[$ and for all function f in $L^p(]0, +\infty[)$, there holds*

$$\int_0^{+\infty} \left|\frac{1}{x}\int_0^x f(y)\,\mathrm{d}y\right|^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} |f(x)|^p \,\mathrm{d}x.$$

Proposition 5.1.4 (Young)**.** *Consider three real numbers p, q, r such that*

$$1 \leqslant p, q, r \leqslant \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}\cdot \tag{5.1.3}$$

Then, for any f in $L^p(\mathbb{R}^d)$ and any g in $L^q(\mathbb{R}^d)$, the integral

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y$$

converges for almost all $x \in \mathbb{R}^d$. In addition, $f * g$ belongs to $L^r(\mathbb{R}^d)$ and

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leqslant \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} .$$

We also recall the definition of Riesz potentials as well as the classical Hardy-Littlewood-Sobolev inequality.

Given a real number $\alpha > 0$, the Riesz potential I_α is the operator defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\,\mathrm{d}y.$$

Notice that $I_\alpha f(x)$ is well-defined for any $\alpha > 0$ and any function f with compact support. To see this, decompose $\mathbb{R}^d$ into two parts: the ball $B(x,1)$ with center x and radius 1, and its complementary. Since f is bounded, the fact that the integral over $B(x,1)$ is well-defined is straightforward since $\alpha > 0$. On the complementary, the integral is well-defined since we are integrating a bounded function with compact support.

Theorem 5.1.5 (Hardy-Littlewood-Sobolev)**.** *Let $d \in \mathbb{N}^*$ and consider three real positive numbers (p, q, α) such that*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d}, \quad 1 < p < \frac{d}{\alpha}.$$

Then there exists a constant $C = C(p, q, d, \alpha)$ such that, for any function f in $C_0^1(\mathbb{R}^d)$, there holds

$$\|I_\alpha f\|_{L^q} \leqslant C \|f\|_{L^p} .$$

5.2 Fourier analysis and Sobolev spaces

5.2.1 Definitions and first properties

Recall the notation

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

Given a real number $s \in [0, +\infty)$, we say that a function $u \in L^2(\mathbb{R}^d)$ belongs to the Sobolev space $H^s(\mathbb{R}^d)$ if

$$\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \, \mathrm{d}\xi < +\infty.$$

Proposition 5.2.1. *Let $s \in [0, +\infty)$. Equipped with the scalar product*

$$(u, v)_{H^s} = (2\pi)^{-d} \int (1 + |\xi|^2)^s \widehat{u}(\xi) \, \overline{\widehat{v}(\xi)} \, \mathrm{d}\xi,$$

and therefore the norm

$$\|u\|_{H^s} = (2\pi)^{-n/2} \left\| (1 + |\xi|^2)^{s/2} \widehat{u} \right\|_{L^2},$$

the Sobolev space $H^s(\mathbb{R}^d)$ is a Hilbert space.

Proof. The application $u \mapsto (2\pi)^{-d/2}(1 + |\xi|^2)^{s/2}\widehat{u}$ is by definition an isometric bijection of $H^s(\mathbb{R}^d)$ on $L^2(\mathbb{R}^d)$. This last space being a Banach space, it is the same for $H^s(\mathbb{R}^d)$ with the norm defined above. □

Proposition 5.2.2. *The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$ for all $s \geq 0$.*

Proof. Let us consider the isometry $u \mapsto (2\pi)^{-d/2}(1 + |\xi|^2)^{s/2}\widehat{u}$ from $H^s(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$. The inverse isometry transforms the dense subspace $\mathcal{S}(\mathbb{R}^d)$ of $L^2(\mathbb{R}^d)$ into a dense subspace of $H^s(\mathbb{R}^d)$. Now this application is a bijection of $\mathcal{S}(\mathbb{R}^d)$ onto itself. We deduce that $\mathcal{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$. □

Proposition 5.2.3. *For any real number $s > d/2$,*

$$H^s(\mathbb{R}^d) \subset C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),$$

with continuous injection.

Proof. According to Cauchy-Schwarz inequality, for any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|f\|_{L^\infty} \leqslant \|\hat{f}\|_{L^1} \leqslant \|\langle \xi \rangle^s\|_{L^2} \|\langle \xi \rangle^s \hat{f}\|_{L^2}, \tag{5.2.1}$$

and we deduce the result by density of $\mathcal{S}(\mathbb{R}^d)$ in $H^s(\mathbb{R}^d)$. □

Theorem 5.2.4. *For any real number $s > d/2$, the product of two elements of $H^s(\mathbb{R}^d)$ belongs to $H^s(\mathbb{R}^d)$. In addition, there is a constant C such that for any u, v in $H^s(\mathbb{R}^d)$,*

$$\|uv\|_{H^s} \leqslant C \, \|u\|_{H^s} \, \|v\|_{H^s} .$$

Proof. The proof rests on the following inequality: for every ξ, η in $\mathbb{R}^d$ we have

$$\forall s \geq 0, \qquad (1+|\xi|^2)^{s/2} \leqslant 2^s \Big\{ (1+|\xi-\eta|^2)^{s/2} + (1+|\eta|^2)^{s/2} \Big\},$$

which is deduced from the triangular inequality and the bound $(a+b)^r \leqslant 2^r(a^r+b^r)$ for any triplet (a,b,r) of positive numbers. Let us write then that for every u, v in $\mathcal{S}(\mathbb{R}^d)$, we have (check the following formula in exercise)

$$\widehat{uv}(\xi) = (2\pi)^{-d} \int \widehat{u}(\xi-\eta)\widehat{v}(\eta)\,\mathrm{d}\eta.$$

Multiplying the two members by $\langle\xi\rangle^s$ and using the previous inequality, we find

$$\begin{aligned} \langle\xi\rangle^s\,|\widehat{uv}(\xi)| \leqslant C &\int \langle\xi-\eta\rangle^s\,|\widehat{u}(\xi-\eta)|\,|\widehat{v}(\eta)|\,\mathrm{d}\eta \\ &+ C\int |\widehat{u}(\xi-\eta)|\,\langle\eta\rangle^s\,|\widehat{v}(\eta)|\,\mathrm{d}\eta. \end{aligned}$$

If $s > d/2$ then $\mathcal{F}(H^s(\mathbb{R}^d)) \subset L^1(\mathbb{R}^d)$ as we have already seen (cf (5.2.1)). We then recognize above two products of convolution between a function of $L^1(\mathbb{R}^d)$ and another of $L^2(\mathbb{R}^d)$, that belong to $L^2(\mathbb{R}^d)$. This implies that $\langle\xi\rangle^s\widehat{uv} \in L^2(\mathbb{R}^d)$, hence the desired result $uv \in H^s(\mathbb{R}^d)$. □

We have seen that, any real number $s > d/2$, the product of two elements of $H^s(\mathbb{R}^d)$ is still in $H^s(\mathbb{R}^d)$. The following proposition shows that we can also define the product φu for everything $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and any $u \in H^s(\mathbb{R}^d)$ with $s \in [0, +\infty[$.

Proposition 5.2.5. *For any $s \in \mathbb{R}$, if $u \in H^s(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ then $\varphi u \in H^s(\mathbb{R}^d)$.*

Proof. The proof uses an inequality, called Peetre's inequality that states that for every ξ, η in $\mathbb{R}^d$, we have

$$\forall s \in \mathbb{R}, \qquad (1+|\xi|^2)^s \leqslant 2^{|s|}(1+|\eta|^2)^s(1+|\xi-\eta|^2)^{|s|}.$$

Let us assume that $s \geq 0$. To obtain this inequality, just use the triangular inequality

$$1+|\xi|^2 \leqslant 1+(|\eta|+|\xi-\eta|)^2 \leqslant 1+2\,|\eta|^2+2\,|\xi-\eta|^2 \leqslant 2(1+|\eta|^2)(1+|\xi-\eta|^2),$$

then raise both sides to the power $s \geq 0$. If $s < 0$, then $-s > 0$ and the previous inequality leads to

$$(1+|\eta|^2)^{-s} \leqslant 2^{-s}(1+|\xi|^2)^{-s}(1+|\xi-\eta|^2)^{-s}.$$

The desired result is obtained by dividing by $(1 + |\eta|^2)^{-s}(1 + |\xi|^2)^{-s}$.

We then proceed as in the proof of the theorem 5.2.4. Indeed, one can still write for $u \in H^s(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\widehat{\varphi u}(\xi)$ as a convolution product. As $\widehat{\varphi}(\zeta)$ is in Schwartz's class, the previous inequality allows the product of convolution of a function to appear. of L^1 and $\langle\eta\rangle^s|\widehat{u}(\eta)|$ which is in L^2. □

5.3 Sobolev embeddings

We will now study the injection of Sobolev spaces $H^s(\mathbb{R}^d)$ into Lebesgue spaces $L^p(\mathbb{R}^d)$.

Theorem 5.3.1. *Let $d \geq 1$ and s be a real such that $0 \leqslant s < d/2$. Then the Sobolev space $H^s(\mathbb{R}^d)$ is continuously embedded into $L^p(\mathbb{R}^d)$ for any p such that*

$$2 \leqslant p \leqslant \frac{2d}{d-2s}.$$

Remark 5.3.2. *The previous theorem states that for any real number s in $[0, d/2[$, we have*

$$\|f\|_{L^{\frac{2d}{d-2s}}} \leqslant C_s \|f\|_{H^s}.$$

In fact, we will show a stronger result (see (5.3.1)) :

$$\|f\|_{L^{\frac{2d}{d-2s}}} \leqslant C \|f\|_{\dot{H}^s} := \left(\int |\xi|^{2s} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}.$$

In particular, for s =1, this gives another proof of the fact that

$$q = \frac{2d}{d-2} \quad \Rightarrow \quad \|f\|_{L^q} \leqslant C \|\nabla f\|_{L^2}.$$

Proof. We will show that there is a constant C such as, for any $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$p = \frac{2d}{d-2s} \quad \Rightarrow \quad \|f\|_{L^p} \leqslant C \|f\|_{\dot{H}^s} := \left(\int |\xi|^{2s} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}. \tag{5.3.1}$$

This is a stronger result than the one stated. Indeed, if $p < 2d/(d-2s)$ then there is $s' \in [0, s)$ such that $p = 2d/(d-2s')$ and hence

$$\|f\|_{L^p} \leqslant C \|f\|_{\dot{H}^{s'}} \leqslant C \|f\|_{H^s}.$$

(A word of caution: one cannot bound $\|f\|_{\dot{H}^{s'}}$ by $\|f\|_{\dot{H}^s}$ because we do not have $|\xi|^{2s'} \leqslant |\xi|^{2s}$ for $|\xi| \leqslant 1$).

We use the proof of Chemin and Xu which is based on the estimate of level sets. We will denote by $\{|f| > \lambda\}$ the set $\left\{x \in \mathbb{R}^d \, : \, |f(x)| > \lambda\right\}$ and $|\{|f| > \lambda\}|$ the Lebesgue measure of this set.

Let us consider a function $f \in \mathcal{S}(\mathbb{R}^d)$. We can assume without loss of generality that $\|f\|_{\dot{H}^s} = 1$. We start from the classical identity

$$\|f\|_{L^p}^p = p \int_0^{+\infty} \lambda^{p-1} \, |\{|f| > \lambda\}| \, \mathrm{d}\lambda.$$

To estimate $|\{|f| > \lambda\}|$, we will use a decomposition in terms of low and high frequencies. For any $\lambda > 0$, we will decompose f into the form

$$f = g_\lambda + h_\lambda$$

where, for a certain constant A_λ to be determined,

$$\begin{aligned} &\widehat{g_\lambda}(\xi) = \widehat{f}(\xi) \quad \text{if} \quad |\xi| \leqslant A_\lambda, && \widehat{g_\lambda}(\xi) = 0 \quad \text{if} \quad |\xi| > A_\lambda \\ &\widehat{h_\lambda}(\xi) = 0 \quad \text{if} \quad |\xi| \leqslant A_\lambda, && \widehat{h_\lambda}(\xi) = \widehat{f}(\xi) \quad \text{if} \quad |\xi| > A_\lambda. \end{aligned}$$

So, according to the triangular inequality,

$$\{|f| > \lambda\} \subset \{|g_\lambda| > \lambda/2\} \cup \{|h_\lambda| > \lambda/2\}\,.$$

We will choose the constant A_λ so that $\{|g_\lambda| > \lambda/2\} = \emptyset$. Then we will have

$$|\{|f| > \lambda\}| \leqslant |\{|h_\lambda| > \lambda/2\}| \leqslant \frac{4}{\lambda^2} \, \|h_\lambda\|_{L^2}^2\,,$$

because

$$\|h_\lambda\|_{L^2}^2 \geq \int_{\{|h_\lambda| > \lambda/2\}} |h_\lambda|^2 \, \mathrm{d}x \geq \frac{\lambda^2}{4} \, |\{|h_\lambda| > \lambda/2\}|\,.$$

Combining the above observations, we conclude

$$\|f\|_{L^p}^p \leqslant 4p \int_0^{+\infty} \lambda^{p-3} \, \|h_\lambda\|_{L^2}^2 \, \mathrm{d}\lambda. \tag{5.3.2}$$

Choice of A_λ. According to the Fourier inversion theorem, we have

$$|g_\lambda(x)| = \left| \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \widehat{g_\lambda}(\xi) \, \mathrm{d}\xi \right| = \left| \frac{1}{(2\pi)^n} \int_{|\xi| \leqslant A_\lambda} e^{ix\cdot\xi} \widehat{f}(\xi) \, \mathrm{d}\xi \right|.$$

As $2s < d$, we can use the Cauchy-Schwarz inequality and write that

$$|g_\lambda(x)| \leqslant \frac{1}{(2\pi)^n}\left(\int_{|\xi|\leqslant A_\lambda} |\xi|^{-2s}\,\mathrm{d}\xi\right)^{\frac{1}{2}}\left(\int |\xi|^{2s}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

If we switch to polar coordinates, we obtain

$$\int_{|\xi|\leqslant A_\lambda} |\xi|^{-2s}\,\mathrm{d}\xi = \int_0^{A_\lambda}\int_{\mathbb{S}^{d-1}} r^{d-1-2s}\,\mathrm{d}\theta\,\mathrm{d}r = \frac{|\mathbb{S}^{d-1}|A_\lambda^{d-2s}}{d-2s}.$$

As $\|f\|_{\dot{H}^s} = 1$ by assumption, we finally get

$$\|g_\lambda\|_{L^\infty} \leqslant C_1(s,d)A_\lambda^{\frac{d}{2}-s}.$$

We then define A_λ by

$$C_1(s,d)A_\lambda^{\frac{d}{2}-s} = \frac{\lambda}{2}.$$

So $\|g_\lambda\|_{L^\infty} \leqslant \lambda/2$. Since g_λ is a continuous function (it is the Fourier transform of an integrable function), we deduce that $\{|g_\lambda| > \lambda/2\} = \emptyset$, which is the desired result.

End of the proof. By definition of h_λ, using the identity (5.3.2) and Plancherel's formula, we find

$$\|f\|_{L^p}^p \leqslant 4p(2\pi)^d\int_0^{+\infty}\int_{|\xi|\geq A_\lambda} \lambda^{p-3}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi\,\mathrm{d}\lambda.$$

By definition of A_λ, if $|\xi| \geq A_\lambda$ then

$$\lambda \leqslant \Lambda(\xi) := 2C_1(s,d)\,|\xi|^{\frac{d}{2}-s},$$

so, using Fubini's theorem, it comes

$$\|f\|_{L^p}^p \leqslant 4p(2\pi)^d\int_{\mathbb{R}^d}\left(\int_0^{\Lambda(\xi)} \lambda^{p-3}d\lambda\right)\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi,$$

from where

$$\|f\|_{L^p}^p \leqslant C_2(s,d)\int_{\mathbb{R}^d} \Lambda(\xi)^{p-2}\left|\widehat{f}(\xi)\right|^2\mathrm{d}\xi,$$

As $\frac{d}{2} - s = \frac{d}{p}$, we have

$$\Lambda(\xi) \leqslant C_1(s,d)\,|\xi|^{\frac{d}{p}}.$$

We finally get

$$\|f\|_{L^p}^p \leqslant C_3(s,d) \int_{\mathbb{R}^d} |\xi|^{\frac{d(p-2)}{p}} \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi,$$

which is the desired result. □

Corollary 5.3.3 (Sobolev embeddings)**.** *Let $d \geq 1$ and $p \in (1,d)$. Define p^* by*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}\cdot$$

Then there exists a constant C such that, for any function $f \in C_0^\infty(\mathbb{R}^d)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \leqslant C \|\nabla f\|_{L^p(\mathbb{R}^d)}\,.$$

Proof. Here we use the following identity

$$f(x) = -\frac{1}{\left|S^{d-1}\right|} \int_{\mathbb{R}^d} \frac{(x-y)\cdot\nabla f(y)}{|x-y|^d}\,\mathrm{d}y.$$

It follows that

$$|f| \leqslant \frac{1}{\left|S^{d-1}\right|} I_1(|\nabla f|),$$

and hence the wanted inequality follows directly from the Hardy-Littlewood-Sobolev inequality. □

Theorem 5.3.4 (Sobolev embedding)**.** *Consider an integer $d \geq 1$ and two real numbers $s \in (0,1)$ and $p \in (0, d/s)$, then set*

$$p^* = \frac{dp}{d-sp}\cdot$$

There exists a constant C such that for all f in $C_b^1(\mathbb{R}^d)$ and all x in $\mathbb{R}^d$ we have

$$|f(x)|^{p^*} \leq C \|f\|_{L^{p^*}}^{p^*-p} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}\,\mathrm{d}y.$$

It follows that

$$\|f\|_{L^{p^*}} \leq C^{\frac{1}{p}} \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}\,\mathrm{d}y\,\mathrm{d}x \right)^{\frac{1}{p}}.$$

Proof. The following nice proof is taken from the book by Ponce [363] where it is credited to Brézis. A similar proof is given by Brué and Nguyen in [98] (see also [99]).

We denote by C several constants that do not have to be depend on d, p or s and whose values can change from one line to another.

Step 1. We first check that the integral

$$\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+ps}}\,\mathrm{d}y$$

is well defined for all f in $\mathcal{S}(\mathbb{R}^d)$ and all x in $\mathbb{R}^d$. To do so, we cut the integral on $\mathbb{R}^d$ in two parts: the integral on $B(x,1)$ and that on $\mathbb{R}^d \setminus B(x,1)$. On $B(x,1)$, we use the estimate

$$|f(x)-f(y)| \le K\,|x-y| \quad \text{with} \quad K = \sup_{m\in\mathbb{R}^d} |\nabla f(m)|\,,$$

while on $\mathbb{R}^d \setminus B(x,1)$ one writes $|f(x)-f(y)| \le 2\sup|f|$.

Step 2. Let us fix $x \in \mathbb{R}^d$ and a real $t > 0$. We denote by C_t the annulus

$$C_t = B(0,2t) - B(0,t) = \{y \in \mathbb{R}^d\,;\, t \le |y| < 2t\},$$

and we denote by $|C_t|$ its Lebesgue measure.

Then

$$|f(x)|^p = \frac{1}{|C_t|}\int_{C_t} |f(x)|^p\,\mathrm{d}h \le \frac{1}{|C_t|}\int_{C_t} \big(\,|f(x+h)-f(h)| + |f(x+h)|\,\big)^p\,\mathrm{d}h.$$

Since

$$|a+b|^p \le (2\max\{|a|,|b|\})^p \le 2^p\big(|a|^p+|b|^p\big),$$

we deduce that

$$|f(x)|^p \le \frac{2^p}{|C_t|}\int_{C_t} |f(x+h)-f(x)|^p\,\mathrm{d}h + \frac{2^p}{|C_t|}\int_{C_t} |f(x+h)|^p\,\mathrm{d}h.$$

Step 3. Hölder's inequality implies that

$$\frac{1}{|C_t|}\int_{C_t} |f(x+h)|^p\,\mathrm{d}h \le C\,\|f\|_{L^{p^*}}^p\,\frac{1}{t^{d-sp}}.$$

Consequently, we see that there exists C such that

$$|f(x)|^p \frac{1}{t^{sp}} \leq \frac{C}{t^d} \|f\|_{L^{p^*}}^p + \frac{C}{t^{sp}|C_t|} \int_{C_t} |f(x+h) - f(x)|^p \, \mathrm{d}h.$$

Step 4. Note that $|C_t| \sim t^d$. Moreover, on C_t we have $t \sim |h|$. It follows that

$$|f(x)|^p \frac{1}{t^{sp}} \leq C \|f\|_{L^{p^*}}^p \frac{1}{t^d} + C \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^p}{|x-y|^{d+ps}} \, \mathrm{d}h.$$

Multiplying the two members of the inequality in the previous question by t^{sp}, we find

$$|f(x)|^p \leq C \|f\|_{L^{p^*}}^p t^{sp-d} + C t^{sp} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^p}{|x-y|^{d+ps}} \, \mathrm{d}h.$$

We then choose t such that

$$\|f\|_{L^{p^*}}^p t^{sp-d} = t^{sp} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^p}{|x-y|^{d+ps}} \, \mathrm{d}h$$

and the desired inequality is inferred. □

5.4 Fractional Sobolev spaces defined on the half-line

Let $I = (0, +\infty)$ and consider a real number $0 < s < 1$. Recall from Lions-Magenes [315, Chapter 1, §10.1] that $H^s(I)$ is defined by interpolation:

$$H^s(I) = [H^1(I), L^2(I)]_\theta \quad , \quad 1 - \theta = s.$$

It is proved in the previous reference that $u \in H^s(I)$ is equivalent to

$$|u|_{H^s(I)} := \left(\|u\|_{L^2(I)}^2 + \iint_{I \times I} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} < +\infty.$$

Moreover, the norms $\|\cdot\|_{H^s(I)}$ and $|\cdot|_{H^s(I)}$ are equivalent. Then we define $H^s(I)$ for $1 < s < 2$ as the space of functions $u \in H^1(I)$ such that $\partial_x u \in H^{s-1}(I)$. And by induction one defines $H^s(I)$ for any $s \geq 0$. We also recall that $C_0^\infty(\overline{I})$ is dense in $H^s(I)$ for all $s \geq 0$.

Proposition 5.4.1. *For a function v defined on $(0,+\infty)$, define v^{ev} and v^{od} to be the even and odd extensions of v to $\mathbb{R}$ defined by*

$$(5.4.1) \quad v^{ev}(y) = \begin{cases} v(-y) & \text{if } y < 0 \\ v(y) & \text{if } y \geq 0 \end{cases}, \quad v^{od}(y) = \begin{cases} -v(-y) & \text{if } y < 0, \\ v(y) & \text{if } y \geq 0. \end{cases}$$

The following properties hold

1. *Assume that $0 \leq s < \frac{3}{2}$. Then the map $v \mapsto v^{ev}$ is continuous from $H^s(0,+\infty)$ to $H^s(\mathbf{R})$.*
2. *Assume that $\frac{3}{2} < s < \frac{7}{2}$. Then the map $v \mapsto v^{ev}$ is continuous from the space $\{v \in H^s(0,+\infty) : v'(0) = 0\}$ to $H^s(\mathbb{R})$.*
3. *Assume that $0 \leq s < \frac{1}{2}$. Then the map $v \mapsto v^{od}$ is continuous from $H^s(0,+\infty)$ to $H^s(\mathbf{R})$.*
4. *Assume that $\frac{1}{2} < s < \frac{5}{2}$. Then the map $v \mapsto v^{od}$ is continuous from the space $\{v \in H^s(0,+\infty) : v(0) = 0\}$ to $H^s(\mathbb{R})$.*
5. *Assume that $\frac{5}{2} < s \leq 4$. Then the map $v \mapsto v^{od}$ is continuous from the space $\{v \in H^s(0,+\infty) : v(0) = v''(0) = 0\}$ to $H^s(\mathbb{R})$.*

Proof. Let $I = (0,+\infty)$.

(1) The case $s = 0$ is trivial since $\|v^{\mathrm{ev}}\|^2_{L^2(\mathbb{R})} = 2\|v\|^2_{L^2(I)}$. Consider now the case $0 < s < 1$. Then the square of the $H^s(\mathbb{R})$-norm of v^{ev} is equivalent to

$$\|v^{\mathrm{ev}}\|^2_{L^2(\mathbb{R})} + A, \quad A =: \iint_{\mathbb{R}\times\mathbb{R}} \frac{|v^{\mathrm{ev}}(x) - v^{\mathrm{ev}}(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Then, by symmetry, we can write

$$A = 2\iint_{I\times I} \frac{|v(x) - v(y)|^2}{|x-y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\iint_{I\times I} \frac{|v(x) - v(y)|^2}{|x+y|^{1+2s}}\,\mathrm{d}x\,\mathrm{d}y := A_1 + A_2.$$

since $(x+y)^{1+2s} \geq |x-y|^{1+2s}$, we have $A_2 \leq A_1$ and hence $A \leq 2A_1$. Now, directly from the equivalence between the norms $\|\cdot\|_{H^s(I)}$ and $|\cdot|_{H^s(I)}$, we have

$$A_1 \leq 2\|v\|^2_{H^s(I)}.$$

This concludes the analysis of the case $0 < s < 1$.

The case $s = 1$ being straightforward, we now move to the case where $1 < s < 3/2$. Set $\sigma = s - 1 \in (0, 1/2)$. Then

$$\|v^{\mathrm{ev}}\|^2_{H^s(\mathbb{R})} = \|v^{\mathrm{ev}}\|^2_{L^2(\mathbb{R})} + \|\partial_x v^{\mathrm{ev}}\|^2_{H^\sigma(\mathbb{R})}.$$

Since $0 < \sigma < 1$ we have

$$\begin{aligned}
&\|\partial_x v^{\mathrm{ev}}\|^2_{H^\sigma(\mathbb{R})} \leq C(A_0 + A_1 + A_2),\\
&A_0 = \|\partial_x v^{\mathrm{ev}}\|^2_{L^2(\mathbb{R})} \leq C_1 \|v'\|^2_{L^2(I)} \leq C_1 \|v\|^2_{H^s(I)}\\
&A_1 = \iint_{I\times I} \frac{|v'(x) - v'(y)|^2}{|x - y|^{1+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y \leq C_2 \|v'\|^2_{H^\sigma(I)} \leq C_2 \|v\|^2_{H^s(I)},\\
&A_2 = \iint_{I\times I} \frac{|v'(x) + v'(y)|^2}{|x + y|^{1+2\sigma}} \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Eventually we have

$$A_2 \leq C_3 \int_I \frac{|v'(x)|^2}{|x|^{2\sigma}}\, dx \leq C_4 \|v'\|_{H^\sigma(I)} \leq C_4 \|v\|_{H^s(I)}$$

by Theorem 11.2 in [315], since $0 < \sigma < \frac{1}{2}$. This completes the proof of statement (1).

(2) Assume that $\frac{3}{2} < s < 2$ and set $\sigma = s - 1 \in (\frac{1}{2}, 1)$. By arguing as above, we see that

$$\|v^{\mathrm{ev}}\|^2_{H^s(\mathbb{R})} \leq C\Big(\|v\|^2_{H^s(I)} + \int_I \frac{|v'(x)|^2}{|x|^{2\sigma}}\,\mathrm{d}x\Big).$$

Now, since $v' \in H^\sigma(I)$ and since $v'(0) = 0$, we are in position to apply the Hardy's inequality given by Theorem 11.3 in [315, Chapter] which ensures that the integral in the right hand side can be estimated by $C\|v\|^2_{H^s(I)}$.

It remains to consider the case where $2 \leq s < 7/2$. Now, since $v'(0) = 0$, we have that $\partial_x^2 v^{\mathrm{ev}} = (\partial_x^2 v)^{\mathrm{ev}}$. So the desired result follows from the first point. We deduce that $\|\partial_x^2 v^{\mathrm{ev}}\|^2_{H^{s-2}(\mathbb{R})} \leq C\|v\|^2_{H^s(I)}$.

The cases (3) to (5) are proved by exactly the same arguments. $\square$

Part III

Study of the Cauchy problem

Chapter 6

Paralinearization of the Dirichlet-to-Neumann operator

6.1 Classical results

If $\eta \in H^\infty(\mathbb{R}^d)$, then it follows from classical elliptic results that, for any $s \geq \frac{1}{2}$, $G(\eta)$ is bounded from $H^s(\mathbb{R}^d)$ into $H^{s-1}(\mathbb{R}^d)$. It is known that this property still holds in the case where η has limited regularity. We have already discussed this point earlier, see Proposition 3.1.3. Namely, for $s \geq 0$ large enough, we have

$$\|G(\eta)\psi\|_{H^s} \leq C\big(\|\eta\|_{H^{s+1}} \big) \|\psi\|_{H^{s+1}} .$$

Various variants of this estimate were obtained by Craig and Nicholls ([169]), Wu ([445, 446]), Günther and Prokert [234], Lannes [300] who proved tame estimates, see also [22] where it is proved that this estimate holds for any $s > d/2$.

On the other hand, it is known since Calderón that, if $\eta \in C^\infty$ is bounded together with all its derivatives, then $G(\eta)$ is a pseudo-differential operator. Recall that given a symbol $a = a(x,\xi)$ we define the pseudo-differential operator $\mathrm{Op}(a)$ by

$$\mathrm{Op}(a)u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi,$$

and then

$$G(\eta)\psi = \mathrm{Op}(\lambda^{(1)})\psi + R_0(\eta)\psi,$$

where

$$\lambda^{(1)} = \sqrt{(1+|\nabla\eta|^2)\,|\xi|^2 - (\nabla\eta\cdot\xi)^2}, \tag{6.1.1}$$

and where the remainder satisfies the following property : there exists K such that for all $s \geq 0$ we have

$$\|R_0(\eta)\psi\|_{H^s} \leq C\left(\|\eta\|_{H^{s+K}}\right)\|\psi\|_{H^s}.$$

This allows to approximate $G(\eta)$ by $\mathrm{Op}(\lambda^{(1)})$ which is an operator of order 1, modulo the remainder $R_0(\eta)$ which is of order 0. Actually, we have an approximation at any order (see [52, 408]).

Theorem 6.1.1. *Assume that $\eta \in C^\infty$ is bounded together with all its derivatives. There exists a sequence of symbols $(\lambda^{(-k)})_{k\in\mathbb{N}}$, where $\lambda^{(-k)}$ is homogeneous of order $-k$ in ξ, such that for all $\ell \in \mathbb{N}$, we have*

$$G(\eta)\psi = \mathrm{Op}\big(\lambda^{(1)} + \lambda^{(0)} + \cdots + \lambda^{(-\ell)}\big)\psi + R_{-\ell}(\eta)\psi,$$

where $R_{-\ell}(\eta)$ satisfies the following property: there exists K_ℓ such that, for all $s \geq \frac{1}{2}$,

$$\|R_{-\ell}(\eta)\psi\|_{H^{s+\ell}} \leq C\left(\|\eta\|_{H^{s+K_\ell}}\right)\|\psi\|_{H^s}. \tag{6.1.2}$$

Remark 6.1.2. *Two observations are in order concerning these symbols.*

(i) If $d = 1$ then $\lambda^{(1)} = |\xi|$ and $\lambda^{(-k)} = 0$ for any $k \in \mathbb{N}$, so

$$\mathrm{Op}\big(\lambda^{(1)} + \lambda^{(0)} + \cdots + \lambda^{(-\ell)}\big) = |D_x|.$$

In general, for $d \geq 2$, we have $\lambda^{(0)} \neq 0$ whenever $\eta \neq 0$. Indeed,

$$\lambda^{(0)} = \frac{1+|\nabla\eta|^2}{2\lambda^{(1)}}\left\{\mathrm{div}\left(\alpha^{(1)}\nabla\eta\right) + i\partial_\xi\lambda^{(1)}\cdot\nabla\alpha^{(1)}\right\}, \tag{6.1.3}$$

with

$$\alpha^{(1)} = \frac{1}{1+|\nabla\eta|^2}\left(\lambda^{(1)} + i\nabla\eta\cdot\xi\right).$$

(ii) Our second observation is that the symbols $\lambda^{(0)}, \lambda^{(-1)}, \ldots$ in Theorem 9.4.3 are defined by induction so that one can check that $\lambda^{(-k)}$ involves only derivatives of η of order $\leq k+2$ (this will be proved below). Then the definition of the full symbol λ of $G(\eta)$ can be extended for $\eta \notin C^\infty$ in the following obvious manner: we consider in the asymptotic expansion

$$\lambda \sim \lambda^{(1)} + \lambda^{(0)} + \lambda^{(-1)} + \cdots$$

only the terms which are meaningful. This means that, for $\eta \in C^{k+2} \setminus C^{k+3}$ with $k \in \mathbb{N}$, we set

$$\lambda = \lambda^{(1)} + \lambda^{(0)} + \cdots + \lambda^{(-k)}. \tag{6.1.4}$$

6.2 Main statement

Notation 6.2.1. *For $\rho \in \mathbb{N}$, we denote by $W^{\rho,\infty}(\mathbb{R}^d)$ the Sobolev spaces of L^∞ functions whose derivatives of order ρ are in L^∞. For $\rho \in]0,+\infty[\setminus\mathbb{N}$, we denote by $W^{\rho,\infty}(\mathbb{R}^d)$ the space functions in $W^{[\rho],\infty}(\mathbb{R}^d)$ whose derivatives of order $[\rho]$ are uniformly Hölder continuous with exponent $\rho - [\rho]$.*

Definition 6.2.2 (homogeneous symbol)**.** *Given real numbers $\rho \geq 0$ and $m \in \mathbb{R}$, we say that $a\colon \mathbb{R}^d \times (\mathbb{R}^d \setminus 0) \to \mathbb{C}$ is an homogeneous symbol of order m and with regularity $W^{\rho,\infty}$ in x provided that: a is homogeneous of degree m and C^∞ with respect to $\xi \neq 0$, and such that, for all $\alpha \in \mathbb{N}^d$ and all $\xi \neq 0$, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbb{R}^d)$ and*

$$\sup_{|\xi|=1} \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}} < +\infty \tag{6.2.1}$$

We denote by $\dot{\Gamma}^m_\rho$ the space of pluri-homogeneous symbols $a(x,\xi)$, they are symbols of the form

$$a = \sum_{0\leq j<\rho} a^{(m-j)} \qquad (j \in \mathbb{N}),$$

where $a^{(m-j)}$ is homogeneous of order $m-j$ in ξ and with regularity $W^{\rho-j,\infty}$ in x (we say that $a^{(m)}$ is the principal symbol of a).

Remark 6.2.3. *Notice that, if a is an homogeneous symbol of order m and with regularity $W^{\rho,\infty}$ in x, then*

$$\exists C_\alpha, \quad \forall\, |\xi| \geq 1/2, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}$$

so that the symbol a belongs to $\Gamma^m_\rho(\mathbb{R}^d)$.

Now fix a C^∞ function ζ such that $\zeta = 0$ on a neighborhood of the origin and $\zeta = 1$ for $|\xi| \geq 1$. We also introduce a C^∞ function χ homogeneous of degree 0 and satisfying, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta,\eta) = 1 \quad \text{if} \quad |\theta| \leq \varepsilon_1 |\eta|, \qquad \chi(\theta,\eta) = 0 \quad \text{if} \quad |\theta| \geq \varepsilon_2 |\eta|.$$

Given a symbol a, we define the paradifferential operator T_a by

$$T_a = \mathrm{Op}(\sigma)$$

where

$$\hat{\sigma}(\eta,\xi) = \chi(\eta,\xi)\hat{a}(\eta,\xi)\zeta(\xi). \tag{6.2.2}$$

where $\widehat{a}(\eta,\xi) = \int e^{-ix\cdot\eta}a(x,\xi)\,\mathrm{d}x$ is the Fourier transform of a with respect to x.

Remark 6.2.4. *We call attention to the fact that, if $Q(D_x)$ is a Fourier multiplier with symbol $q(\xi)$, then we do not have $Q(D_x) = T_q$, because of the cut-off function ζ. However, this is obviously almost true since we have $Q(D_x) = T_q + R$ where $R = Q(D_x)(I - \zeta(D_x))$ is a smoothing operator (it maps H^t to H^∞ for all $t \in \mathbb{R}$).*

Our goal in this chapter is to prove a sharp version of Proposition 3.1.4.

Theorem 6.2.5. *Let $d \geq 1$ and consider a real number $s > 3 + d/2$. There exists a nondecreasing function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the following property holds: For any $\eta \in H^s(\mathbb{R}^d)$ and $\psi \in H^s(\mathbb{R}^d)$,*

$$G(\eta)\psi = T_{\lambda_{(s)}}\big(\psi - T_B\eta\big) - T_V\cdot\nabla\eta - T_{\operatorname{div}V}\eta + F(\eta)\psi,$$

where

$$\lambda_{(s)} = \lambda^{(1)} + \lambda^{(0)} + \cdots + \lambda^{(-k)} \quad \textit{with} \quad k := \left[s - 2 - \frac{d}{2}\right],$$

the coefficients B and V are as in (1.2.5), and $F(\eta)\psi$ satisfies

$$\|F(\eta)\psi\|_{H^{2s-K_0}} \leq C\left(\|\eta\|_{H^s}\right)\|\psi\|_{H^s},$$

where $K_0 = 5/2 + d/2$.

We will also establish a variant of the previous result which allows to identify quadratic and cubic terms in the remainder term $F(\eta)\psi$.

Theorem 6.2.6. *Let $d \geq 1$ and fix $s_0 > 3 + d/2$. For any $s > s_0$ there exists a nondecreasing function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the following property holds: For any $\eta \in H^s(\mathbb{R}^d)$ and $\psi \in H^s(\mathbb{R}^d)$,*

$$\begin{aligned} G(\eta)\psi = {} & \big(|D_x| + T_{\lambda_{s_0} - |\xi|}\big)\big(\psi - T_B\eta\big) - T_V\cdot\nabla\eta - T_{\operatorname{div}V}\eta \\ & + F_{\leq 2}(\eta)\psi + F_{\geq 3}(\eta)\psi, \end{aligned}$$

where

$$\lambda_{(s_0)} = \lambda^{(1)} + \lambda^{(0)} + \cdots + \lambda^{(-k)} \quad \textit{with} \quad k := \left[s_0 - 2 - \frac{d}{2}\right],$$

the coefficients B and V are as in (1.2.5)*,*

$$F_{\le 2}(\eta)\psi = -|D_x|R_{\mathcal{B}}\big(|D_x|\psi,\eta\big) - \operatorname{div} R(\nabla\psi,\eta)$$

where $R_{\mathcal{B}}(\cdot,\cdot)$ *is as defined by* (3.6.4) *and where* $F_{\ge 3}(\eta)\psi$ *satisfies*

$$\|F_{\ge 3}(\eta)\psi\|_{H^{s+s_0-K_0}} \le C\left(\|\eta\|_{H^{s_0}}\right)\left\{\|\eta\|^2_{H^{s_0}}\|\psi\|_{H^s} + \|\eta\|_{H^{s_0}}\|\psi\|_{H^{s_0-\frac{1}{2}}}\|\eta\|_{H^s}\right\},$$

where $K_0 = 5/2 + d/2$*.*

6.3 The good unknown of Alinhac

Let us discuss the role of the "good unknown" of Alinhac ω defined by

$$\omega = \psi - T_B\eta.$$

This tool was introduced in [29]. The idea of working with ω is rooted in a cancellation first observed by Lannes [300] for the water waves equations linearized around a non trivial solution. Here, we want to explain that ω appears naturally when one introduces the operator of paracomposition of Alinhac [36] associated to the change of variables that flattens the boundary $y = \eta(x)$ of the domain. Though we shall not use general results about paracomposition operators, we explain here the ideas that underly the computations that will be made later.

To study the elliptic equation $\Delta_{x,y}\phi = 0$ in $\Omega = \{(x,y) \in \mathbb{R}^{d+1}\,;\, y < \eta(x)\}$, we shall reduce the problem to the half-space $\mathbb{R}^d \times (-\infty,0)$ through the change of coordinates $\kappa\colon (x,z) \mapsto (x, z+\eta(x))$. Then $\phi(x,y)$ solves $\Delta_{x,y}\phi = 0$ if and only if $v = \phi\circ\kappa = \phi(x,z+\eta(x))$ is a solution of $P\varphi = 0$ in $z < 0$, where

$$P = (1+|\nabla\eta|^2)\partial_z^2 + \Delta^2 - 2\nabla\eta\cdot\nabla\partial_z - (\Delta\eta)\partial_z. \tag{6.3.1}$$

The boundary condition on $\phi|_{y=\eta(x)}$ becomes $v(x,0) = \psi(x)$ and $G(\eta)$ is given by

$$G(\eta)\psi = \big[(1+|\nabla\eta|^2)\partial_z v - \nabla\eta\cdot\nabla v\big]\big|_{z=0}.$$

We first explain the main difficulty to handle a diffeomorphism with limited regularity. Let us use the notation $D = -i\partial$ and introduce the symbol

$$p(x,\xi,\zeta) = (1+|\nabla\eta(x)|^2)\zeta^2 + |\xi|^2 - 2\nabla\eta(x)\cdot\xi\zeta + i(\Delta\eta(x))\zeta.$$

Notice that $P = -p(x, D_x, D_z)$ use write T_p as a short notation for $T_{1+|\nabla(x)|^2}D_z^2 + |D_x|^2 - 2T_{\nabla\eta} \cdot D_x D_z + T_{\Delta\eta}\partial_z$. Starting from $p(x, D_x, D_z)v = 0$, by using standard results for paralinearization of products, we find that $T_p v = f_1$ for some source term f_1 which is continuous in z with values in H^{s-2} if η is in H^s and the first and second order derivatives in x, z of v are bounded. The key point is that one can associate to κ a paracomposition operator, denoted by κ^*, such that $T_p(\kappa^*\phi) = f_2$ for some smoother remainder term f_2. That is for some function f_2 continuous in z with values in $H^{s+\gamma-K(d)}$, if η is in H^s and if the derivatives in x, z of order less than γ of v are bounded (the key difference between f_1 and f_2 is that one cannot improve the regularity of f_1 by assuming that v is smoother).

We shall not define κ^*, instead we recall the two main properties of paracomposition operators (we refer to the original article [36] for the general theory). First, modulo a smooth remainder, one has

$$\kappa^*\phi = \phi \circ \kappa - T_{\phi' \circ \kappa}\kappa$$

where ϕ' denotes the differential of ϕ. On the other hand, there is a symbolic calculus formula which allows to compute the commutator of κ^* to a paradifferential operator. This formula implies that

$$\kappa^*\Delta - T_p\kappa^*$$

is a smoothing operator (that is an operator bounded from H^μ to $H^{\mu+m}$ for any real number μ, where m is a positive number depending on the regularity of κ). Since $\Delta_{x,y}\phi = 0$, this implies that $T_p\left(\phi \circ \kappa - T_{\phi' \circ \kappa}\kappa\right)$ is a smooth remainder term as asserted above.

Now observe that

$$\omega = \left(\phi \circ \kappa - T_{\phi' \circ \kappa}\kappa\right)\big|_{z=0}.$$

This is the reason why the good unknown enters into the analysis.

6.4 Paralinearization

This section contains the core of the analysis. We start by flattening the boundary and recall some classical elliptic estimates. We then paralinearize the equations obtained after this change of variables.

6.4.1 Flattening of the free boundary

We begin by reducing the problem to a domain which does not depend on η. Set $I = (-\infty, 0]$ and

$$\rho(x,z) = z + \eta(x).$$

Now introduce the function $v\colon \mathbb{R}^d \times I \to \mathbb{R}$ defined by $v(x,z) = \phi(x, \rho(x,z))$. From $\Delta_{x,y}\phi = 0$ we deduce that v satisfies the elliptic equation

$$\left(\frac{1}{\partial_z\rho}\partial_z\right)^2 v + \left(\nabla - \frac{\nabla\rho}{\partial_z\rho}\partial_z\right)^2 v = 0. \tag{6.4.1}$$

This yields

$$\alpha\partial_z^2 v + \Delta v + \beta\cdot\nabla\partial_z v - \gamma\partial_z v = 0, \tag{6.4.2}$$

where

$$\alpha := 1 + |\nabla\eta|^2, \quad \beta := -2\nabla\eta, \quad \gamma := \Delta\eta.$$

Note also that $v|_{z=0} = \psi$ and

$$G(\eta)\psi = (1 + |\nabla\eta|^2)\partial_z v - \nabla\eta\cdot\nabla v\,\big|_{z=0}.$$

Our purpose is to give $\partial_z v|_{z=0}$ in terms of tangential derivatives.

6.4.2 Elliptic regularity

Lemma 6.4.1. *Set $I = (-\infty, 0]$. There holds*

$$\nabla_{x,z}v \in L^\infty_z(I; H_x^{s_0-\frac{3}{2}}) \cap L^2_z(I; H_x^{s_0-1}), \quad \partial_z^2 v \in L^2_z(H_x^{s_0-2}), \quad \partial_z^3 v \in L^2(I; H^{s_0-3}),$$

together with the estimates

$$\left\|\nabla_{x,z}v\right\|_{L^\infty_z(I;H_x^{s_0-\frac{3}{2}})\cap L^2_z(I;H_x^{s_0-1})} \le C\big(\|\eta\|_{H^{s_0}}\big)\,\|\psi_0\|_{H^{s_0-\frac{1}{2}}}, \tag{6.4.3}$$

$$\left\|\partial_z^2 v\right\|_{L^2_z(H_x^{s_0-2})} \le C\big(\|\eta\|_{H^{s_0}}\big)\,\|\psi_0\|_{H^{s_0-\frac{1}{2}}}, \tag{6.4.4}$$

$$\left\|\partial_z^3 v\right\|_{L^2(I;H^{s_0-3})} \le C\big(\|\eta\|_{H^{s_0}}\big)\,\|\psi_0\|_{H^{s_0-\frac{1}{2}}}, \tag{6.4.5}$$

$$\|\partial_z v - |D_x|v\|_{L^2_z(I;H_x^{s_0-1})} \le C\big(\|\eta\|_{H^{s_0}}\big)\,\|\eta\|_{H^{s_0}}\,\|\psi_0\|_{H^{s_0-\frac{1}{2}}}, \tag{6.4.6}$$

for some universal non-decreasing function $C\colon \mathbb{R}_+ \to \mathbb{R}_+$.

Remark 6.4.2. *Notice that the right-hand side is quadratic in* (η, ψ) *while the one of* (6.4.3) *is sub-linear.*

Proof. These results follow from quite classical results in the analysis of elliptic problems. In [22], it is proved that, for $s_0 > 1 + d/2$, there holds

$$\nabla_{x,z} v \in L^\infty_z([-1,0]; H^{s_0-\frac{3}{2}}_x) \cap L^2_z([-1,0]; H^{s_0-1}_x),$$

together with the estimate

$$\left\| \nabla_{x,z} v \right\|_{L^\infty_z([-1,0];H^{s_0-\frac{3}{2}}_x) \cap L^2_z([-1,0];H^{s_0-1}_x)} \leq C\big(\| \eta \|_{H^{s_0}} \big) \left\| \psi_0 \right\|_{H^{s_0-\frac{1}{2}}} . \tag{6.4.7}$$

Notice that the latter estimate is different from (6.4.3). The fact that the estimate (6.4.7) holds when $I = (-\infty, 0]$ is replaced by $[-1, 0]$ can be deduced from the analysis in the present paper, using in addition an induction argument already used in [22] or [24]. We will not reproduce this induction argument here for the sake of conciseness. Similarly, the estimate (6.4.6) follows from the results alluded to above and the fact that the domain has infinite depth.

Let us recall a product estimate in Sobolev spaces (see Chapter 8 in [244]):

- If $u_j \in H^{s_j}(\mathbb{R}^d)$, $j = 1, 2$, and $s_1 + s_2 > 0$ then $u_1 u_2 \in H^{s_0}(\mathbb{R}^d)$ and

  $$\| u_1 u_2 \|_{H^{s_0}} \leq K \| u_1 \|_{H^{s_1}} \| u_2 \|_{H^{s_2}} , \tag{6.4.8}$$

  if

  $$s_0 \leq s_j, \quad j = 1, 2, \quad \text{and} \quad s_0 \leq s_1 + s_2 - d/2,$$

  where the last inequality is strict if s_1 or s_2 or $-s_0$ is equal to $d/2$.

Now, writing

$$\partial_z^2 v = -\frac{1}{1 + |\nabla \eta|^2} \big(\Delta v - 2 \nabla \eta \cdot \nabla \partial_z v - \Delta \eta \partial_z v \big), \tag{6.4.9}$$

and using the product rule in Sobolev spaces (6.4.8), since $s_0 > 1 + d/2$ we easily deduce that $\partial_z^2 v$ belongs to $L^2_z(H^{s_0-2}_x)$, and the estimate (6.4.4) follows from (6.4.3).

Then, by differentiating (6.4.9) in z and using again the product rule in Sobolev spaces, since $s_0 > 2 + d/2$, one deduces the estimate (6.4.5) from (6.4.3) and (6.4.4). □

6.4.3 Paralinearization of the interior equation

Following the approach in [29], the key technical point is to obtain an equation for

$$W = v - T_{\partial_z v}\eta.$$

Hereafter we consider symbols $a(z, x, \xi)$ depending on the phase space variables $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ and on the parameter $z \in I$. Then $(T_a u)(z) = T_{a(z)} u(z)$.

Lemma 6.4.3. *The unknown W satisfies the paradifferential equation*

$$(I + T_{|\nabla\eta|^2})\partial_z^2 W + \Delta W - 2T_{\nabla\eta} \cdot \nabla\partial_z W - T_{\Delta\eta}\partial_z W = Q_1 + C_1, \tag{6.4.10}$$

where Q_1 is a smooth quadratic term given by

$$Q_1 := R_{\mathcal{B}}(2\nabla\eta, \nabla\partial_z v) + R_{\mathcal{B}}(\Delta\eta, \partial_z v)$$

and C_1 is a smooth cubic term, such that

$$\|C_1\|_{L^2(I;H^{s+s_0-K})} \le C\big(\|\eta\|_{H^{s_0}}\big)\, \|\eta\|_{H^{s_0}}\, \|\psi\|_{H^{s_0-\frac{1}{2}}}\, \|\eta\|_{H^s} \tag{6.4.11}$$

with $K = 3 + \frac{d}{2}$.

Proof. For this proof, we use the notation $f_1 \sim f_2$ to say that $\|f_1 - f_2\|_{L^2(I;H^{s+s_0-K})}$ is bounded by the right-hand side of (6.4.11).

Introduce the operators

$$E := (1 + |\nabla\eta|^2)\partial_z^2 + \Delta - 2\nabla\eta \cdot \nabla\partial_z - (\Delta\eta)\partial_z,$$

and

$$P := (I + T_{|\nabla\eta|^2})\partial_z^2 + \Delta - 2T_{\nabla\eta} \cdot \nabla\partial_z - T_{\Delta\eta}\partial_z.$$

We shall prove that $PW \sim Q_1$. To do so, we begin with the paralinearization formula for products. Write

$$Ev = Pv + T_{\partial_z^2 v}|\nabla\eta|^2 - 2T_{\nabla\partial_z v} \cdot \nabla\eta - T_{\partial_z v}\Delta\eta + g_1$$

with

$$g_1 = R_{\mathcal{B}}(|\nabla\eta|^2, \partial_z^2 v) - 2R_{\mathcal{B}}(\nabla\eta, \nabla\partial_z v) - R_{\mathcal{B}}(\Delta\eta, \partial_z v).$$

Write

$$g_1 = -Q_1 + R_{\mathcal{B}}(|\nabla\eta|^2, \partial_z^2 v)$$

where Q_1 is as given by the statement of the lemma.

Let us recall the following

Proposition 6.4.4. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha + \beta > 0$. If $a \in H^{\alpha}(\mathbb{R}^d)$ and $b \in H^{\beta}(\mathbb{R}^d)$, then $R_{\mathcal{B}}(a, b) \in H^{\alpha+\beta-\frac{d}{2}}(\mathbb{R}^d)$. Moreover,*

$$\|R_{\mathcal{B}}(a,b)\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbb{R}^d)} \leq C \|a\|_{H^{\alpha}(\mathbb{R}^d)} \|b\|_{H^{\beta}(\mathbb{R}^d)}, \tag{6.4.12}$$

for some positive constant C depending only on α, β, d.

Since $s - 1 > d/2$, according to Proposition 6.4.4 we have

$$\left\|R_{\mathcal{B}}(|\nabla\eta|^2, \partial_z^2 v)\right\|_{L^2_z(H_x^{s+s_0-K})} \lesssim \left\||\nabla\eta|^2\right\|_{H^{s-1}} \left\|\partial_z^2 v\right\|_{L^2_z(H^{s_0-2})}.$$

Using the estimate $\left\||\nabla\eta|^2\right\|_{H^{s-1}} \lesssim \|\nabla\eta\|_{H^{s_0-1}} \|\nabla\eta\|_{H^{s-1}}$ for $s \geq s_0 > d/2 + 1$, and the estimate (6.4.4) we end up with

$$\left\|R_{\mathcal{B}}(|\nabla\eta|^2, \partial_z^2 v)\right\|_{L^2_z(H_x^{s+s_0-K})} \lesssim \|\eta\|_{H^{s_0}} \|\psi_0\|_{H^{s_0-\frac{1}{2}}} \|\eta\|_{H^s}.$$

Now, writing $Ev = 0$ and $v = W + T_{\partial_z v}\eta$, this yields

$$PW + PT_{\partial_z v}\eta + T_{\partial_z^2 v}|\nabla\eta|^2 - 2T_{\nabla\partial_z v} \cdot \nabla\eta - T_{\partial_z v}\Delta\eta \sim Q_1.$$

Hence, we need only prove that

$$PT_{\partial_z v}\eta + T_{\partial_z^2 v}|\nabla\eta|^2 - 2T_{\nabla\partial_z v} \cdot \nabla\eta - T_{\partial_z v}\Delta\eta \sim 0. \tag{6.4.13}$$

By using the Leibniz' rule (that is $\partial_x T_p = T_{\partial_x p} + T_p\partial_x$ and $\partial_z T_p = T_{\partial_z p} + T_p\partial_z$ for any function $p = p(x, z)$), one obtains

$$PT_{\partial_z v}\eta = A + B$$

with

$$A := (I + T_{|\nabla\eta|^2})T_{\partial_z^3 v}\eta + T_{\Delta\partial_z v}\eta - 2T_{\nabla\eta} \cdot T_{\nabla\partial_z^2 v}\eta - T_{\Delta\eta}T_{\partial_z^2 v}\eta$$
$$B := +2T_{\nabla\partial_z v} \cdot \nabla\eta - 2T_{(\partial_z^2 v)\nabla\eta} \cdot \nabla\eta + T_{\partial_z v}\Delta\eta.$$

Now, we use the following result to contract two paraproducts:

Proposition 6.4.5. *For any $\rho \geq 0$, there holds*

$$\|T_a T_b - T_{ab}\|_{H^{\mu} \to H^{\mu+\rho}} \leq K \|a\|_{W^{\rho,\infty}} \|b\|_{W^{\rho,\infty}}, \tag{6.4.14}$$

provided that a and b belong to $W^{\rho,\infty}(\mathbb{R})$.

Now (6.4.14) applied[1] with $\rho = s_0 - K = s_0 - 3 - d/2$ that implies that

$$\left\| T_{|\nabla\eta|^2} T_{\partial_z^3 v}\eta - T_{|\nabla\eta|^2\partial_z^3 v}\eta \right\|_{H^{s+s_0-K}} \lesssim \left\| |\nabla\eta|^2 \right\|_{W^{s_0-K,\infty}} \left\| \partial_z^3 v \right\|_{W^{s_0-K,\infty}} \|\eta\|_{H^s}$$
$$\lesssim \|\nabla\eta\|_{H^{s_0-3}}^2 \left\| \partial_z^3 v \right\|_{H^{s_0-3}} \|\eta\|_{H^s}$$

and hence $T_{|\nabla\eta|^2} T_{\partial_z^3 v}\eta \sim T_{|\nabla\eta|^2\partial_z^3 v}\eta$. Similarly, one deduces that $T_{\nabla\eta} \cdot T_{\nabla\partial_z^2 v}\eta \sim T_{\nabla\eta\cdot\nabla\partial_z^2 v}\eta$ and $T_{\Delta\eta} T_{\partial_z^2 v}\eta \sim T_{\Delta\eta\partial_z^2 v}\eta$. We conclude that

$$A \sim T_{E\partial_z v}\eta.$$

The key observation is that the right-hand side vanishes since $E\partial_z v = \partial_z E v = 0$ for $Ev = 0$. On the other hand, writing

$$|\nabla\eta|^2 = 2T_{\nabla\eta}\cdot\nabla\eta + R_{\mathcal{B}}(\nabla\eta,\nabla\eta),$$

it follows from (6.4.12) and (3.6.3) that

$$T_{\partial_z^2 v}|\nabla\eta|^2 \sim 2T_{(\partial_z^2 v)\nabla\eta}\cdot\nabla\eta$$

and the desired result (6.4.13) easily follows. □

6.4.4 Paralinearization of the boundary condition

We now express $G(\eta)\psi$ in terms of the unknown W.

Lemma 6.4.6. *There holds*

(6.4.15)
$$G(\eta)\psi = \Big((I + T_{|\nabla\eta|^2})\partial_z W - T_{\nabla\eta}\cdot\nabla W \Big)\Big|_{z=0} - \operatorname{div}\left(T_V\eta\right) + Q_2 + C_2.$$

where $Q_2 := -R_{\mathcal{B}}(\nabla\psi,\nabla\eta)$ and C_2 is a smooth cubic term, such that

(6.4.16)
$$\|C_2\|_{H^{s+s_0-\frac{5}{2}-\frac{d}{2}}} \le C\left(\|\eta\|_{H^{s_0}}\right)\|\eta\|_{H^{s_0}}\|\psi\|_{H^{s_0-\frac{1}{2}}}\|\eta\|_{H^s}.$$

Proof. Set $\mathcal{U} := (1 + |\nabla\eta|^2)\partial_z v - \nabla\eta\cdot\nabla v$ and recall that, by definition of v, $G(\eta)\psi = \mathcal{U}|_{z=0}$. As before, we find that

$$\mathcal{U} = \partial_z v + T_{|\nabla\eta|^2}\partial_z v + T_{\partial_z v}|\nabla\eta|^2 - T_{\nabla\eta}\cdot\nabla v - T_{\nabla v}\cdot\nabla\eta + Q_2 + R_1,$$

[1] Assuming that ρ is not an integer. Otherwise we work with $\rho' = \rho - \varepsilon$ with ε small.

where

$$\mathcal{Q}_2 := -R_{\mathcal{B}}(\nabla v, \nabla\eta), \quad R_1 := R_{\mathcal{B}}\big(|\nabla\eta|^2, \partial_z v\big).$$

Then

$$\mathcal{U} = \partial_z v + T_{|\nabla\eta|^2}\partial_z v + 2T_{\partial_z v\nabla\eta}\cdot\nabla\eta - T_{\nabla\eta}\cdot\nabla v - T_{\nabla v}\cdot\nabla\eta + \mathcal{Q}_2 + R_1 + R_2,$$

with

$$R_2 = T_{\partial_z v}|\nabla\eta|^2 - 2T_{\partial_z v\nabla\eta}\cdot\nabla\eta.$$

We next replace $\partial_z v$ (resp. ∇v) by $\partial_z(W + T_{\partial_z v}\eta)$ (resp. $\nabla(W + T_{\partial_z v}\eta)$) in the right hand-side, to obtain, after a few computations,

$$\begin{aligned}\mathcal{U} &= (I + T_{|\nabla\eta|^2})\partial_z W + \Big(T_{\partial_z^2 v} + T_{|\nabla\eta|^2}T_{\partial_z^2 v}\Big)\eta\\ &\quad + 2T_{\partial_z v\nabla\eta}\cdot\nabla\eta - T_{\nabla v}\cdot\nabla\eta - T_{\nabla\eta}\cdot\nabla T_{\partial_z v}\eta - T_{\nabla\eta}\cdot\nabla W + \mathcal{Q}_2 + R_1 + R_2\\ &= (I + T_{|\nabla\eta|^2})\partial_z W + T_{(1+|\nabla\eta|^2)\partial_z^2 v - \nabla\eta\cdot\nabla\partial_z v}\eta\\ &\quad - T_{\nabla v - (\partial_z v)\nabla\eta}\cdot\nabla\eta - T_{\nabla\eta}\cdot\nabla W + \mathcal{Q}_2 + R_1 + R_2 + R_3\end{aligned}$$

with

$$\begin{aligned}R_3 := -\Big(T_{|\nabla\eta|^2\partial_z^2 v} - T_{|\nabla\eta|^2}T_{\partial_z^2 v}\Big)\eta - T_{\nabla\eta}\cdot T_{\partial_z\nabla v}\eta + T_{\nabla\eta\cdot\partial_z\nabla v}\eta\\ - T_{(\partial_z v)\nabla\eta}\cdot\nabla\eta + T_{\nabla\eta}\cdot T_{\partial_z v}\nabla\eta.\end{aligned}$$

Since

$$(1 + |\nabla\eta|^2)\partial_z^2 v - \nabla\eta\cdot\nabla\partial_z v = -\operatorname{div}\big(\nabla v - (\partial_z v)\nabla\eta\big)$$

we deduce

$$\mathcal{U} = (I + T_{|\nabla\eta|^2})\partial_z W - T_{\nabla\eta}\cdot\nabla W - \operatorname{div}\Big(T_{\nabla v - (\partial_z v)\nabla\eta}\eta\Big) + \mathcal{Q}_2 + R_1 + R_2 + R_3.$$

We thus obtain

$$\begin{aligned}G(\eta)\psi = \Big((I + T_{|\nabla\eta|^2})\partial_z W - T_{\nabla\eta}\cdot\nabla W - \operatorname{div}\Big(T_{\nabla v - (\partial_z v)\nabla\eta}\eta\Big)\Big)\Big|_{z=0}\\ + Q_2 + C_2.\end{aligned} \tag{6.4.17}$$

with $Q_2 = \mathcal{Q}_2|_{z=0}$ and $C_2 = (R_1 + R_2 + R_3)|_{z=0}$. Since

$$\operatorname{div}\big(T_{\nabla v - (\partial_z v)\nabla\eta}\eta\big)\big|_{z=0} = \operatorname{div}\big(T_V\eta\big),$$

to complete the proof it remains only to estimate C_2. We claim that

$$\|R_1 + R_2 + R_3\|_{C_z^0(H^{s+s_0-\frac{5}{2}-\frac{d}{2}})} \le C\big(\|\eta\|_{H^{s_0}}\big)\,\|\eta\|_{H^{s_0}}\,\|\psi\|_{H^{s_0-\frac{1}{2}}}\,\|\eta\|_{H^s},$$

so that the desired bound for C_2 follows directly by taking the trace on $z = 0$. As in the proof of the previous lemma, we estimate R_1 (resp. R_2, R_3) directly from (6.4.12) (resp. (6.4.12) and (3.6.3)) and the bounds (6.4.3) and (6.4.4) for $\nabla_{x,z}v$ and $\partial_z^2 v$. □

6.5 Reduction to the boundary

Lemma 6.5.1. *Set*

$$b = \frac{1}{1+|\nabla\eta|^2}\cdot$$

The unknown W satisfies the paradifferential equation

$$\partial_z^2 W + T_{-b|\xi|^2}W - 2T_{-ib\xi\cdot\nabla\eta}\partial_z W - T_{b\Delta\eta}\partial_z W = R, \tag{6.5.1}$$

where

$$\|R\|_{L^2(I;H^{2s-K})} \le C\big(\|\eta\|_{H^s}\big)\,\|\psi\|_{H^{s-\frac{1}{2}}} \tag{6.5.2}$$

with $K = 3 + \frac{d}{2}$.

Proof. It follows from Lemma 6.4.3 applied with $s = s_0$ that

$$(I + T_{|\nabla\eta|^2})\partial_z^2 W + \Delta W - 2T_{\nabla\eta}\cdot\nabla\partial_z W - T_{\Delta\eta}\partial_z W = Q_1 + C_1, \tag{6.5.3}$$

where the $L^2(I;H^{2s-K})$-norm of C_1 satisfies (6.5.2) and where

$$Q_1 := R_{\mathcal{B}}(2\nabla\eta, \nabla\partial_z v) + R_{\mathcal{B}}(\Delta\eta, \partial_z v).$$

In view of the estimate (6.4.3) for $\partial_z v$, the classical estimate for Bony's remainder term (see Proposition 6.4.4) implies that the $L^2(I;H^{2s-K})$-norm of Q_1 satisfies (6.5.2).

We now make act T_b on (6.5.3). We obtain

$$\Pi\partial_z^2 W + T_b\Delta W - 2T_b T_{\nabla\eta}\cdot\nabla\partial_z W - T_b T_{\Delta\eta}\partial_z W = T_b(Q_1 + C_1),$$

where $\Pi = T_b(I + T_{|\nabla\eta|^2})$. Remembering that for all $\mu\in\mathbb{R}$ we have, $\|T_b u\|_{H^\mu} \le K(\mu)\,\|b\|_{L^\infty}\,\|u\|_{H^\mu}$, we see that the $L^2(I;H^{2s-K})$-norm of $T_b(Q_1 + C_1)$ is bounded by the right-hand side of (6.5.2).

On the other hand, as explained in Remark 6.2.4, we have $I + T_{|\nabla\eta|^2} = T_b + S$ where S is a smoothing operator. Also, it follows from Proposition 6.4.5 that

$$T_b T_{1/b} = T_1 + S_1 = \mathrm{Id} + S_1', \qquad T_b T_{\nabla\eta} = T_{b\nabla\eta} + S_2 \qquad T_b T_{\Delta\eta} = T_{b\Delta\eta} + S_3,$$

where S_1, S_1', S_2, S_3 are operators of order $-\rho$ with $\rho = s - 3 - d/2$. This immediately implies the wanted result. □

We now perform a full decoupling into a forward and a backward elliptic evolution equations. Recall that the classes $\dot{\Gamma}^m_\rho(\mathbb{R}^d)$ have been defined in §6.2.2.

Lemma 6.5.2. *There exist two symbols $a, A \in \dot{\Gamma}^1_{s-1-d/2}(\mathbb{R}^d)$ such that,*

$$(\partial_z - T_a)(\partial_z - T_A)v = f \in L^2_z\big(I; H^{2s-3-\frac{d}{2}}(\mathbb{R}^d)\big). \tag{6.5.4}$$

Proof. We will use the symbolic calculus for paradifferential operators. For the reader convenience, we recall the statement here:

Theorem 6.5.3 (Composition)**.** *Let m, m' be real numbers and $\rho > 0$. If $a \in \Gamma^m_\rho(\mathbb{R}^d)$ and $b \in \Gamma^{m'}_\rho(\mathbb{R}^d)$ then $T_aT_b - T_{a\#_\rho b}$ is of order $\leq m + m' - \rho$ with*

$$a\#_\rho b = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial^\alpha_\xi a \partial^\alpha_x b.$$

We seek a and A in the form

$$a(x,\xi) = \sum_{0\leq j<t} a_{1-j}(x,\xi), \qquad A(x,\xi) = \sum_{0\leq j<t} A_{1-j}(x,\xi), \tag{6.5.5}$$

where

$$t := s - 3 - d/2,$$

and where a_m and A_m are homogeneous symbols in ξ of order m with regularity $W^{t+1-m,\infty}$ in x. As we have seen earlier in this chapter, this implies that a_m and A_m belong to $\Gamma^m_{t+1-m}(\mathbb{R}^d)$ (see Definition 6.2.2 and the remark that follows).

We want to solve the system

$$\begin{aligned} T_a \circ T_A &= -T_{b|\xi|^2} + R, \\ T_a + T_A &= 2T_{ib\cdot\xi + b\Delta\eta}, \end{aligned} \tag{6.5.6}$$

for some remainder term R of order $\leq -t$.

Assume that we have defined a and A such that (6.5.17) is satisfied, and let us then prove the desired result (6.5.21). We use here the following elementary property: if $Q(D_x)$ is a Fourier multiplier with symbol $q(\xi)$, then for any symbol $a \in \Gamma^m_0(\mathbb{R}^d)$ for some $m \in \mathbb{R}$, we have

$$T_{a(x,\xi)}Q(D_x) = T_{a(x,\xi)q(\xi)}$$

Firstly, the first equation implies that

$$T_a T_A v - b\Delta v \in L^2(H^{s+t}) = L^2(H^{2s-3-\frac{d}{2}}).$$

Notice that A does noes depend on z. Hence $\partial_z T_A = T_A \partial_z$. It follows from the second equation that

$$\partial_z T_A + T_a \partial_z u = \big(2T_{b\nabla\eta} \cdot \nabla \partial_z u - T_{b\Delta\eta}\partial_z u\big).$$

We thus obtain the desired result (6.5.21).

We now have to solve (6.5.17). We claim that $T_a \circ T_A - T_\tau$ is of order $\leq -t$ where

$$\tau := \sum_{|\alpha| < s-3-\frac{d}{2}+k+\ell} \sum \sum \frac{1}{i^\alpha \alpha!} \partial_\xi^\alpha a_k \partial_x^\alpha A_\ell. \tag{6.5.7}$$

To verify this claim, we apply Theorem 6.5.3 to infer that

$$\begin{aligned}
&T_{a_1} T_{A_1} - T_{a_1 \sharp_{s-1-\frac{d}{2}} A_1} \quad \text{is of order } \leq 1 + 1 - (s - 1 - \frac{d}{2}) = -t,\\
&T_{a_1} T_{A_0} - T_{a_1 \sharp_{s-2-\frac{d}{2}} A_0} \quad \text{is of order } \leq 1 + 0 - (s - 2 - \frac{d}{2}) = -t,\\
&T_{a_0} T_{A_1} - T_{a_0 \sharp_{s-2-\frac{d}{2}} A_1} \quad \text{is of order } \leq 0 + 1 - (s - 2 - \frac{d}{2}) = -t,
\end{aligned}$$

and, for $-t \leq k, \ell \leq 0$,

$$T_{a_k} T_{A_\ell} - T_{a_k \sharp_{t+k+\ell} A_\ell} \text{ is of order } \leq -t.$$

We then write $\tau = \sum \tau_m$ where τ_m is of order m. Together with the second equation in (6.5.17), this yields a cascade of equations that allows to determine a_m and A_m by induction.

We are now in position to determine a and A. We proceed in several steps. We first solve the principal system:

$$\begin{aligned}
a_1 A_1 &= -b\,|\xi|^2\,,\\
a_1 + A_1 &= 2ib\nabla\eta \cdot \xi,
\end{aligned}$$

by setting

$$\begin{aligned}
a_1(x,\xi) &= ib\nabla\eta \cdot \xi - \sqrt{b|\xi|^2 - (b\nabla\eta \cdot \xi)^2},\\
A_1(x,\xi) &= ib\nabla\eta \cdot \xi + \sqrt{b|\xi|^2 - (b\nabla\eta \cdot \xi)^2}.
\end{aligned}$$

Note that $b|\xi|^2 - (b\nabla\eta\cdot\xi)^2 \geq b^2|\xi|^2$ so that the symbols a_1, A_1 are well defined and belong to $\Gamma^1_{s-1-d/2}(\mathbb{T}^d)$.

We next solve the sub-principal system

$$
\begin{aligned}
&a_0 A_1 + a_1 A_0 + \frac{1}{i}\partial_\xi a_1 \partial_x A_1 = 0,\\
&a_0 + A_0 = b\Delta\eta.
\end{aligned}
$$

It is found that

$$
a_0 = \frac{i\partial_\xi a_1\cdot\partial_x A_1 - b\Delta\eta a_1}{A_1 - a_1}, \quad A_0 = \frac{i\partial_\xi a_1\cdot\partial_x A_1 - b\Delta\eta A_1}{a_1 - A_1}.
$$

Once the principal and sub-principal symbols have been defined, one can define the other symbols by induction. By induction, for $-t+1 \leq m \leq 0$, suppose that $a_1,\ldots,a_m$ and $A_1,\ldots,A_m$ have been determined. Then define a_{m-1} and A_{m-1} by

$$
A_{m-1} = -a_{m-1},
$$

and

$$
a_{m-1} = \frac{1}{a_1 - A_1}\sum\sum\sum \frac{1}{i^\alpha \alpha!}\partial_\xi^\alpha a_k \partial_x^\alpha A_\ell
$$

where the sum is over all triple $(k,\ell,\alpha)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}^d$ such that

$$
m \leq k \leq 1, \quad m\leq \ell \leq 1, \quad |\alpha| = k+\ell-m.
$$

By definition, one has $a_m, A_m \in \Gamma^m_{t+1+m}$ for $-t+1 < m \leq 0$. Also, we obtain that $\tau = -b\,|\xi|^2$ and $a + A = 2b(i\nabla\eta\cdot\xi) + b\Delta\eta$.

This completes the proof. □

6.5.1 Formal computations

The aim of this paragraph is to compute, formally, various quadratic terms. For this paragraph only, as a notation, we say that

$$
A(\eta,\psi)\approx B(\eta,\psi) \quad\Longleftrightarrow\quad A(\eta,\psi) - B(\eta,\psi) = O(|(\eta,\psi)|^3)\,.
$$

Lemma 6.5.4. *There holds*

$$
\partial_z W|_{z=0} \approx A_+ W|_{z=0} - Q_2 + F_{\leq 2}(\eta)\psi
$$

with

$$
A_+ := |D_x| - |D_x|(T_\eta |D_x|\cdot) - T_\eta\Delta.
$$

Proof. We begin by recalling the computation of the second order approximation of $G(\eta)\psi$, namely

$$G(\eta)\psi \approx \Big(G(0) + G'(0)[\eta]\Big)\psi.$$

To do so, we use the *shape-derivative* formula:

$$G'(\eta)[\widehat{\eta}]\psi = -G(\eta)\big(\widehat{\eta}B(\eta)\psi\big) - \operatorname{div}\big(\widehat{\eta}V(\eta)\psi\big)\,, \tag{6.5.8}$$

where

$$B(\eta)\psi = \frac{G(\eta)\psi + \nabla\eta\cdot\nabla\psi}{1+|\nabla\eta|^2}\,, \qquad V(\eta)\psi = \nabla\psi - \big(B(\eta)\psi\big)\nabla\eta\,.$$

We have

$$G(0) = |D_x|\,, \quad B(0) = G(0)\,, \quad V(0) = \nabla\,, \tag{6.5.9}$$

moreover applying (6.5.8), (6.5.9) we get

$$G'(0)[\eta]\psi = -G(0)\big(\eta B(0)\psi\big) - \operatorname{div}\big(\eta V(0)\psi\big) = -|D_x|\big(\eta|D_x|\psi\big) - \operatorname{div}(\eta\nabla\psi)\,.$$

Consequently,

$$G(\eta)\psi \approx G(0)\psi + G'(0)[\eta]\psi = |D_x|\psi - |D_x|\big(\eta|D_x|\psi\big) - \operatorname{div}\big(\eta\nabla\psi\big). \tag{6.5.10}$$

We next paralinearize the quadratic part, that is we replace products ab by the sum of paraproducts $T_ab + T_ba$ and a remainder term $R_{\mathcal{B}}(a,b)$. We have

$$\begin{aligned} G(\eta)\psi &\approx |D_x|\psi - |D_x|\big(T_\eta|D_x|\psi\big) - |D_x|\big(T_{|D_x|\psi}\eta\big) - |D_x|R_{\mathcal{B}}(\eta, G(0)\psi) \\ &\quad - \operatorname{div}\big(T_\eta\nabla\psi\big) - \operatorname{div}\big(T_{\nabla\psi}\eta\big) - R_{\mathcal{B}}(\eta,\nabla\psi) \\ &= |D_x|\Big(\psi - T_{|D_x|\psi}\eta\Big) - \operatorname{div}\big(T_{\nabla\psi}\eta\big) - |D_x|(T_\eta|D_x|\psi) - \operatorname{div}(T_\eta\nabla\psi) \\ &\quad + F_{\le 2}(\eta)\psi \end{aligned} \tag{6.5.11}$$

where $F_{\le 2}(\eta)\psi$ is as in (**??**).

Now by comparing (6.4.17) with (6.5.11) we deduce that, modulo cubic terms,

$$\begin{aligned} &\Big((I + T_{|\nabla\eta|^2})\partial_z W - T_{\nabla\eta}\cdot\nabla W\Big)\Big|_{z=0} - \operatorname{div}\big(T_V\eta\big) + Q_2 \\ &\qquad \approx |D_x|\Big(\psi - T_{|D_x|\psi}\eta\Big) - \operatorname{div}(T_{\nabla\psi}\eta) - |D_x|(T_\eta|D_x|\psi) - \operatorname{div}(T_\eta\nabla\psi) + F_{\le 2}(\eta)\psi. \end{aligned}$$

Since

$$\begin{aligned}
&\mathrm{div}\,(T_V\eta) \approx \mathrm{div}(T_{\nabla\psi}\eta),\\
&T_{\nabla\eta}\cdot\nabla W|_{z=0} \approx T_{\nabla\eta}\cdot\nabla\psi,\\
&(I+T_{|\nabla\eta|^2})\partial_z W|_{z=0} \approx \partial_z W|_{z=0},
\end{aligned}$$

we infer that

$$\begin{aligned}
\partial_z W|_{z=0} \approx{}& |D_x|\big(\psi - T_{|D_x|\psi}\eta\big) + T_{\nabla\eta}\cdot\nabla\psi - |D_x|(T_\eta|D_x|\psi) - \mathrm{div}(T_\eta\nabla\psi)\\
&- Q_2 + F_{\le 2}(\eta)\psi,
\end{aligned}$$

and, simplifying $T_{\nabla\eta}\cdot\nabla\psi - \mathrm{div}(T_\eta\nabla\psi)$,

$$\partial_z W|_{z=0} \approx |D_x|\big(\psi - T_{|D_x|\psi}\eta\big) - |D_x|(T_\eta|D_x|\psi) - T_\eta\Delta\psi - Q_2 + F_{\le 2}(\eta)\psi.$$

Modulo quadratic term, ψ is given by $W|_{z=0}$ and we easily complete the proof. □

Remark 6.5.5. *To compare* (**??**) *and* (6.5.11)*, notice that the quadratic part of* $T_V\cdot\nabla\eta + T_{\mathrm{div}\,V}\eta$ *is given by*

$$T_{\nabla\psi}\cdot\nabla\eta + T_{\Delta\psi}\eta = \mathrm{div}(T_\eta\nabla\psi).$$

It remains to compare $-|D_x|(T_\eta|D_x|\psi) - \mathrm{div}(T_\eta\nabla\psi)$ *with the quadratic part of* $T_{\lambda-|\xi|}\big(\psi - T_B\eta\big)$*. Since* $\lambda^{(1)} - |\xi|$ *is quadratic in* η*, only the sub-principal symbols contribute. Notice that, when* $d=1$*, then* $-|D_x|(T_\eta|D_x|\psi) - \mathrm{div}(T_\eta\nabla\psi) = 0$ *(see [24, Lemma A.1.11]) and this cancellation corresponds to the ones of the subprincipal symbols (see Remark 6.1.2).*

We now study the quadratic term in the equation (6.5.1) for the unknown W. Write the quadratic part of the left-hand side of this equation under the form $\mathcal{P}_2 W$ with

$$\mathcal{P}_2 := \partial_z^2 + \Delta - 2T_{\nabla\eta}\cdot\nabla\partial_z - T_{\Delta\eta}\partial_z.$$

We next seek an operator A_- such that

$$(\partial_z - A_-)(\partial_z - A_+)W \approx \mathcal{P}_2 W.$$

Lemma 6.5.6. *The previous identity holds with* $A_- := -A_+ + 2T_{\nabla\eta}\cdot\nabla + T_{\Delta\eta}$*.*

Proof. It is sufficient to prove that $A_-A_+W \approx \Delta W$. Now observe that

$$A_- = -|D_x| + |D_x|(T_\eta|D_x|\cdot) + \Delta(T_\eta\cdot)$$

and hence

$$A_-A_+ = -|D_x|^2 + |D_x|^2 T_\eta |D_x| + |D_x| T_\eta \Delta + |D_x| T_\eta |D_x|^2 + \Delta T_\eta |D_x| - \mathcal{R}_A$$

where $\mathcal{R}_A$ is given by

$$\mathcal{R}_A = |D_x| T_\eta |D_x|^2 T_\eta |D_x| + |D_x| T_\eta |D_x| T_\eta \Delta + \Delta T_\eta |D_x| T_\eta |D_x| + \Delta T_\eta T_\eta \Delta. \tag{6.5.12}$$

Since $\Delta = -|D_x|^2$ one has $A_-A_+ = \Delta + \mathcal{R}_A$. Now $\mathcal{R}_A W \approx 0$ which completes the proof. □

6.5.2 Factorization

Recall that we proved that

$$(I + T_{|\nabla\eta|^2})\partial_z^2 W + \Delta W - 2T_{\nabla\eta}\cdot\nabla\partial_z W - T_{\Delta\eta}\partial_z W = Q_1 + C_1,$$

where Q_1 is a smooth quadratic term given by

$$Q_1 := R_{\mathcal{B}}(2\nabla\eta, \nabla\partial_z v) + R_{\mathcal{B}}(\Delta\eta, \partial_z v)$$

and C_1 is a smooth cubic term. Moreover, in the previous paragraph, we have found two operators A_- and A_+ such that

$$(\partial_z - A_-)(\partial_z - A_+)W = (\partial_z^2 + \Delta - 2T_{\nabla\eta}\cdot\nabla\partial_z - T_{\Delta\eta}\partial_z)W + \text{ cubic terms.}$$

In this paragraph, we shall prove that one can factor out the full elliptic equation, modulo smooth cubic terms.

Proposition 6.5.7. *There exist two symbols $q_- = q_-(x,\xi)$ and $q_+ = q_+(x,\xi)$ of order 1 in ξ and depending quadratically in η, satisfying*

$$N^1_{\rho-1}(q_-) + N^1_{\rho-1}(q_+) \le C\big(\|\eta\|_{H^{s_0}}\big)\,\|\eta\|^2_{H^{s_0}} \tag{6.5.13}$$

with $\rho = s_0 - \frac{d}{2}$, and such that

$$(\partial_z - A_- - T_{q_-})(\partial_z - A_+ - T_{q_+})W = Q_1 + C_3 \tag{6.5.14}$$

where Q_1 is as above and C_3 is a smooth cubic term, such that

$$\|C_3\|_{L^2(I;H^{s+s_0-K})} \le C\big(\|\eta\|_{H^{s_0}}\big)\,\|\eta\|_{H^{s_0}}\,\|\psi\|_{H^{s_0-\frac{1}{2}}}\,\|\eta\|_{H^s} \tag{6.5.15}$$

with $K = 3 + \frac{d}{2}$.

Proof. The key point is to use Calderón factorization as in the previous section. The main new point is that we identify the quadratic and cubic components of the remainder. To sort terms according to the homogeneity, it is convenient to introduce the following notations.

Notation 6.5.8. *Given two functions f, g, we write $f \equiv g \mod \text{linear}$ to say that there exists an increasing function C such that*

$$\|f-g\|_{H^{s+s_0-K}} \le C\big(\|\eta\|_{H^{s_0}}\big)\, \|\psi\|_{H^{s_0-\frac12}}$$

where $K = 3 + d/2$. Similarly, $f \equiv g \mod \text{quadratic}$ (resp. $f \equiv g \mod \text{cubic}$) means that there exists an increasing function C such that

$$\|f-g\|_{H^{s+s_0-K}} \le C\big(\|\eta\|_{H^{s_0}}\big)\, \|\eta\|_{H^{s_0}}\, \|\psi\|_{H^{s_0-\frac12}}$$

(resp. $\|f-g\|_{H^{s+s_0-K}} \le C\big(\|\eta\|_{H^{s_0}}\big)\, \|\eta\|^2_{H^{s_0}}\, \|\psi\|_{H^{s_0-\frac12}}$). We say then that f is equal to g modulo a smooth quadratic (resp. cubic) remainder term.

The strategy of the proof is the following. We first seek two symbols $p = p(x,\xi)$ and $q = q(x,\xi)$ depending on η, q_-, q_+ such that

$$(\partial_z - A_- - T_{q_-})(\partial_z - A_+ - T_{q_+})W \equiv T_p W + T_q \partial_z W + Q_1 \mod \text{cubic}$$

and then to find q_- and q_+ which are approximate solutions of $p = 0$ and $q = 0$.

Since $(\partial_z - A_-)(\partial_z - A_+)W = \mathcal{P}_2 W + \mathcal{R}_A W$ and since $\partial_z T_{q_+} = T_{q_+}\partial_z$, one has

$$\begin{aligned}&(\partial_z - A_- - T_{q_-})(\partial_z - A_+ - T_{q_+})W\\ &\qquad = \big(\mathcal{P}_2 + \mathcal{R}_A - (T_{q_-} + T_{q_+})\partial_z + A_- T_{q_+} + T_{q_-}A_+ + T_{q_-}T_{q_+}\big)W.\end{aligned}$$

Step 1. We begin by rewriting $\mathcal{P}_2 W$ under the form $T_p W + T_q \partial_z W + Q_1$ modulo a smooth cubic term. It follows from (6.5.1) that

$$\mathcal{P}_2 W \equiv -T_{|\nabla\eta|^2}\partial_z^2 W + Q_1 \mod \text{cubic}.$$

So the question is to write $\partial_z^2 W$ under the form $T_p W + T_q \partial_z W$ modulo a smooth cubic term. To do so, we use again (6.5.1). By replacing the factor $I + T_{|\nabla\eta|^2}$ multiplying $\partial_z^2 W$ by $T_{1+|\nabla\eta|^2} + (I - T_1)$, it follows from (6.5.1) that

$$T_{1+|\nabla\eta|^2}\partial_z^2 W = -\Delta W + 2T_{\nabla\eta}\cdot\nabla\partial_z W + T_{\Delta\eta}\partial_z W + (T_1 - I)\partial_z^2 W + Q_1 + C_1.$$

Now it follows from the estimates for Q_1, C_1 (see (6.5.11) for C_1, while Q_1 is estimated by means of (6.4.12)) and the one for $\partial_z^2 W$ (which follows directly from the definition $\partial_z^2 W = \partial_z^2 v - T_{\partial_z^3 v}\eta$ and the estimates (6.4.4), (6.4.5) as well as (**??**)) that

$$T_{1+|\nabla\eta|^2}\partial_z^2 W \equiv -\Delta W + 2T_{\nabla\eta}\cdot\nabla\partial_z W + T_{\Delta\eta}\partial_z W \quad \operatorname{mod}\text{linear}.$$

Moreover, using the estimate (3.6.3), one has

$$\left\|T_{(1+|\nabla\eta|^2)^{-1}}T_{1+|\nabla\eta|^2}u\right\|_{H^{\mu+s_0-1-\frac{d}{2}}} \le C\left(\|\eta\|_{H^{s_0}}\right)\|u\|_{H^\mu} \tag{6.5.16}$$

and hence

$$\partial_z^2 W \equiv T_{(1+|\nabla\eta|^2)^{-1}}\Big(-\Delta W + 2T_{\nabla\eta}\cdot\nabla\partial_z W + T_{\Delta\eta}\partial_z W\Big) \quad \operatorname{mod}\text{linear}.$$

Applying $T_{|\nabla\eta|^2}$ and using again (3.6.3) to obtain estimates similar to (6.5.16), this yields

$$T_{|\nabla\eta|^2}\partial_z^2 W \equiv -T_\alpha\Delta W + 2T_{\alpha\nabla\eta\cdot(i\xi)+\alpha\Delta\eta}\partial_z W \quad \operatorname{mod}\text{cubic}, \text{ with } \alpha = \frac{|\nabla\eta|^2}{1+|\nabla\eta|^2}.$$

We conclude that

$$\mathcal{P}_2 W \equiv -T_{\alpha|\xi|^2}W - 2T_{\alpha\nabla\eta\cdot(i\xi)+\alpha\Delta\eta}\partial_z W + Q_1 \quad \operatorname{mod}\text{cubic}.$$

Step 2. We next rewrite $\mathcal{R}_A W$ under the form $T_p W$ modulo a smooth cubic term. Directly from the definition (6.5.12) one has

$$\mathcal{R}_A W = \left(|D_x|T_\eta|D_x| + \Delta T_\eta\right)\left(|D_x|T_\eta|D_x| + T_\eta\Delta\right)$$

thus

$$\mathcal{R}_A W = |D_x|\left[T_\eta, |D_x|\right]\left[|D_x|, T_\eta\right]|D_x|.$$

Since $T_{|\xi|} - |D_x|$ is regularizing, one can replace $|D_x|$ by $T_{|\xi|}$ to the prize of smooth errors which are cubic since $\mathcal{R}_A$ is quadratic in η:

$$\mathcal{R}_A W \equiv T_{|\xi|}\left[T_\eta, T_{|\xi|}\right]\left[T_{|\xi|}, T_\eta\right]T_{|\xi|}W \quad \operatorname{mod}\text{cubic}.$$

Now we claim that, for some symbol r,

$$\mathcal{R}_A W \equiv T_r W \quad \operatorname{mod}\text{cubic}.$$

This follows from (3.5.8) setting

$$r = |\xi|\sharp_{\rho-1}\left(\eta\sharp_\rho|\xi| - |\xi|\sharp_\rho\eta\right)\sharp_{\rho-1}\left(|\xi|\sharp_\rho\eta - \eta\sharp_\rho|\xi|\right)|\xi|.$$

Step 3. We next rewrite $A_-T_{q_+}+T_{q_-}A_++T_{q_-}T_{q_+}$ under the form T_pW modulo a smooth cubic term. Replacing $|D_x|$ by $T_{|\xi|}+(|D_x|-T_{|\xi|})$ and recalling that $(|D_x|-T_{|\xi|})$ is a smoothing operator, one easily finds symbols a_- and a_+ such that $A_-=T_{a_-}$ and $A_+=T_{a_+}$ modulo remainder terms whose operator norms are easily estimated from (3.5.8), namely

$$a_+=|\xi|-\big(|\xi|\sharp_\rho\eta-\eta\sharp_\rho|\xi|\big)|\xi|,\qquad a_-=-a_++2\nabla\eta\cdot(i\xi)+\Delta\eta.$$

Notice that a_+ and a_- belong to $\Sigma^1_{\rho-1}$. Then assuming that (6.5.13) holds, using again (3.5.8), one obtains that

$$A_-T_{q_+}+T_{q_-}A_++T_{q_-}T_{q_+}W\equiv T_\wp W\quad \operatorname{mod}\text{cubic}$$

where

$$\wp=a_-\sharp_{\rho-1}q_++q_-\sharp_{\rho-1}a_++q_-\sharp_{\rho-1}q_+.$$

Notice that $\wp$ belongs to $\Sigma^2_{\rho-1}$.

Step 4. We thus have found that

$$(\partial_z-A_--T_{q_-})(\partial_z-A_+-T_{q_+})W\equiv T_pW+T_q\partial_zW+Q_1\quad\operatorname{mod}\text{cubic}$$

where

$$\begin{aligned}p&=-\alpha|\xi|^2+r+\wp,\\ q&=-q_--q_+-2\alpha\nabla\cdot(i\xi)+\alpha\Delta\eta.\end{aligned}$$

Now notice that since $A_-A_+=\Delta+\mathcal{R}_A$ we have $a_-\sharp_{\rho-1}a_+=-|\xi|^2+r$ (at least if one removes symbols smoother than the remainders) and hence we conclude that to solve the system $(p=0,q=0)$ it is sufficient that the symbols $p_\pm:=a_\pm+q_\pm$ solve

(6.5.17)
$$\begin{aligned}p_-\sharp p_+&=-\frac{|\xi|^2}{1+|\nabla\eta|^2},\\ p_-+p_+&=2\frac{i\nabla\eta\cdot\xi}{1+|\nabla\eta|^2}+\frac{\Delta\eta}{1+|\nabla\eta|^2}.\end{aligned}$$

The previous system is solved in the previous section. So one obtains the desired symbol $q_\pm$ writing $q_\pm=p_\pm-a_\pm$. □

Corollary 6.5.9. *Set*

$$\underline{w}:=(\partial_z-A_+-T_{q_+})W.$$

Then

$$\big\|\underline{w}|_{z=0}-|D_x|R_{\mathcal{B}}(|D_x|\psi,\eta)+\operatorname{div}R_{\mathcal{B}}(\nabla\psi,\eta)\big\|_{H^{s+s_0-K}}\le C\big(\|\eta\|_{H^{s_0}}\big)\|\eta\|_{H^{s_0}}\|\psi\|_{H^{s_0-\frac12}}\|\eta\|_{H^s}.$$

with $K=3+\frac{d}{2}$.

Proof. The proof is in three steps.

Step 1. Directly, from the definition of $\underline{w}$, we begin by proving that

$$\left\|\underline{w}\right\|_{L^2(I;H^{s_0-1})} \leq C\left(\|\eta\|_{H^{s_0}}\right) \|\eta\|_{H^{s_0}} \|\psi\|_{H^{s_0-\frac{1}{2}}} . \tag{6.5.18}$$

To obtain this estimate, we shall write $\underline{w}$ under the form $\partial_z v - |D_x|v + Q$ where Q is a quadratic term. Namely, using the definition $W = v - T_{\partial_z v}\eta$ and the one of $A_+ = |D_x| - \left[|D_x|, T_\eta\right]|D_x|$, one has $\underline{w} = (\partial_z - |D_x|)v + Q$ with

$$Q = -(\partial_z - |D_x|)T_{\partial_z v}\eta + \left(\left[|D_x|, T_\eta\right]|D_x| - T_{q_+} \right)W.$$

The first term $(\partial_z - |D_x|)v$ is estimated by means of (6.4.6). Let us estimate Q. Directly from (3.5.7) and (3.5.8), one has

$$\begin{aligned}
&\left\|\partial_z T_{\partial_z v}\eta\right\|_{H^{s_0-1}} = \left\|T_{\partial_z^2 v}\eta\right\|_{H^{s_0-1}} \lesssim \left\|\partial_z^2 v\right\|_{L^\infty} \|\eta\|_{H^{s_0-1}} \lesssim \left\|\partial_z^2 v\right\|_{H^{s_0-1}} \|\eta\|_{H^{s_0-1}} ,\\
&\left\||D_x|T_{\partial_z v}\eta\right\|_{H^{s_0-1}} \lesssim \|\partial_z v\|_{L^\infty} \|\eta\|_{H^{s_0}} \lesssim \|\partial_z v\|_{H^{s_0}} \|\eta\|_{H^{s_0}} ,\\
&\left\|\left[|D_x|, T_\eta\right]W\right\|_{H^{s_0}} \lesssim \|\eta\|_{W^{1,\infty}} \|W\|_{H^{s_0}} \lesssim \|\eta\|_{H^{s_0}} \|W\|_{H^{s_0}} ,\\
&\left\|T_{q_+}W\right\|_{H^{s_0}} \lesssim M_0^1(q_+) \|W\|_{H^{s_0}} \lesssim C\left(\|\eta\|_{H^{s_0}}\right) \|\eta\|_{H^{s_0}}^2 \|W\|_{H^{s_0}} ,
\end{aligned}$$

where we used (6.5.13) in the last inequality. Notice that, in view of the cut-off function ζ in the definition (6.2.2) of paradifferential operators, the last two-estimates hold with $\|W\|_{H^{s_0}}$ replaced by $\|\nabla_x W\|_{H^{s_0-1}}$. To estimate the latter norm, directly from the definition of $W = v - T_{\partial_z v}\eta$, one obtains

$$\|\nabla_x W\|_{H^{s_0-1}} \lesssim \|\nabla_x v\|_{H^{s_0-1}} + \|\partial_z v\|_{L^\infty} \|\eta\|_{H^{s_0}} . \tag{6.5.19}$$

By combining the previous estimates and using (6.4.3) and (6.4.4) for the $L^2(I;H^{s_0})$-norm of $\partial_z^k v$, one concludes that the $L^2(I;H^{s_0-1})$-norm of Q is bounded by the right-hand side of (6.5.18).

Step 2. In this step we prove that

$$\left\|\underline{w}\right\|_{L^2(I;H^{s+s_0-K})} \leq C\left(\|\eta\|_{H^{s_0}}\right) \|\psi\|_{H^{s_0-\frac{1}{2}}} \|\eta\|_{H^s} \tag{6.5.20}$$

By definition of $\underline{w}$, it follows from Proposition 6.5.7 that

$$(\partial_z - A_- - T_{q_-})\underline{w} = Q_1 + C_3 \tag{6.5.21}$$

where Q_1 is as above and C_3 is a smooth cubic term. The $L^2(I;H^{s+s_0-K})$ of Q_1 is estimated directly from (6.4.12) and we obtain

$$\|Q_1\|_{L^2(I;H^{s+s_0-K})} \leq C\left(\|\eta\|_{H^{s_0}}\right) \|\psi\|_{H^{s_0-\frac{1}{2}}} \|\eta\|_{H^s} \tag{6.5.22}$$

Notice that (6.5.22) obviously implies that the same estimate holds for C_3.

Now, as already mentioned $A_- - T_{a_-}$ is a smoothing operator satisfying, for any real numbers μ and m,

$$\left\|(A_- - T_{a_-})u\right\|_{H^{\mu+m}} \lesssim (1 + \|\eta\|_{H^{s_0}}) \|u\|_{H^\mu}.$$

In particular, it follows from (6.5.18) that

$$\left\|(A_- - T_{a_-})\underline{w}\right\|_{H^{s+s_0-K}} \leq C\left(\|\eta\|_{H^{s_0}}\right) \|\eta\|_{H^{s_0}} \|\psi\|_{H^{s_0-\frac{1}{2}}}.$$

Setting $p_- = a_- + q_-$, it thus follows from previous estimate that

$$\partial_z \underline{w} - T_{p_-}\underline{w} = f$$

with $\|f\|_{L^2(I;H^{s+s_0-K})} \leq C\left(\|\eta\|_{H^{s_0}}\right) \|\psi\|_{H^{s_0-\frac{1}{2}}} \|\eta\|_{H^s}$. Since $\Re p_- \leq -|\xi|$ this is a parabolic evolution equation. The desired estimate for $\underline{w}$ now follows easily from parabolic estimate as in [29, 22] (see for instance Prop. 4.10 in [29] or Prop. 2.18 in [22]).

Step 3. Once (6.5.20) is established, one deduces that

$$A_-\underline{w} = -|D_x|\underline{w} + C$$

with

$$\|C\|_{L^2(I;H^{s+s_0-K})} \leq C\left(\|\eta\|_{H^{s_0}}\right) \|\eta\|_{H^{s_0}} \|\psi\|_{H^{s_0-\frac{1}{2}}} \|\eta\|_{H^s}.$$

Now write

$$\partial_z \underline{w} + |D_x|\underline{w} = Q_1 + C.$$

Now we use a trick which consists in observing that, since

$$Q_1 := R_{\mathcal{B}}(2\nabla\eta, \nabla\partial_z v) + R_{\mathcal{B}}(\Delta\eta, \partial_z v),$$

one can write Q_1 under the form $Q_1 = |D_x|\alpha + \partial_z\beta$ with

$$\alpha = -|D_x| R_{\mathcal{B}}(\partial_z v, \eta), \qquad \beta = -R_{\mathcal{B}}(\Delta v, \eta).$$

Then

$$(\partial_z + |D_x|)(\underline{w} - \alpha - \beta) = |D_x| R_{\mathcal{B}}(\partial_z^2 v + \Delta v, \eta) + C.$$

Since $\partial_z^2 v + \Delta v$ is quadratic in (η, v), it follows from (6.4.12) that the right-hand side satisfies the same estimate as C does. Using parabolic estimate for the solution of $\partial_z u + |D_x| u = f$, one deduces that the $(C^0 \cap L^\infty)(I; H^{s+s_0-K})$-norm of $\underline{w} - \alpha - \beta$ is bounded by $C\left(\|\eta\|_{H^{s_0}}\right) \|\eta\|_{H^{s_0}} \|\psi\|_{H^{s_0-\frac{1}{2}}} \|\eta\|_{H^s}$. Using the obvious estimate $|h(0)| \leq \sup_{z\in I} |h(z)|$, this implies that

$$\left\|\underline{w} - \alpha - \beta|_{z=0}\right\|_{H^{s+s_0-K}} \leq C\left(\|\eta\|_{H^{s_0}}\right) \|\eta\|_{H^{s_0}} \|\psi\|_{H^{s_0-\frac{1}{2}}} \|\eta\|_{H^s}.$$

Evaluating α and β for $z = 0$ concludes the proof. $\square$

By combining the previous result with (6.4.17) concludes the proof of Theorem 6.2.6.

6.6 References

To flatten the boundary, it might be tempting to use a general change of variables of the form $y = \rho(x, z)$ (as in [167, 168, 292, 300]). However, these changes of variables do not modify the behavior of the functions on $z = 0$ and hence they do not modify the Dirichlet to Neumann operator (see the discussion in [430]). Therefore, the fact that we use the most simple change of variables one can think of (that is $(x, y) \to (x, y - \eta(x))$) is an interesting feature of the approach developed in this chapter.

Chapter 7

The water-wave problem with surface tension

We have seen in Chapter 1 (see Proposition 1.2.5) that the water-wave equations with surface tension can be written under the form:

$$
\left\{
\begin{aligned}
&\frac{\partial \eta}{\partial t} - G(\eta)\psi = 0,\\
&\frac{\partial \psi}{\partial t} + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\psi\cdot\nabla\eta + G(\eta)\psi)^2}{1+|\nabla\eta|^2} + g\eta + \kappa(\eta) = 0.
\end{aligned}
\right.
\tag{7.0.1}
$$

where the unknowns are $\eta = \eta(t,x) \in \mathbb{R}$, $\psi = \psi(t,x) \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \geq 1$, $G(\eta)$ is the Dirichlet-to-Neumann operator (studied in chapters 4 and 6), $g > 0$ is the acceleration of gravity and $\kappa(\eta)$ is the mean curvature defined by

$$
\kappa(\eta) = -\operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right).
$$

7.1 The Cauchy problem

As we said in a previous chapter, in the framework of Sobolev spaces, without any smallness hypothesis on the initial data, the well-posedness of the Cauchy problem has been shown by Beyer-Günther ([80]) in the case with surface tension ($\lambda = 1$). Many extensions of the result of Beyer and Günther have been obtained, by different methods, by Ogawa and Tani [354], Ambrose-Masmoudi [45, 46, 47], Schneider-Wayne [385], Zhang-Zhang [456], Schweizer [387], Iguchi [257, 255, 256], Shatah-Zeng [388, 389, 390], Ming-Zhang [344], Coutand-Shkoller [159], Guo-Tice [237, 238], Rousset-Tzvetkov [378], Christianson-Hur-Staffilani [129], Lannes [302], Germain-Masmoudi-Shatah [218], Alazard-Burq-Zuily [19, 20], Chen-Marzuola-Spirn-Wright [124], de Poyferré and Nguyen [187, 186], Ifrim-Tataru [252], Ionescu-Pusateri [259], Zhu [459], Berti-Delort [74], Rimah-Said [381].

Theorem 7.1.1 (from [19])**.** *Let $d \geq 1$, $\mathbb{D} = \mathbb{R}^d$ or $\mathbb{T}^d$ and $s > 2 + d/2$. For any initial data $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{D}) \times H^s(\mathbb{D})$, there exists a time $T > 0$ such that the Cauchy problem for (7.0.1) with initial data (η_0, ψ_0) has a unique solution*

$$(\eta, \psi) \in C^0\big([0, T]; H^{s+\frac{1}{2}}(\mathbb{D}) \times H^s(\mathbb{D})\big).$$

Remark 7.1.2. *The assumption $s > 2 + d/2$ corresponds to a natural threshold. Indeed, if η and ψ belong to $H^s(\mathbb{R}^d)$, then $\phi \in H^{s+\frac{1}{2}}(\Omega)$ where ϕ denotes the harmonic extension of ψ in the domain Ω occupied by the fluid. This is equivalent to the fact that the velocity $u = \nabla_{x,y}\phi$ belongs to $H^{s-\frac{1}{2}}(\Omega)$. Now $u \in H^{s-\frac{1}{2}}(\Omega)$ implies $u \in Lip(\Omega)$ if and only if $s - \frac{1}{2} - 1 > \frac{d+1}{2}$, that is $s > 2 + \frac{d}{2}$. Therefore, the hypothesis $s > 2 + d/2$ is the minimal assumption (in terms of Sobolev spaces based on L^2) to guarantee that the initial velocity is Lipschitz.*

One could want to solve the Cauchy problem in other spaces, for instance in Hölder spaces. The next proposition shows that this is never not possible for a dispersive equation, without loss of derivatives.

Proposition 7.1.3. *Let $\alpha \in (0, +\infty)$ with $\alpha \neq 1$ and set $S(t) = e^{-it|D_x|^\alpha}$. Consider two real numbers $s, \sigma \in \mathbb{R}$ and assume that there exists $t_0 \neq 0$ such that $S(t_0)$ is continuous from the Zygmund space $C^\sigma_*(\mathbb{R}^d)$ to $C^s_*(\mathbb{R}^d)$. Then $s \leq \sigma - \frac{d\alpha}{2}$.*

Moreover we have also shown that there is continuity with respect to the initial data in the space where we take the initial data.

Theorem 7.1.4 ([19])**.** *Let $d \geq 1$, $\mathbb{D} = \mathbb{R}^d$ or $\mathbb{T}^d$ and $s > 2+d/2$. Consider a solution $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{D}) \times H^s(\mathbb{D}))$ of (7.0.1) and a sequence $(\eta_{n,0}, \psi_{n,0})_{n\in\mathbb{N}^*}$ converging in $H^{s+\frac{1}{2}}(\mathbb{D}) \times H^s(\mathbb{D})$ to $(\eta, \psi)\mid_{t=0}$. Then, for n large enough, the solutions $(\eta_n, \psi_n) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{D}) \times H^s(\mathbb{D}))$ with data $(\eta_{n,0}, \psi_{n,0})$ exist on the time interval $[0, T]$ and satisfy*

$$\lim_{n\to+\infty} \|(\eta_n, \psi_n) - (\eta, \psi)\|_{C^0([0,T];H^{s+\frac{1}{2}}\times H^s)} = 0. \tag{7.1.1}$$

7.2 A priori estimates

We will only prove *a priori* estimates and refer to the original article [19] for the rest of the proof. Namely, we shall prove the following proposition.

Proposition 7.2.1. *Let $d \geq 1$ and $s > 2 + d/2$. Then there exist a non-decreasing function C such that, for all $T \in]0,1]$ and all solution (η, ψ) of (7.0.1) such that*

$$(\eta, \psi) \in C^1([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)),$$

the norm

$$M(T) = \|(\eta, \psi)\|_{L^\infty(0,T; H^{s+\frac{1}{2}} \times H^s)},$$

satisfies the estimate

$$M(T) \leq C(M_0) + TC(M(T)),$$

with $M_0 := \|(\eta_0, \psi_0)\|_{H^{s+\frac{1}{2}} \times H^s}$.

7.2.1 Paralinearization

In this paragraph the time variable plays no role and we skip it.

Define the symbol

$$\lambda = \lambda^{(1)} + \lambda^{(0)}, \tag{7.2.1}$$

where

$$\begin{aligned} \lambda^{(1)} &= \sqrt{(1 + |\nabla\eta|^2)\,|\xi|^2 - (\nabla\eta \cdot \xi)^2}, \\ \lambda^{(0)} &= \frac{1 + |\nabla\eta|^2}{2\lambda^{(1)}} \left\{ \operatorname{div}\left(\alpha^{(1)} \nabla\eta\right) + i\partial_\xi \lambda^{(1)} \cdot \nabla\alpha^{(1)} \right\} \qquad \text{with} \\ \alpha^{(1)} &= \frac{1}{1 + |\nabla\eta|^2} \left(\lambda^{(1)} + i\nabla\eta \cdot \xi\right). \end{aligned} \tag{7.2.2}$$

In our case the function η will be at least C^2, so that all these symbols are well defined.

The following observation contains one of the key dichotomy the cases $d = 1$ and $d = 2$: If $d = 1$ then

$$\lambda^{(1)}(x, \xi) = |\xi|, \quad \lambda^{(0)}(x, \xi) = 0.$$

In particular, $\lambda = |\xi|$. In this case T_λ is equal to $|D|$ up to a smoothing remainder. Also, directly from (7.2.2), one can check the following formula

$$\operatorname{Im} \lambda^{(0)} = -\frac{1}{2}(\partial_\xi \cdot \partial_x)\lambda^{(1)}, \tag{7.2.3}$$

which reflects the fact that the Dirichlet-Neumann operator is a symmetric operator.

Proposition 7.2.2. *Let $d \geq 1$ and $s > 2 + d/2$. Assume that*

$$(\eta, \psi) \in H^{s+\frac{1}{2}}(\mathbb{D}) \times H^s(\mathbb{D}).$$

and that η is such that $\operatorname{dist}(\Sigma, \Gamma) > 0$. *Then*

$$G(\eta)\psi = T_\lambda\big(\psi - T_B\eta\big) - T_V \cdot \nabla\eta + f(\eta, \psi), \tag{7.2.4}$$

where λ is given by (7.2.2) *and* (7.2.1)*,*

$$B := \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1 + |\nabla\eta|^2}, \qquad V := \nabla\psi - B\nabla\eta,$$

and $f(\eta, \psi) \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$. Moreover, we have the estimate

$$\|f(\eta, \psi)\|_{H^{s+\frac{1}{2}}} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}}\right)\|\psi\|_{H^{s-1}},$$

for some non-decreasing function C depending only on $\operatorname{dist}(\Sigma, \Gamma) > 0$.

Lemma 7.2.3. *There holds $\kappa(\eta) = T_\ell\eta + f$, where $\ell = \ell^{(2)} + \ell^{(1)}$ with*

$$\begin{aligned} \ell^{(2)} &= \left(1 + |\nabla\eta|^2\right)^{-\frac{1}{2}}\left(|\xi|^2 - \frac{(\nabla\eta \cdot \xi)^2}{1 + |\nabla\eta|^2}\right), \\ \ell^{(1)} &= -\frac{i}{2}(\partial_x \cdot \partial_\xi)\ell^{(2)}, \end{aligned} \tag{7.2.5}$$

and $f \in H^{2s-2-d/2}(\mathbb{D})$ is such that

$$\|f\|_{H^{2s-2-\frac{d}{2}}} \leq C(\|\eta\|_{H^{s+1/2}}), \tag{7.2.6}$$

for some non-decreasing function C.

Proof. Recall the following paralinearization formula (see Theorem 3.6.3): if $\alpha > d/2$, then

$$\left\|F(u) - T_{F'(u)}u\right\|_{H^{2\alpha-d/2}} \leq C(\|u\|_{H^\alpha})\,\|u\|_{H^\alpha}, \tag{7.2.7}$$

It follows that

$$\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}} = T_M\nabla\eta + \widetilde{f}, \qquad M = \frac{1}{\sqrt{1+|\nabla\eta|^2}}I - \frac{\nabla\eta\otimes\nabla\eta}{(1+|\nabla\eta|^2)^{3/2}},$$

with $\big\|\widetilde{f}\big\|_{H^{2s-1-\frac{d}{2}}} \le C(\|\eta\|_{H^{s+\frac{1}{2}}})$. Since

$$\operatorname{div}(T_M\nabla\eta) = T_{-M\xi\cdot\xi+i\operatorname{div}M\xi}\eta,$$

we obtain the desired result with $\ell^{(2)} = M\xi\cdot\xi$, $\ell^{(1)} = -i\operatorname{div}M\xi$ and $f = \operatorname{div}\tilde{f}$. □

We also need to paralinearize the nonlinear terms that appear in the equation for appear in the equation for ψ.

Lemma 7.2.4. *We have*

$$\frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = T_V\cdot\nabla\psi - T_BT_V\cdot\nabla\eta - T_BG(\eta)\psi + f',$$

where B and V are as defined in Definition 1.2.1 and f' satisfies

$$\|f'\|_{H^{2s-2-\frac{d}{2}}} \le C\left(\|(\eta,\psi)\|_{H^{s+\frac{1}{2}}\times H^s}\right),$$

for some non-decreasing function C.

Proof. Again, we use the paralinearization formula (7.2.7) applied with the function

$$F(a,b,c) = \frac{1}{2}\frac{(a\cdot b+c)^2}{1+|a|^2} \quad (a\in\mathbb{R}^d, b\in\mathbb{R}^d, c\in\mathbb{R}).$$

Notice that

$$\partial_a F = \frac{(a\cdot b+c)}{1+|a|^2}\left(b - \frac{(a\cdot b+c)}{1+|a|^2}a\right),\ \partial_b F = \frac{(a\cdot b+c)}{1+|a|^2}a,\ \partial_c F = \frac{(a\cdot b+c)}{1+|a|^2}.$$

We use these identities for $a = \nabla\eta$, $b = \nabla\psi$ and $c = G(\eta)\psi$. Then

$$\frac{(a\cdot b+c)}{1+|a|^2} = B, \quad b - \frac{(a\cdot b+c)}{1+|a|^2}a = V,$$

where B and V are (as always) defined by Definition 1.2.1. It follows that

$$\frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = \left\{T_{VB}\cdot\nabla\eta + T_{B\nabla\eta}\nabla\psi + T_BG(\eta)\psi\right\} + r,$$

with $r \in H^{2s-2-\frac{d}{2}}(\mathbb{R}^d)$ satisfies the desired estimate. Since $V = \nabla\psi - B\nabla\eta$, this yields

$$\frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta\cdot\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = \{T_V\cdot\nabla\psi - T_{VB}\cdot\nabla\eta - T_B G(\eta)\psi\} + r'$$

with $r' \in H^{2s-2-\frac{d}{2}}(\mathbb{R}^d)$. Since by symbolic calculus

$$T_{BV} - T_B T_V \quad \text{is of order } -\left(s-1-\frac{d}{2}\right),$$

this completes the proof. □

7.2.2 A class of symbols

Definition 7.2.5. *Given $m \in \mathbb{R}$, E^m denotes the class of symbols $a = a(t,x,\xi)$ of the form*

$$a = a^{(m)} + a^{(m-1)}$$

with

$$a^{(m)}(t,x,\xi) = F(\nabla\eta(t,x),\xi),$$
$$a^{(m-1)}(t,x,\xi) = \sum_{|\alpha|=2} G_\alpha(\nabla\eta(t,x),\xi)\partial_x^\alpha\eta(t,x),$$

such that

(i) T_a maps real-valued functions to real-valued functions;

(ii) F is a C^∞ real-valued function of $(\zeta,\xi) \in \mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$, homogeneous of order m in ξ; and such that there exists a continuous function $K = K(\zeta) > 0$ such that

$$F(\zeta,\xi) \geq K(\zeta)\,|\xi|^m,$$

for all $(\zeta,\xi) \in \mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$;

(iii) G_α is a C^∞ complex-valued function of $(\zeta,\xi) \in \mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$, homogeneous of order $m-1$ in ξ.

Remark 7.2.6. *Let us discuss a delicate technical point which comes from the fact that we are interested to symbols that are not very regular. To study the equations* *with surface tension, as we have seen it is necessary to work everywhere at the sub principal level. But the sub principal symbols depend on the second derivatives of second derivatives of η which are just $C^{1/2}$ in x. An important technical remark is that the subprincipal symbols $\lambda^{(0)}, \ell^{(1)}$ as well as the symbols $p^{(-1/2)}$ and $\gamma^{(1/2)}$ below, depend linearly of $\nabla^2\eta$.*

Proposition 7.2.7. *Let $m \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Then there exists a function C such that for all symbol $a \in \Sigma^m$ and all $t \in [0,T]$,*

$$\left\| T_{a(t)} u \right\|_{H^{\mu-m}} \leq C(\|\eta(t)\|_{H^{s-1}}) \|u\|_{H^\mu} .$$

Remark 7.2.8. *This result is obvious for $s > 3 + d/2$ since the L^∞-norm of $a(t,\cdot,\xi)$ is controlled by $\|\eta(t)\|_{H^{s-1}}$ in this case. As alluded to above, this proposition solves the technical difficulty which appears since we only assume $s > 2 + d/2$.*

Proof. By abuse of notations, we omit the dependence in time.

a) Consider a symbol $p = p(x,\xi)$ homogeneous of degree r in ξ such that

$$x \mapsto \partial_\xi^\alpha p(\cdot,\xi) \quad \text{belongs to } H^{s-3}(\mathbb{R}^d) \quad \forall \alpha \in \mathbb{N}^d.$$

Let q be defined by

$$\widehat{q}(\theta,\xi) = \frac{\chi_1(\theta,\xi)\psi_1(\xi)}{|\xi|} \widehat{p}(\theta,\xi)$$

where $\chi_1 = 1$ on $\operatorname{supp}\chi$, $\psi_1 = 1$ on $\operatorname{supp}\psi$ (see (6.2.2)), $\psi_1(\xi) = 0$ for $|\xi| \leq \frac{1}{3}$, $\chi_1(\theta,\xi) = 0$ for $|\theta| \geq |\xi|$ and $\widehat{f}(\theta,\xi) = \int e^{-ix\cdot\theta} f(x,\xi)\,dx$. Then

$$T_q\,|D_x| = T_p, \tag{7.2.8}$$

and

$$\left| \partial_\xi^\alpha \widehat{q}(\theta,\xi) \right| \lesssim \langle\theta\rangle^{-1} \sum_{\beta\leq\alpha} \left| \partial_\xi^\beta \widehat{p}(\theta,\xi) \right|.$$

Therefore we have

$$\left\| \partial_\xi^\alpha q(\cdot,\xi) \right\|_{H^{s-2}} \lesssim \sum_{\beta\leq\alpha} \left\| \partial_\xi^\beta p(\cdot,\xi) \right\|_{H^{s-3}} . \tag{7.2.9}$$

Now, it follows from the above estimate and the embedding $H^{s-2}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ that q is L^∞ in x and hence $q \in \Gamma_0^{r-1} \subset \Gamma_0^r$. Then, according to (3.5.7) applied with $m = r$ (and not $m = r-1$), we have for all $\sigma \in \mathbb{R}$,

$$\left\| T_q v \right\|_{H^{\sigma-r}} \lesssim \sup_{|\alpha|\leq\frac{d}{2}+1} \sup_{|\xi|\geq\frac{1}{2}} |\xi|^{|\alpha|-r} \left\| \partial_\xi^\alpha q(\cdot,\xi) \right\|_{L^\infty} \|v\|_{H^\sigma} .$$

Applying this inequality with $v = |D_x| u$, $\sigma = \mu - 1$ and using again the Sobolev embedding and (7.2.8), (7.2.9), we obtain

$$\left\| T_p u \right\|_{H^{\mu-r-1}} \lesssim \sup_{|\alpha| \leq \frac{d}{2}+1} \sup_{|\xi|=1} \left\| \partial_\xi^\alpha p(\cdot, \xi) \right\|_{H^{s-3}} \|u\|_{H^\mu} . \tag{7.2.10}$$

b) Consider a symbol $a \in \Sigma^m$ of the form

$$a = a^{(m)} + a^{(m-1)} = F(\nabla \eta, \xi) + \sum_{|\alpha|=2} G_\alpha(\nabla \eta, \xi) \partial_x^\alpha \eta. \tag{7.2.11}$$

Up to substracting the symbol of a Fourier multiplier of order m, we can assume without loss of generality that $F(0, \xi) = 0$.

It follows from the previous estimates that

$$\left\| T_{a^{(m)}} u \right\|_{H^{\mu-m}} \lesssim \sup_{|\xi|=1} \| a^{(m)}(\cdot, \xi) \|_{H^{s-2}} \|u\|_{H^\mu} ,$$

$$\left\| T_{a^{(m-1)}} u \right\|_{H^{\mu-m}} \lesssim \sup_{|\xi|=1} \| a^{(m-1)}(\cdot, \xi) \|_{H^{s-3}} \|u\|_{H^\mu} .$$

Now since $s > 2 + d/2$ it follows from the usual nonlinear estimates in Sobolev spaces that

$$\sup_{|\xi|=1} \| a^{(m)}(\cdot, \xi) \|_{H^{s-2}} = \sup_{|\xi|=1} \| F(\nabla \eta, \xi) \|_{H^{s-2}} \leq C \left(\|\eta\|_{H^{s-1}} \right) .$$

On the other hand, by using the product rule (6.4.8) with $(s_0, s_1, s_2) = (s-3, s-2, s-3)$ we obtain

$$\begin{aligned} \| a^{(m-1)}(\cdot, \xi) \|_{H^{s-3}} &\leq \sum_{|\alpha|=2} \left\| G_\alpha(\nabla \eta, \xi) \partial_x^\alpha \eta \right\|_{H^{s-3}} \\ &\lesssim \Big(|G_\alpha(0, \xi)| + \sum_{|\alpha|=2} \| G_\alpha(\nabla \eta, \xi) - G_\alpha(0, \xi) \|_{H^{s-2}} \Big) \left\| \partial_x^\alpha \eta \right\|_{H^{s-3}} , \end{aligned}$$

for all $|\xi| \leq 1$. We conclude that $\| a^{(m-1)}(\cdot, \xi) \|_{H^{s-3}} \leq C(\|\eta\|_{H^{s-1}})$, which completes the proof. □

Similarly we have the following result about elliptic regularity where one controls the various constants by the H^{s-1}-norm of η only.

Proposition 7.2.9. *Let $m \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Then there exists a function C such that for all $a \in E^m$ and all $t \in [0,T]$, we have*

$$\|u\|_{H^{\mu+m}} \leq C(\|\eta(t)\|_{H^{s-1}}) \left\{ \left\|T_{a(t)}u\right\|_{H^\mu} + \|u\|_{L^2} \right\}.$$

Remark 7.2.10. *It is classical that, for all elliptic symbol $a \in \Gamma^m_\rho(\mathbb{R}^d)$ with $\rho > 0$, there holds*

$$\|f\|_{H^m} \leq K \{\|T_a f\|_{L^2} + \|f\|_{L^2}\},$$

where K depends only on $M^m_\rho(a)$. Hence, if we use the natural estimate

$$M^{m-1}_\rho(a^{(m-1)}(t)) \leq C(\|\eta(t)\|_{W^{2+\rho}}) \leq C(\|\eta(t)\|_{H^s})$$

for $\rho > 0$ small enough, then we obtain an estimate which is worse than the one just stated for $2 + d/2 < s < 3 + d/2$.

Proof. Again, by abuse of notations, we omit the dependence in time.

Introduce $b = 1/a^{(m)}$ and consider ε such that

$$0 < \varepsilon < \min\{s - 2 - d/2, 1\}.$$

By applying (3.5.8) with $\rho = \varepsilon$ we find that $T_b T_{a^{(m)}} = I + r$ where r is of order $-\varepsilon$ and satisfies

$$\|ru\|_{H^{\mu+\varepsilon}} \leq C(\|\nabla\eta\|_{W^{\varepsilon,\infty}}) \|u\|_{H^\mu} \leq C(\|\eta\|_{H^{s-1}}) \|u\|_{H^\mu}.$$

Then

$$u = T_b T_a u - ru - T_b T_{a^{(m-1)}}.$$

Denoting by $R = -r - T_b T_{a^{(m-1)}}$, we have

$$(I - R)u = T_b T_a u.$$

We claim that there exists a function C such that

$$\left\|T_{a^{(m-1)}}u\right\|_{H^{\mu-m+\varepsilon}} \leq C(\|\eta\|_{H^{s-1}}) \|u\|_{H^\mu}.$$

To see this, notice that the previous proof applies with the decomposition $T_p = T_q |D_x|^{1-\varepsilon}$ where

$$\widehat{q}(\theta,\xi) = \frac{\chi_1(\theta,\xi)\psi_1(\xi)}{|\xi|^{1-\varepsilon}}\widehat{p}(\theta,\xi).$$

Once this claim is granted, since T_b is of order $-m$, we find that R satisfies

$$\|Ru\|_{H^{\mu+\varepsilon}} \leq C(\|\eta\|_{H^{s-1}}) \|u\|_{H^{\mu}} .$$

Writing

$$(I + R + \cdots + R^N)(I - R)u = (I + R + \cdots + R^N)T_b T_a u$$

we get

$$u = (I + R + \cdots + R^N)T_b T_a u + R^{N+1}u.$$

The first term in the right hand side is estimated by means of the obvious inequality

$$\begin{aligned} &\left\|(I + R + \cdots + R^N)T_b\right\|_{H^{\mu}\to H^{\mu+m}} \\ &\qquad \leq \left\|(I + R + \cdots + R^N)\right\|_{H^{\mu+m}\to H^{\mu+m}} \|T_b\|_{H^{\mu}\to H^{\mu+m}} , \end{aligned}$$

so that

$$\left\|(I + R + \cdots + R^N)T_b T_a u\right\|_{H^{\mu+m}} \leq C(\|\eta\|_{H^{s-1}}) \|T_a u\|_{H^{\mu}} .$$

Choosing N so large that $(N + 1)\varepsilon > \mu + m$, we obtain that

$$\left\|R^{N+1}\right\|_{H^{\mu}\to H^{\mu+m}} \lesssim \|R\|_{H^{\mu+m-\varepsilon}\to H^{\mu+m}} \cdots \|R\|_{H^{\mu}\to H^{\mu+\varepsilon}} \leq C(\|\eta\|_{H^{s-1}}),$$

which yields the desired estimate for the second term. □

7.2.3 Symmetrization of equations

Zakharov showed that the surface wave system can be written as

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta},$$

where H is the total energy. Zakharov and his collaborators have extensively studied the properties related to the Hamiltonian nature of Hamiltonian nature of the equations (see [454] for many references). Noting by H_0 the Hamiltonian associated to the linearized system around the equilibrium $(\eta, \psi) = (0, 0)$, we have

$$H_0 = \frac{1}{2}\int \left[|\xi|\, |\widehat{\psi}|^2 + (g + |\xi|^2)|\widehat{\eta}|^2\right] \mathrm{d}\xi.$$

An important observation is that the canonical transformation $(\eta, \psi) \mapsto a$ defined by

$$\widehat{a} = \frac{1}{\sqrt{2}}\left\{\left(\frac{g + \lambda\,|\xi|^2}{|\xi|}\right)^{1/4}\widehat{\eta} - i\left(\frac{|\xi|}{g + \lambda\,|\xi|^2}\right)^{1/4}\widehat{\psi}\right\},$$

diagonalizes H_0 and thus reduces the study to that of the complex equation (cf [454])

$$\frac{\partial \widehat{a}}{\partial t} + i\frac{\partial H}{\partial \bar{\widehat{a}}} = 0.$$

We will explain in this part how to obtain such a diagonalization for the non linear problem. This is the main technical step of the paper [19].

For this we will start by paralinearizing the system (7.0.1).

Notation 7.2.11. *In the sequence we will consider time as a parameter. Given a function $u = u(t,x)$, it is classical to note $u(t)$ the function $x \mapsto u(t,x)$. Given a symbol $a(t,x,\xi)$ which depends on time, we will note T_a the operator defined by*

$$(T_a u)(t) = T_{a(t)} u(t).$$

We have seen previously that the unknown

$$U = \psi - T_B \eta$$

plays a key role in the paralinearization of the operator $G(\eta)$. The following result states that this unknown is also involved in the paralinearization of the equation on ψ.

Proposition 7.2.12. *Under the previous regularity assumptions, we have*

$$\begin{cases} \partial_t \eta + T_V \cdot \nabla \eta - T_\lambda U = f_1, \\ \partial_t U + T_V \cdot \nabla U + T_\ell \eta = f_2, \end{cases}$$

where the remainders are estimated by

$$\|(f_1, f_2)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s)} \leq C\left(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}} \times H^s)}\right).$$

Consider a solution (η, ψ) of (7.0.1) which has the following regularity

$$\eta \in C^0\big([0,T]; H^{s+\frac{1}{2}}(\mathbb{D})\big), \quad \psi \in C^0\big([0,T]; H^s(\mathbb{D})\big),$$

for some index $s > 2 + d/2$.

Our main result on the symmetrization of equations is as follows.

Proposition 7.2.13. *There exists a symbol $p \in E^{1/2}$, a symbol $\gamma \in E^{3/2}$ and a symbol $q \in E^0$ such that the unknown (complex-valued) $\Phi \in C^0([0,T]; H^s(\mathbb{D}))$ defined by*

$$\Phi = T_p\,\eta + iT_q\,U,$$

(recall that $U = \psi - T_B\eta$) satisfies

$$\partial_t\Phi + T_V \cdot \nabla\Phi + iT_\gamma\Phi = F,$$

where the remainder F is such that

$$\|F\|_{L^\infty(0,T;H^s)} \leq C\left(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}\right).$$

Proof. Let ℓ denote the symbol of the operator $\kappa(\eta)$. We can find p, γ, q by a systematic method by using the relations

$$\ell^{(1)} = -\frac{i}{2}(\partial_x \cdot \partial_\xi)\ell^{(2)}, \quad \operatorname{Im}\lambda^{(0)} = -\frac{1}{2}(\partial_x \cdot \partial_\xi)\lambda^{(1)},$$

and

$$\ell^{(2)} = \left(c\lambda^{(1)}\right)^2 \quad \text{with} \quad c = \left(1 + |\nabla\eta|^2\right)^{-\frac{3}{4}}.$$

The first two identities reflect the fact that the operators are self-adjoint. The third identity has a geometric meaning.

Notation 7.2.14. *Let $m \in \mathbb{R}$ and consider two families of operators order m,*

$$\{A(t)\ :\ t \in [0,T]\}, \quad \{B(t)\ :\ t \in [0,T]\}.$$

We shall say that $A \sim B$ if $A - B$ is of order $m - 3/2$ and satisfies the following estimate: for all $\mu \in \mathbb{R}$, there exists a continuous function C such that for all $t \in [0,T]$,

$$\|A(t) - B(t)\|_{H^\mu \to H^{\mu - m + \frac{3}{2}}} \leq C\left(\|\eta(t)\|_{H^{s+\frac{1}{2}}}\right).$$

Namely, we shall prove that there exist three symbols p, q, γ such that

$$T_pT_\lambda \sim T_\gamma T_q, \qquad T_qT_\ell \sim T_\gamma T_p, \qquad T_\gamma \sim (T_\gamma)^*. \tag{7.2.12}$$

We want to explain how we find p, q, γ by a systematic method. We first observe that if (7.2.12) holds true then γ is of order $3/2$. To be definite, we chose q of order 0, and then necessarily p is of order $1/2$. Therefore we seek p, q, γ under the form

$$p = p^{(1/2)} + p^{(-1/2)}, \quad q = q^{(0)} + q^{(-1)}, \quad \gamma = \gamma^{(3/2)} + \gamma^{(1/2)}, \tag{7.2.13}$$

where $a^{(m)}$ is a symbol homogeneous in ξ of order $m \in \mathbb{R}$.

Let us list some necessary constraints on these symbols. Firstly, we seek real elliptic symbols such that,

$$p^{(1/2)} \geq K\,|\xi|^{1/2}, \quad q^{(0)} \geq K, \quad \gamma^{(3/2)} \geq K\,|\xi|^{3/2},$$

for some positive constant K. Secondly, in order for T_p, T_q, T_γ to map real valued functions to real valued functions, we must have

$$\overline{p(t,x,\xi)} = p(t,x,-\xi), \quad \overline{q(t,x,\xi)} = q(t,x,-\xi), \quad \overline{\gamma(t,x,\xi)} = \gamma(t,x,-\xi).$$

In order for T_γ to satisfy the last identity in (7.2.12), $\gamma^{(1/2)}$ must satisfy

$$\operatorname{Im}\gamma^{(1/2)} = -\frac{1}{2}(\partial_\xi \cdot \partial_x)\gamma^{(3/2)}. \tag{7.2.14}$$

Our strategy is then to seek q and γ such that

$$T_q T_\ell T_\lambda \sim T_\gamma T_\gamma T_q. \tag{7.2.15}$$

The idea is that if this identity is satisfied then the first two equations in (7.2.12) are compatible; this means that if any of these two equations is satisfied, then the second one is automatically satisfied. Therefore, once q and γ are so chosen that (7.2.15) is satisfied, then one can define p by solving either one of the first two equations. The latter task being immediate.

Introduce the notations

$$\begin{aligned}
\ell\sharp\lambda &= \ell^{(2)}\lambda^{(1)} + \ell^{(1)}\lambda^{(1)} + \ell^{(2)}\lambda^{(0)} + \frac{1}{i}\partial_\xi \ell^{(2)} \cdot \partial_x \lambda^{(1)},\\
\gamma\sharp\gamma &= \left(\gamma^{(3/2)}\right)^2 + 2\gamma^{(1/2)}\gamma^{(3/2)} + \frac{1}{i}\partial_\xi\gamma^{(3/2)} \cdot \partial_x\gamma^{(3/2)}.
\end{aligned}$$

By symbolic calculus, it is enough to find q and γ such that

$$\begin{aligned}
q^{(0)}(\ell\sharp\lambda) &+ q^{(-1)}\ell^{(2)}\lambda^{(1)} + \frac{1}{i}\partial_\xi q^{(0)} \cdot \partial_x(\ell^{(2)}\lambda^{(1)})\\
&= (\gamma\sharp\gamma)q^{(0)} + \left(\gamma^{(3/2)}\right)^2 q^{(-1)} + \frac{1}{i}\partial_\xi(\gamma^{(3/2)}\gamma^{(3/2)}) \cdot \partial_x q^{(0)}.
\end{aligned} \tag{7.2.16}$$

We set

$$\gamma^{(3/2)} = \sqrt{\ell^{(2)}\lambda^{(1)}},$$

so that the leading symbols of both sides of (7.2.16) are equal. Then $\operatorname{Im}\gamma^{(1/2)}$ has to be fixed by means of (7.2.14). We set

$$\operatorname{Im}\gamma^{(1/2)} = -\frac{1}{2}(\partial_x \cdot \partial_\xi)\gamma^{(3/2)}.$$

With these choices of $\gamma^{(3/2)}$ and $\operatorname{Im}\gamma^{(1/2)}$, (7.2.16) is equivalent to the following equation (where the unknowns are $q^{(0)}, q^{(-1)}$ and $\operatorname{Re}\gamma^{(1/2)}$):

$$\begin{aligned} q^{(0)}(\ell\sharp\lambda - \gamma\sharp\gamma) &= \frac{1}{i}\partial_\xi(\ell^{(2)}\lambda^{(1)}) \cdot \partial_x q^{(0)} - \frac{1}{i}\partial_\xi q^{(0)} \cdot \partial_x(\ell^{(2)}\lambda^{(1)}) \\ &= \frac{1}{i}\{\ell^{(2)}\lambda^{(1)}, q^{(0)}\} \end{aligned} \tag{7.2.17}$$

where

$$\begin{aligned} &\ell\sharp\lambda - \gamma\sharp\gamma := \tau \\ &= \frac{1}{i}\partial_\xi\ell^{(2)} \cdot \partial_x\lambda^{(1)} + \ell^{(1)}\lambda^{(1)} + \ell^{(2)}\lambda^{(0)} - 2\gamma^{(1/2)}\gamma^{(3/2)} + i\partial_\xi\gamma^{(3/2)} \cdot \partial_x\gamma^{(3/2)}. \end{aligned} \tag{7.2.18}$$

Since $q^{(-1)}$ does not appear in this equation, one can freely set $q^{(-1)} = 0$. Since $\ell^{(2)}, \lambda^{(1)}$ are real-valued symbols, we see easily that (7.2.17) will be satisfied (with $q^{(0)}$ real) as soon as

$$\operatorname{Re}\tau = 0, \qquad q^{(0)}\operatorname{Im}\tau = -\left\{\ell^{(2)}\lambda^{(1)}, q^{(0)}\right\}. \tag{7.2.19}$$

The first condition is satisfied if $\operatorname{Re}\gamma^{(1/2)}$ solves the equation

$$\ell^{(2)}\operatorname{Re}\lambda^{(0)} = 2\gamma^{(3/2)}\operatorname{Re}\gamma^{(1/2)},$$

that is

$$\operatorname{Re}\gamma^{(1/2)} = \frac{\ell^{(2)}\operatorname{Re}\lambda^{(0)}}{2\gamma^{(3/2)}} = \sqrt{\frac{\ell^{(2)}}{\lambda^{(1)}}}\frac{\operatorname{Re}\lambda^{(0)}}{2}.$$

It remains to solve the second equation in (7.2.19). Let us first recall that

$$\ell^{(1)} = -\frac{i}{2}(\partial_x \cdot \partial_\xi)\ell^{(2)},\ \operatorname{Im}\lambda^{(0)} = -\frac{1}{2}(\partial_x \cdot \partial_\xi)\lambda^{(1)},\ \operatorname{Im}\gamma^{(1/2)} = -\frac{1}{2}(\partial_x \cdot \partial_\xi)\gamma^{(3/2)},$$

and consequently

$$\begin{aligned} \operatorname{Im}\tau = -\partial_\xi\ell^{(2)} \cdot \partial_x\lambda^{(1)} - \frac{1}{2}\lambda^{(1)}(\partial_\xi \cdot \partial_x)\ell^{(2)} - \frac{1}{2}\ell^{(2)}(\partial_\xi \cdot \partial_x)\lambda^{(1)} \\ + \gamma^{(3/2)}(\partial_\xi \cdot \partial_x)\gamma^{(3/2)} + \partial_\xi\gamma^{(3/2)} \cdot \partial_x\gamma^{(3/2)}. \end{aligned}$$

Writing

$$\gamma^{(3/2)}(\partial_\xi \cdot \partial_x)\gamma^{(3/2)} + \partial_\xi \gamma^{(3/2)} \cdot \partial_x \gamma^{(3/2)} = \frac{1}{2}\partial_x \cdot \partial_\xi \left(\gamma^{(3/2)}\right)^2 = \frac{1}{2}\partial_x \cdot \partial_\xi (\ell^{(2)}\lambda^{(1)}),$$

we thus obtain

$$\operatorname{Im}\tau = \frac{1}{2}\partial_\xi \lambda^{(1)} \cdot \partial_x \ell^{(2)} - \frac{1}{2}\partial_\xi \ell^{(2)} \cdot \partial_x \lambda^{(1)},$$

and hence the second equation in (7.2.19) simplifies to

$$\frac{1}{2}\left\{\lambda^{(1)}, \ell^{(2)}\right\} q^{(0)} + \left\{\ell^{(2)}\lambda^{(1)}, q^{(0)}\right\} = 0. \tag{7.2.20}$$

The key observation is the following relation between $\ell^{(2)}$ and $\lambda^{(1)}$:

$$\ell^{(2)} = \left(c\lambda^{(1)}\right)^2 \quad \text{with} \quad c = \left(1 + |\nabla\eta|^2\right)^{-\frac{3}{4}}.$$

Consequently (7.2.20) reduces to

$$-\frac{1}{2}q^{(0)}(\lambda^{(1)})^2 \partial_x c^2 \cdot \partial_\xi \lambda^{(1)} + 3c^2(\lambda^{(1)})^2 \partial_\xi \lambda^{(1)} \cdot \partial_x q^{(0)} - \partial_\xi q^{(0)} \cdot \partial_x \left(c^2(\lambda^{(1)})^3\right) = 0.$$

Seeking a solution $q^{(0)}$ which does not depend on ξ, we are led to solve

$$\frac{\partial_\xi \lambda^{(1)} \cdot \partial_x q^{(0)}}{q^{(0)}} = -\frac{1}{3}\frac{\partial_\xi \lambda^{(1)} \cdot \partial_x c}{c}.$$

We find the following explicit solution:

$$q^{(0)} = c^{-\frac{1}{3}} = \left(1 + |\nabla\eta|^2\right)^{\frac{1}{4}r}.$$

Then, we define p by solving the equation

$$T_q T_\ell \sim T_\gamma T_p.$$

By symbolic calculus, this yields

$$q\ell^{(2)} + q\ell^{(1)} = \gamma^{(3/2)}p^{(1/2)} + \gamma^{(1/2)}p^{(1/2)} + \gamma^{(3/2)}p^{(-1/2)} + \frac{1}{i}\partial_\xi \gamma^{(3/2)} \cdot \partial_x p^{(1/2)}.$$

Therefore, by identifying terms with the same homogeneity in ξ, we successively find that

$$p^{(1/2)} = \frac{q^{(0)}\ell^{(2)}}{\gamma^{(3/2)}} = q^{(0)}\sqrt{\frac{\ell^{(2)}}{\lambda^{(1)}}} = \left(1 + |\nabla\eta|^2\right)^{-\frac{5}{4}}\sqrt{\lambda^{(1)}},$$

and

$$p^{(-1/2)} = \frac{1}{\gamma^{(3/2)}} \left\{ q^{(0)} \ell^{(1)} - \gamma^{(1/2)} p^{(1/2)} + i\partial_\xi \gamma^{(3/2)} \cdot \partial_x p^{(1/2)} \right\}. \tag{7.2.21}$$

Note that the precise value of $p^{(-1/2)}$ is meaningless since we have freely imposed $q^{(-1)} = 0$. □

7.2.4 *A priori* estimates

In this paragraph, we complete the proof of Proposition 7.2.1. To do so, we use Proposition 7.2.13 and work with the function $\Phi = T_p \eta + iT_q U$, solution to an equation of the form

$$\partial_t \Phi + T_V \cdot \nabla \Phi + iT_\gamma \Phi = F. \tag{7.2.22}$$

To obtain estimates in Sobolev, we shall commute the previous equation with an elliptic operator of order s, then use an L^2-energy estimate and eventually derive estimates for the original unknowns η and ψ.

We will also make extensive use of symbolic calculus. Let us recall that if T_a and T_b are two paradifferential operators with symbols $a \in \Gamma^m_\rho$ and $b \in \Gamma^{m'}_\rho$, then

$$i) \quad [T_a, T_b] \quad \text{is of order} \quad m + m' - \min\{\rho, 1\}, \tag{7.2.23}$$

$$ii) \quad [T_a, T_b] - T_{\frac{1}{i}\{a,b\}} \quad \text{is of order} \quad m + m' - \min\{\rho, 2\}, \tag{7.2.24}$$

where the Poisson bracket notation is defined by

$$\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = \sum_{j=1}^{d} (\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b).$$

A first thing to do is to find an operator of order s that commutes well to the equation. We look for an elliptic P operator of order s such that the commutator $[T_\gamma, P]$ is of order s. One cannot simply commute the operator $(I - \Delta)^{s/2}$ to the equation because, in general, the commutator $[T_\gamma, (I - \Delta)^{s/2}]$ is of order $3/2 + s - 1 = s + 1/2$. One possibility is to commute powers of the operator T_γ. Then the commutator vanishes. However this requires to consider indices s which are multiples of $3/2$. Note that this idea corresponds to commuting time derivatives with the equations (which is also a classical strategy [80, 378, 388]). The paradifferential calculus allows to solve this problem in a systematic (and optimal) way. For this we introduce the operator

$$T_\beta \quad \text{with } \beta = \left(\gamma^{(3/2)}\right)^{\frac{2s}{3}},$$

where we denote $\gamma^{(3/2)}$ the main symbol of γ, which is given by $\gamma^{(3/2)} = \sqrt{\ell^{(2)}\lambda^{(1)}}$. It is an operator of order s and such that the Poisson bracket

$$\left\{\beta, \gamma^{(3/2)}\right\} = 0.$$

Thus under the only assumption that $\nabla\eta$ is of regularity $C^{3/2}$ in x, it follows from (7.2.24) that the commutator $[T_{\gamma^{(3/2)}}, T_\beta]$ is of order $3/2 + s - 3/2 = s$.

On the other hand, directly from (7.2.23) applied with $\rho = 1/2$, we verify that $[T_{\gamma^{(3/2)}}, T_\beta]$ is of order $1/2 + s - 1/2 = s$. Therefore, the commutator $\left[T_\beta, T_\gamma\right]$ is of order s.

Also, by using (7.2.23) with $\rho = 1$, we see that the commutator $\left[T_\beta, T_V \cdot \nabla\right]$ is clearly of order s. It remains to study the commutator $[T_\beta, T_{\partial_t}] = -T_{\partial_t \beta}$. For this, observe that the symbol β is of the form $\beta = B(\nabla\eta, \xi)$. Then the most direct estimate shows that the $L^\infty_x(\mathbb{R}^d)$-norm of $\partial_t\beta$ is estimated by the $L^\infty_x(\mathbb{R}^d)$-norm of $(\nabla\eta, \partial_t\nabla\eta)$ and hence by $C(M(T))$ in view of $\partial_t\eta = G(\eta)\psi$ and the Sobolev embedding $H^{s-1}(\mathbb{R}^d) \subset W^{1,\infty}(\mathbb{R}^d)$. We thus end up with the following estimates

$$\left\|\left[T_\beta, T_\gamma\right]\right\|_{H^s\to L^2} + \left\|\left[T_\beta, \partial_t\right]\right\|_{H^s\to L^2} + \left\|\left[T_\beta, T_V \cdot \nabla\right]\right\|_{H^s\to L^2} \le C(M(T)),$$

for some non-decreasing function C.

Therefore, by commuting the equation (7.2.22) with T_β, we find that the function $v := T_\beta\Phi$ satisfies

$$(\partial_t + T_V \cdot \nabla)\, v + iT_\gamma v = F', \tag{7.2.25}$$

with

$$\|F'\|_{L^\infty(0,T;L^2)} \le C(M(T)),$$

for some non-decreasing function C.

d) Since by assumption $(\eta, \psi) \in C^1(0, T); H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$, we have

$$v \in C^1([0, T]; L^2(\mathbb{R}^d)),$$

and hence we can write
$$\frac{d}{dt}\langle v,v\rangle = 2\,\mathrm{Re}\,\langle \partial_t v, v\rangle\,,$$
where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}^d;\mathbb{C})$. Therefore, (7.2.25) implies that
$$\frac{d}{dt}\langle v,v\rangle = 2\,\mathrm{Re}\left\langle -T_V\cdot\nabla v - iT_\gamma v + F', v\right\rangle$$
and hence
$$\frac{d}{dt}\langle v,v\rangle = \langle \mathscr{R} v, v\rangle + 2\,\mathrm{Re}\,\langle F', v\rangle\,,$$
where
$$\mathscr{R} := -\{(T_V\cdot\nabla)^* + T_V\cdot\nabla\} - \left\{(iT_\gamma)^* + (iT_\gamma)\right\}.$$
Since $(T_\gamma)^*\sim T_\gamma$, we easily verify that
$$\sup_{t\in[0,T]} \|\mathscr{R}(t)\|_{L^2\times L^2\to L^2\times L^2} \le C(M(T)).$$
Therefore, integrating in time we conclude that for all $t\in[0,T]$,
$$\|v(t)\|_{L^2}^2 - \|v(0)\|_{L^2}^2 \le C(M(T))\int_0^T \left(\|v\|_{L^2}^2 + \|F'\|_{L^2}^2\right) \mathrm{d}t',$$
which immediately implies that
$$\|v\|_{L^\infty(0,T;L^2)} \le C(M_0) + TC(M(T)).$$
By definition of v, this yields

(7.2.26) $$\left\|T_\beta T_p\eta\right\|_{L^\infty(0,T:L^2)} + \left\|T_\beta T_q U\right\|_{L^\infty(0,T;L^2)} \le C(M_0) + TC(M(T)).$$

First of all, we use Proposition 7.2.9 to obtain

(7.2.27) $$\|\eta\|_{L^\infty(0,T;H^{s+\frac12})} \le K\left\{\left\|T_\beta T_p\eta\right\|_{L^\infty(0,T;L^2)} + \|\eta\|_{L^\infty(0,T;H^{\frac12})}\right\},$$

(7.2.28) $$\|\psi\|_{L^\infty(0,T;H^s)} \le K\left\{\left\|T_\beta T_q\psi\right\|_{L^\infty(0,T;L^2)} + \|\psi\|_{L^\infty(0,T;L^2)}\right\},$$

where K depends only on $\|\eta\|_{L^\infty(0,T;H^{s-1})}$.

Let us prove that the constant K satisfies an inequality of the form

(7.2.29) $$K \le C(M_0) + TC(M(T)).$$

To see this, notice that one can assume without loss of generality that

$$K \le F(\|\eta\|^2_{L^\infty(0,T;H^{s-1})})$$

for some non-decreasing function $F \in C^1(\mathbb{R})$. Set $C(t) = F(\|\eta(t)\|^2_{H^{s-1}})$. We then obtain the desired bound (7.2.29) from

$$K \le C(0) + \int_0^T |C'(t)| \,\mathrm{d}t \le F(M_0) + \int_0^T 2F'(\|\eta\|^2_{H^{s-1}}) \,\|\partial_t\eta\|_{H^{s-1}} \,\|\eta\|_{H^{s-1}} \,\mathrm{d}t,$$

and using the equation $\partial_t\eta = G(\eta)\psi$ to estimate $\|\eta\|_{H^{s-1}}$.

Consequently, (7.2.26) and (7.2.27) imply that we have

$$\|\eta\|_{L^\infty(0,T;H^{s+\frac{1}{2}})} \le C(M_0) + TC(M(T)).$$

It remains to prove an estimate for ψ. To do this, we begin by noticing that, since $\psi = U + T_B\eta$, we have

$$\begin{aligned}&\left\|T_\beta T_q \psi\right\|_{L^\infty(0,T;L^2)}\\ &\qquad \le \left\|T_\beta T_q U\right\|_{L^\infty(0,T;L^2)} + \left\|T_\beta T_q T_B\right\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\to L^2)} \left\|\eta\right\|_{L^\infty(0,T;H^{s+\frac{1}{2}})}.\end{aligned}$$

Now we have

$$\begin{aligned}&\left\|T_\beta T_q T_B\right\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\to L^2)}\\ &\qquad \le \sup_{t\in[0,T]}\sup_{|\xi|=1} \|\beta(t,\cdot,\xi)\|_{L^\infty_x} \|q\|_{L^\infty(0,T;L^\infty)} \|B\|_{L^\infty(0,T;H^{\frac{d-1}{2}})}\end{aligned}$$

and hence

$$\|\psi\|_{H^s} \le K'\left\{\left\|T_q U\right\|_{H^s} + \|\psi\|_{L^2} + \|\eta\|_{H^{s+\frac{1}{2}}}\right\}, \tag{7.2.30}$$

where K' depends only on $\|(\eta,\psi)\|_{L^\infty(0,T;H^{s-1}\times H^{s-3/2})}$. Notice that $\|\psi\|_{L^2}$ is estimated from the equation for ψ. Then by using the inequality (7.2.26) and the previous estimate for η, and the fact that K' satisfies the same estimate as K does, we conclude that

$$\|\psi\|_{L^\infty(0,T;H^s)} \le C(M_0) + TC(M(T)).$$

We end up with the wanted inequality $M(T) \le C(M_0) + TC(M(T))$. This completes the proof of Proposition 7.2.1.

7.3 The smoothing effect

Theorem 7.3.1. *Assume that $d = 1$ and let $s > 5/2$ and $T > 0$. Consider a solution (η, ψ) of (7.0.1) on the time interval $[0, T]$. If*

$$(\eta, \psi) \in C^0\big([0, T]; H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})\big),$$

then for any $\delta > 0$

$$\langle x\rangle^{-\frac{1}{2}-\delta}(\eta, \psi) \in L^2\big(0, T; H^{s+\frac{3}{4}}(\mathbb{R}) \times H^{s+\frac{1}{4}}(\mathbb{R})\big).$$

7.3.1 Reduction to an L^2 estimate

Let Φ be as defined in Proposition 7.2.13, such that

$$\partial_t \Phi + T_V \partial_x \Phi + iT_\gamma \Phi = F, \tag{7.3.1}$$

with $F \in L^\infty(0, T; H^s(\mathbb{R}^d))$. Recall that, if $d = 1$ then

$$\lambda^{(1)} = |\xi|, \quad \lambda^{(0)} = 0, \quad \ell^{(2)} = c^2 |\xi|^2,$$

with

$$c = (1 + |\partial_x \eta|^2)^{-\frac{3}{4}}.$$

Therefore, directly from the definition of γ wee see that if $d = 1$ then γ simplifies to

$$\gamma = c |\xi|^{\frac{3}{2}} - \frac{3i}{4} \xi |\xi|^{-\frac{1}{2}} \partial_x c,$$

and hence modulo an error term of order 0, T_γ is given by $|D_x|^{\frac{3}{4}} T_c |D_x|^{\frac{3}{4}}$.

In this paragraph we shall prove that one can deduce Theorem 7.3.1 from the following proposition.

Proposition 7.3.2. *Assume that $\varphi \in C^0([0, T]; L^2(\mathbb{R}))$ satisfies*

$$\partial_t \varphi + T_V \partial_x \varphi + iT_\gamma \varphi = f,$$

with $f \in L^1(0, T; L^2(\mathbb{R}))$. Then, for all $\delta > 0$,

$$\langle x\rangle^{-\frac{1}{2}-\delta} \varphi \in L^2(0, T; H^{\frac{1}{4}}(\mathbb{R})).$$

We postpone the proof of Proposition 7.3.2 to the next paragraph.

The fact that one can deduce Theorem 7.3.1 from the above proposition, though elementary, contains the idea that one simplify hardly all the nonlinear analysis by means of paradifferential calculus.

Proof of Theorem 7.3.1 given Proposition 7.3.2. Following the proof of Proposition 7.2.1, we consider the symbol

$$\beta := c^{\frac{2}{3}s} \left|\xi\right|^{s}. \tag{7.3.2}$$

Then one can easily verify that the commutators $[T_\beta, \partial_t]$, $\left[T_\beta, T_\gamma\right]$ and $[T_\beta, T_V\partial_x]$ are of order s. Consequently, (7.3.1) implies that

$$\left(\partial_t + T_V\partial_x + iT_\gamma\right) T_\beta \Phi \in L^\infty(0,T;L^2(\mathbb{R})),$$

and hence,

$$\left(\partial_t + T_V\partial_x + iT_\gamma\right) T_\beta \Phi \in L^1(0,T;L^2(\mathbb{R})).$$

Therefore it follows from Proposition 7.3.2 that

$$\langle x\rangle^{-\frac{1}{2}-\delta} T_\beta \Phi \in L^2(0,T;H^{\frac{1}{4}}(\mathbb{R})).$$

Since, by definition, $\Phi = T_p\eta + iT_qU$ where $T_p\eta$ and T_qU are real valued functions, this yields

$$\langle x\rangle^{-\frac{1}{2}-\delta} T_\beta T_p\eta \in L^2(0,T;H^{\frac{1}{4}}(\mathbb{R})), \quad \langle x\rangle^{-\frac{1}{2}-\delta} T_\beta T_q U \in L^2(0,T;H^{\frac{1}{4}}(\mathbb{R})),$$

and hence, since $\psi = U + T_{\mathfrak{B}}\eta$,

$$\langle x\rangle^{-\frac{1}{2}-\delta} T_\beta T_p\eta \in L^2(0,T;H^{\frac{1}{4}}(\mathbb{R})), \quad \langle x\rangle^{-\frac{1}{2}-\delta} T_\beta T_q \psi \in L^2(0,T;H^{\frac{1}{4}}(\mathbb{R})), \tag{7.3.3}$$

Since $\langle x\rangle^{-\frac{1}{2}-\delta} \in \Gamma^0_\rho(\mathbb{R}^d)$ for any $\rho \geq 0$, Theorem 3.5.13 implies that the commutators

$$\left[\langle x\rangle^{-\frac{1}{2}-\delta}, T_\beta T_p\right], \quad \left[\langle x\rangle^{-\frac{1}{2}-\delta}, T_\beta T_q\right]$$

are of order $s - 1/2$ and $s - 1$, respectively. Therefore, directly from (7.3.3) and the assumption

$$\eta \in C^0([0,T];H^{s+\frac{1}{2}}(\mathbb{R})), \quad \psi \in C^0([0,T];H^s(\mathbb{R})),$$

we obtain

$$T_\beta T_p \langle x\rangle^{-\frac{1}{2}-\delta}\eta \in L^2(0,T;H^{\frac{1}{4}}(\mathbb{R})), \quad T_\beta T_q \langle x\rangle^{-\frac{1}{2}-\delta}\psi \in L^2(0,T;H^{\frac{1}{4}}(\mathbb{R})).$$

Now since β, p, q are elliptic symbols of order s, $1/2$, 0, respectively, we conclude (cf the proof of Proposition 7.2.9)

$$\langle x\rangle^{-\frac{1}{2}-\delta}\eta \in L^2(0,T;H^{s+\frac{3}{4}}(\mathbb{R})), \quad \langle x\rangle^{-\frac{1}{2}-\delta}\psi \in L^2(0,T;H^{s+\frac{1}{4}}(\mathbb{R})).$$

This proves Theorem 7.3.1. □

7.3.2 Proof of Proposition 7.3.2

To complete the proof of Theorem 7.3.1, it remains to prove Proposition 7.3.2. To do so, following Doi's approach, we begin with the following lemma which follows from the observation that $\partial_\xi(\frac{\xi}{|\xi|}) = 0$ for $\xi \in \mathbb{R} \setminus \{0\}$ (and the fact that c is uniformly bounded from below).

Lemma 7.3.3. *Let $\delta > 0$ and consider*

$$a(x, \xi) = \frac{\xi}{|\xi|} \int_0^x \frac{1}{\langle y \rangle^{1+\delta}} \, \mathrm{d}y.$$

Then

$$a \in \dot{\Gamma}^0_\infty(\mathbb{R}) := \bigcap_{\rho \geq 0} \dot{\Gamma}^0_\rho(\mathbb{R}),$$

and there exists $K > 0$ such that

$$\left\{ c\, |\xi|^{\frac{3}{2}}, a \right\} (t, x, \xi) \geq K \langle x \rangle^{-1-\delta} |\xi|^{\frac{1}{2}},$$

for all $t \in [0, T]$, $x \in \mathbb{R}$, $\xi \in \mathbb{R} \setminus \{0\}$.

We are now in position to prove Proposition 7.3.2.

Proof of Proposition 7.3.2. We will prove *a priori* estimates only, which means that we assume that $\varphi \in C^1(I; L^2(\mathbb{R}))$ (one can reduce to this case by using well-chosen mollifiers, see [19]).

This allows us to write

$$\begin{aligned}
\frac{d}{dt} \langle T_a \varphi, \varphi \rangle &= \left\langle T_{\partial_t a} \varphi, \varphi \right\rangle + \langle T_a \partial_t \varphi, \varphi \rangle + \langle T_a \varphi, \partial_t \varphi \rangle \\
&= \left\langle T_{\partial_t a} \varphi, \varphi \right\rangle \\
&\quad - \left\langle T_a T_V \partial_x \varphi + T_a i T_\gamma \varphi - T_a f, \varphi \right\rangle \\
&\quad - \left\langle T_a \varphi, + T_V \partial_x \varphi + i T_\gamma \varphi - f \right\rangle,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 scalar product. Introduce the commutator

$$C := \left[i T_\gamma, T_a \right].$$

Since $\partial_t a = 0$, the previous identity yields

$$\begin{aligned}\frac{d}{dt}\langle T_a\varphi,\varphi\rangle &= \langle C\varphi,\varphi\rangle + \left\langle i(T_\gamma^* - T_\gamma)T_a\varphi,\varphi\right\rangle \\ &\quad + \langle \partial_x(T_V T_a\varphi) - T_a T_V\partial_x\varphi,\varphi\rangle \\ &\quad + \langle T_a f,\varphi\rangle + \langle T_a\varphi, f\rangle\end{aligned} \tag{7.3.4}$$

Since $a \in \dot{\Gamma}^0_0$, it follows from the usual estimates for paradifferential operators that

$$|\langle T_a\varphi,\varphi\rangle| \lesssim \|\varphi\|_{L^2}^2,$$

and

$$|\langle T_a\varphi, f\rangle| + |\langle T_a f,\varphi\rangle| \leq K\,\|\varphi\|_{L^2}^2 + K\,\|f\|_{L^2}^2,$$

for some positive constant K. One easily obtains similar bounds for the second and third terms in the right hand-side of (7.3.4). Indeed, by definition of γ we know that $T_\gamma^* - T_\gamma$ is of order 0. On the other hand, as already seen, it follows from Theorem 3.5.13 that $\partial_x(T_V T_a\cdot) - T_a T_V\partial_x$ is of order 0. Therefore, integrating (7.3.4) in time, we end up with

$$\int_0^T \langle C\varphi,\varphi\rangle\,dt \leq M\left\{\|\varphi(0)\|_{L^2}^2 + \|\varphi(T)\|_{L^2}^2 + \int_0^T \left(\|\varphi\|_{L^2}^2 + \|f\|_{L^2}^2\right)dt\right\},$$

where M depends only on the $L^\infty(0,T;H^{s+\frac{1}{2}}(\mathbb{R})\times H^s(\mathbb{R}))$-norm of (η,ψ).

Hence to complete the proof it remains only to obtain a lower bound for the left hand-side. To do so, write

$$iT_\gamma = iT_c\,|D_x|^{\frac{3}{2}} + \frac{3}{4}T_{\frac{\xi}{|\xi|}\partial_x c}\,|D_x|^{\frac{1}{2}},$$

and recall that, by definition of a (see Lemma 7.3.3) there exists a constant K such that

$$\left\{c(t,x)\,|\xi|^{\frac{3}{4}}, a(x,\xi)\right\} \geq K\langle x\rangle^{-1-2\delta}\,|\xi|^{\frac{1}{2}},$$

for some positive constant $K > 0$. Since

$$\left[T_a, T_{\frac{\xi}{|\xi|}\partial_x c}\,|D_x|^{\frac{1}{2}}\right] \text{ is of order 0,}$$

Proposition 7.3.4 below then implies that

$$\langle C\varphi,\varphi\rangle \geq a\left\|\langle x\rangle^{-\frac{1}{2}-\delta}\varphi\right\|_{H^{\frac{1}{4}}}^2 - A\,\|\varphi\|_{L^2}^2,$$

This completes the proof of Proposition 7.3.2 and hence of Theorem 7.3.1. □

Proposition 7.3.4. *Let $d \geq 1$ and $\delta > 0$. Assume that $d \in \Gamma^{1/2}_{1/2}(\mathbb{R}^d)$ is such that, for some positive constant K, we have*

$$d(x,\xi) \geq K\langle x\rangle^{-1-2\delta} |\xi|^{\frac{1}{2}},$$

for all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$. Then there exist two positive constants $0 < a < A$ such that

$$\langle T_d u, u\rangle \geq a \left\| \langle x\rangle^{-\frac{1}{2}-\delta} u \right\|^2_{H^{\frac{1}{4}}} - A \left\| u \right\|^2_{L^2}.$$

We refer to [19] for the proof. Related results were previously proved by Bony [88] in the much more general setting of sharp Gårding or Fefferman-Phong inequalities. Notice however that these results require much more regularity than Proposition 7.3.4 (where the symbol is only assumed to be $C^{\frac{1}{2}}$).

7.4 Strichartz estimates

The following two theorems correspond to Strichartz estimates. We have two complementary results. The first one gives a gain of regularity for solutions of low regularity whose existence is ensured by Theorem 7.1.1. The second one gives a gain of regularity which is better (instead of gaining $1/4$ of derivative we gain $3/8$) but only applies to solutions which are *a priori* more regular.

Theorem 7.4.1 (semi-classical Strichartz estimate, from [20])**.** *Let $s > 5/2$ and $T > 0$. Let us consider a solution (η,ψ) of (7.0.1) defined on $I = [0,T]$. If*

$$(\eta,\psi) \in C^0\big(I, H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})\big),$$

then

$$(\eta,\psi) \in L^4\big(I, B^{s+\frac{1}{4}}_{\infty,2}(\mathbb{R}) \times B^{s-\frac{1}{4}}_{\infty,2}(\mathbb{R})\big).$$

Remark 7.4.2. *The Besov space $B^{\sigma}_{\infty,2}(\mathbb{R})$ is defined by*

$$u \in B^{\sigma}_{\infty,2}(\mathbb{R}) \Longleftrightarrow \sum_{j\in\mathbb{N}} 2^{2j\sigma} \left\| \Delta_j(u) \right\|^2_{L^\infty(\mathbb{R})} < +\infty,$$

where $u = \sum_j \Delta_j(u)$ is the Littlewood-Paley decomposition of u. If $\sigma \notin \mathbb{N}$, then $B^{\sigma}_{\infty,2}(\mathbb{R}) \subset W^{\sigma,\infty}(\mathbb{R})$, where $W^{\sigma,\infty}(\mathbb{R})$ denotes the usual Hölder space of order σ.

Theorem 7.4.3 (classical Strichartz estimate, from [20]). *Let $s > 11/2$ and $T > 0$. Let us consider a solution (η, ψ) of (7.0.1) defined on $I = [0, T]$. If*

$$(\eta, \psi) \in C^0\big(I, H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})\big),$$

then

$$(\eta, \psi) \in L^4\big(I, B^{s+\frac{3}{8}}_{\infty,2}(\mathbb{R}) \times B^{s-\frac{1}{8}}_{\infty,2}(\mathbb{R})\big).$$

Remark 7.4.4. (i) *The gain of regularity given by Theorem 7.4.3 is optimal, in the sense that it is exactly the one we have for the linearized system.*

(ii) *A key point is that the result of Theorem 7.4.1 remains true if $\mathbb{R}$ is replaced by the torus $\mathbb{T}$.*

7.4.1 Semi-classical strichartz

In this section we place ourselves in the 1 dimension. As explained before, one of the difficulties of the analysis is linked to the presence of the derivative term $T_W \partial_x$, and to the weak regularity of this one.

Symbol smoothing

The first step in order to demonstrate the dispersion estimates is to regularize this term, following an idea introduced by Lebeau [309] and also used by Smith [398] and Bahouri-Chemin [56].

Recall that the paraproduct T_W is defined by

$$T_W = \sum_{j \geq 4} S_{(j-3)}(W)\Delta_j.$$

We will replace the operator T_W by the operator

$$T_W^\delta = \sum_{j \geq 4} S_{[\delta(j-3)]}(W)\Delta_j,$$

Lemma 7.4.5. *The operator $T_W - T_W^\delta$ is of order $-\delta(s-\frac{3}{2})$ (and is therefore continuous from $H^\mu(\mathbb{R}^d)$ into $H^{\mu-s+\frac{3}{2}}(\mathbb{R}^d)$).*

We deduce that for $s > 2 + \frac{1}{2}$, there exists $\delta < \frac{1}{2}$ such that the function u verifies the equation

$$\partial_t u + T_W^\delta \partial_x u + i|D_x|^{3/2} u = G \in L^\infty(0, T; H^{s-\frac{1}{2}}).$$

Recall that $u \in L^\infty(0, T; H^s(\mathbb{R}))$ by assumption. The regularity of the source term G is half a derivative less than that of the unknown. However, such a loss of half a derivative is admissible if we only want to prove Strichartz estimates.

Construction of an approximate solution

We want to construct an approximate solution of the homogeneous equation

$$\partial_t u + T_W^\delta \partial_x u + i|D_x|^{3/2} u = 0.$$

We look for an approximate solution in the form

$$\widetilde{U}_h(t, x) = \frac{1}{2\pi h} \iint e^{\frac{i}{h}(\phi(t,x,\xi,h) - z\xi)} \widetilde{B}(t, x, z, \xi, h) u_{0,h}(z) \, \mathrm{d}z \, \mathrm{d}\xi,$$

where ϕ will satisfy an eïkonal equation and

$$\widetilde{B}(t, x, z, \xi, h) = B(t, x, \xi, h)$$

will be given by solving transport equations.

Let $v(t, x, \xi, x) = u(h^{-\frac{1}{2}} t, x, \xi, x)$. Then we have

$$h\partial_\tau v + h^{\frac{1}{2}} T_W^\delta (h\partial_x) v + i \, |hD_x|^{\frac{3}{2}} \, v = 0, \tag{7.4.1}$$

and we want to prove

$$\|v(\tau)\|_{L^\infty} \leq \frac{K}{h} \left(\frac{h}{\tau}\right)^{\frac{1}{2}} \|v(0)\|_{L^1}.$$

The factor K/h corresponds to the Sobolev injection, and the factor $\sqrt{h/\tau}$ corresponds to the gain due to the dispersion. With $h = 2^{-j/2}$ we have

$$v_0 = \chi(hD_x) v_0 = \frac{1}{2\pi h} \int_{\frac{1}{2} \leq |\eta| \leq 2} e^{\frac{i}{h}(x-y)\eta} \chi(\eta) v_0(y) \, \mathrm{d}y$$

and we search for v in the form of an oscillating integral

$$v(\tau, x) = \frac{1}{2\pi h} \iint_{\frac{1}{2} \leq |\eta| \leq 2} e^{\frac{i}{h}\left((x-y)\eta - \tau|\eta|^{\frac{3}{2}} + h^{\frac{1}{2}} \phi(\tau, x, \eta)\right)} b(\tau, x, \eta, h) v_0(y) \, d\eta dy,$$

where

$$b(\tau, x, \eta, h) = \sum_{k=0}^{K} h^{k\mu} b_k(\tau, x, \eta; h), \qquad \mu = \frac{1}{2} - \delta,$$

and

$$\partial_x \phi, b_j \in S^0_\delta = \{f \in C^\infty; |\partial_\eta^\beta \partial_x^\alpha f| \leq C h^{-\delta(|\alpha|+|\beta|-1)_+}\}$$

If we apply (formally) the operators $T_W^\delta \partial_x$ and $i\,|D_x|^{\frac{3}{2}}$ and cancel the terms corresponding to the different powers of h that we obtain, we find that ϕ must verify

$$\partial_\tau \phi - \partial_\eta(|\eta|^{\frac{3}{2}})\partial_x \phi = -S_{\delta j}(W)(h^{\frac{1}{2}}\tau, x)\eta,$$

and the symbols b_k are determined by transport equations:

$$\begin{cases} \dfrac{\partial b_0}{\partial \tau} - \partial_\eta(|\eta|^{\frac{3}{2}})\dfrac{\partial b_0}{\partial x} + i\partial_\eta^2(|\eta|^{\frac{3}{2}})(\partial_x \phi)^2 b_0 = 0, \\ b_0|_{\tau=0} = \chi(\eta), \end{cases}$$

$$\begin{cases} \dfrac{\partial b_j}{\partial \tau} - \partial_\eta(|\eta|^{\frac{3}{2}})\partial_x b_j + i\partial_\eta^2(|\eta|^{\frac{3}{2}})(\partial_x \phi)^2 b_j = -h^\mu \displaystyle\sum_{\ell=0}^{M-1} e_\ell (h^{\delta+3\varepsilon}\partial_x)^\ell b_{j-1}, \\ b_j|_{\tau=0} = 0. \end{cases}$$

We can then verify by very classical estimations on the solutions of ordinary differential equations, using $\phi = 0$ for $\tau = 0$,

$$\left\| \partial_x^\alpha \partial_\eta^\beta \phi(\tau, x, \eta) \right\|_{L^\infty(\mathbb{R})} \leq K\tau h^{-\delta(|\alpha|+|\beta|-1)_+}.$$

and similarly that b_j satisfy

$$\left\| \partial_x^\alpha \partial_\eta^\beta b_k(\tau, x, \eta) \right\|_{L^\infty(\mathbb{R})} \leq K h^{-\delta(|\alpha|+|\beta|-1)_+}.$$

As $\frac{1}{2} - \delta > 0$, we have an asymptotic calculation which allows us to justify the previous formal calculations and show that $v(x,\tau)$ is indeed an approximate solution of of the dilated equation (7.4.1).

Dispersion estimate

We can now obtain the dispersion estimate by making a stationary phase argument in the η variable by using that for $1/2 \leq |\eta| \leq 2$ we have

$$\left| \partial_\eta^2 \left((x-y)\cdot\eta - \tau\, |\eta|^{\frac{3}{2}} + h^{\frac{1}{2}}\phi(\tau, x, \eta) \right) \right| = \left| \tau \partial_\eta^2\, |\eta|^{\frac{3}{2}} + O(h^{\frac{1}{2}} h^{-\delta}\tau) \right| \geq c\,|\tau|\,.$$

For this we show in [19] a variant of the lemma of Van der Corput which leads to

$$\begin{aligned}|v(\tau,x)| &= \left|\frac{1}{2\pi h}\iint_{\frac{1}{2}\le|\eta|\le 2} e^{\frac{i}{h}\left((x-y)\eta-\tau|\eta|^{\frac{3}{2}}+h^{\frac{1}{2}}\phi(\tau,x,\eta)\right)} b(\tau,x,\eta,h)v_0(y)\,\mathrm{d}\eta\,\mathrm{d}y\right| \\ &\le \frac{K}{h}\left(\frac{h}{\tau}\right)^{\frac{1}{2}}\int |v_0(y)|\,\mathrm{d}y.\end{aligned}$$

Obtaining this inequality is the main step in the demonstration.

Reconciliation of estimates

The previous dispersion estimate implies, via the TT^* argument , the following Strichartz estimate: if u is a solution of

$$\partial_t u + \frac{1}{2}(S_{\delta j}(W)\partial_x + (\partial_x S_{\delta j}(W))u) + i\,|D_x|^{3/2}\,u = f, \quad u|_{t=0} = 0$$

with supp $\widehat{f} \subset \{\frac{1}{2}h^{-1} \le |\xi| \le 2h^{-1}\}$, then

$$\|u\|_{L^4((0,h^{\frac{1}{2}}),L^\infty(\mathbb{R}))} \le Kh^{-1/8}\,\|f\|_{L^1((0,h^{\frac{1}{2}}),L^2(\mathbb{R}))}.$$

The idea, coming from the works of Bahouri and Chemin [56, 57] and Burq, Gérard and Tzvetkov [102], is that this Strichartz estimate in very short time is sufficient to demonstrate a Strichartz estimate on a time interval of size 1. The principle is to apply this estimate to functions

$$u_{j,k} = \varphi\Big(\frac{t-kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\Big)\Delta_j u \quad h = 2^{-j},\ -1 \le k \le Th^{-\frac{1}{2}},$$

where $\varphi \in C_0^\infty(0,2)$, is equal to 1 on $(1/2, 3/2)$. We can easily find the equation verified by $u_{j,k}$ by commuting the truncation in time $\varphi\big(h^{-\frac{1}{2}}(t-kh^{\frac{1}{2}})\big)$ to the equation verified by $\Delta_j u$. We are then able to apply the previous estimate; then estimate the $L^4([0,T], B^{s-\frac{1}{4}}_{\infty,2}(\mathbb{R}))$-norm of $\Delta_j u$ by putting the estimates back together: more precisely, by writing that

$$\|\Delta_j u\|^4_{L^4((0,T),\,L^\infty(\mathbb{R}))} \le \sum_{k=-1}^{Th^{-\frac{1}{2}}} \|u_{h,k}\|^4_{L^4((kh^{\frac{1}{2}},(k+2)h^{\frac{1}{2}}),\,L^\infty(\mathbb{R}))},$$

we find, with always the notation $h = 2^{-j}$,

$$\|\Delta_j u\|^4_{L^4((0,T),\,L^\infty(\mathbb{R}))} \le Ch^{-\frac{1}{2}}h^{4(s-\frac{1}{8})}(h^\varepsilon + \|u_h\|_{L^\infty((0,T),H^s(\mathbb{R}))})^4.$$

To obtain an estimate on u from the estimates of the dyadic blocks, it remains to write that

$$\begin{aligned}
\|u\|_{L^4((0,T),\,B^{s-\frac{1}{4}}_{\infty,2}(\mathbb{R}))} &= \|\big(\sum_{j\in\mathbb{N}} 2^{2j(s-\frac{1}{4})}\|\Delta_j u\|^2_{L^\infty(\mathbb{R})}\big)^{\frac{1}{2}}\|_{L^4(0,T)} \\
&= \|\sum_{j\in\mathbb{N}} 2^{2j(s-\frac{1}{4})}\|\Delta_j u\|^2_{L^\infty(\mathbb{R})}\|^{\frac{1}{2}}_{L^2(0,T)} \\
&\le \Big(\sum_{j\in\mathbb{N}} \|2^{2j(s-\frac{1}{4})}\|\Delta_j u\|^2_{L^\infty(\mathbb{R})}\|_{L^2(0,T)}\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{j\in\mathbb{N}} 2^{2j(s-\frac{1}{4})}\|\Delta_j u\|^2_{L^4((0,T),L^\infty(\mathbb{R}))}\Big)^{\frac{1}{2}}.
\end{aligned}$$

This concludes the proof.

7.4.2 Classical Strichartz estimation

To conclude this chapter, I just want to say a few words here about the strategy of the proof of the theorem 7.4.3, which is long and quite difficult. The big difference with the proof of the semi-classical estimates is that we need a long time parametrix (of the order of $O(h^{-1/2})$) for the dilated equation (7.4.1) (see the previous work of Robbiano-Zuily [374, 375] who write such a parametrix in a much more regular framework). The previous method does not work anymore because of the term $\partial^2_\xi(|\xi|^{3/2})(\partial_x\phi)^2$ which intervenes in the transport equations. We have shown that we can get around this problem by defining the phase as a solution of the equation **nonlinear**

$$\begin{cases} \dfrac{\partial\phi}{\partial\sigma} + \dfrac{1}{h^{\frac{1}{2}}}\left(|\xi + h^{\frac{1}{2}}\partial_x\phi|^{\frac{3}{2}} - |\xi|^{\frac{3}{2}}\right) + h^{\frac{1}{2}}S_{\delta j}W(h^{\frac{1}{2}}t,x)\partial_x\phi = -S_{\delta j}W(h^{\frac{1}{2}}t,x)\xi, \\ \psi(0,x,\xi,h) = 0. \end{cases}$$

This allows to introduce a shift in the BKW cascade which determines the coefficients b_j. The difficulty is that we need to understand how to show uniform estimates in large time and also how to order the computations to show that we obtain an approximate solution. Similarly, adaptations are needed in the stationary phase argument. We show that the coefficient b can be written in the form $b = e^{\theta}$ with $\theta = \sum h^{j\mu_0}\theta_j, \mu_0 > 0$, and we use the stationary phase lemma with the complex phase $x\xi - \sigma|\xi|^{3/2} + h^{\frac{1}{2}}\phi + \frac{h}{i}\theta$. This approach applies under the constraint $1/(s - \frac{3}{2}) < \frac{1}{4}$ i.e. $s > 11/2$.

Chapter 8

The dynamics of the roots of polynomials under differentiation

8.1 Introduction

Stefan Steinerberger studied in [402] (see also [356]) the following question: Considering a polynomial p_n of degree n having all its roots on the real line distributed according to a smooth function $u_0(x)$, and a real number $t \in (0,1)$, how is the distribution of the roots of the derivatives $\partial_x^k p_n$ with $k = \lfloor t \cdot n \rfloor$? This question led him to discover a nice non-local nonlinear equation which reads as follows

$$\partial_t u + \frac{1}{\pi}\partial_x \left(\arctan\left(\frac{Hu}{u}\right)\right) = 0, \tag{8.1.1}$$

where the unknown $u = u(t,x)$ is a positive real-valued function.

Besides its aesthetic aspect, this equation has many interesting features. Shlyakhtenko and Tao [392] derived the same equation in the context of free probability and random matrix theory (see also [403]). However, our motivation comes from the links between this equation and many models studied in fluid dynamics.

In this chapter, we assume that the space variable x belongs to the circle $\mathbb{S} = \mathbb{R}/(2\pi\mathbb{Z})$, and H is the circular Hilbert transform (which acts on periodic functions), defined by

$$Hu(x) = \frac{1}{2\pi}\,\mathrm{pv}\int_{\mathbb{S}} \frac{g(x) - g(x-\alpha)}{\tan(\alpha/2)}\,\mathrm{d}\alpha, \tag{8.1.2}$$

where the integral is understood as a principal value. Granero-Belinchón ([221]) proved the local existence of smooth solutions for positive initial data u_0 in the Sobolev space $H^2(\mathbb{S}) = \{u \in L^2(\mathbb{S})\,;\, \partial_x^2 u \in L^2(\mathbb{S})\}$, as well as the global existence under a condition in appropriate Wiener spaces. Then, Kiselev and Tan [289] proved that the Cauchy problem for (8.1.1) is globally well-posed for positive initial data in the Sobolev space $H^s(\mathbb{S})$ for all $s > 3/2$, with $H^s(\mathbb{S}) = \{u \in L^2(\mathbb{S})\,;\, \Lambda^s u \in L^2(\mathbb{S})\}$ where Λ denotes the fractional Laplacian:

$$\Lambda = \partial_x H = (-\partial_{xx})^{\frac{1}{2}}.$$

In fact, the equation (8.1.1) enters the family of fractional parabolic equations, which has attracted a lot of attention in recent years. To see this, introduce the coefficients

$$V = -\frac{1}{\pi}\frac{Hu}{u^2 + (Hu)^2} \quad , \quad \gamma = \frac{1}{\pi}\frac{u}{u^2 + (Hu)^2}.$$

Then the equation (8.1.1) has the following form

$$\partial_t u + V\partial_x u + \gamma \Lambda u = 0. \tag{8.1.3}$$

We can see that this last equation shares many characteristics with the Hele-Shaw equation, the Muskat equation or the dissipative surface quasi-geostrophic equation, to name a few.

Inspired by these results, our goal here is to solve the Cauchy problem for (8.1.1) in the critical Sobolev space $H^{\frac{1}{2}}(\mathbb{S})$. Several interesting difficulties appear at that level of regularity.

The main result of this chapter is the following.

Theorem 8.1.1 (A., Lazar, Nguyen)**.** *For all initial data u_0 in $H^{\frac{1}{2}}(\mathbb{S})$ such that $\inf_{x\in\mathbb{S}} u_0 > 0$, there exists a time T such that the Cauchy problem has a unique solution $u \in C^0([0,T]; H^{\frac{1}{2}}(\mathbb{S})) \cap L^2((0,T); \dot{H}^1(\mathbb{S}))$, satisfying $\inf_{x\in\mathbb{S}} u(t,x) \geq \inf_{x\in\mathbb{S}} u_0(x)$ for $t \in (0,T]$.*

Remark 8.1.2. *Moreover the solution is smooth for positive times. Consequently, it follows from the global regularity result of Kiselev and Tan that the previous solution exists globally in time.*

8.2 A toy model

To prove Theorem 9.3.1, the main difficulty is that the coefficient γ is not bounded from below by a positive constant. This means that (8.1.3) is a degenerate parabolic equation.

Although it is not essential for the rest of the paper, it helps if we begin by examining a model equation with some non-degenerate dissipative term. Our goal here is to introduce a basic commutator estimate which allows to deal with equations of the form (8.1.3).

Consider the equation

$$\partial_t u + a(u, Hu)\Lambda u = b(u, Hu)H\Lambda u, \tag{8.2.1}$$

where a and b are two C^∞ real-valued functions defined on $\mathbb{R}^2$, satisfying $a \geq m > 0$ for some given positive constant m, together with

$$\sup_{(x,x',y,y')\in\mathbb{R}^4} \frac{|b(x,y) - b(x',y')|}{|x - x'| + |y - y'|} < +\infty.$$

Proposition 8.2.1. *There exists a constant $C > 0$ such that, for all $T > 0$ and for all $u \in C^1([0,T]; \dot{H}^{\frac{1}{2}}(\mathbb{S}))$ solution to (8.2.1), there holds*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{\dot{H}^{\frac{1}{2}}} + m \int_{\mathbb{S}} |\Lambda u|^2 \,\mathrm{d}x \leq C \|u\|_{\dot{H}^{\frac{1}{2}}} \|\Lambda u\|^2_{L^2}. \tag{8.2.2}$$

Remark 8.2.2. *Using classical arguments, it is then possible to infer from the* a priori *estimate (8.2.2) a global well-posedness result for initial data which are small enough in $H^{\frac{1}{2}}(\mathbb{S})$. However, the study of the local well-posedness of the Cauchy problem for large data is more difficult and requires and extra argument which is explained in the next section.*

Proof. Let us use the short notations $a = a(u, Hu)$ and $b = b(u, Hu)$. Multiplying the equation (8.2.1) by Λu and integrating over $\mathbb{S}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{\dot{H}^{\frac{1}{2}}} + \int_{\mathbb{S}} a\,|\Lambda u|^2 \,\mathrm{d}x = I := \int_{\mathbb{S}} b(H\Lambda u)(\Lambda u)\,\mathrm{d}x. \tag{8.2.3}$$

To estimate I, we use the fact that H is skew-symmetric to write

$$|I| = \frac{1}{2}\left|\int_{\mathbb{S}} \big([b, H]\Lambda u\big)\Lambda u \,\mathrm{d}x\right|. \tag{8.2.4}$$

Now we claim that

$$\left\| \left[b, H \right] \Lambda u \right\|_{L^2} \lesssim \| b \|_{\dot{H}^{\frac{1}{2}}} \| \Lambda u \|_{L^2} . \tag{8.2.5}$$

Indeed, this follows from the Sobolev embedding $\dot{H}^{\frac{1}{2}} \subset \mathrm{BMO}$ and the classical commutator estimate

$$\| [H, f] v \|_{L^2} \lesssim \| f \|_{\mathrm{BMO}} \| v \|_{L^2} . \tag{8.2.6}$$

(Alternatively one can prove (8.2.5) directly using the definition of the Hilbert transform as a singular integral and the Gagliardo semi-norm.) It follows that

$$I \lesssim \| b \|_{\dot{H}^{\frac{1}{2}}} \| \Lambda u \|_{L^2}^2 . \tag{8.2.7}$$

Now we estimate the $\dot{H}^{\frac{1}{2}}$-norm of b by means of the following elementary estimate.

Lemma 8.2.3. *Consider a C^∞ function $\sigma \colon \mathbb{R}^2 \to \mathbb{R}$ satisfying*

$$\forall (x, x', y, y') \in \mathbb{R}^4, \qquad | \sigma(x, y) - \sigma(x', y') | \le K |x - x'| + K |y - y'| .$$

Then, for all $s \in (0, 1)$ and all $u \in \dot{H}^s(\mathbb{R})$, one has $\sigma(u, Hu) \in \dot{H}^s(\mathbb{R})$ together with the estimate

$$\| \sigma(u, Hu) \|_{\dot{H}^s} \le 2K \| u \|_{\dot{H}^s} . \tag{8.2.8}$$

Proof. By assumption, for any $\alpha \in \mathbb{R}$, we have

$$\| \delta_\alpha \sigma(u, Hu) \|_{L^2} \le K (\| \delta_\alpha u \|_{L^2} + \| \delta_\alpha H u \|_{L^2}).$$

Then by using the Gagliardo semi-norms, we get

$$\| \sigma(u, Hu) \|_{\dot{H}^s} \le K \| u \|_{\dot{H}^s} + K \| Hu \|_{\dot{H}^s} ,$$

and the desired result follows since $\| Hu \|_{\dot{H}^s} = \| u \|_{\dot{H}^s}$. □

The previous lemma implies that

$$\| b \|_{\dot{H}^{\frac{1}{2}}} \lesssim \| u \|_{\dot{H}^{\frac{1}{2}}} + \| Hu \|_{\dot{H}^{\frac{1}{2}}} \lesssim \| u \|_{\dot{H}^{\frac{1}{2}}} ,$$

and we deduce from (8.2.7) that

$$I \lesssim \| u \|_{\dot{H}^{\frac{1}{2}}} \| \Lambda u \|_{L^2}^2 .$$

Therefore the wanted result (8.2.2) follows from (8.2.3). □

8.3 Local well-posedness

We construct solutions to (8.1.1) as limits of solutions to a sequence of approximate nonlinear systems. We divide the analysis into three parts.

1. We start by proving that the Cauchy problem for these systems systems are well posed globally in time and satisfy the maximum principles. In particular, the approximate solutions are bounded by a positive constant.

2. Then, we show that the solutions of the approximate systems are bounded in $C^0([0,T];\dot{H}^{\frac{1}{2}}(\mathbb{S}))$ on a uniform time interval that depends on the profile of the initial data (and not only on their norm).

3. The third task is to show that these approximate solutions converge to a limit which is a solution of the original equation. To do this, we use interpolation and compactness arguments.

8.3.1 Approximate systems

Fix $\delta \in (0,1]$ and consider the following approximate Cauchy problem:

$$\begin{cases} \partial_t u + \dfrac{1}{\pi}\dfrac{u\Lambda u - (Hu)\partial_x u}{u^2 + (Hu)^2} - \delta\partial_x^2 u = 0, \\ u|_{t=0} = e^{\delta\partial_x^2} u_0. \end{cases} \tag{8.3.1}$$

The following elementary lemma states that this Cauchy problem has smooth positive solutions (see Granero in [221] for the maximum principle).

Lemma 8.3.1. *For any initial data $u_0 \in L^2(\mathbb{S})$ with $\inf_{x\in\mathbb{S}} u_0 > 0$ and for any $\delta > 0$, the initial value problem (8.3.1) has a unique solution u in $C^1([0,+\infty); H^\infty(\mathbb{S}))$. This solution is such that, for all $t \in [0,+\infty)$,*

$$\inf_{x\in\mathbb{S}} u(t,x) \geq \inf_{x\in\mathbb{S}} u_0(x). \tag{8.3.2}$$

In addition, if $\max_{x\in\mathbb{S}} u_0(x) < +\infty$ then $\max_{x\in\mathbb{S}} u(t,x) \leq \max_{x\in\mathbb{S}} u_0(x)$.

8.3.2 Uniform estimates

Fix $\delta > 0$ and c_0 and consider an initial data u_0 in $H^{\frac{1}{2}}(\mathbb{S})$ with $\inf_{x\in\mathbb{S}} u_0(x) \geq c_0$. As we have seen in the previous paragraph, there exists a unique function $u \in C^1([0,+\infty); H^\infty(\mathbb{S}))$ satisfying

$$
\begin{cases}
\partial_t u + \dfrac{1}{\pi}\dfrac{u\Lambda u - (Hu)\partial_x u}{u^2+(Hu)^2} - \delta\partial_x^2 u = 0,\\
u|_{t=0} = e^{\delta\partial_x^2}u_0,\\
\inf\limits_{x\in\mathbb{S}} u(t,x) \geq c_0.
\end{cases}
\tag{8.3.3}
$$

We shall prove estimates which are uniform with respect to $\delta \in (0,1]$ and this is why we are writing simply u instead of u_δ, to simplify notations.

Notice that

$$
\partial_t u + \frac{1}{\pi}\arctan\left(\frac{Hu}{u}\right) - \delta\partial_x^2 u = 0.
$$

This implies that the mean value of u is preserved and hence it will be sufficient to estimate the homogeneous $\dot{H}^{\frac{1}{2}}(\mathbb{S})$-norm of u. For this, we proceed as follows. Given $\varepsilon > 0$ to be determined, we want to estimate

$$
v := u - e^{(\varepsilon+\delta)\partial_x^2}u_0.
$$

Set

$$
u_{0,\varepsilon} = e^{(\varepsilon+\delta)\partial_x^2}u_0,
$$

and introduce the coefficients

$$
\begin{aligned}
\gamma &= \frac{1}{\pi}\frac{u}{u^2+(Hu)^2},\\
V &= -\frac{1}{\pi}\frac{Hu}{u^2+(Hu)^2},\\
\rho &= \sqrt{u^2+(Hu)^2}.
\end{aligned}
$$

With the previous notations, we have

$$
\partial_t v + V\partial_x v + \gamma\Lambda v - \delta\partial_x^2 v = R_\varepsilon(u,u_0)
\tag{8.3.4}
$$

where

$$
R_\varepsilon(u,u_0) = -\gamma\Lambda u_{0,\varepsilon} - V\partial_x u_{0,\varepsilon} + \delta\partial_x^2 u_{0,\varepsilon}.
$$

The following result is our main technical estimate.

Lemma 8.3.2. *For any $u_0 \in H^{\frac{1}{2}}(\mathbb{S})$ with $\inf_{x\in\mathbb{S}} u_0(x) > 0$, there exist a constant ε_0 and a function $T\colon (0,1] \to (0,1)$ with*

$$\lim_{\varepsilon\to 0} T(\varepsilon) = 0,$$

such that the following result holds: for all $\delta \in (0,1]$, all $u \in C^1([0,+\infty); H^\infty(\mathbb{S}))$ satisfying (8.3.3) with initial data $u|_{t=0} = e^{\delta\partial_x^2}u_0$, and for all $\varepsilon \in (0,\varepsilon_0]$, the function $v = u - e^{(\varepsilon+\delta)\partial_x^2}u_0$ satisfies

(8.3.5)

$$\sup_{t\in[0,T(\varepsilon)]} \|v(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \int_0^{T(\varepsilon)}\int_{\mathbb{S}} \frac{u\,|\Lambda v|^2}{u^2+(Hu)^2}\,\mathrm{d}x\,\mathrm{d}t + \delta\int_0^{T(\varepsilon)} \|v\|^2_{\dot{H}^{\frac{3}{2}}}\,\mathrm{d}t \le \mathcal{F}(\varepsilon),$$

for some function $\mathcal{F}\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon\to 0}\mathcal{F}(\varepsilon) = 0$.

Proof. Hereafter, C denotes various constants which depend only on the constant c_0 (remembering that c_0 is some given constant such that $\inf u(t,x) \ge \inf u_0 \ge c_0$) and we use the notation $A \lesssim_{c_0} B$ to indicate that $A \le CB$ for such a constant C.

Consider a parameter $\kappa \in (0,1]$ whose value is to be determined. Then decompose the Hilbert transform as $H = H_{\kappa,1} + H_{\kappa,2}$ where

$$\begin{aligned}
H_{\kappa,1}g(x) &= \frac{1}{2\pi}\int_{\mathbb{S}} g(x-\alpha)\chi\left(\frac{\alpha}{\kappa}\right)\frac{\mathrm{d}\alpha}{\tan(\alpha/2)},\\
H_{\kappa,2}g(x) &= \frac{1}{2\pi}\int_{\mathbb{S}} g(x-\alpha)\left(1-\chi\left(\frac{\alpha}{\kappa}\right)\right)\frac{\mathrm{d}\alpha}{\tan(\alpha/2)},
\end{aligned}$$

for some cut-off function $\chi \in C^\infty$ satisfying $\chi = 1$ in $[-1,1]$ and $\chi = 0$ in $\mathbb{R}\setminus[-2,2]$.

Multiply equation (8.3.4) by Λv and then integrate over $\mathbb{S}$, to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{\dot{H}^{\frac{1}{2}}} + \int_{\mathbb{S}} \gamma(\Lambda v)^2\,\mathrm{d}x = A + B + R + R' \tag{8.3.6}$$

where

$$\begin{aligned}
A &= \int_{\mathbb{S}} V(H_{\kappa,1}\Lambda v)(\Lambda v)\,\mathrm{d}x,\\
B &= \int_{\mathbb{S}} V(H_{\kappa,2}\Lambda v)(\Lambda v)\,\mathrm{d}x,\\
R &= \int_{\mathbb{S}} (-\gamma\Lambda u_{0,\varepsilon} - V\partial_x u_{0,\varepsilon})(\Lambda v)\,\mathrm{d}x,\\
R' &= \int_{\mathbb{S}} \delta(\partial_x^2 u_{0,\varepsilon})\Lambda v\,\mathrm{d}x.
\end{aligned}$$

Set

$$W := \sqrt{\gamma}\Lambda v,$$

so that the dissipative term in (8.3.6) is of the form

$$\int_{\mathbb{S}} \gamma(\Lambda v)^2 \,\mathrm{d}x = \int_{\mathbb{S}} W^2 \,\mathrm{d}x.$$

Step 1: Estimate of B, R and R'. Directly from the definition of γ and V, we have

$$\gamma \le \frac{1}{\pi c_0} \quad , \quad |V| \le \frac{1}{2\pi c_0}. \tag{8.3.7}$$

One important feature of the critical problem is that the dissipative term is degenerate. This means that the coefficient γ is not bounded from below by a fixed positive constant. As a result, we do not control the L^2-norm of Λv. Instead, we merely control the L^2-norm of $W = \sqrt{\gamma}\Lambda v$. Hereafter, we will systematically write Λv under the form

$$\Lambda v = \frac{1}{\sqrt{\gamma}}\sqrt{\gamma}\Lambda v = \frac{1}{\sqrt{\gamma}} W.$$

To absorb the contribution of the factor $1/\sqrt{\gamma}$ in the estimates for B and R, it will be sufficient to notice that we have the pointwise bound

$$\left|\frac{V}{\sqrt{\gamma}}\right| = \frac{1}{u\sqrt{\delta + u^2 + (Hu)^2}} |Hu| \lesssim_{c_0} |Hu| \,.$$

In particular, remembering that the Hilbert transform is bounded from $L^p(\mathbb{S})$ to $L^p(\mathbb{S})$ for any $p \in (1, +\infty)$ and using the Sobolev embedding $H^s(\mathbb{S}) \subset L^{2/(1-2s)}(\mathbb{S})$, we deduce that

$$\left\|V/\sqrt{\gamma}\right\|_{L^4} \lesssim_{c_0} \|Hu\|_{L^4} \lesssim_{c_0} \|u\|_{\dot{H}^{\frac{1}{2}}} \,. \tag{8.3.8}$$

Then it follows from Hölder's inequality that

$$|B| \le \int_{\mathbb{S}} \frac{V}{\sqrt{\gamma}} \left|H_{\kappa,2}\Lambda v\right| \left|\sqrt{\gamma}\Lambda v\right| \mathrm{d}x \lesssim_{c_0} \|u\|_{\dot{H}^{\frac{1}{2}}} \left\|H_{\kappa,2}\Lambda v\right\|_{L^4} \|W\|_{L^2} \,.$$

On the other hand,

$$\left\|H_{\kappa,2}\Lambda v\right\|_{L^4} = \left\|H_{\kappa,2}\partial_x H v\right\|_{L^4} \lesssim \kappa^{-1} \|Hv\|_{L^4} \lesssim \kappa^{-1} \|v\|_{\dot{H}^{\frac{1}{2}}} \,,$$

where we used the definition of $H_{\kappa,2}$, noting that $g_x(x-\alpha) = \partial_\alpha(g(x) - g(x-a))$ and integrating by parts in α.

By combining the previous estimates, we conclude that

$$|B| \lesssim_{c_0} \kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2}.$$

The estimate of R is similar. Recall that

$$R = \int_{\mathbb{S}} (-\gamma\Lambda u_{0,\varepsilon} - V\partial_x u_{0,\varepsilon})(\Lambda v)\,\mathrm{d}x.$$

To estimate the contribution of the first term, we write

$$\left|\int_{\mathbb{S}} \gamma(\Lambda u_{0,\varepsilon})(\Lambda v)\,\mathrm{d}x\right| \le \left\|\sqrt{\gamma}\right\|_{L^\infty} \left\|\Lambda u_{0,\varepsilon}\right\|_{L^2} \left\|\sqrt{\gamma}\Lambda v\right\|_{L^2} \lesssim_{c_0} \varepsilon^{-\frac{1}{2}} \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2},$$

where we have used the elementary inequality

$$\left\|u_{0,\varepsilon}\right\|_{\dot{H}^1} \lesssim (\varepsilon+\delta)^{-\frac{1}{4}} \left\|u_{0,\varepsilon}\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \varepsilon^{-\frac{1}{4}} \|u_0\|_{\dot{H}^{\frac{1}{2}}},$$

since the Fourier transform of $u_{0,\varepsilon} = e^{(\varepsilon+\delta)\partial_x^2} u_0$ is essentially localized in the interval $|\xi| \lesssim \sqrt{\varepsilon+\delta}$. With regards to the second term, we use again the estimate (8.3.8) to get

$$\begin{aligned}\left|\int_{\mathbb{S}} V(\partial_x u_{0,\varepsilon})(\Lambda v)\,\mathrm{d}x\right| &\le \left\|V/\sqrt{\gamma}\right\|_{L^4} \left\|\partial_x u_{0,\varepsilon}\right\|_{L^4} \left\|\sqrt{\gamma}\Lambda v\right\|_{L^2} \\ &\lesssim_{c_0} \varepsilon^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2}.\end{aligned}$$

Eventually, we have

$$\left\|\delta\partial_x^2 u_{0,\varepsilon}\right\|_{\dot{H}^{\frac{1}{2}}} = \left\|\delta\partial_x^2 e^{(\varepsilon+\delta)\partial_x^2} u_0\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\|\delta\partial_x^2 e^{\delta\partial_x^2} u_0\right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{2}}}.$$

Hence

$$|R'| \lesssim \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}}.$$

So, by combining the previous inequalities, we conclude that

$$\begin{aligned}|B+R+R'| \lesssim_{c_0} \kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} &+ \varepsilon^{-\frac{1}{2}}(1 + \|u\|_{\dot{H}^{\frac{1}{2}}}) \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} \\ &+ \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}},\end{aligned}$$

hence, replacing u by $v + u_{0,\varepsilon}$ in the right-hand side, we conclude that

$$
\begin{aligned}
|B + R + R'| \lesssim_{c_0} &\left(\kappa^{-1} \|u\|_{\dot H^{\frac12}} + \varepsilon^{-\frac12} \|u_0\|_{\dot H^{\frac12}}\right) \|v\|_{\dot H^{\frac12}} \|W\|_{L^2} \\
&+ \varepsilon^{-\frac12}\left(\|u_0\|_{\dot H^{\frac12}} + \|u_0\|^2_{\dot H^{\frac12}} \right) \|W\|_{L^2} + \|u_0\|_{\dot H^{\frac12}} \|v\|_{\dot H^{\frac12}} .
\end{aligned}
\tag{8.3.9}
$$

Step 2: Estimate of A. Since $H_{\kappa,1}$ is skew-symmetric, one easily verifies that

$$
A = \frac12 \int_{\mathbb{S}} \big(\big[V, H_{\kappa,1}\big]\Lambda v\big)\Lambda v \,\mathrm{d}x.
$$

Then, we multiply and divide by $\sqrt{\gamma}$ and use the Cauchy-Schwarz inequality, to find

$$
A \le \left(\int_{\mathbb{S}} \gamma^{-1}\big|[V, H_{\kappa,1}](\Lambda v)\big|^2 \,\mathrm{d}x\right)^{\frac12} \big\|\sqrt{\gamma}\Lambda v\big\|_{L^2} .
\tag{8.3.10}
$$

We now have to estimate the commutator $[V, H_{\kappa,1}]$. Directly from the definition of $H_{\kappa,1}$, we have

$$
\begin{aligned}
[V, H_{\kappa,1}](\Lambda v) &= \frac{1}{2\pi}\,\mathrm{pv}\int_{\mathbb{S}} \frac{V(x)(\delta_\alpha \Lambda v)(x) - \delta_\alpha (V\Lambda v)(x)}{\tan(\alpha/2)} \chi\left(\frac{\alpha}{\kappa}\right) \mathrm{d}\alpha \\
&= -\frac{1}{2\pi}\,\mathrm{pv}\int_{\mathbb{S}} \frac{(\delta_\alpha V)(x)(\Lambda v)(x-\alpha)}{\tan(\alpha/2)} \chi\left(\frac{\alpha}{\kappa}\right) \mathrm{d}\alpha \\
&= -\frac{1}{2\pi}\,\mathrm{pv}\int_{\mathbb{S}} \frac{(\delta_\alpha V)(x)}{\sqrt{\gamma}(x-\alpha)} W(x-\alpha)\chi\left(\frac{\alpha}{\kappa}\right) \frac{\mathrm{d}\alpha}{\tan(\alpha/2)},
\end{aligned}
$$

where we replaced Λv by $W/\sqrt{\gamma}$ to obtain the last identity.

Therefore,

$$
\begin{aligned}
&\int_{\mathbb{S}} \gamma^{-1}\big|[V, H_{\kappa,1}](\Lambda v)\big|^2 \,\mathrm{d}x \\
&\qquad\lesssim \int_{\mathbb{S}} (\gamma(x))^{-1} \left(\int_{|\alpha|\le 2\kappa} |\delta_\alpha V(x)|\, \gamma(x-\alpha)^{-1/2}\, |W(x-\alpha)| \,\frac{\mathrm{d}\alpha}{|\tan(\alpha/2)|}\right)^2 \mathrm{d}x \\
&\qquad\lesssim_{c_0} \int_{\mathbb{S}} (\rho^2(x) \left(\int_{|\alpha|\le 2\kappa} |\delta_\alpha V(x)|\, \rho(x-\alpha)\, |W(x-\alpha)| \,\frac{\mathrm{d}\alpha}{|\tan(\alpha/2)|}\right)^2 \mathrm{d}x.
\end{aligned}
$$

Lemma 8.3.3. *Introduce the notation*

$$
Q_\alpha(g) := |\delta_\alpha g| + |\delta_\alpha H g|.
$$

Then there holds

$$(8.3.11) \qquad |\delta_\alpha V| \lesssim_{c_0} \frac{Q_\alpha(v)(1+Q_\alpha(v))^2}{\rho^2} + \varepsilon^{-3}(1+\|u_0\|_{H^{\frac{1}{2}}})^3 |\alpha| .$$

Proof. One has

$$\begin{aligned} |\delta_\alpha V(x)| &\lesssim \frac{|\delta_\alpha Hu(x)|}{\rho^2(x)} + |Hu(x-\alpha)| \frac{|\delta_\alpha u(x)|(|u(x)|+|u(x-\alpha)|)}{\rho^2(x-\alpha)\rho^2(x)} \\ &\lesssim_{c_0} \frac{|\delta_\alpha Hu(x)|}{\rho^2(x)} + \frac{|\delta_\alpha u(x)|^2 + |\delta_\alpha Hu(x)|^2}{\rho(x-\alpha)\rho(x)} + \frac{|\delta_\alpha u(x)| + |\delta_\alpha Hu(x)|}{\rho(x-\alpha)\rho(x)}. \end{aligned}$$

Since

$$\frac{1}{\rho(x-\alpha)} \lesssim \frac{1}{\rho(x)}(1+Q_\alpha(u)),$$

we obtain

$$|\delta_\alpha V| \lesssim_{c_0} \frac{Q_\alpha(u)(1+Q_\alpha(u))^2}{\rho^2}.$$

To get the wanted result (8.3.11) from this, we replace u by $v + u_{0,\varepsilon}$ and use the two following elementary ingredients:

$$\frac{Q_\alpha(u_{0,\varepsilon})(1+Q_\alpha(u_{0,\varepsilon}))^2}{\rho^2} \lesssim_{c_0} Q_\alpha(u_{0,\varepsilon})(1+Q_\alpha(u_{0,\varepsilon}))^2,$$

$$Q_\alpha(u_{0,\varepsilon}) \le \left(\left\|\partial_x u_{0,\varepsilon}\right\|_{L^\infty} + \left\|\partial_x (Hu_{0,\varepsilon})\right\|_{L^\infty} \right) |\alpha| \lesssim \left\|u_{0,\varepsilon}\right\|_{H^2} |\alpha| \lesssim \varepsilon^{-\frac{3}{4}} \|u_0\|_{H^{\frac{1}{2}}} |\alpha| .$$

Since $|\alpha|^3 \lesssim |\alpha|$, this completes the proof. □

Set $K(\varepsilon) := \varepsilon^{-3}(1 + \left\|u_{0,\varepsilon}\right\|_{H^{\frac{1}{2}}})^3$. It follows from the previous lemma and the preceding inequality that

$$\int_{\mathbb{S}} \gamma^{-1} \left|[V, H_{\kappa,1}](\Lambda v)\right|^2 \, \mathrm{d}x \lesssim_{c_0} (I) + (II),$$

where

$$(I) := \int_{\mathbb{S}} \rho(x)^2 \bigg(\int_{|\alpha|\le 2\kappa} \frac{Q_\alpha(v)(x)(1+Q_\alpha(v)(x))^2}{\rho(x)^2} \rho(x-\alpha) \, |W(x-\alpha)| \, \frac{\mathrm{d}\alpha}{|\tan(\frac{\alpha}{2})|} \bigg)^2 \, \mathrm{d}x,$$

$$(II) := K(\varepsilon)^2 \int_{\mathbb{S}} \rho(x)^2 \bigg(\int_{|\alpha|\le 2\kappa} \rho(x-\alpha) \, |W(x-\alpha)| \, \mathrm{d}\alpha \bigg)^2 \, \mathrm{d}x.$$

Using the Cauchy-Schwarz inequality, we see that

$$(I) \lesssim \left(\iint_{\mathbb{S}^2} Q_\alpha(v)(x)^2 (1 + Q_\alpha(v)(x))^4 \frac{\rho^2(x-\alpha)}{\rho^2(x)} \frac{\mathrm{d}\alpha\, \mathrm{d}x}{|\alpha|^2} \right) \|W\|_{L^2}^2 .$$

Since

$$\frac{\rho^2(x-\alpha)}{\rho^2(x)} \lesssim 1 + |Q_\alpha(u)(x)|^2,$$

we end up with

$$(I) \lesssim \left(\iint_{\mathbb{S}^2} Q_\alpha(v)(x)^2 (1 + Q_\alpha(v)(x))^4 (1 + Q_\alpha(u)(x))^2 \frac{\mathrm{d}\alpha\, \mathrm{d}x}{|\alpha|^2} \right) \|W\|_{L^2}^2 .$$

On the other hand,

$$\begin{aligned}
&\iint_{\mathbb{S}^2} Q_\alpha(v)(x)^2 (1 + Q_\alpha(v)(x))^4 (1 + Q_\alpha(u)(x))^2 \frac{\mathrm{d}\alpha\, \mathrm{d}x}{|\alpha|^2} \\
&\lesssim \iint_{\mathbb{S}^2} Q_\alpha(v)(x)^2 (1 + Q_\alpha(v)(x))^4 \frac{\mathrm{d}\alpha\, \mathrm{d}x}{|\alpha|^2} \\
&+ \left(\iint_{\mathbb{S}^2} Q_\alpha(v)(x)^4 (1 + Q_\alpha(v)(x))^8 \frac{\mathrm{d}\alpha\, \mathrm{d}x}{|\alpha|^2} \right)^{\frac{1}{2}} \left(\iint_{\mathbb{S}^2} Q_\alpha(u)(x)^4 \frac{\mathrm{d}\alpha\, \mathrm{d}x}{|\alpha|^2} \right)^{\frac{1}{2}} \\
&\lesssim \|v\|_{\dot{H}^{\frac{1}{2}}}^2 \left(1 + \|v\|_{\dot{H}^{\frac{1}{2}}}\right)^4 \left(1 + \|u\|_{\dot{H}^{\frac{1}{2}}}\right)^2,
\end{aligned}$$

where we used the fact that $Q_\alpha(f)(x) = |\delta_\alpha f(x)| + |\delta_\alpha H f(x)|$ and

$$\iint_{\mathbb{S}^2} \left(|\delta_\alpha f(x)|^{2p} + |\delta_\alpha H f(x)|^{2p} \right) \frac{\mathrm{d}\alpha\, \mathrm{d}x}{|\alpha|^2} \lesssim ||f||_{\dot{H}^{\frac{1}{2}}}^{2p}$$

for any $p \geq 1$.

This gives

$$(I) \lesssim \|v\|_{\dot{H}^{\frac{1}{2}}}^2 \left(1 + \|v\|_{\dot{H}^{\frac{1}{2}}}\right)^4 \left(1 + \|u\|_{\dot{H}^{\frac{1}{2}}}\right)^2 \|W\|_{L^2}^2 .$$

On the other hand,

$$(II) \lesssim K(\varepsilon)^2 \kappa^{\frac{1}{2}} \|u\|_{\dot{H}^{\frac{1}{2}}}^4 \|W\|_{L^2}^2 .$$

Therefore, it follows from (8.3.10) that

$$
\begin{aligned}
A &\le \left(\int_{\mathbb{S}} \gamma^{-1} \big|[V, H_{\kappa,1}](\Lambda v)\big|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \|W\|_{L^2} \\
&\lesssim \big((I) + (II)\big)^{\frac{1}{2}} \|W\|_{L^2} \\
&\lesssim \|v\|_{\dot{H}^{\frac{1}{2}}} \big(1 + \|v\|_{\dot{H}^{\frac{1}{2}}}\big)^2 \big(1 + \|u\|_{\dot{H}^{\frac{1}{2}}}\big) \|W\|_{L^2}^2 + K(\varepsilon)\kappa^{\frac{1}{4}} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 \|W\|_{L^2}^2 .
\end{aligned}
$$

By combining this with (8.3.9), we get from (8.3.6) that there exists a constant C depending only on c_0 such that

(8.3.12)

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|v\|_{\dot{H}^{\frac{1}{2}}}^2 + \|W\|_{L^2}^2 \le{}& C \|v\|_{\dot{H}^{\frac{1}{2}}} \big(1 + \|v\|_{\dot{H}^{\frac{1}{2}}}\big)^2 \big(1 + \|u\|_{\dot{H}^{\frac{1}{2}}}\big) \|W\|_{L^2}^2 \\
&+ CK(\varepsilon)\kappa^{\frac{1}{4}} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 \|W\|_{L^2}^2 \\
&+ C\left(\kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} + C\varepsilon^{-\frac{1}{2}} \|u_0\|_{\dot{H}^{\frac{1}{2}}}\right) \|v\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} \\
&+ C\varepsilon^{-\frac{1}{2}} \Big(\|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2 \Big) \|W\|_{L^2} + \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|W\|_{L^2} \\
&+ C \|u_0\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}} .
\end{aligned}
$$

Using the Young's inequality, this immediately yields an inequality of the form

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|v\|_{\dot{H}^{\frac{1}{2}}}^2 + \Upsilon \|W\|_{L^2}^2 \le M \|v\|_{\dot{H}^{\frac{1}{2}}}^2 + F, \tag{8.3.13}
$$

where

$$
\begin{aligned}
M &:= 4C^2\varepsilon^{-1}\Big(\|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2 \Big)^2 + 1, \\
F &:= 4C^2\varepsilon^{-1}\Big(\|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|u_0\|_{\dot{H}^{\frac{1}{2}}}^2 \Big)^2, \\
\Upsilon &:= \frac{1}{4} - C \|v\|_{\dot{H}^{\frac{1}{2}}} \big(1 + \|v\|_{\dot{H}^{\frac{1}{2}}}\big)^2 \big(1 + \|u\|_{\dot{H}^{\frac{1}{2}}}\big) \\
&\quad - CK(\varepsilon)\kappa^{\frac{1}{4}} \|u\|_{\dot{H}^{\frac{1}{2}}}^2 \\
&\quad - C^2 \left(\kappa^{-1} \|u\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon^{-\frac{1}{2}} \|u_0\|_{\dot{H}^{\frac{1}{2}}}\right)^2 \|v\|_{\dot{H}^{\frac{1}{2}}}^2 .
\end{aligned}
$$

In particular, as long as $\Upsilon \geq 0$, we have

$$\|v(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq e^{2Mt} \|v(0)\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{e^{tM} - 1}{M} F.$$

If one further assumes that $tM \leq 1$, it follows that

$$\|v(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq e^{2Mt} \|v(0)\|^2_{\dot{H}^{\frac{1}{2}}} + tF.$$

Introduce the parameter

$$\nu(\varepsilon) := 2 \|v|_{t=0}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{S})} = 2 \left\| u_0 - e^{\varepsilon \partial_x^2} u_0 \right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{S})}.$$

Then choose ε small enough, so that

$$C\nu(\varepsilon)\big(1 + \nu(\varepsilon)\big)^2 \big(1 + 2 \|u_0\|_{\dot{H}^{\frac{1}{2}}}\big) \leq \frac{1}{16}.$$

We then fix κ small enough to that

$$CK(\varepsilon)\kappa^{\frac{1}{4}} (2 \|u_0\|_{\dot{H}^{\frac{1}{2}}})^2 \leq \frac{1}{16},$$

where recall that $K(\varepsilon) := \varepsilon^{-3}(1 + \left\|u_{0,\varepsilon}\right\|_{H^{\frac{1}{2}}})^3$.

We then deduce the wanted uniform estimate by an elementary continuation argument. □

8.3.3 Compactness

Previously, we have proved *a priori* estimates for the spatial derivatives. In this paragraph, we collect results from which we will derive estimates for the time derivative as well as for the nonlinearity. These estimates are used to pass to the limit in the equation.

Recall the notations introduced in the previous section, as well as the estimates proved there. Fix $c_0 > 0$. Given $\delta \in (0, 1]$ and an initial data $u_0 \in H^{\frac{1}{2}}(\mathbb{S})$ satisfying $u_0 \geq c_0$, we have seen that there exists a (global in time) solution u_δ to the Cauchy problem:

$$\begin{cases} \partial_t u_\delta + \dfrac{1}{\pi} \dfrac{u_\delta \Lambda u_\delta - (Hu_\delta)\partial_x u_\delta}{u_\delta^2 + (Hu_\delta)^2} - \delta \partial_x^2 u_\delta = 0, \\ u_\delta|_{t=0} = e^{\delta \partial_x^2} u_0. \end{cases} \tag{8.3.14}$$

Moreover, we have proved that one can fix ε small enough such that one can write u_δ under the form

$$u_\delta(t,x) = (e^{(\varepsilon+\delta)\partial_x^2} u_0)(x) + v_\delta(x),$$

and there exist $T > 0$ and $M > 0$ depending on u_0 such that, for all $\delta \in (0,1]$,

$$\sup_{t\in[0,T]} \|v_\delta(t)\|^2_{H^{\frac{1}{2}}} + \int_0^T \int_{\mathbb{S}} \frac{u_\delta \, |\Lambda v_\delta|^2}{u_\delta^2 + (Hu_\delta)^2} \,\mathrm{d}x\,\mathrm{d}t + \delta \int_0^T \|v_\delta\|^2_{\dot{H}^{\frac{3}{2}}} \,\mathrm{d}t \le M. \tag{8.3.15}$$

Now, to pass to the limit in the equation (8.3.14), we need to estimate the time derivative. Since $\partial_t u_\delta = \partial_t v_\delta$, it is sufficient to estimate the latter quantity. It is given by

$$\partial_t v_\delta = -V_\delta \partial_x v_\delta - \gamma_\delta \Lambda v_\delta + \delta \partial_x^2 v_\delta + R_\delta, \tag{8.3.16}$$

where

$$\gamma_\delta = \frac{1}{\pi} \frac{u_\delta}{u_\delta^2 + (Hu_\delta)^2} \quad , \quad V_\delta = -\frac{1}{\pi} \frac{Hu_\delta}{u_\delta^2 + (Hu_\delta)^2},$$
$$R_\delta = -\gamma_\delta(\Lambda e^{(\varepsilon+\delta)\partial_x^2} u_0) - V_\delta(\partial_x e^{(\varepsilon+\delta)\partial_x^2} u_0) + \delta \partial_x^2 e^{(\varepsilon+\delta)\partial_x^2} u_0.$$

As already seen in (8.3.7), we have $\gamma_\delta \lesssim_{c_0} 1$ and $|V_\delta| \lesssim_{c_0} 1$. By combining this with the fact that $e^{\varepsilon\partial_x^2}$ is a smoothing operator, we immediately see that

$$\|R_\delta\|_{L^\infty([0,T];L^2)} \lesssim_{c_0,\varepsilon} \|u_0\|_{\dot{H}^{\frac{1}{2}}} .$$

Here the implicit constant depends on ε, but this is harmless since ε is fixed now. On the other, directly from (8.3.15), we get that

$$\|\gamma_\delta \Lambda v_\delta\|_{L^2([0,T];L^2)} \lesssim_{c_0} \left\|\sqrt{\gamma_\delta}\Lambda v_\delta\right\|_{L^2([0,T];L^2)} \lesssim_{c_0} M,$$

and

$$\delta \left\|\partial_x^2 v_\delta\right\|_{L^2([0,T];H^{-\frac{1}{2}})} \le \sqrt{\delta} \left\|\partial_x^2 v_\delta\right\|_{L^2([0,T];H^{-\frac{1}{2}})} \le M.$$

It remains only to estimate the contribution of $V_\delta \partial_x v_\delta$. For this, we begin by proving that $(v_\delta)_{\delta\in(0,1]}$ is bounded in $L^p([0,T(\varepsilon)];\dot{H}^1(\mathbb{S}))$ for any $1 \le p < 2$. Indeed, we can write

(8.3.17)

$$
\begin{aligned}
\|v_\delta\|_{\dot H^1}^2 &\le \left\| \frac{u_\delta^2 + (Hu_\delta)^2}{u_\delta} \right\|_{L^\infty} \int_{\mathbb{S}} \frac{u_\delta\, |\Lambda v_\delta|^2}{u_\delta^2 + (Hu_\delta)^2}\,\mathrm{d}x \\
&\lesssim_{c_0} \|(u_\delta, Hu_\delta)\|_{L^\infty}^2 \int_{\mathbb{S}} \frac{u_\delta\, |\Lambda v|^2}{u_\delta^2 + (Hu_\delta)^2}\,\mathrm{d}x \\
&\lesssim_{c_0} \left(\|(v_\delta, Hv_\delta)\|_{L^\infty}^2 + \varepsilon^{-\frac{1}{2}} \|u_0\|_{H^{\frac{1}{2}}}^2 \right) \int_{\mathbb{S}} \frac{u_\delta\, |\Lambda v|^2}{u_\delta^2 + (Hu_\delta)^2}\,\mathrm{d}x \\
&\lesssim_{c_0} \left(\|v_\delta\|_{H^{\frac{1}{2}}}^2 \log\left(2 + \|v_\delta\|_{\dot H^1}\right) + \varepsilon^{-\frac{1}{2}} \|u_0\|_{H^{\frac{1}{2}}}^2 \right) \int_{\mathbb{S}} \frac{u_\delta\, |\Lambda v_\delta|^2}{u_\delta^2 + (Hu_\delta)^2}\,\mathrm{d}x,
\end{aligned}
$$

to conclude that

$$
\frac{\|v_\delta\|_{\dot H^1}^2}{\log(2 + \|v_\delta\|_{\dot H^1})} \lesssim \int_{\mathbb{S}} \frac{u_\delta |\Lambda v_\delta|^2}{u_\delta^2 + (Hu_\delta)^2}\,\mathrm{d}x,
$$

where the implicit constant depends on $c_0, \varepsilon, \|u_0\|_{H^{\frac{1}{2}}}$ and M. Remembering that $|V_\delta| \lesssim_{c_0} 1$, it immediately follows that, for any $p \in [1,2)$, $(V_\delta \partial_x v_\delta)_{\delta \in (0,1]}$ is bounded in $L^p([0,T]; L^2)$.

Now, by combining all the previous estimates, we see that $(v_\delta)_{\delta\in(0,1]}$ is bounded in the space

$$
X_p = \left\{ u \in C^0([0,T]; H^{\frac{1}{2}}(\mathbb{S})) \cap L^p([0,T]; H^1(\mathbb{S}))\,;\, \partial_t u \in L^p([0,T]; H^{-\frac{1}{2}}(\mathbb{S})) \right\}.
$$

Since $H^{\frac{1}{2}}(\mathbb{S})$ (resp. $H^1(\mathbb{S})$) is compactly embedded into $H^s(\mathbb{S})$ (resp. $H^{\frac{1}{2}+s}(\mathbb{S})$ for any $s < 1/2$, by the classical Aubin-Lions lemma, this in turn implies that one can extract a sequence $(u_{\delta_n})_{n\in\mathbb{N}}$ which converges strongly in

$$
C^0([0,T]; H^s(\mathbb{S})) \cap L^p([0,T]; H^{\frac{1}{2}+s}(\mathbb{S})).
$$

Then it is elementary to pass to the limit in the equation.

Exercise 8.3.4. *The goal of this exercise is to prove Lemma 8.3.1.*

We begin by considering a modified system, namely

$$
\begin{cases}
\partial_t u + \dfrac{1}{\pi} \dfrac{u\Lambda u - (Hu)\partial_x u}{u^2 + (Hu)^2} - \delta \partial_x^2 u = 0, \\
u|_{t=0} = e^{\delta \partial_x^2} u_0.
\end{cases}
\tag{8.3.18}
$$

- *Write the Cauchy problem (8.3.18) under the following form*

$$\partial_t u - \delta\partial_x^2 u = F_\delta, \quad u|_{t=0} = e^{\delta\partial_x^2}u_0, \tag{8.3.19}$$

where

$$F_\delta = -\frac{1}{\pi}\frac{u\Lambda u - (Hu)\partial_x u}{\delta + u^2 + (Hu)^2}\cdot$$

Prove that for $\nu > 0$,

$$\begin{aligned}\|F_\delta(u) - F_\delta(v)\|_{L^2} \lesssim_\delta & \left(\|u\|_{H^{\frac{1}{2}+\nu}} + \|v\|_{H^{\frac{1}{2}+\nu}} \right) \|u - v\|_{H^1} \\ & + \left(\|u\|_{H^1} + \|v\|_{H^1} \right) \|u - v\|_{H^{\frac{1}{2}+\nu}},\end{aligned}$$

where the notation $\lesssim_\delta$ is intended to indicate that the implicit constant depends on δ.

- *Deduce that the Cauchy problem for (8.3.18) has a unique mild solution in $C^0([0,T_\delta);L^2(\mathbb{S})) \cap L^2(0,T_\delta;H^1(\mathbb{S}))$.*

- *Show that we have the following alternative: either*

$$T_\delta = +\infty \qquad or \qquad \limsup_{t\to T_\delta}\|u(t)\|_{L^2} = +\infty. \tag{8.3.20}$$

- *Use*

$$\|F_\delta\|_{L^2} \lesssim_\delta \|u\|_{H^1},$$

to obtain that the solution exists globally in time.

- *By using the nonlinear estimates in Sobolev spaces, show that the Cauchy problem is well-posed on $H^s(\mathbb{S})$ for all $s > 3/2$.*

- *Recall why at a point x_t where the function $u(t,\cdot)$ reaches its minimum, we have*

$$\partial_x u(t,x_t) = 0, \quad \partial_x^2 u(t,x_t) \geq 0, \quad \Lambda u(t,x_t) \leq 0.$$

(Hint: use an expression for Λ in terms of singular integrals.)

Deduce that $\inf_{x\in\mathbb{S}} u(t,x) \geq \inf_{x\in\mathbb{S}} u_0(x)$.

By similar arguments, we obtain the second inequality $\sup_{x\in\mathbb{S}} u(t,x) \leq \sup_{x\in\mathbb{S}} u_0(x)$.

Then explain how to deal with the original equation (8.3.1).

Chapter 9

Control and Stabilization of water-waves

In this chapter, we give applications of the techniques developed in Chapter 7 to study the observability, controllability, and stabilization of water wave equations with surface tension. The question we want to understand is: which water waves can be generated or damped by means of pressure perturbations applied over a small portion of the free surface.

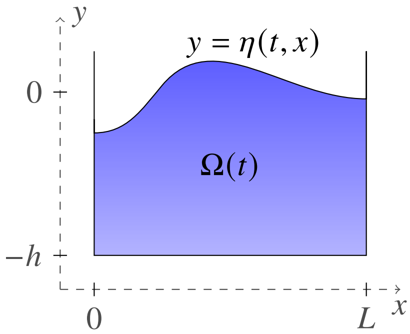
9.1 Introduction

We introduced the water wave equations in Chapter 1 (see in particular Proposition 1.2.5). In this section, we introduce a more general situation where: i) the waves evolve inside a container with lateral boundaries and ii) we take into consideration an external pressure. To simplify the presentation, we assume that the fluid domain is two-dimensional; however, the results proved in this chapter are valid in an arbitrary dimension.

More precisely, we consider the incompressible Euler equations for a potential flow in a fluid domain located between with a free surface, two vertical walls and a flat bottom, which is at time t of the form

$$\Omega(t) = \{ (x, y) \in [0, L] \times \mathbb{R} \, : \, -h < y < \eta(t, x) \},$$

where L is the length of the basin, h is its depth and η, the free surface elevation, is an unknown function.



The Eulerian velocity field v is assumed to be irrotational. It follows that $v = \nabla_{x,y}\phi$ for some time-dependent potential ϕ satisfying

$$\Delta_{x,y}\phi = 0, \quad \partial_t\phi + \frac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + P + gy = 0, \tag{9.1.1}$$

where $g > 0$ is the gravity acceleration, P is the pressure, $\nabla_{x,y} = (\partial_x, \partial_y)$ and $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. In addition we assume that the velocity satisfies the solid wall boundary condition on the bottom and the lateral walls:

$$\begin{aligned} \partial_y\phi = 0 \quad &\text{on } y = -h, \\ \partial_x\phi = 0 \quad &\text{on } x = 0 \text{ or } x = L, \end{aligned} \tag{9.1.2}$$

The water waves equations are then given by two boundary conditions on the free surface: the classical kinematic boundary condition, describing the deformations of the domain,

$$\partial_t\eta = \sqrt{1 + (\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta},$$

where ∂_n is the outward normal derivative, so $\sqrt{1 + (\partial_x\eta)^2}\,\partial_n\phi = \partial_y\phi - (\partial_x\eta)\partial_x\phi$. Secondly, the balance of forces across the free surface reads

$$P|_{y=\eta(t,x)} = P_{\text{ext}}(t, x) + \lambda\kappa(\eta),$$

where λ is a positive constant, P_{ext} is an external source term (whose sign does not matter for our purposes) and $\kappa(\eta)$ is the curvature of the free surface:

$$\kappa(\eta) := -\partial_x\left(\frac{\partial_x\eta}{\sqrt{1 + (\partial_x\eta)^2}}\right) = -\frac{\partial_x^2\eta}{(1 + (\partial_x\eta)^2)^{3/2}}.$$

Controllability

In this chapter, we present results which are concerned with the generation and the absorption of water waves by means of pressure disturbances applied above a small portion of the free surface.

The first question is the following : given

- a time $T > 0$,
- an initial state (η_{in}, ϕ_{in}) and a final state $(\eta_{final}, \phi_{final})$ in some space,
- a small subset $\omega = (a, b) \subset [0, L]$,

is-it possible to find a function $P_{ext}(t, x)$ supported in $[0, T] \times \omega$ such that the solution to the water-wave equations with initial data (η_{in}, ϕ_{in}) satisfies $(\eta, \phi)|_{t=T} = (\eta_{final}, \phi_{final})$?

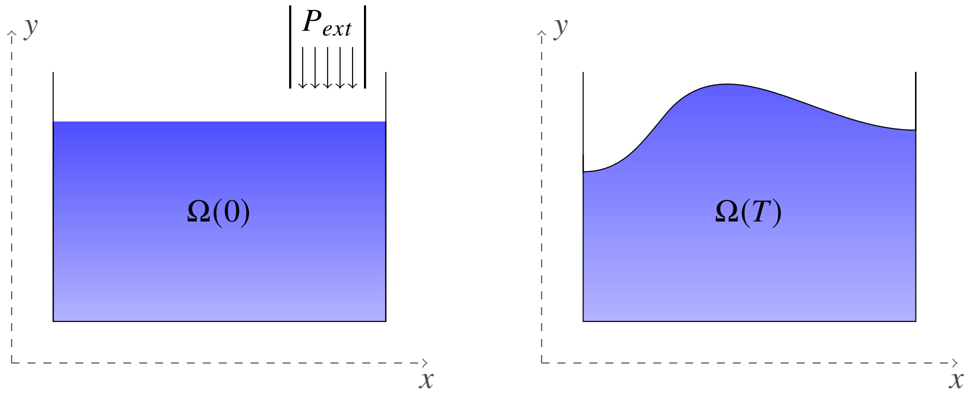


Figure 9.1: Generation of a water wave in time T starting from a fluid at rest, which means that $(\eta_{in}, \phi_{in}) = (0, 0)$.

Stabilization

Think of a rectangular wave pool, with vertical walls, with a wave generator at one end. The waves generated by the wave generator will be be reflected on the opposite side and then interact with the wave produced by the wave generator. Therefore, to experimentally simulate the propagation in the open sea, it is necessary to introduce wave absorbers to minimize the reflection of the waves. The same problem arises for the numerical analysis of the wave equations. Indeed, for computational reasons, one must work in a bounded domain. To simulate the propagation in an unbounded domain such as the open sea, one can damp the outgoing waves in an absorption zone surrounding the boundary of the boundary. The mathematical study of the damping properties of these absorbers corresponds to the question of the stabilization of the water wave equations.

To state this problem, let us introduce the energy $\mathcal{E} = \mathcal{E}(t)$, defined by

$$(9.1.3)\quad \mathcal{E} = \frac{g}{2}\int_0^L \eta^2 \,\mathrm{d}x + \lambda \int_0^L \left(\sqrt{1+\eta_x^2} - 1\right) \mathrm{d}x + \frac{1}{2}\int_0^L \int_{-h}^{\eta(t,x)} \left|\nabla_{x,y}\phi\right|^2 \mathrm{d}y \,\mathrm{d}x.$$

This is the sum of the gravitational potential energy, a surface energy due to stretching of the surface and the kinetic energy. Recall that the energy is conserved when there is no external pressure, which means that if $P_{ext} = 0$ then $\mathcal{E}(t) = \mathcal{E}(0)$ for all time. The stabilization problem for the water-wave equations consists in finding a pressure law, relating P_{ext} to the unknown (η, ψ), such that:

i) $\mathcal{E}$ is decreasing and converges to zero;

ii) $\partial_x P_{ext}$ is supported inside a small subset of $[0, L]$.

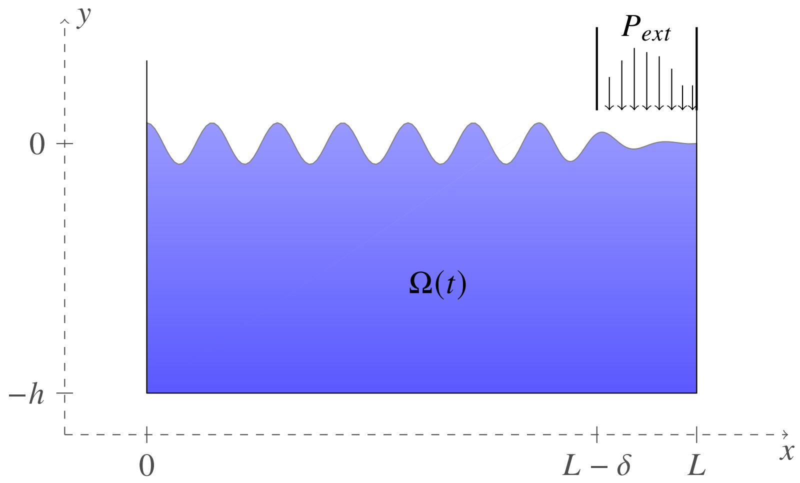


Figure 9.2: Absorption of water waves in the neighborhood of $x = L$ by means of an external counteracting pressure produced by blowing above the free surface.

9.2 Extension to periodic functions

We pause in this section to explain how, instead of working in a domain with lateral boundaries, one can reduce the study to a periodic problem in x.

This is a classical idea that goes back to to Boussinesq (see [93, page 37]). The idea is to extend the initial data to periodic functions, solve the Cauchy problem for these extended initial data and then obtain a solution to the water wave equations in a bounded container by considering the restrictions of these solutions.

To do this, following a classical idea, we use a reflection/periodization procedure with respect to the normal variable at the boundary of the container, as illustrated below.

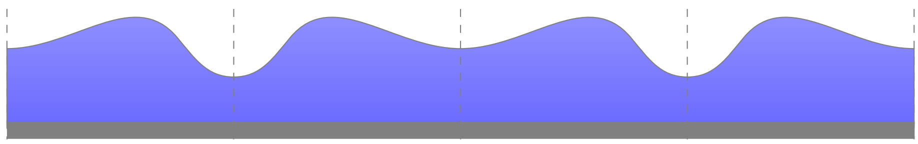


Figure 9.3: Periodization of the domain (see Boussinesq [93]).

A question naturally arises about the regularity of the functions obtained by this process. Indeed, if we consider the graph obtained by this process, starting from from the smooth function $[0, \pi] \ni x \mapsto x \in [0, \pi]$, then we obtain a new function which is simply Lipschitz. To avoid these singularities, we must assume that the normal derivative vanishes.

Since this method is of independent interest, we work with 2D or 3D fluid domains. Let $d \in \{1, 2\}$, $Q = [0, L_1]$ if $d = 1$ and $Q = [0, L_1] \times [0, L_2]$ if $d = 2$. One denotes by ν the outward unit normal to Q ($\nu = (1, 0)$ if $x_1 = L_1$, $\nu = (0, -1)$ if $x_2 = 0$,...).

Definition 9.2.1. *Given $1 \leq d \leq 2$ and $\sigma > 3/2$, one denotes by $H_e^\sigma(Q)$ the space*

$$H_e^\sigma(Q) = \{v \in H^\sigma(Q) : \partial_\nu v = 0 \text{ on } \partial Q\}.$$

(We refer the reader to Lions-Magenes [315] or to the proof of Proposition 9.2.4 for comments about the fractional spaces $H^\sigma(Q)$.)

Definition 9.2.2. *Let $v : Q \to \mathbb{R}$. If $d = 2$, we define $\widetilde{v} : \mathbb{R}^2 \to \mathbb{R}$ as the unique* *extension of v satisfying*

$$\widetilde{v}(x) = v(x) \quad \forall x \in Q, \tag{9.2.1}$$

$$\widetilde{v}(-x_1, x_2) = \widetilde{v}(x_1, x_2) = \widetilde{v}(x_1, -x_2) \quad \forall x \in \mathbb{R}^2, \tag{9.2.2}$$

$$\widetilde{v}(x_1 + 2L_1, x_2) = \widetilde{v}(x) = \widetilde{v}(x_1, x_2 + 2L_2) \quad \forall x \in \mathbb{R}^2. \tag{9.2.3}$$

Similarly, when $d = 1$, $\widetilde{v} \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\widetilde{v}(x) = v(x) \quad \forall x \in Q, \tag{9.2.4}$$

$$\widetilde{v}(-x) = \widetilde{v}(x) \quad \forall x \in \mathbb{R}, \tag{9.2.5}$$

$$\widetilde{v}(x + 2L_1) = \widetilde{v}(x) \quad \forall x \in \mathbb{R}. \tag{9.2.6}$$

Definition 9.2.3. *Given $\sigma \in \mathbb{R}$, denote by $H^\sigma_e(\mathbb{T}^d)$ the Sobolev space of those periodic functions which are even (satisfying (9.2.2)-(9.2.3) when $d = 2$ and (9.2.5)-(9.2.6) for $d = 1$).*

Now consider the case $d = 1$ (to fix notations) and $u \in H^\sigma_e(\mathbb{T})$ with $\sigma > d/2 + 1 = 3/2$. Then, $\partial_x u(x)$ is C^0 and odd which implies that $\partial_x u(0) = 0$. Moreover, one has $u(L_1 + \varepsilon) = u(-L_1 + \varepsilon) = u(L_1 - \varepsilon)$ and hence one has also $\partial_x u(L_1) = 0$ (then $\partial_x u(nL_1) = 0$ for any $n \in \mathbb{Z}$). We have a similar result when $d = 2$. This proves that

$$\forall \sigma > \frac{d}{2} + 1, \ \forall v \in H^\sigma_e(\mathbb{T}^d), \qquad v|_Q \in H^\sigma_e(Q). \tag{9.2.7}$$

Conversely, the following result shows that any function v in $H^\sigma_e(Q)$, with $\sigma \in (3/2, 7/2)$, is the restriction to Q of a function belonging to $H^\sigma_e(\mathbb{T}^d)$.

Proposition 9.2.4. *Let $1 \le d \le 2$ and $\frac{3}{2} < \sigma < \frac{7}{2}$. Then the map $v \mapsto \widetilde{v}$ is continuous from $H^\sigma_e(Q) = \{v \in H^\sigma(Q) : \partial_\nu v = 0 \text{ on } \partial Q\}$ to $H^\sigma_e(\mathbb{T}^d)$.*

To prove the previous proposition, notice that $\partial_{x_2} \widetilde{v} = \widetilde{\partial_{x_2} v}$. This means that the result is clearly a one dimensional result and it is enough to prove it for the one dimensional case, in which case it is a direct consequence of the following

9.3 Controllability

We begin by discussing the controllability problem. We use the Craig-Sulem-Zakharov formulation as explained in Proposition 1.2.5. Recall the notations and this formulation. Set

$$\psi(t, x) := \phi(t, x, \eta(t, x)),$$

and define the Dirichlet to Neumann operator, denoted by $G(\eta)$, relating ψ to the normal derivative of the potential by

$$(G(\eta)\psi)(t,x) = \sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta(t,x)}.$$

Then (η,ψ) solves

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + g\eta + \dfrac{1}{2}(\partial_x\psi)^2 - \dfrac{1}{2}\dfrac{\big(G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)\big)^2}{1+(\partial_x\eta)^2} + \lambda\kappa(\eta) + P_{\text{ext}} = 0. \end{cases} \tag{9.3.1}$$

This system is augmented with initial data

$$\eta|_{t=0} = \eta_{in}, \quad \psi|_{t=0} = \psi_{in}. \tag{9.3.2}$$

We consider the case when η and ψ are 2π-periodic in the space variable x and we set $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. Recall that the mean value of η is conserved in time and can be taken to be 0 without loss of generality. We thus introduce the Sobolev spaces $H_0^\sigma(\mathbb{T})$ of functions with mean value 0. Our main result asserts that, given any control domain ω and any arbitrary control time $T > 0$, the equation (9.3.1) is controllable in time T for small enough data.

The following result asserts that, in arbitrarily small time, one can generate any small amplitude, two-dimensional, gravity-capillary water waves.

Theorem 9.3.1 (Alazard, Baldi, Han-Kwan [17] and Zhu [458]). *i) Assume that $\lambda > 0$. Let $T > 0$ and consider a non-empty open subset $\omega \subset \mathbb{T}$. There exist σ large enough and a positive constant M_0 small enough such that, for any two pairs of functions (η_{in},ψ_{in}), $(\eta_{final},\psi_{final})$ in $H_0^{\sigma+\frac{1}{2}}(\mathbb{T})\times H^\sigma(\mathbb{T})$ satisfying*

$$\|\eta_{in}\|_{H^{\sigma+\frac{1}{2}}} + \|\psi_{in}\|_{H^\sigma} < M_0, \quad \|\eta_{final}\|_{H^{\sigma+\frac{1}{2}}} + \|\psi_{final}\|_{H^\sigma} < M_0,$$

there exists P_{ext} in $C^0([0,T];H^\sigma(\mathbb{T}))$, supported in $[0,T]\times\omega$, that is

$$\operatorname{supp} P_{ext}(t,\cdot) \subset \omega, \quad \forall t \in [0,T],$$

such that the Cauchy problem (9.3.1)-(9.3.2) has a unique solution

$$(\eta,\psi) \in C^0([0,T];H_0^{\sigma+\frac{1}{2}}(\mathbb{T})\times H^\sigma(\mathbb{T})),$$

and the solution (η,ψ) satisfies $(\eta|_{t=T},\psi|_{t=T}) = (\eta_{final},\psi_{final})$.

ii) The result extends in any dimension provided that the domain ω satisfies the geometric control condition (see [458]).

Remark 9.3.2. *i) This result holds for any $T > 0$ and not only for T large enough. Compared to the Cauchy problem, for the control problem it is more difficult to work on short time intervals than on large time intervals.*

ii) This result holds also in the infinite depth case.

In the rest of this section, our goal is to sketch the proof of the controllability result.

Step 1: Reduction to a dispersive equation

The proof is based on the fact that the water waves equation is a dispersive equation.

Recall that $G(0)$ is the Fourier multiplier $|D_x|\tanh(h\,|D_x|)$. Removing quadratic and higher order terms in the equations, System (9.3.1) becomes

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi + g\eta - \lambda \partial_x^2 \eta = P_{\text{ext}}. \end{cases}$$

Introduce the Fourier multiplier (of order $3/2$)

$$L := \big((g - \lambda\partial_x^2)G(0)\big)^{\frac{1}{2}}.$$

The operator $G(0)^{-1}$ is well-defined on periodic functions with mean value zero. Then $u = \psi - iLG(0)^{-1}\eta$ satisfies the dispersive equation

$$\partial_t u + iLu = P_{ext}.$$

It is proved in Chapter 7 that there are symbols $p = p(t,x,\xi)$ and $q = q(t,x,\xi)$ with p of order 0 in ξ and q of order $1/2$, such that $u = T_p\psi + iT_q\eta$ satisfies an equation of the form

$$P(u)u = P_{ext} \quad \text{with} \quad P(u) := \partial_t + T_{V(u)}\partial_x + iL^{\frac{1}{2}}\big(T_{c(u)}L^{\frac{1}{2}}\cdot\big),$$

where $L^{\frac{1}{2}} = \big((g - \kappa\partial_x^2)G(0)\big)^{\frac{1}{4}}$, $T_{V(u)}$ and $T_{c(u)}$ are paraproducts. Here V, c depend on the unknown u with $V(0) = 0$ and $c(0) = 1$, and hence $P(0) = \partial_t + iL$ is the linearized operator around the null solution. Furthermore, the mapping $(\eta,\psi) \mapsto u$ is invertible (under a smallness assumption) and, up to modifying the sub-principal symbols of p and q, one can further require that

$$\int_{\mathbb{T}} \operatorname{Im} u(t,x)\,\mathrm{d}t = 0. \tag{9.3.3}$$

Here we have simplified the result (neglecting remainder terms and simplifying the dependence of V, c on u) and we refer the original article [17] for the full statement.

Step 2: Quasi-linear scheme

Since the water waves system (9.3.1) is quasi-linear, one cannot deduce the controllability of the nonlinear equation from the one of $P(0)$. Instead of using a fixed point argument, we use a quasi-linear scheme and seek P_{ext} as the limit of *real-valued* functions P_n determined by means of approximate control problems. To guarantee that P_{ext} will be real-valued we seek P_n as the real part of some function. To insure that $\operatorname{supp} P_n \subset \omega$ we seek P_n under the form

$$P_n = \chi_\omega \operatorname{Re} f_n.$$

Hereafter, we fix ω, a non-empty open subset of $\mathbb{T}$, and a C^∞ cut-off function χ_ω, supported on ω, such that $\chi_\omega(x) = 1$ for all x in some open interval $\omega_1 \subset \omega$.

The approximate control problems are defined by induction as follows: we choose f_{n+1} by requiring that the unique solution u_{n+1} of the Cauchy problem

$$P(u_n)u_{n+1} = \chi_\omega \operatorname{Re} f_{n+1}, \quad u_{n+1}|_{t=0} = u_{in}$$

satisfies $u_{n+1}(T) = u_{final}$. Our goal is to prove that

- this scheme is well-defined (that is one has to prove a controllability result for $P(u_n)$);
- the sequences (f_n) and (u_n) are bounded in $C^0([0,T]; H^\sigma(\mathbb{T}))$;
- the series $\sum(f_{n+1} - f_n)$ and $\sum(u_{n+1} - u_n)$ converge in $C^0([0,T]; H^{\sigma-\frac{3}{2}}(\mathbb{T}))$.

It follows that (f_n) and (u_n) are Cauchy sequences in $C^0([0,T]; H^{\sigma-\frac{3}{2}}(\mathbb{T}))$ (and in fact, by interpolation, in $C^0([0,T]; H^{\sigma'}(\mathbb{T}))$ for any $\sigma' < \sigma$).

To use the quasi-linear scheme, we need to study a sequence of linear approximate control problems. The key point is to study the control problem for the linear operator $P(\underline{u})$ for some given function $\underline{u}$. Our goal is to prove the following result.

Proposition 9.3.3. *Let $T > 0$. There exists s_0 such that, if $\|\underline{u}\|_{C^0([0,T];H^{s_0})}$ is small enough, depending on T, then the following properties hold.*

i) (Controllability) For all $\sigma \geq s_0$ and all

$$u_{in}, u_{final} \in \tilde{H}^\sigma(\mathbb{T}) := \left\{ w \in H^\sigma(\mathbb{T})\, ;\, \operatorname{Im} \int_{\mathbb{T}} w(x)\, \mathrm{d}x = 0 \right\},$$

there exists f satisfying $\|f\|_{C^0([0,T];H^\sigma)} \leq K(T)(\|u_{in}\|_{H^\sigma} + \|u_{final}\|_{H^\sigma})$ such that the unique solution u to

$$P(\underline{u})u = \chi_\omega \operatorname{Re} f \quad ;\ u_{|t=0} = u_{in},$$

satisfies $u(T) = u_{final}$.

ii) (Stability) Consider another state $\underline{u}'$ with $\|\underline{u}'\|_{C^0([0,T];H^{s_0})}$ small enough and denote by f' the control associated to $\underline{u}'$. Then

$$\|f - f'\|_{C^0([0,T];H^{\sigma-\frac{3}{2}})} \leq K'(T)(\|u_{in}\|_{H^\sigma} + \|u_{final}\|_{H^\sigma}) \|\underline{u} - \underline{u}'\|_{C^0([0,T];H^{s_0})}.$$

Again, we simplified the assumptions and refer the reader to the original article for the full statement.

Step 3: Reduction to a regularized problem

We next reduce the analysis by proving that it is sufficient

- to consider a *classical* equation instead of a *paradifferential* equation;
- to prove a L^2-result instead of a Sobolev-result.

This is obtained by commuting $P(\underline{u})$ with some well-chosen elliptic operator $\Lambda_{h,s}$ of order s with

$$s = \sigma - \frac{3}{2}$$

and depending on a small parameter denoted by $\hbar$ to avoid confusion with the depth h (the reason to introduce $\hbar$ is explained below). In particular $\Lambda_{\hbar,s}$ is chosen so that the operator

$$P(\underline{u}) := \Lambda_{\hbar,s} P(\underline{u}) \Lambda_{\hbar,s}^{-1}$$

satisfies

(9.3.4) $$P(\underline{u}) = P(\underline{u}) + R(\underline{u})$$

where $R(\underline{u})$ is a remainder term of order 0. For instance, if $s = 3m$ with $m \in \mathbb{N}$, set

$$\Lambda_{\hbar,s} = I + \hbar^s \mathcal{L}^{\frac{2s}{3}} \quad \text{where } \mathcal{L} := L^{\frac{1}{2}}\big(T_c L^{\frac{1}{2}} \cdot \big).$$

With this choice one has $[\Lambda_{\hbar,s}, \mathcal{L}] = 0$ so (9.3.4) holds with $R(\underline{u}) = [\Lambda_{\hbar,s}, T_{V(\underline{u})}]\Lambda_{\hbar,s}^{-1}$. It follows from symbolic calculus that $\left\| R(\underline{u}) \right\|_{\mathcal{L}(L^2)} \lesssim \|V\|_{W^{1,\infty}}$ uniformly in $\hbar$.

Moreover, since $V(\underline{u})$ and $c(\underline{u})$ are continuous in time with values in $H^{s_0}(\mathbb{T})$ with s_0 large, one can replace paraproducts by usual products, up to remainder terms in $C^0([0,T]; \mathcal{L}(L^2))$. We have

$$P(\underline{u}) = \partial_t + V(\underline{u})\partial_x + iL^{\frac{1}{2}}\big(c(\underline{u})L^{\frac{1}{2}} \cdot \big) + R_2(\underline{u})$$

where

$$R_2(\underline{u}) := R(\underline{u}) + \big(T_{V(\underline{u})} - V(\underline{u})\big)\partial_x + iL^{\frac{1}{2}}\Big(\big(T_{c(\underline{u})} - c(\underline{u})\big)L^{\frac{1}{2}} \cdot \Big).$$

The remainder $R_2(\underline{u})$ belongs to $C^0([0,T]; \mathcal{L}(L^2))$ uniformly in $\hbar$. On the other hand,

$$\left\| [\Lambda_{\hbar,s}, \chi_\omega]\Lambda_{\hbar,s}^{-1} \right\|_{\mathcal{L}(L^2)} = O(h), \tag{9.3.5}$$

which is the reason to introduce the parameter $\hbar$. The key point is that one can reduce the proof of Proposition 9.3.3 to the proof of the following result.

Proposition 9.3.4. *Let $T > 0$. There exists s_0 such that, if $\left\|\underline{u}\right\|_{C^0([0,T];H^{s_0})}$ is small enough, then the following properties hold.*

i) (Controllability) For all $v_{in} \in L^2(\mathbb{T})$ there exists f with $\|f\|_{C^0([0,T];L^2)} \le K(T)\,\|v_{in}\|_{L^2}$ such that the unique solution v to $P(\underline{u})v = \chi_\omega \operatorname{Re} f$, $\quad v_{|t=0} = v_{in}$ is such that $v(T)$ is an imaginary constant:

$$\exists b \in \mathbb{R} \,/\, \forall x \in \mathbb{T}, \quad v(T,x) = ib.$$

ii) (Regularity) Moreover $\|f\|_{C^0([0,T];H^{\frac{3}{2}})} \le K(T)\,\|v_{in}\|_{H^{\frac{3}{2}}}$.

iii) (Stability) Consider another state $\underline{u}'$ with $\left\|\underline{u}'\right\|_{C^0([0,T];H^{s_0})}$ small enough and denote by f' the control associated to $\underline{u}'$. Then

$$\|f - f'\|_{C^0([0,T];L^2)} \le K'(T)\,\|v_{in}\|_{H^{\frac{3}{2}}} \left\|\underline{u} - \underline{u}'\right\|_{C^0([0,T];H^{s_0})}.$$

Let us explain how to deduce Proposition 9.3.3 from the latter proposition. Consider u_{in}, u_{final} in $\tilde{H}^\sigma(\mathbb{T})$ and seek $f \in C^0([0,T]; H^\sigma(\mathbb{T}))$ such that

$$P(\underline{u})u = \chi_\omega \operatorname{Re} f, \quad u(0) = u_{in} \quad \Longrightarrow \quad u(T) = u_{final}.$$

Since the equation is reversible in time, it is sufficient to consider the case where $u_{final} = 0$. Now, to deduce this result from Proposition 9.3.4, the main difficulty is that the conjugation with $\Lambda_{\hbar,s}$ introduces a nonlocal term: indeed, $\Lambda_{\hbar,s}^{-1}(\chi_\omega f)$ is not compactly supported in general. This is a possible source of difficulty since we seek a localized control term. We overcome this problem by considering the control problem for $P(\underline{u})$ associated to some well-chosen initial data v_{in}. Proposition 9.3.4 asserts that for all $v_{in} \in H^{\frac{3}{2}}(\mathbb{T})$ there is $\widetilde{f} \in C^0([0,T]; H^{\frac{3}{2}}(\mathbb{T}))$ such that

$$P(\underline{u})v_1 = \chi_\omega \operatorname{Re} \widetilde{f}, \quad v_1|_{t=0} = v_{in} \Longrightarrow v_1(T,x) = ib, \ b \in \mathbb{R}.$$

Define $\mathcal{K} v_{in} = v_2(0)$ where v_2 is the solution to

$$P(\underline{u})v_2 = \left[\Lambda_{\hbar,s}, \chi_\omega\right]\Lambda_{\hbar,s}^{-1} \operatorname{Re} \widetilde{f}, \quad v_2|_{t=T} = 0.$$

Using (9.3.5)ne can prove that the $\mathcal{L}(H^{\frac{3}{2}})$-norm of $\mathcal{K}$ is $O(\hbar)$ and hence $I + \mathcal{K}$ is invertible for $\hbar$ small. So, v_{in} can be so chosen that $v_{in} + \mathcal{K} v_{in} = \Lambda_{\hbar,s} u_{in}$. Then, setting $f := \Lambda_{\hbar,s}^{-1} \widetilde{f}$ and $u := \Lambda_{\hbar,s}^{-1}(v_1 + v_2)$, one checks that

$$P(\underline{u})u = \chi_\omega \operatorname{Re} f, \quad u(0) = u_{in}, \quad u(T,x) = ib, \ b \in \mathbb{R}.$$

It remains to prove that $u(T)$ is not only an imaginary constant, but it is 0. This follows from the property (9.3.3). Indeed, P can be so defined that if $P(\underline{u})u$ is a real-valued function, then $\frac{d}{dt} \int_{\mathbb{T}} \operatorname{Im} u(t,x)\,dx = 0$. Since $\int_{\mathbb{T}} \operatorname{Im} u(0,x)\,dx = 0$ by assumption, one deduces that $\int_{\mathbb{T}} \operatorname{Im} u(T,x)\,dx = 0$ and hence $u(T) = 0$.

Step 4: Reduction to a constant coefficient equation

The controllability of $P(\underline{u})$ will be deduced from the classical HUM method. A key step in the HUM method consists in proving that some bilinear mapping is coercive. To determine the appropriate bilinear mapping, we follow an idea introduced in [16] and conjugate $P(\underline{u})$ to a constant coefficient operator modulo a remainder term of order 0.

To do so, we use a change of variables and a pseudo-differential change of unknowns to find an operator $M(\underline{u})$ such that

$$M(\underline{u})P(\underline{u})M(\underline{u})^{-1} = \partial_t + iL + \mathcal{R}(\underline{u}),$$

where $\left\|\mathcal{R}(\underline{u})\right\|_{\mathcal{L}(L^2)} \lesssim \left\|\underline{u}\right\|_{H^{s_0}}$ (and hence $\mathcal{R}(\underline{u})$ is a *small*erturbation of order 0).

To find $M(\underline{u})$, we begin by considering three changes of variables of the form

$$(1 + \partial_x \kappa(t,x))^{\frac{1}{2}}\, h(t, x + \kappa(t,x)), \quad h(a(t), x), \quad h(t, x - b(t)),$$

to replace $P(\underline{u})$ with

$$Q(\underline{u}) = \partial_t + W\partial_x + iL + R_3, \tag{9.3.6}$$

where $W = W(t,x)$ satisfies $\int_{\mathbb{T}} W(t,x)\,\mathrm{d}x = 0$, $\|W\|_{C^0([0,T];H^{s_0-d})} \lesssim \left\|\underline{u}\right\|_{C^0([0,T];H^{s_0})}$ where $d > 0$ is a universal constant, and R_3 is of order zero. This is not trivial since the equation is nonlocal and also because this exhibits a cancellation of a term of order $1/2$. Indeed, in general the conjugation of $L^{\frac{1}{2}}\big(c(\underline{u})L^{\frac{1}{2}}\,\cdot\big)$ and a change of variables generates also a term of order $3/2-1$. This term disappears here since we consider transformations which preserve the $L^2(dx)$ scalar product.

We next seek an operator A such that $i\big[A, |D_x|^{\frac{3}{2}}\big] + W\partial_x A$ is a zero order operator. This leads to consider, following [16], a pseudo-differential operator $A = \mathrm{Op}(a)$ for some symbol $a = a(x,\xi)$ in the Hörmander class $S^0_{\rho,\rho}$ with $\rho = \frac{1}{2}$, namely $a = \exp(i|\xi|^{\frac{1}{2}}\beta(t,x))$ for some function β depending on W.

Step 5: Observability

Then, we establish an observability inequality. That is, we prove that there exists $\varepsilon > 0$ such that for any initial data v_0 whose mean value $\langle v_0\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} v_0(x)\,\mathrm{d}x$ satisfies

$$|\mathrm{Re}\langle v_0\rangle| \geq \frac{1}{2}\,|\langle v_0\rangle| - \varepsilon\,\|v_0\|_{L^2}\,, \tag{9.3.7}$$

the solution v of

$$\partial_t v + iLv = 0, \quad v(0) = v_0$$

satisfies

$$\int_0^T \int_\omega |\mathrm{Re}(Av)(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \geq K \int_{\mathbb{T}} |v_0(x)|^2\,\mathrm{d}x. \tag{9.3.8}$$

To prove this inequality with the real-part in the left-hand side allows to prove the existence of a real-valued control function.

The observability inequality is deduced using a variant of Ingham's inequality. Recall that Ingham's inequality is an inequality for the L^2-norm of a sum of oscillatory functions which generalizes Parseval's inequality (it applies to pseudo-periodic functions and not only to periodic functions). For example, one such result asserts that for any $T > 0$ there exist two positive constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that

$$C_1 \sum_{n\in\mathbb{Z}} |w_n|^2 \leq \int_0^T \left| \sum_{n\in\mathbb{Z}} w_n e^{in|n|^{\frac{1}{2}}t} \right|^2 \mathrm{d}t \leq C_2 \sum_{n\in\mathbb{Z}} |w_n|^2 \tag{9.3.9}$$

for all sequences (w_n) in $\ell^2(\mathbb{C})$. The fact that this result holds for any $T > 0$ (and not only for T large enough) is a consequence of a general result due to Kahane on lacunary series (see [**?**]).

Note that, since the original problem is quasi-linear, we are forced to prove an Ingham type inequality for sums of oscillatory functions whose phases differ from the phase of the linearized equation. For our purposes, we need to consider phases that do not depend linearly on t, of the form

$$\operatorname{sign}(n)\left[\ell(n)^{\frac{3}{2}}t + \beta(t,x)|n|^{\frac{1}{2}}\right], \qquad \ell(n) := \Big((g + |n|^2)\,|n| \tanh(h\,|n|)\Big)^{\frac{1}{2}},$$

where x plays the role of a parameter. Though it is a sub-principal term, to take into account the perturbation $\beta(t,x)|n|^{\frac{1}{2}}$ requires some care since $e^{i\beta(t,x)|n|^{1/2}} - 1$ is not small. In particular we need to prove upper bounds for expressions in which we allow some amplitude depending on time (and whose derivatives in time of order k can grow as $|n|^{k/2}$).

Step 6: HUM method

Inverting A, we deduce from (9.3.8) an observability result for the adjoint operator $Q(\underline{u})^*$ ($Q(\underline{u})$ is as given by (9.3.6)). Then the controllability will be deduced from the classical HUM method (we need in fact a version that makes it possible to consider a real-valued control). The idea is that the observability property implies that some bilinear form is coercive and hence the existence of the control follows from the Riesz's theorem and a duality argument. A possible difficulty is that the control P_{ext} is acting only on the equation for ψ. To explain this, consider the case where $(\eta_{final}, \psi_{final}) = (0,0)$. Since the HUM method is based on orthogonality arguments, the fact that the control is not acting on both equations means for our problem that the final state is orthogonal to a co-dimension 1 space. The fact that this final state can be chosen to be 0 will be obtained by choosing this co-dimension 1 space in an appropriate way, introducing an auxiliary function $M = M(x)$ which is chosen later on.

Consider any real function $M = M(x)$ with $M - 1$ small enough, and introduce

$$L^2_M := \left\{ \varphi \in L^2(\mathbb{T};\mathbb{C}) \; ; \; \mathrm{Im} \int_{\mathbb{T}} M(x)\varphi(x)\,\mathrm{d}x = 0 \right\}.$$

Notice that L^2_M is an $\mathbb{R}$-Hilbert space. Also, for any $v_0 \in L^2_M$, the condition (9.3.7) holds. Then, using a variant of the HUM method in this space, one deduces that for all $v_{in} \in L^2$ (not necessarily in L^2_M) there is $f \in C^0([0,T];L^2)$ such that, if

$$Q(\underline{u})w = \partial_t w + W\partial_x w + iLw + R_3 w = \chi_\omega \,\mathrm{Re}\, f, \quad w(0) = w_{in},$$

then

$$w(T,x) = ibM(x)$$

for some constant $b \in \mathbb{R}$. Now

$$Q(\underline{u}) = \Phi(\underline{u})^{-1} P(\underline{u}) \Phi(\underline{u}),$$

where $\Phi(\underline{u})$ is the composition of the transformations in (9.3.5). Since $\Phi(\underline{u})$ and $\Phi(\underline{u})^{-1}$ are local operators, one easily deduce a controllability result for $P(\underline{u})$ from the one proved for $Q(\underline{u})$. Now, choosing $M = \Phi(\underline{u}(T,\cdot))(1)$ where 1 is the constant function 1, we deduce from $w(T,x) = ibM(x)$ that $u(T,x)$ is an imaginary constant, as asserted in statement *i*) of Proposition 9.3.3. Concerning M, notice that $M \neq 1$ because of the factor $(1 + \partial_x \kappa(t,x))^{\frac{1}{2}}$ multiplying $h(t, x + \kappa(t,x))$ in (9.3.5).

Step 7: Convergence of the scheme

To prove that the scheme converges, we prove that (f_n) and (u_n) are Cauchy sequences. This is where we need statement *ii*) in Proposition 9.3.3, to estimate the difference of two controls associated with different coefficients. To prove this stability estimate we introduce an auxiliary control problem which, loosely speaking, interpolates the two control problems. Since the original nonlinear problem is quasilinear, there is a loss of derivative (this reflects the fact that the flow map is expected to be merely continuous and not Lipschitz on Sobolev spaces). We overcome this loss by proving and using a regularity property of the control, see statement *ii*) in

Proposition 9.3.4. This regularity result is proved by adapting an argument used by Dehman-Lebeau [188] and Laurent [304]. We also need to study how the control depends on T or on the function M.

9.4 Stabilization of the water-wave equations

9.4.1 The multiplier method for the water-wave equations

We now discuss the multiplier method for the water-wave problem, as introduced in [11] for the nonlinear water-wave equations. The motivation was to prove an observability inequality. Namely, the question studied in [11] is the following: is-it possible to estimate the energy of gravity water waves by looking only at the motion of some of the curves of contact between the free surface and the vertical walls? From the point of view of control theory, this is the question of boundary observability of gravity water waves.

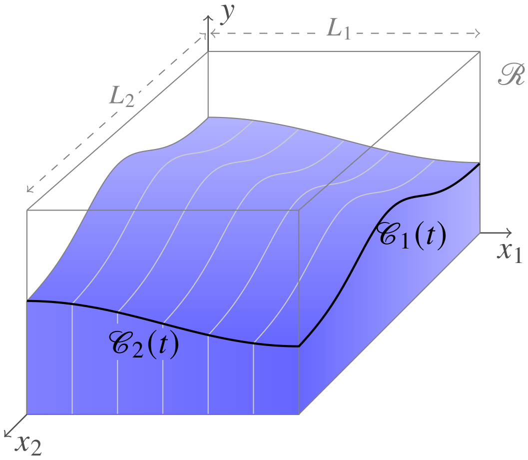


Figure 9.4: The boundary observability problem consists in bounding from below the energy by looking only at the motion of the curves $\mathscr{C}_1$ and $\mathscr{C}_2$.

For the sake of readability, we begin by recalling some well-known results for the linear wave equation

$$\partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad u|_{\partial\Omega} = 0. \tag{9.4.1}$$

The multiplier method, introduced by Morawetz, consists in multiplying the equations by $m(x) \cdot \nabla u(t,x)$, for some well-chosen function m, and to integrate by parts

in space and time. For instance, by considering a smooth extension $m\colon \Omega \longrightarrow \mathbb{R}^n$ of the normal $\nu(x)$ to the boundary $\partial\Omega$, one obtains
(9.4.2)
$$\int_0^T \int_{\partial\Omega} (\partial_n u)^2 \,\mathrm{d}\sigma \,\mathrm{d}t \leq K(T)\mathcal{E}(u) \text{ where } \mathcal{E}(u) := \|u(0,\cdot)\|^2_{H^1_0(\Omega)} + \|\partial_t u(0,\cdot)\|^2_{L^2(\Omega)}\,.$$

This is the so-called *hidden regularity* property. The name comes from the fact that, using energy estimates, one controls only the square of the $C^0([0,T];L^2(\Omega))$-norm of $\nabla_x u$ by means of the right-hand side of (9.4.2), which is insufficient to control the left-hand side of (9.4.2) by means of classical trace theorems.

Another key estimate is the so-called *boundary observability inequality*, which is, compared to (9.4.2), a reverse inequality where one can bound the norms of the initial data by the integral of $\partial_n u$ restricted to a domain $\Gamma_0 \subset \partial\Omega$. Let us recall the proof of such an inequality in the simplest case. Consider the one dimensional linear wave equation with Dirichlet boundary condition:

$$\partial_t^2 u - \partial_x^2 u = 0, \quad u(t,0) = u(t,1) = 0. \tag{9.4.3}$$

Multiply the equation by $x\partial_x u$ and integrate by parts, to obtain
(9.4.4)
$$\frac{1}{2}\int_0^T (\partial_x u(t,1))^2 \,\mathrm{d}t = \int_0^1 (\partial_t u)(x\partial_x u)\,\mathrm{d}x \Big|_0^T + \frac{1}{2}\iint_S \left[(\partial_t u)^2 + (\partial_x u)^2\right] \mathrm{d}x\,\mathrm{d}t$$

where $S = (0,T)\times(0,1)$. Since

$$\left|\int_0^1 (\partial_t u)(x\partial_x u)\,\mathrm{d}x\right| \leq \mathcal{E} := \frac{1}{2}\int_0^1 \left[(\partial_t u)^2 + (\partial_x u)^2\right] dx, \tag{9.4.5}$$

by using the conservation of energy ($d\mathcal{E}/dt = 0$), we deduce

$$\int_0^T (\partial_x u(t,1))^2 \,\mathrm{d}t \geq (T-2)\int_0^1 \left[(\partial_t u)^2 + (\partial_x u)^2\right](0,x)\,dx. \tag{9.4.6}$$

This inequality implies that, for $T > 2$, one can bound the energy by means of an observation at the boundary. For more details, generalizations and extensions to other equations, we refer the reader to the SIAM Review article by Lions [316]. Similar results are known for many other wave equations and we only mention the paper by Machtyngier and Zuazua [321] for the Schrödinger equation $i\partial_t u + \Delta u = 0$. Biccari [81] used recently the multiplier method to analyze the interior controllability problem for the fractional Schrödinger equation $i\partial_t u + (-\Delta)^s u = 0$ with $s \geq 1/2$ in a $C^{1,1}$ bounded domain with Dirichlet boundary condition.

In [11] we proved the following exact identity analogous to (9.4.4) for the *nonlinear* water-wave equations without surface tension. (The result in [11] holds for 3D waves but here we give a statement for 2D waves to simplify notations.)

Theorem 9.4.1. *Consider the equations without surface tension (that is assume that* $\kappa = 0$*) and without external pressure (*$P_{ext} = 0$*). Consider a smooth enough solution defined on the time interval* $[0, T]$ *and introduce*

$$m(t) = \eta(t, L).$$

Then

(9.4.7)

$$\begin{aligned}\frac{L}{2}\int_0^T \left[g m(t)^2 - m(t) m'(t)^2 \right] \mathrm{d}t = \frac{T}{2}\mathcal{H} \\ &+ \frac{L}{2}\int_0^T \int_{-h}^{m(t)} (\partial_y \phi)^2(t, L, y)\,\mathrm{d}y\,\mathrm{d}t \\ &+ \frac{1}{2}\int_0^T \int_0^L \left(h + \frac{7}{4}\eta\right)(\partial_x\phi)^2(t, x, -h)\,\mathrm{d}x\,\mathrm{d}t \\ &- \frac{1}{4}\int_0^L \eta\psi\, dx\,\Big|_{t=0}^{t=T} - \int_0^L x\eta\partial_x\psi\,\mathrm{d}x\,\Big|_{t=0}^{t=T} \\ &- \frac{7}{4}\int_0^T \iint_{\Omega(t)} (\partial_x\eta)(\partial_x\phi)(\partial_y\phi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,\end{aligned}$$

where $\int f\,dx|_{t=0}^{t=T}$ *stands for* $\int f(T, x)\,dx - \int f(0, x)\,dx$ *and*

$$\mathcal{H} = \frac{g}{2}\int_{-L}^{L} \eta^2(t, x)\,\mathrm{d}x + \frac{1}{2}\iint_{\Omega(t)} \left|\nabla_{x,y}\phi(t, x, y)\right|^2 \mathrm{d}x\,\mathrm{d}y. \tag{9.4.8}$$

The proof uses the Zakharov's formulation of the water-wave problem as a Hamiltonian system (see [453]) and the observation by Craig and Sulem [171] that the equations and the hamiltonian are most naturally expressed in terms of the Dirichlet to Neumann operator $G(\eta)$. The main ingredients of the proof of Theorem 9.3.1 are then: *i*) a Pohozaev identity for the Dirichlet to Neumann operator (that is a computation of $\int (G(\eta)\psi) x\partial_x\psi\,\mathrm{d}x$) which shows that the contributions due to the boundary conditions are positive and *ii*) some computations inspired by the analysis of Benjamin and Olver [68] of the conservation laws for water waves. In the appendix of [11], we give another proof of (9.4.7) which exploits the hamiltonian structure of the water-wave equations.

9.4.2 Stabilization

We now consider the stabilization problem.

Recall that the energy $\mathcal{E} = \mathcal{E}(t)$ is defined by

$$\mathcal{E} = \frac{g}{2}\int_0^L \eta^2 \,\mathrm{d}x + \kappa \int_0^L \left(\sqrt{1+\eta_x^2} - 1\right) \mathrm{d}x + \frac{1}{2}\int_0^L \int_{-h}^{\eta(t,x)} \left|\nabla_{x,y}\phi\right|^2 \mathrm{d}y\,\mathrm{d}x.$$

Our goal is to prove that, for some pressure law relating P_{ext} to the unknown (η, ψ), there holds:

i) $\mathcal{E}$ is decreasing and converges to zero exponentially in time;

ii) $\partial_x P_{ext}$ is supported inside a small subset of $[0, L]$.

In the numerical literature, a popular choice is to assume that $P_{ext} = \chi\partial_t\eta$ for some cut-off function $\chi \geq 0$ supported[1] in $[L - \delta, L]$ for some $\delta > 0$. To explain this choice, we start by recalling that, as observed by Zakharov [453], the equations have the hamiltonian form

$$\frac{\partial\eta}{\partial t} = \frac{\delta\mathcal{E}}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta\mathcal{E}}{\delta\eta} - P_{ext}.$$

Consequently,

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \int \left(\frac{\delta\mathcal{E}}{\delta\eta}\frac{\partial\eta}{\partial t} + \frac{\delta\mathcal{E}}{\delta\psi}\frac{\partial\psi}{\partial t}\right)\mathrm{d}x = -\int \frac{\partial\eta}{\partial t} P_{ext}\,\mathrm{d}x, \tag{9.4.9}$$

and hence, if $P_{ext} = \chi\partial_t\eta$ with $\chi \geq 0$, we deduce that $\mathrm{d}\mathcal{E}/\mathrm{d}t \leq 0$. It is thus easily seen that the energy decays. However, it is much more complicated to prove that the energy converges exponentially to zero. To study this problem, we first need to pause to clarify the question since, in general, solutions of the water-wave equations do not exist globally in time (they might blow-up in finite time, see [115, 160]). Our goal is in fact to prove that there exists a constant C such that, if a regular solution exists on the time interval $[0, T]$, then

$$\mathcal{E}(T) \leq \frac{C}{T}\mathcal{E}(0). \tag{9.4.10}$$

[1] We consider a damping located near $x = L$ only, and not also near $x = 0$, since we imagine that water waves are generated near $x = 0$. To do so, as explained above, one could use also the variations of an external pressure.

Since the equation is invariant by translation in time, one can iterate this inequality. Consequently, if the solution exists on time intervals of size nT_0 with $T_0 \geq 2C$, then $\mathcal{E}(nT_0) \leq 2^{-n}\mathcal{E}(0)$, which is the desired exponential decay.

We now define the pressure law.

Definition 9.4.2. *We assume that*

$$P_{ext}(t,x) = \chi(x)\partial_t\eta(t,x) - \frac{1}{2L}\int_{-L}^{L} \chi(x)\partial_t\eta(t,x)\,\mathrm{d}x,$$

where the cut-off function χ is defined as follows. Fix $\delta > 0$ and consider a $2L$-periodic C^∞ function κ, satisfying $0 \leq \kappa \leq 1$ and such that (see Figure 9.5)

$$\kappa(x) = \kappa(-x), \quad x\kappa'(x) \leq 0 \text{ for } x \in [-L, L], \quad \kappa(x) = \begin{cases} 1 \text{ if } x \in [0, L-\delta], \\ 0 \text{ if } x \in \left[L - \dfrac{\delta}{2}, L\right]. \end{cases}$$

We successively set

$$m(x) = x\kappa(x),$$

and

$$\chi(x) = 1 - m_x(x).$$

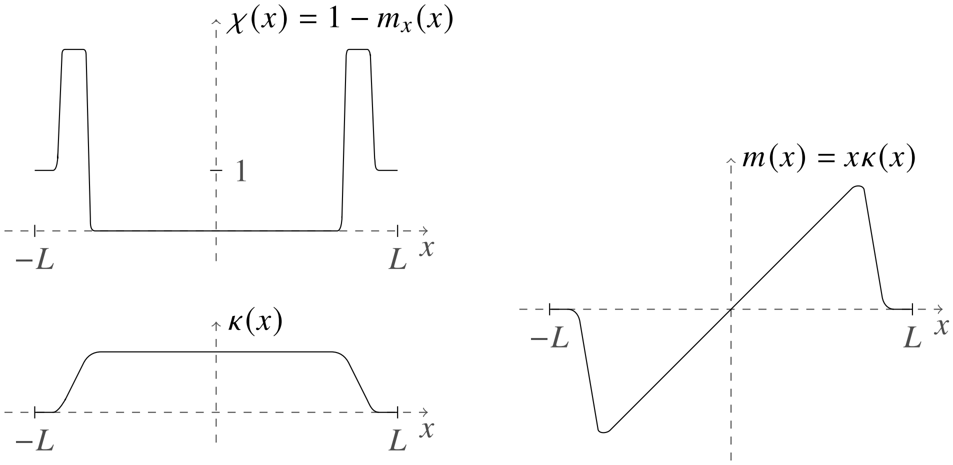


Figure 9.5: The cutoff function χ and the multiplier m.

Here is our main result.

Theorem 9.4.3. *Assume that* $\kappa > 0$ *and that*

$$\kappa \sup_{[-L,L]} m_{xx}(x)^2 \le g.$$

Then there exists a positive constant C*, depending only on the physical parameters* κ, g, h, L*, such that the following result holds. Let* $T > 0$ *and consider a regular solution* (η, ψ, P_{ext}) *defined on the time interval* $[0, T]$ *with* P_{ext} *as given by Definition* 9.4.2 *and set*

$$\rho(t,x) = (m(x) - x)\eta_x(t,x) + \frac{9}{4}\eta(t,x) - \frac{1}{2}m_x(x)\eta(t,x).$$

If, for all $(t,x) \in [0,T] \times [-L,L]$*,*

i) $\qquad \rho(t,x) \ge -\dfrac{h}{4}, \qquad |\rho_x(t,x)| < \dfrac{1}{4},$

ii) $\qquad \displaystyle\int_0^L (1 - m_x(x))\eta(t,x)\,\mathrm{d}x \le \frac{h}{12}, \qquad |m_x(x)|\,|\eta_x(t,x)|^2 \le 2, \quad |\eta_x(t,x)| \le \frac{1}{2},$

then one has the estimate

$$\mathcal{E}(T) \le \frac{C}{T}\mathcal{E}(0).$$

Remark 9.4.4. (i) *An important remark is that the constant* C *can be given by an explicit formula in terms of* κ, g, h, L*.*

(ii) *By combining this result with the local controllability result for small data proved by Alazard, Baldi and Han-Kwan in [17], this in turn implies that one can deduce controllability for larger data, but in large time (following an argument due to Dehman-Lebeau-Zuazua [189]).*

(iii) *We studied a similar problem in [12] for the case without surface tension. Assuming that* P_{ext} *is given by*

$$\partial_x P_{ext} = \chi(x) \int_{-h}^{\eta(t,x)} \phi_x(t,x,y)\,\mathrm{d}y, \tag{9.4.11}$$

where $\chi \ge 0$ *is a cut-off function, we proved in [12] an inequality of the form*

$$\mathcal{E}(T) \le \frac{C(N)}{\sqrt{T}}\mathcal{E}(0), \tag{9.4.12}$$

where the constant $C(N)$ depends on the frequency localization[2] *of the solution (η,ψ). The fact that this constant must depend on the frequency localization can be easily understood by considering the linearized equations (see [12]).*

To prove Theorem 9.4.3, it is sufficient to prove that there exists a positive constant C such that

$$\int_0^T \mathcal{E}(t)\,\mathrm{d}t \le C\mathcal{E}(0). \tag{9.4.13}$$

Indeed, since $\mathcal{E}$ is a decreasing function, this will imply the wanted inequality

$$\mathcal{E}(T) \le \frac{1}{T}\int_0^T \mathcal{E}(t)\,\mathrm{d}t \le \frac{C}{T}\mathcal{E}(0).$$

The proof of (9.4.13) is in two steps.

First step. The first step consists in deriving an exact *identity* which involves the integral in time of the energy. Here one can explain one of the main difficulties one has to cope with to stabilize the water-wave equations: one cannot decouple the problem of the observability and the question of the stabilization. Compared to what is done for the usual wave equation for instance, since the water-wave system is quasi-linear, one cannot write the solution as the sum of the two different problems. Another difficulty is that we do not know how to deduce an internal observability inequality from a boundary observability inequality (for the wave equation or the Schrödinger equation, this is possible thanks to a hidden regularity result). To overcome these two problems, we prove directly an internal observability result for the water-wave system with an external source term, by considering a multiplier $m(x)\partial_x$ with $m(x) = x\kappa(x)$ where κ is a cut-off function satisfying $\kappa(x) = 1$ for $0 \le x \le L-\delta$ and $\kappa(x) = 0$ for $L-\delta/2 \le x \le L$.

We proceed by establishing an exact identity similar to the one given above for the case without surface tension. With surface tension, one has to handle many remainder terms. In the end, instead of stating an identity (as we did in Theorem 9.4.1 for the case without surface tension), we will obtain an inequality.

Consider a smooth function $m = m(x)$ with $m(0) = m(L) = 0$, and set

$$\zeta = \partial_x(m\eta) + \frac{3}{2}(1-m_x)\eta - \frac{1}{4}\eta,$$

[2] The quantity N measuring the frequency localization is of the ratio of two Sobolev norms. One can think of the ratio $\|u\|_{H^1}/\|u\|_{L^2}$, which is proportional to N for a typical function oscillating at frequency N, like $u(x) = \cos(2\pi Nx/L)$.

our first main task will be to derive the following inequality

$$\begin{aligned}
&\frac{1}{4}\int_0^T \mathcal{E}(t)\,\mathrm{d}t \le O + W + B - I \qquad \text{where}\\
&I := \frac{h}{4}\int_0^T\int_{-L}^{L} \phi_x(t,x,-h)^2\,\mathrm{d}x\,\mathrm{d}t,\\
&O := \int_0^T\int_{-L}^{L}\left(\frac{3}{2}(1-m_x)\psi + (x-m)\psi_x\right)G(\eta)\psi\,\mathrm{d}x\,\mathrm{d}t,\\
&W := -\int_0^T\int_{-L}^{L} P_{ext}\,\zeta\;\mathrm{d}x\,\mathrm{d}t,\\
&B := \int_{-L}^{L}\zeta(0,x)\psi(0,x)\,\mathrm{d}x - \int_{-L}^{L}\zeta(T,x)\psi(T,x)\,\mathrm{d}x.
\end{aligned} \tag{9.4.14}$$

These four quantities play different roles. Their key properties are the following:

- $I \ge 0$ and hence (9.4.14) gives a bound for the horizontal component of the velocity at the bottom (this plays a key role to control the velocity in terms of the pressure, see (9.4.16)).
- W is the only term which involves the pressure.
- O corresponds to an *observation*, this means that this term depends only on the behavior of the solutions near $\{x = -L\}$ or $\{x = L\}$ when m is as given by Definition 9.4.2. Indeed, $x - m$ and $1 - m_x$ vanish when $x \in [-L+\delta, L-\delta]$ by definition of m.
- B is not an integral in time, by contrast with the other terms and, in addition, it is easily estimated by $K\mathcal{E}(0)$.

Second step. The goal of the second step is to deduce the wanted result (9.4.13) from the inequality (9.4.14). To do so, it is sufficient to prove that there exists a constant K depending only on g, κ, h, L such that

$$O + W + B - I \le K\mathcal{E}(0) + a\int_0^T \mathcal{E}(t)\,\mathrm{d}t \quad \text{for some} \quad a < \frac{1}{4}. \tag{9.4.15}$$

Indeed, by combining (9.4.14) and (9.4.15) one obtains that

$$\int_0^T \mathcal{E}(t)\,\mathrm{d}t \le \frac{K}{1/4 - a}\mathcal{E}(0),$$

which is the wanted result (9.4.13). To prove (9.4.15), recalling that $I \geq 0$, we need to estimate the terms B, W, O.

The estimate of the term B is easy so we begin by explaining how to estimate W. Recall that the hamiltonian structure of the equation implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\int_{-L}^{L} P_{ext}\partial_t \eta \,\mathrm{d}x.$$

Since $\partial_t \eta = G(\eta)\psi$, one deduces the inequality

$$\int_0^T \int_{-L}^{L} \chi(\partial_t \eta)^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{-L}^{L} \chi(G(\eta)\psi)^2 \,\mathrm{d}x\,\mathrm{d}t \leq \mathcal{E}(0).$$

This gives an estimate for the L^2-norm of $\chi\partial_t\eta$, which is the main contribution to the definition of P_{ext}. A more tricky inequality, relying on the special choice $\chi(x) = 1 - m_x(x)$, is that

$$-\int_0^T \left[\left(\int_{-L}^{L} \chi\partial_t \eta \,\mathrm{d}x\right)\int_{-L}^{L} \zeta \,\mathrm{d}x\right] \mathrm{d}t \leq \frac{L}{g}\left\|(1-m_x)^2\right\|_{L^\infty}\mathcal{E}(0).$$

By combining the above inequalities, we will be able to estimate the term W. The main difficulty is to bound the observation term O, and in particular to estimate

$$\iint (x-m)\psi_x G(\eta)\psi \,\mathrm{d}x\,\mathrm{d}t.$$

Using the Cauchy-Schwarz inequality, the key point is to estimate the L^2-norm of ψ_x in terms of the two quantities which are under control, that is the integral of $\chi(G(\eta)\psi)^2$ and the positive term I. In this direction, we will prove the following inequality, which is of independent interest: there exists a constant A, depending only on $\|\eta_x\|_{L^\infty_x}$, such that

$$\begin{aligned}\int_{-L}^{L} \chi\psi_x^2 \,\mathrm{d}x \leq A\int_{-L}^{L} \chi(G(\eta)\psi)^2 \,\mathrm{d}x + A\int_{-L}^{L} \phi_x^2|_{y=-h} \,\mathrm{d}x \\ - A\int_{-L}^{L}\int_{-h}^{\eta(t,x)} \chi_x\phi_x\phi_y \,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{9.4.16}$$

Notice that this result holds in fact for any smooth cut-off function χ and one can also take $\chi = 1$. In this case this inequality simplifies since the last term in the right-hand side vanishes. This gives a way to control the L^2-norm of ψ_x by the L^2-norm of $G(\eta)\psi$. When η is smooth, this can be obtained by delicate commutator estimates. However, it is possible to give a simple proof which applies for any Lipschitz domain.

9.5 References

There are many results about the controllability or the stabilization of linear or nonlinear equations describing water waves in some asymptotic regimes like Benjamin-Ono, KdV or nonlinear Schrödinger equations (see the Coron's book [155]). However, one cannot easily adapt these studies to the water-wave system since it is quasi-linear (instead of semi-linear) and since it is a pseudo-differential system, involving the Dirichlet-Neumann operator which is nonlocal and also depends nonlinearly on the unknown. The first results about the possible applications of control theory to the linearized water-wave equations are due to Reid and Russell [373] and Reid [371, 372], who studied the linearized equations at the origin (see also Miller [343], Lissy [318] and Biccari [81] for other control results about dispersive equations involving a fractional Laplacian). The first study about the controllability of the *nonlinear* water-wave problem is due to Alazard, Baldi and Han-Kwan ([17]).

As explained in this chapter, to experimentally simulate the propagation of water waves in a numerical wave tank, for computational reasons, one must work in a bounded domain. To simulate the propagation in an unbounded domain such as the open sea, one can use artificial boundary conditions as explained in [268, 428, 273, 272]. Another possibility consists in damping the outgoing waves in an absorption zone surrounding the boundary of the boundary (cf. [268, 428]). For the water wave equations, the idea of using this last method goes back to Le Méhauté [307] in 1972. This approach is used in many numerical studies ([227, 133, 200, 86, 131, 201]).

Chapter 10

The Cauchy problem for the Hele-Shaw and Muskat equations

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