Free surface flows in fluid dynamics
Lectures notes for a UCB course

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Description of the course

A classical subject in the mathematical theory of hydrodynamics consists in studying the evolution of the free surface separating the air from a perfect incompressible fluid. We will examine this question for two important sets of equations: the water-wave equations and the Hele-Shaw equations, including the Muskat problem. They are of a different nature, dispersive or parabolic, but we will see that they can be studied by related tools.

These courses are intended for graduate students with a general interest in analysis and no prerequisites for advanced theory are required. A second and major part of the courses will consist of short self-contained introductions to the following topics: paradifferential calculus, study of the fractional Laplacian and Morawetz and Lions multiplier methods. I will give a detailed analysis of the Cauchy problem for the water-wave, Hele-Shaw and Muskat equations, and discuss some qualitative properties of their solutions.

Grading: 2-3 homeworks.

No required text. Lecture notes will be posted at the url

http://talazard.perso.math.cnrs.fr/CL.pdf

These lectures will be included in a book in preparation with Quoc-Hung Nguyen. It will rely on several recent works in collaboration with him, as well as with Pietro Baldi, Didier Bresch, Nicolas Burq, Jean-Marc Delort, Daniel Han-Kwan, Mihaela Ifrim, Omar Lazar, Guy Métivier, Nicolas Meunier, Didier Smets, Daniel Tataru and Claude Zuily.
Course syllabus

I. *The main equations*

a. The incompressible Euler equations and the water-wave problem
b. Darcy’s law and the Hele-Shaw and Muskat equations

II. *Methods in PDEs*

a. An introduction to paradifferential calculus
b. Sobolev embedding theorem and the fractional Laplacian
c. The multipliers method, from Rellich to Morawetz and J.-L. Lions

III. *On the Cauchy problem*

a. Study of the Dirichlet-to-Neumann operator
b. The Cauchy problem for the water wave and Hele-Shaw equations
c. The Cauchy problem for the Muskat equation in critical spaces

IV. *Exact identities and nonlinear methods*

a. Hamiltonian, Lagrangian, Momentum, Entropy. Conservation laws vs gradient flows
b. Morawetz estimates for the water-wave equations
c. Lyapounov functionals and entropies for the Hele-Shaw equations
Part I

The equations
Chapter 1

The water-wave equations

The purpose of this chapter is to introduce the free surface Euler equation. We will begin by briefly explaining the meaning of these equations. I will also introduce the Dirichlet-to-Neumann operator and state some results about the Cauchy theory.

1.1 The equations

A heavy fluid mass, originally at rest, and of indefinite depth, has been set in motion by the effect of a given cause. One asks, after a given time, the shape of the external surface of the fluid and the velocity of each of the molecules located on the same surface.

Une masse fluide pesante, primitivement en repos, et d'une profondeur indéfinie, a été mise en mouvement par l'effet d'une cause donnée. On demande, au bout d'un temps déterminé, la forme de la surface extérieure du fluide et la vitesse de chacune des molécules situées à cette même surface.

Figure 1.1: Question by the French Academy of Sciences in 1813.

Let us consider an ocean of infinite depth primarily at rest. For the sake of simplicity, let us suppose that this ocean is unbounded laterally as well as in depth. To visualize this, let us assimilate the space domain to $\mathbb{R}^3$, which is
assumed to be divided into two distinct parts: the upper half-space occupied by air and the lower half-space occupied by water. Let us imagine that the wind is blowing over this ocean. Under certain conditions, this wind will generate waves, called wind waves. It is observed that they can propagate over immense distances until they reach the shore. We are going to be interested in the propagation of these waves far from the wind zone and far from the shore. We then speak of swell or gravity waves, because the waves propagate thanks to the restoring force of gravity.

To describe a wave, we need to introduce some notations. In the following, the fluid will always be referred to the rectangular coordinates of \( x_1, x_2, y \), the plane \((Ox_1 x_2)\) being horizontal, and coinciding with the surface of the fluid when in equilibrium, the axis \((Oy)\) being directed upwards. We write \( x = (x_1, x_2) \) and assume that the gravity is oriented along the axis \((Oy)\).

We introduce also the elevation of the surface of the sea, denoted by \( \eta \) and the velocity field of the fluid, denoted by \( u \). The unknown \( \eta \) depends on the time \( t \) and the spatial variable \( x = (x_1, x_2) \in \mathbb{R}^2 \). At a given time \( t \), the graph of the function \( \eta \), denoted by \( \Sigma(t) \), is what we call the free surface. The domain occupied by water at time \( t \) is the half-space, denoted by \( \Omega(t) \), located
Figure 1.2: 3D and 2D waves

underneath the free surface\footnote{For the sake of simplicity, we assumed that the free surface $\Sigma(t)$ is a graph, the domain $\Omega(t)$ is bottomless and that $\Omega$ has no lateral boundary ($x \in \mathbb{R}^2$). We refer to the review paper by Lannes [256] for the general case.}:

\[
\begin{align*}
\Sigma(t) &= \{ (x, y) \in \mathbb{R}^2 \times \mathbb{R} : y = \eta(t, x) \}, \\
\Omega(t) &= \{ (x, y) \in \mathbb{R}^2 \times \mathbb{R} : y < \eta(t, x) \}.
\end{align*}
\]

In the following we will use the following notations.

**Notation 1.1.1.**

1. Given a function $f = f(x, y)$ and a function $h = h(x)$, we use $f|_{y=h}$ as a short notation for the function $x \mapsto f(x, h(x))$.

2. We set

\[
\nabla = (\partial_{x_1}, \partial_{x_2}), \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2, \quad \Delta_{x,y} = \Delta + \partial_y^2.
\]

Let us insist on the fact that the operators $\nabla$ and $\Delta$ contain only derivatives with respect to the horizontal variable $x$.

3. Given a function $f = f(t, x, y)$ and a time $t$, we have

\[
\int_{\Omega(t)} f(t, x, y) \ dy \ dx = \int_{\mathbb{R}^2} \left( \int_{-\infty}^{\eta(t,x)} f(t, x, y) \ dy \right) \ dx.
\]
1.1.1 A word of caution

In this chapter we perform only formal computations and consider smooth enough functions. This is intended to mean that the functions are sufficiently regular, and decay sufficiently fast at spatial infinity, so that all the computations are justified.

In this direction, introduce the space

\[ H^\infty(\mathbb{R}^d) = \{ u \in C^\infty(\mathbb{R}^d) ; \partial^\alpha_x u \in L^2(\mathbb{R}^d) \text{ for any } \alpha \in \mathbb{N}^d \} . \]

1.1.2 The incompressible Euler equations

The fluid we will study will be assumed to be subjected to the force of gravity and/or surface tension. Moreover, we assume that the eulerian velocity field \( u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3 \) is solution to the incompressible Euler equations of fluid mechanics:

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial y} &= -\frac{\partial P}{\partial x_1}, \\
\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial y} &= -\frac{\partial P}{\partial x_2}, \\
\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x_1} + u_2 \frac{\partial u_3}{\partial x_2} + u_3 \frac{\partial u_3}{\partial y} &= -g - \frac{\partial P}{\partial y}, \\
\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial y} &= 0,
\end{align*}
\]

where \( g > 0 \) is the acceleration of gravity and \( P : \Omega \to \mathbb{R} \) is the pressure. This is better formulated under the form

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla_{x,y}) u + \nabla_{x,y} (P + gy) &= 0, \\
\text{div}_{x,y} u &= 0,
\end{align*}
\]

where \( \cdot \) is the scalar product in \( \mathbb{R}^3 \), so that \( u \cdot \nabla_{x,y} = u_1 \partial_{x_1} + u_2 \partial_{x_2} + u_3 \partial_{y} \).

1.1.3 Boundary conditions on the free surface

We need two boundary conditions on the free surface. The first one corresponds to a balance of forces: it states that the pressure jump across the surface
which separates the fluid from the air is proportional to the mean curvature of the interface. If in addition we assume that the air pressure above the fluid is constant (and then this constant can be chosen to be 0 without loss of generality) this results in the condition

\[(1.1.3) \quad P\|_{y=\eta} = \lambda \kappa(\eta) \quad ; \quad \kappa(\eta) = - \text{div} \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right), \]

where \(\lambda \in [0, 1]\) is a coefficient that measures the importance of surface tension.

The second boundary condition has to do with the normal velocity of the free surface. To introduce it, we need to introduce the outward pointing unit normal \(n\) to \(\Omega\). At a point \(X = (x, \eta(t, x))\) of \(\Sigma(t)\), it is given by

\[n(X) = \frac{1}{\sqrt{1 + |\nabla \eta(t, x)|^2}} \begin{pmatrix} -\nabla \eta(t, x) \\ 1 \end{pmatrix}.\]

The dynamics of the free surface \(\Sigma(t) = \{y = \eta(t, x)\}\) is coupled to that of the fluid through the following kinematic boundary condition:

\[(1.1.4) \quad \frac{\partial \eta}{\partial t} = \sqrt{1 + |\nabla \eta|^2} U \cdot n \quad \text{where} \quad U = u|_{y=\eta}.\]

This simplifies to

\[\frac{\partial \eta}{\partial t} + U_1 \partial_{x_1} \eta + U_2 \partial_{x_2} \eta = U_3.\]

Let us make two elementary observations about this equation.

**Definition 1.1.2.** A fluid particle is a curve \(\mathbb{R} \ni t \mapsto m(t) \in \mathbb{R}^3\) solution to \(\dot{m}(t) = \frac{d}{dt} m(t) = u(t, m(t))\).

**Proposition 1.1.3.** Any fluid particle which is on the free surface of the fluid at the initial time will remain on the free surface for any further time.

**Proof.** Consider a fluid particle \(m(t) = (x_1(t), x_2(t), y(t))\) and introduce the function \(\theta(t) = y(t) - \eta(t, x(t))\). Then \(\theta\) vanishes at \(t = 0\) by assumption. Moreover, using (1.1.4),

\[\frac{d}{dt} \theta = \dot{y}(t) - (\partial_t \eta)(t, m(t)) - \dot{x}_1(t)(\partial_{x_1} \eta)(t, m(t)) - \dot{x}_2(t)(\partial_{x_2} \eta)(t, m(t)) = (U_3 - U_1 \partial_{x_1} \eta - U_2 \partial_{x_2} \eta - \partial_t \eta) \big|_{(x, y) = m(t)} = 0,\]

and the result follows. \(\square\)
The following lemma allows in many situations to compute the evolution of global quantities.

**Lemma 1.1.4.** Consider two regular functions \( \eta, u \) satisfying (1.1.4). Then for any regular function \( f = f(t, x, y) \), one has

\[
\frac{d}{dt} \int_{\Omega(t)} f(t, x, y) \, dy \, dx = \int_{\Omega(t)} (\partial_t + u \cdot \nabla_{x,y}) f \, dy \, dx.
\]

**Proof.** Write

\[
\frac{d}{dt} \int_{\Omega(t)} f(t, x, y) \, dy \, dx = \frac{d}{dt} \int_{\mathbb{R}^2} \left( \int_{-\infty}^{\eta(t,x)} f(t, x, y) \, dy \right) \, dx
\]

\[
= \int_{\Omega(t)} \partial_t f(t, x, y) \, dy \, dx + \int_{\mathbb{R}^2} (\partial_t \eta) f(t, x, \eta) \, dx
\]

and then use (1.1.4) and Stokes’ theorem\(^3\) to verify that

\[
\int_{\mathbb{R}^2} (\partial_t \eta) f(t, x, \eta) \, dx = \int_{\mathbb{R}^2} (U \cdot n) f(t, x, \eta) \sqrt{1 + |\nabla \eta|^2} \, dx
\]

\[
= \int_{\partial \Omega(t)} n \cdot (fu) \, d\sigma = \int_{\Omega(t)} \nabla_{x,y} (fu) \, dy \, dx,
\]

where \( d\sigma = \sqrt{1 + |\nabla \eta|^2} \, dx \). Since \( \nabla_{x,y} u = 0 \), this implies that

\[
\int_{\mathbb{R}^2} (\partial_t \eta) f(t, x, \eta) \, dx = \int_{\Omega(t)} u \cdot \nabla_{x,y} f \, dy \, dx,
\]

which concludes the proof. \( \square \)

\(^2\)See the warning in §1.1.1. \(^3\) Let us recall a formulation of Stokes’ theorem.

**Definition 1.1.5.** We say that a domain \( D \subset \mathbb{R}^d \) is \( C^1 \) if for each point \( m \) belonging to the boundary \( \partial D \) there is a cartesian coordinate system, a radius \( r > 0 \) and a \( C^1 \) function \( \zeta: \mathbb{R}^d \to \mathbb{R} \) with compact support such that

\[
D \cap B(m, r) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} ; \zeta(x) < y\} \cap B(m, r).
\]

**Theorem 1.1.6.** Suppose that \( D \) is a bounded \( C^1 \) domain and consider a function \( u \in C^1(\overline{D}) \). Then

\[
\int_D \nabla u \, dx = \int_{\partial D} n \cdot u \, d\sigma,
\]

where \( n \) is the outward pointing unit normal.

We can apply this result in the unbounded domain \( \Omega(t) \) provided that the functions at stake decay sufficiently at spatial infinity to justify a standard truncation argument.
1.1.4 Conserved quantities

Consider a solution \((\eta, u)\) of the free surface Euler equations. At a given time \(t\), we define its kinetic energy \(E_k(t)\) and its potential energy \(E_p(t)\) by

\[
E_k(t) = \frac{1}{2} \int_{\Omega(t)} |u(t, x, y)|^2 \, dy \, dx \quad (|\cdot| \text{ denotes the Euclidean norm on } \mathbb{R}^3)
\]

\[
E_p(t) = \frac{g}{2} \int_{\mathbb{R}^2} \eta(t, x)^2 \, dx + \lambda \int_{\mathbb{R}^4} \left( \sqrt{1 + |\nabla \eta|^2} - 1 \right) \, dx.
\]

The total energy is then by definition the function \(E = E_k + E_p\).

**Proposition 1.1.7.** For any regular solution, there holds \(\frac{d}{dt} E = 0\).

**Proof.** It follows from Lemma 1.1.4 that

\[
\frac{d}{dt} E_k = \int_{\Omega(t)} \frac{1}{2} \left( \partial_t + u \cdot \nabla_{x,y} \right) |u|^2 \, dy \, dx.
\]

Now observe that

\[
\frac{1}{2} \left( \partial_t + u \cdot \nabla_{x,y} \right) |u|^2 = u \cdot (\partial_t u + (u \cdot \nabla_{x,y}) u) \quad \text{(Leibniz rule)}
\]

\[
= -u \cdot \nabla_{x,y} (P + gy) \quad \text{(by (1.1.2))}
\]

\[
= - \text{div}_{x,y} ((P + gy) u) \quad \text{(since div}_{x,y} u = 0).
\]

Consequently, Stokes’ theorem implies that

\[
\frac{d}{dt} E_k = - \int_{\Omega(t)} \text{div}_{x,y} ((P + gy) u) \, dy \, dx = - \int_{\partial \Omega(t)} (P + gy) u \cdot n \, d\sigma.
\]

Now, by assumption on the pressure (see (1.1.3)), on the free surface \(\partial \Omega(t)\) we have \(P + gy = g\eta + \lambda \kappa\). Also the kinematic boundary condition (1.1.4) implies that \(u \cdot n \, d\sigma = \partial_t \eta \, dx\) (since \(d\sigma = \sqrt{1 + |\nabla \eta|^2} \, dx\)). It follows that

\[
\frac{d}{dt} E_k = - \int_{\mathbb{R}^2} (g\eta + \lambda \kappa(\eta)) \partial_t \eta \, dx = - \frac{d}{dt} E_p.
\]

The identity is proven. \(\square\)

**Exercise 1.1.8.** Write \(u = (u_1, u_2, u_3)\), and introduce for \(j \in \{1, 2\}\) the momentum

\[
\mathcal{M}_j = \int_{\Omega(t)} u_j \, dy \, dx
\]

Prove that they are also conserved quantity (by formal computations).
1.1.5 Wave propagation

Another observation is in order. In the introduction of an article very famous [319] where he describes his discovery of the solitary wave, J. Scott Russell explains why it is fundamental to understand that this is a wave propagation problem. One of his arguments is

![Figure 1.3: From Russell [319]](image)

In other words: we can see that there is energy transfer without mass transfer, which is a circumstance of all wave phenomena.

Let us illustrate Russell’s argument with a figure which represents the movement of a floating object (red dot) at successive times at the passage of a wave on the surface of the water moving to the right (naively represented by the graph of the function $y = \cos(x - t)$):

![Figure 1.4: Russell’s argument](image)
1.2 Zakharov equations

1.2.1 Velocity potential

ABSTRACT: We study the stability of steady nonlinear waves on the surface of an infinitely deep fluid [1, 2]. In section 1, the equations of hydrodynamics for an ideal fluid with a free surface are transformed to canonical variables: the shape of the surface $\eta(x, t)$ and the hydrodynamic potential $\Psi(x, t)$ at the surface are expressed in terms of these variables. By introducing canonical variables, we can consider the problem of the stability of surface waves as part of the more general problem of nonlinear waves in media with dispersion [3, 4]. The results of the rest of the paper are also easily applicable to the general case.

Figure 1.5: From Zakharov’ first paper [383]

In the sequel, we will always assume that the fluid was initially irrotational and furthermore has been set in motion by the action of conservative forces. This implies that the flow will remain irrotational for all time, that is

$$\text{curl}_{x,y} u = \nabla_{x,y} \wedge u = 0.$$  

Since the domain is simply connected and that the fluid is incompressible, there is a function $\phi$ defined on $\Omega$ with real values such that

$$u = \nabla_{x,y} \phi \quad ; \quad \Delta_{x,y} \phi = 0 \quad \text{in} \quad \Omega.$$  

The function $\phi$ is called the velocity potential. The equation $\Delta_{x,y} \phi = 0$ means that $\phi$ is an harmonic function, which is a crucial property.

In view of (1.1.2), we see that up to translating $\phi$ by a factor that depends only on time, $\phi$ satisfies the following equation in $\Omega$,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla_{x,y} \phi|^2 + P + gy = 0.$$  

This equation is called the Bernoulli’s equation. When $u = \nabla_{x,y} \phi$, the kinematic boundary condition (1.1.4) can be written under the form

$$ \frac{\partial \eta}{\partial t} = \left[ \frac{\partial \phi}{\partial y} - \nabla \eta \cdot \nabla \phi \right] \bigg|_{y=\eta}.$$
1.2.2 The Dirichlet-to-Neumann operator

Rather than studying the system in \((\eta, \phi)\), Zakharov ([383, 384]) suggests working with the unknown \((\eta, \psi)\) where

\[ \psi(t, x) = \phi(t, x, \eta(t, x)) \]

is the trace of \(\phi\) at the free surface \(\Sigma\). The observation is that one can then subsequently deduct the properties of \(\phi\) and \(u\) given that the function \(\phi\) is harmonic in \(\Omega\) (cf. (1.2.1)). An interest of this notation is that now the problem only depends on two unknowns which are functions of time \(t\) and \(x \in \mathbb{R}^2\).

To state the equations that govern the propagation of \((\eta, \psi)\), we need to introduce the Dirichlet–Neumann operator. This operator intervenes in many problems in analysis (harmonic analysis, inverse problem, spectral theory...). It plays a central role in the study of the water-waves problem since the work of Craig\(^4\), C. Sulem [143]. By definition, this is the operator that associates to a function \(f\), defined on the boundary of an open set \(\Omega\), the normal derivative of its harmonic extension. Here, it is more convenient to introduce a coefficient in the definition.

**Definition 1.2.1.** i) Consider two smooth functions \(\eta, \psi\) defined on \(\mathbb{R}^2\) with real values. Introduce the domain \(\Omega := \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}; y < \eta(x)\}\) and denote by \(\phi\) the harmonic extension of \(\psi\) in \(\Omega\), so that \(\phi = \phi(x, y)\) satisfies

\[ \Delta_{x,y} \phi = 0 \quad \text{in} \quad \Omega, \]

\[ \phi(x, \eta(x)) = \psi(x), \quad \nabla_{x,y} \phi(x, y) \rightarrow 0 \quad \text{when} \quad y \rightarrow -\infty. \]

Then the Dirichlet-to-Neumann operator, denoted by \(G(\eta)\), is defined by

\[ G(\eta) \psi(x) = \sqrt{1 + |\nabla \eta|^2} \partial_n \phi|_{y=\eta(x)}, \]

where \(\partial_n\) is the normal derivative: \(\partial_n = n \cdot \nabla_{x,y}\).

ii) We also define the operators

\[ B(\eta) \psi = (\partial_y \phi)|_{y=\eta(x)}, \]

\[ V(\eta) \psi = (\nabla_x \phi)|_{y=\eta(x)}. \]

\(^4\)See the memorial tribute here.
Remark 1.2.2. i) We will always use $B$ and $V$ as compact notations for $B(\eta)\psi$ and $V(\eta)\psi$.

ii) To clarify the notations, note that

\[
\sqrt{1 + |\nabla \eta|^2} \partial_n \phi \bigg|_{y=\eta(x)} = \partial_y \phi - \nabla \eta \cdot \nabla \phi \bigg|_{y=\eta(x)} = (\partial_y \phi)(x, \eta(x)) - \nabla \eta(x) \cdot (\nabla \phi)(x, \eta(x)).
\]

(iii) If $\eta$ and $\psi$ belong to $H^\infty(\mathbb{R}^d)$, then $G(\eta)\psi$ is well-defined and belongs to $H^\infty(\mathbb{R}^d)$. We shall see later on that one can consider functions with limited regularity.

A important observation is that the traces of the derivatives of an harmonic function satisfy several identities. Recall that $B$ and $V$ as compact notations for $B(\eta)\psi$ and $V(\eta)\psi$. We will make extensive use of the following

Lemma 1.2.3. For any smooth functions, there holds

\[
B = \frac{G(\eta)\psi + \nabla \eta \cdot \nabla \psi}{1 + |\nabla \eta|^2},
\]

(1.2.5)

\[
V = \nabla \psi - B \nabla \eta,
\]

(1.2.6)

\[
G(\eta)B = - \text{div} \ V.
\]

Proof. i) The chain rule implies that

\[
\nabla \psi = \nabla (\phi(x, \eta(x))) = (\nabla \phi + (\partial_y \phi) \nabla \eta)|_{y=\eta} = V + B \nabla \eta
\]

which implies that $V = \nabla \psi - B \nabla \eta$. On the other hand, by definition of the operator $G(\eta)$, one has

\[
G(\eta)\psi = (\partial_y \phi - \nabla \eta \cdot \nabla \phi)|_{y=\eta} = B - V \cdot \nabla \eta,
\]

so the identity for $B$ in (1.2.5) follows from $V = \nabla \psi - B \nabla \eta$.

ii) By definition, one has $B = (\partial_y \phi)|_{y=\eta}$. Therefore $\Phi(x, y) = \partial_y \phi(x, y)$ satisfies

\[
\Delta_{x,y} \Phi = 0, \quad \Phi|_{y=h} = B.
\]

Directly from the definition of $G(\eta)$, we have

\[
G(\eta)B = \partial_y \Phi - \nabla \eta \cdot \nabla \Phi|_{y=\eta}.
\]
So it suffices to show that \( \partial_y \Phi - \nabla \eta \cdot \nabla \Phi \big|_{y=\eta} = - \text{div} V \). To do that we first write that \( \partial_y \Phi = \partial_y^2 \phi = - \Delta \phi \) to obtain

\[
(\partial_y \Phi - \nabla \eta \cdot \nabla \Phi) \big|_{y=\eta} = -(\Delta \phi + \nabla \eta \cdot \nabla \partial_y \phi) \big|_{y=\eta} = - \text{div}(\nabla \phi \big|_{y=\eta}),
\]

which proves statement \( ii \). \( \square \)

Exercise 1.2.4. Assume that \( d = 1 \) and consider two functions \( \eta \) and \( \psi \) in \( H^\infty(\mathbb{R}) \) and set \( B = B(\eta)\psi \) and \( V = V(\eta)\psi \).

i) Prove that \( \partial_x B = G(\eta)V \).

ii) Consider an harmonic function \( \varphi = \varphi(x,y) \). Verify that \((\partial_x \varphi)^2 - (\partial_y \varphi)^2 \) is also harmonic.

iii) Denote by \( \phi \) the harmonic extension of \( \psi \). The result of the previous question implies that \((\partial_x \phi)^2 - (\partial_y \phi)^2 \) is the harmonic extension of \( V^2 - B^2 \). Use this information to obtain that

\[
G(\eta)(V^2 - B^2) = 2 \partial_x (B\psi).
\]

Craig and Sulem [144, 143] wrote the equations on \( \eta \) and \( \psi \) in an explicit form.

Proposition 1.2.5. The water-wave equations can be written under the form

\[
\begin{cases}
\frac{\partial \eta}{\partial t} - G(\eta)\psi = 0, \\
\frac{\partial \psi}{\partial t} + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \left( \nabla \psi \cdot \nabla \eta + G(\eta)\psi \right)^2 + g\eta + \lambda \kappa(\eta) = 0.
\end{cases}
\]

Proof. By definition of \( G(\eta)\psi \), we get the first equation from (1.1.4) and (1.2.4).

Now, by evaluating the equation

\[
\partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + P + gy = 0,
\]
on the free surface, we obtain

\[
(\partial_t \psi - B \partial_t \eta) + \frac{1}{2} |V|^2 + \frac{1}{2} B^2 + g\eta + \lambda \kappa(\eta) = 0.
\]

Since \( \partial_t \eta = G(\eta)\psi = B - V \cdot \nabla \eta \), it follows that

\[
\partial_t \psi + g\eta + \frac{1}{2} |V|^2 - \frac{1}{2} B^2 + BV \cdot \nabla \eta = 0,
\]

which gives the second equation. \( \square \)
1.2.3 Arbitrary space dimension

It is immediate to generalize the problem in case \( x \) belongs to \( \mathbb{R}^d \) instead of \( \mathbb{R}^2 \). The case \( d = 1 \) is particularly useful to describe waves that do not depend, say, on \( x_2 \). We then speak of two-dimensional waves because, at a given instant \( t \), the domain \( \Omega(t) \) can be described by two variables only (here \( x_1 \) and \( y \)). The equations being invariant by rotation, we can reduce to this case the general case where all phenomena are identical in planes parallel to a fixed vertical plane. In all the following we will use the following notations:

\[
\nabla = (\partial_{x_1}, \ldots, \partial_{x_d}), \quad \nabla_{x,y} = (\nabla, \partial_y), \quad \Delta = \sum_{1 \leq j \leq d} \partial^2_{x_j}, \quad \Delta_{x,y} = \Delta_{x,y} + \partial^2_y.
\]

1.2.4 The linearized equation around the rest state

It is important to determine the Dirichlet-to-Neumann operator in the simplest case where \( \Omega \) is the half-space

\[
\Omega = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y < 0\},
\]

that is for the case \( \eta = 0 \).

**Proposition 1.2.6.** There holds

\[
G(0)\psi = |D|\psi,
\]

where \( |D| \) is the square root of the opposite of the Laplacian, defined by

\[
|D| e^{i\xi \cdot x} = |\xi| e^{i\xi \cdot x} \quad \text{where} \quad |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_d^2}.
\]

(Notice that \( |D|^2 e^{i\xi \cdot x} = -\Delta e^{i\xi \cdot x} \).)

**Remark 1.2.7.** More generally we define \( |D|^s \) for \( s \geq 0 \) as the Fourier multiplier with symbol \( |\xi|^s \). This means that if \( u \) is in the Schwartz class \( \mathcal{S}(\mathbb{R}^d) \), then

\[
(|D|^s u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} |\xi|^s \hat{u}(\xi) \, d\xi.
\]

**Proof.** Let us introduce the Fourier transform of \( \phi \) with respect to \( x \):

\[
u(\xi, y) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \phi(x, y) \, dx.
\]
Then
\[ \phi(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} u(\xi, y) \, d\xi. \]
Since \( \Delta_{x,y} \phi = 0 \), we have \( \partial_y^2 u - |\xi|^2 u = 0 \). The solution \( e^{-y|\xi|} \) being to be excluded because \( y < 0 \), we find that \( u = e^{y|\xi|} u(\xi, 0) \) which leads to
\[ (\partial_y u)(\xi, 0) = |\xi| u(\xi, 0), \]
which in turn implies that \( (\partial_y \phi)(x, 0) = (|D| \phi)(x, 0) \) and thus the desired result. \( \square \)

We can now derive the linearized equation around the trivial solution. Neglecting terms that are at least quadratic with respect to the unknowns, we obtain the following system
\[
\begin{cases}
\partial_t \eta = G(0) \psi, \\
\partial_t \psi + g\eta - \lambda \Delta \eta = 0.
\end{cases}
\]
Using that
\[ G(0) = |D| \quad \Delta = -|D|^2, \]
and deriving in time we get the equation we are looking for:
\[ \partial^2_t \eta + g |D| \eta + \lambda |D|^3 \eta = 0. \tag{1.2.8} \]
Notice that the Cauchy problem for this equation is well posed on Sobolev’s spaces in two cases : i) as soon as \( \lambda \) is strictly positive, for any real value \( g \); ii) in the case of \( \lambda = 0 \), the Cauchy Problem remains well posed under the assumption that \( g \) is positive or zero. To verify this it is sufficient to note that the equation (1.2.8) is solved using the Fourier transform. For example, if \( \lambda = 0 \) and if \( \partial_t \eta|_{t=0} = 0 \), we find that
\[ \hat{\eta}(t, \xi) = e^{i\sqrt{|D|}|\xi|^{1/2}} \hat{\eta}(0, \xi) \quad \text{if } g > 0, \]
\[ \hat{\eta}(t, \xi) = e^{i\sqrt{-g}|\xi|^{1/2}} \hat{\eta}(0, \xi) \quad \text{if } g < 0. \]
Note the stabilizing role of gravity : the case \( g > 0 \) is stable while the case \( g < 0 \) is unstable; there is an exponential amplification of high frequencies. (The case \( g < 0 \) corresponds to the very unstable situation where a heavy fluid is placed over a light fluid).
1.2.5 Dispersive estimates

Assume that \( d = 1, \, g = 0 \) and \( \lambda = 1 \) so that the water-wave equations read

\[
\begin{align*}
\partial_t \eta - |D| \psi &= 0, \\
\partial_t \psi + |D|^2 \eta &= 0.
\end{align*}
\]

We symmetrize this system by introducing \( \varphi = |D|^{1/2} \eta + i\psi \), so that

\[
(1.2.9) \quad \partial_t \varphi + A \varphi = 0 \quad \text{where} \quad A := i |D|^3/2.
\]

Notice that \( A^* = -A \), which means that

\[
\langle u, Av \rangle = \int_{\mathbb{R}} u(x) \overline{Av(x)} \, dx = -\int_{\mathbb{R}} Au(x) \overline{v(x)} \, dx = -\langle Au, v \rangle.
\]

This is another way to obtain the conservation of energy: take the \( L^2 \)-scalar product with \( \varphi \) and then take the real part to obtain

\[
\int_{\mathbb{R}} |\varphi(t, x)|^2 \, dx = \int_{\mathbb{R}} |\varphi(0, x)|^2 \, dx.
\]

A first dispersive estimate is given by the following

**Proposition 1.2.8.** Let \( T > 0 \). For any \( \delta > 0 \), there is \( C > 0 \) such that, for any solution \( \varphi \) of (1.2.9) with initial data \( \varphi_0 \in L^2(\mathbb{R})^2 \) we have,

\[
\int_{-T}^{+T} \int_{\mathbb{R}} \langle x \rangle^{-\frac{1}{2} - \delta} \, |D_x|^{1/4} \varphi(t, x) |^2 \, dx \, dt \leq C \int_{\mathbb{R}} |\varphi_0(x)|^2 \, dx,
\]

where \( \langle x \rangle = (1 + |x|^2)^{1/2} \).

**Sketch of the proof.** Denote by \( \langle \cdot, \cdot \rangle \) the scalar product on \( L^2(\mathbb{R}) \). Consider an operator \( C \in \mathcal{L}(L^2(\mathbb{R})) \), independent of time. We compute

\[
\frac{d}{dt} \langle C \varphi, \varphi \rangle = \langle C \partial_t \varphi, \varphi \rangle + \langle C \varphi, \partial_t \varphi \rangle = -\langle CA \varphi, \varphi \rangle - \langle C \varphi, A \varphi \rangle.
\]

\[
= \langle [A, C] \varphi, \varphi \rangle \quad \text{where} \quad [A, C] = A \circ C - C \circ A.
\]

The operator \([A, C]\) is called the commutator. It vanishes if \( A \) and \( C \) commute.

The idea consists in constructing an operator \( C \) (bounded on \( L^2(\mathbb{R}) \)) such that
\[ [A,C] \] is positive (the so-called positive commutator method). Then we will conclude the proof by writing
\[
\left| \int_0^T \langle [A,C] \varphi , \varphi \rangle \, dt \right| = \left| \int_0^T \frac{d}{dt} \langle C \varphi , \varphi \rangle \, dt \right| \lesssim \| \varphi(T) \|_{L^2}^2 + \| \varphi(0) \|_{L^2}^2 .
\]

We seek \( C \) under the form of a pseudo-differential operator, that is an operator of the form
\[
C u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} c(x, \xi) \widehat{u}(\xi) \, d\xi ,
\]
for some function \( c = c(x, \xi) \) to be chosen. We say that \( C \) is a pseudo-differential operator associated to the symbol \( c \) and we write \( C = \text{Op}(c) \). There is a general theory which allows to handle such operators and we will study it in a forthcoming chapter. It follows from the later that the commutator \([A,C]\) is given, up to some remainder term, by a pseudo-differential operator associated with a symbol \( \gamma \) given by
\[
\gamma = \frac{1}{i} \left( \partial_\xi (i|\xi|^\frac{3}{2}) (\partial_x c) - \partial_x (i|\xi|^\frac{3}{2}) (\partial_\xi c) \right) = \frac{3}{2} \left( \partial_x c(x, \xi) \right) \frac{\xi}{|\xi|} |\xi|^\frac{1}{2} .
\]

Let us choose
\[
c(x, \xi) = \frac{\xi}{|\xi|} \int_0^x \frac{dy}{(y)^{1+\delta}} .
\]
Then
\[
\gamma = (\partial_x c(x, \xi)) \frac{\xi}{|\xi|} |\xi|^\frac{1}{2} = \frac{|\xi|^\frac{1}{2}}{(x)^{1+\delta}} .
\]

Then, the general theory alluded to above allows to show that the pseudo-differential operator \( \text{Op}(\gamma) \) with symbol \( \gamma \) is equal, modulo an admissible error, to the operator
\[
E = |D|^{\frac{1}{4}} \left( \frac{1}{(x)^{1+\delta}} |D|^{\frac{1}{4}} \right) .
\]

Now, notice that
\[
\langle E \varphi , \varphi \rangle = \int_{\mathbb{R}} \langle x \rangle^{-(1+\delta)} |D|^{\frac{1}{2}} \varphi |^2 \, dx ,
\]
whence the result. \( \square \)
1.3 The Cauchy problem

The water-wave equations read as follows:

\[
\begin{aligned}
\begin{cases}
\partial_t \eta - G(\eta) \psi &= 0, \\
\partial_t \psi + g \eta + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \left( \frac{\nabla \eta \cdot \nabla \psi + G(\eta) \psi^2}{1 + |\nabla \eta|^2} \right) + \lambda \kappa(\eta) &= 0,
\end{cases}
\end{aligned}
\]

where the unknowns are \( \eta = \eta(t, x) \), \( \psi = \psi(t, x) \) (\( x \in \mathbb{R}^d \), \( d \in \{1, 2\} \)), \( G(\eta) \) is the Dirichlet–Neumann operator we introduced in the previous paragraph and \( g > 0 \) is the acceleration of gravity, \( \lambda \geq 0 \) is a coefficient and \( \kappa(\eta) \) is the mean curvature given by

\[
\kappa(\eta) = -\text{div} \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right).
\]

There is an abundant literature on Cauchy problem for water wave equations. We refer to the references in §1.5.1 and §1.5.2. We will examine this issue in detail in a next chapter. In this introduction, we simply cite two results concerning the existence and uniqueness of smooth solutions. The first result states that the Cauchy problem is well-posed locally in time for any smooth initial data. The second gives a global well-posedness result for initial data which are small and localized.

**Theorem 1.3.1** (local existence). Let \( d \geq 1 \) and assume that either \((g, \lambda) \in \mathbb{R} \times (0, +\infty)\) or \( g > 0 \) and \( \lambda = 0 \). For any \((\eta_0, \psi_0) \in H^\infty(\mathbb{R}^d)\), the Cauchy problem for (1.3.1) with initial data \((\eta_0, \psi_0)\) has a unique maximal solution \((\eta, \psi) \in C^\infty(I; H^\infty(\mathbb{R}^d))\), \( I \subset \mathbb{R}\).

**Theorem 1.3.2** (global existence). Let \( d \geq 1 \) and assume that either \((g, \lambda) = (1, 0)\) or \((g, \lambda) = (0, 1)\). Let us consider \((\eta_0, \psi_0) \in C_0^\infty(\mathbb{R}^2)\). There is \( \varepsilon_0 > 0 \) such that for every \( 0 \leq \varepsilon < \varepsilon_0 \), the Cauchy problem for (1.3.1) with initial data \((\varepsilon \eta_0, \varepsilon \psi_0)\) has a unique solution \((\eta, \psi) \in C^\infty(\mathbb{R}; H^\infty(\mathbb{R}^d))\).

We will prove Theorem 1.3.1 later in this book. For Theorem 1.3.2 we refer to the original papers (see §1.5.1 and §1.5.2).
1.4 Stokes waves

In this section we discuss the existence of travelling waves on the surface of a heavy liquid, neglecting surface tension. This is a very old problem, going back to Stokes for periodic waves and to Boussinesq and Rayleigh for waves that decrease to 0 to infinity, called solitary waves. The problem consists in looking for waves propagating without alteration of shape, such that

\[ \eta(t, x) = \tilde{\eta}(x - ct), \]

for some function \( \tilde{\eta} \) and some constant vector \( c \in \mathbb{R}^2 \). People used to talk about permanent waves, today we talk about progressive waves.

We begin by studying this problem neglecting all the nonlinearities. The linear theory of gravity waves was much studied in the eighteenth and nineteenth centuries, notably by Laplace, Lagrange, Cauchy and Poisson. Recall that in this linear theory \( \eta \) solves

\[ \frac{\partial^2 \eta}{\partial t^2} + g |D| \eta = 0, \tag{1.4.1} \]

where \( g > 0 \) (recall that we neglect surface tension). One easily calculates that if \( \varepsilon \cos(k \cdot (x - ct)) \) is solution of the equation (1.4.1), then

\[ |c| = \sqrt{\frac{g}{|k|}}. \tag{1.4.2} \]

For a periodic progressive wave of the form \( \varepsilon \cos(k \cdot (x - ct)) \), we have a double periodicity: at a given instant, the quantity considered is spatially periodic, and at a given location, it oscillates periodically over time. The relation (1.4.2) gives a relationship between the spatial period and the temporal period.

As \( |c| \) corresponds to the speed of the wave, this means that harmonics of different wavelengths propagate at different speeds and so they tend to disperse.
Roughly summarizing what is found in oceanography books, this explains that at a certain place above the ocean, far from the wind zone, the ocean surface is very regular because it is essentially only possible to represent it with a single harmonic.\footnote{This is one of the great consequences of the memoir (\textsuperscript{[?]}) by Cauchy: "As to the state of the fluid after a given time, it will be itself very irregular in the different points of the mass fluid originally subject to the immediate influence of causes that produced the movement. But, if we move away from these same points at increasingly greater distances, we will see the movement become more and more regular." We refer to page 83 of his memoir for a precise description of this result.}

Nevertheless, the oceanographer observes that if the ocean surface has a sinuous shape, it is not the graph of a sinusoid. There are slightly marked edges and curves that suggest the presence of several harmonics. This corresponds to what we observe on the graph of the function $\cos(x) + 0.2\cos(2x)$ drawn below:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{waves.png}
\end{figure}

However, we have just seen that the linear theory predicts that two harmonics with different wavelengths will travel at different speeds. As a result their sum cannot be written as a function of $x - ct$. Stokes solved this problem. To explain the shape of the waves, he took into account terms neglected in linear theory.

Stokes’ basic idea is that the relation of dispersion $\omega^2 = g|k|$ cannot be exact and that it should depend on the amplitude of the solution. By so doing, he
discovered approximate solutions of the form:

\[ \eta(t, x) = \varepsilon \cos \theta + \frac{1}{2} |k| \varepsilon^2 \cos(2\theta) + \frac{3}{8} |k|^2 \varepsilon^3 \cos(3\theta) + O(\varepsilon^4), \]

where the phase \( \theta \) and the angular velocity \( \omega \) are given by

\[ \theta = kx - \omega t, \quad \omega = \sqrt{g |k| (1 + \frac{1}{2} \varepsilon^2 |k|^2 + O(\varepsilon^4))}, \quad x \in \mathbb{R}. \]

Stokes’ work is a formal work (his result is that if a solution exists, then it admits the previous development). It was Levi-Civita and Nekrasov who much later managed to show the existence of exact solutions in the vicinity of these approximate solutions, having this development (see the original articles \[267, 298\] as well as the texts of Strauss \[341\] and Toland \[357\] for many extensions).

To observe experimentally the nonlinear dependence of \( \omega \) on \( \varepsilon \), let us go back to the figure 1.4. In fact, at the end of a cycle, the object has not returned
exactly to its initial position, but is slightly shifted to the right, even in the absence of current; we speak of drift, which is related to the fact that the equations are non-linear. This phenomenon, discovered by Stokes, has been studied by Constantin [117] and Constantin and Escher [117, 118].

Let us conclude with two remarks on the original article by Stokes [340].

Bifurcation theory explains Stokes’ fundamental observation that the dispersion relation depends on the amplitude. It is interesting to understand Stokes’ calculations. His method can be compared to the one used by Lindstedt (then Poincaré) to find know a good approximation for large times of periodic solutions of differential equations (a classical and relevant problem in astronomy, which arises to predict the position of the planets). The most classical example of an equation to implement this method is the Duffing equation. Let us consider the Cauchy problem

\[ \ddot{u} + u^3 = 0, \quad u(0) = \varepsilon, \quad \dot{u}(0) = 0. \]

We look for \( u \) in the form \( u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots \), then

\[ \begin{align*}
\ddot{u}_1 + u_1 &= 0, \quad u_1(0) = 1, \quad \dot{u}_1(0) = 0, \\
\ddot{u}_2 + u_2 &= 0, \quad u_2(0) = 0, \quad \dot{u}_2(0) = 0, \\
\ddot{u}_3 + u_3 + (u_1)^3 &= 0, \quad u_3(0) = 0, \quad \dot{u}_3(0) = 0.
\end{align*} \]

We calculate successively that

\[ u_1 = \cos(t), \quad u_2 = 0, \quad u_3^1 = \frac{1}{4} \cos(3t) + \frac{3}{4} \cos(t) \]

and

\[ u_3 = -\frac{3}{8} t \sin(t) + \frac{1}{32} \cos(3t), \]

where to calculate \( u_3 \) we used that the solution of \( \ddot{u} + \omega_0^2 u = a \cos(\omega t) \) is

\[ \frac{a}{\omega_0^2 - \omega^2} \cos(\omega t) \text{ if } \omega \neq \omega_0, \quad \frac{a}{2\omega_0} t \sin(\omega_0 t) \text{ if } \omega = \omega_0. \]

The disturbance \( u_3 \) contains an unbounded term, \( t \sin(t) \), called the secular term. The idea of Stokes and Lindstedt is that in using a Taylor expansion of the cosine, one obtains an expression without secular term, of the form

\[ u = \varepsilon \cos \left( (1 + \frac{3}{8} \varepsilon^2 t) \right) + O(\varepsilon^3). \]

We can in fact give an explicit formula for the solution that involves the amplitude function of Jacobi and the elliptic cosine of Jacobi, see the article by Baldi [49].
Still about Stokes’ calculations, there is a remark to be made about the formula (1.4.3). This is, with one sign difference, the formula given by Stokes [340], page 211. In the original article, the equation for the free surface is written in the form

\[ y = a \cos(mx) - \frac{1}{2} ma^2 \cos(2mx) + \frac{3}{8} m^2 a^3 \cos(3mx). \]  

This contradiction in the signs is explained by the fact that for Stokes, the \((Oy)\) axis is oriented downwards, whereas we have chosen it oriented upwards. We find the formula (1.4.3) by changing \(y\) to \(-y\) and \(a\) to \(-a\) (or \(x\) in \(x + \pi\) if we want to give a physical meaning to the value of \(a\)).

### 1.5 References (work in progress)

#### 1.5.1 References about the equations

For an introduction to linear gravity wave theory in physics, we refer the reader to paragraph §12 of Landau and Lifshitz [255], Chapters 3,4,5 of the book of Holthuijsen [206] and Chapter 16 of Stewart’s book [339]. An aspect often highlighted is that the equations are Hamiltonian (see Zakharov [383]). A natural question is whether other conservation laws exist. Benjamin and Olver [56] have found 8 conservation laws for two-dimensional waves; Olver showed that under certain conditions in the definition of a conservation law, there are no other [302]. This contrasts with the discovery of Lax [262] for Korteweg de Vries’ equation.

The equations can be written in much more general forms. One can thus consider an initial velocity that does not derive from a potential (see among others [113, 269, 385]), which is important if viscosity is to be taken into account. The case of a viscous fluid is of course also very interesting, but we will not talk about it because viscosity is a simplifying hypothesis for the problems we will consider. For the study of wave propagation, one can also see that the viscosity is negligible. Indeed, it can travel immense distances without being attenuated (for example 10000 kms) between the place where it is generated and the shore. There is therefore no energy dissipation by viscous friction (the argument can be faulty in other situations). Let us just quote the work of Solonnikov [336] and Beale [53] and the recent works of Guo.
and Tice [200], Bresch and Noble [81]. The reader will find in the article by Longuet-Higgins [271], on page 537, arguments that explain why it is sometimes important to consider data that are not irrotational. A very large number of adaptations of the results on the Stokes waves in the so-called rotation are described in the review article by Strauss [341].

There are many possible formulations of the equations. One can model the water wave equations in holomorphic coordinates: this is described in detail in the papers [214] for the infinite depth case, respectively [205] for the finite depth case (see also [169]). Another idea that has been very fruitful in the field has been to introduce tools from geometric analysis, like the systematic use of the lagrangian formulation of the equations by Ambrose and Masmoudi [37], Coutand-Shkoller [133, 132], Kukavica and Tuffaha [252, 251, 253], or the tools of Riemannian geometry (parallel transport, covariant derivative) by Shatah and Zeng [327, 328, 329]. Geometrical methods have also been used by Christodoulou and Lindblad [113]. This approach was further developed by de Poyferré to initiate the study of the water-wave equation with emerging bottom [156].

Let us also mention that many papers focused on the numerical analysis of the Dirichlet-to-Neumann operator, we refer to [48, 143, 374, 311] and the references there in.

1.5.2 References about the Cauchy problem

The first results on the Cauchy problem go back to the pioneering works of Nalimov [296], Shinbrot [330], Reeder and Shinbrot [313, 314], [382] and Craig [136] (see also Tani [345], Hou, Teng and Zhang [212] and Beale, Hou and Lowengrub [54]). The first results without any hypothesis of smallness or analyticity are due to to Beyer and Günther [69] (for the case with surface tension, in any dimension) as well as Wu (for the case without surface tension, in dimension $d = 1$ first in [375], then in any dimension in [376]).

Numerous extensions of their results have been obtained by different methods: see Ogawa-Tani [301], Ambrose-Masmoudi [37, 38, 39], Schneider-Wayne [324, 325], Zhang-Zhang [385], Schweizer [326], Iguchi [221, 219, 220], Shatah-Zeng [327, 328, 329], Ming-Zhang [292], Coutand-Shkoller [133], Guo-Tice [201, 202], Masmoudi-Rousset [276], Rousset and Tzvetkov [318], Christian-
son, Hur and Staffilani [112], Alazard-Burq-Zuily [13], Lannes [259] or Berti and Delort [63] among others. The regularity of the flow map was studied by Chen, Marzuola, Spirn and Wright [107] and by Rimah-Said [320].

The main difference between the case $\lambda = 1$ (with surface tension) and the case $\lambda = 0$ (without surface tension) is that in the latter case we must make an assumption of positivity on the Taylor coefficient (named after Geoffrey Taylor [350]) which is defined by

$$a(t, x) = - (\partial_y P)(t, x, \eta(t, x)).$$

One of the main results in the articles [375, 376] by Wu states that this assumption is automatically satisfied for a domain of infinite depth. Lannes then showed that this result remains true for a regular bottom which is a small disturbance of a flat bottom ([257]); see also [170, 113].

There have been many recent results about rough solutions. One direction of research is the study of angled crested type water waves: See the recent papers by Kinsey-Wu [245] and Wu [379]. In the latter reference, Sijue Wu proved the well-posedness of the 2D water-wave equations in a regime that allows for non-$C^1$ interfaces. In this regime, only a degenerate Taylor inequality $a \geq 0$ holds. Agrawal [4] proved that initial interface with angled crests remains angled crested, the Euler equations hold pointwise even on the boundary, the particle at the tip remains at the tip, and the acceleration at the tip is the one due to gravity. Also, Agrawal extended the study of angled crested water waves to the case with surface tension (see [3, 2]). On the other hand, several recent papers are devoted to the study of the Cauchy problem with rough initial data: starting from [15] and continued in [214, 16, 158, 157, 6, 7]. The best results at the time these notes are written are due to Ai-Ifrim-Tataru [8, 9] and Wu [380].

Let us briefly discuss some additional points concerning the regularity thresholds for the water-wave equations. As explained in §1.1.4, a well-known property of smooth solutions is that their energy is conserved

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega(t)} |\nabla_{x,y} \phi(t, x, y)|^2 \, dy \, dx + \frac{g}{2} \int_{\mathbb{R}^d} \eta(t, x)^2 \, dx \right\} = 0.$$ 

However, it is not known if weak solutions exist at this level of regularity (even the meaning of the equations is not clear). This is the only known coercive quantity (see [57]). Another regularity threshold is given by the scaling
invariance, which is the following property: If $\phi$ and $\eta$ are solutions of the gravity water waves equations, then $\phi_\lambda$ and $\eta_\lambda$ defined by

$$
\phi_\lambda(t,x,y) = \lambda^{-3/2} \phi(\sqrt{\lambda} t, \lambda x, \lambda y), \quad \eta_\lambda(t,x) = \lambda^{-1} \eta(\sqrt{\lambda} t, \lambda x),
$$
solve the same system of equations. The (homogeneous) Hölder spaces invariant by this scaling (the scaling critical spaces) correspond to $\eta_0$ Lipschitz and $\phi_0$ in $\dot{W}^{3/2,\infty}$ (one can replace the Hölder spaces by other spaces having the same invariance by scalings). One could expect that the problem exhibits some kind of “ill-posedness” for initial data such that the free surface is not Lipschitz. See e.g. [93, 111] for such ill-posedness results for semi-linear equations. However, the water waves equations are not semi-linear (see [320]) and it is not clear whether the scaling argument is the only relevant regularity threshold to determine the optimal regularity in the analysis of the Cauchy problem (One key result in this direction is the resolution of the bounded $L^2$ curvature conjecture by Klainerman, Rodnianski and Szeftel [249]). In particular, it remains an open problem to prove an ill-posedness result for the gravity water waves equations. We refer to the recent paper by Chen, Marzuola, Spirn and Wright [107] for a related result in the presence of surface tension.

There are very few results about the possible blow-up of water-waves. One reference here is the paper by Castro, Córdoba, Fefferman, Gancedo and Gómez Serrano [98] (see also [100, 134]). The authors consider the more general case where the surface is not a graph, but a curve drawn in $\mathbb{R}^2$, which does not self-intersect at initial time. The authors show the existence of an initial configuration such that in finite time the free surface will self-intersect; forming what is called in the literature a splash singularity.

On the contrary, in the case without surface tension, it has been recently demonstrated by Wu [378] and also by Germain, Masmoudi and Shatah [184] that we have global existence with small data for spatially localized three-dimensional waves ($d = 2$). This result was later extended by Alazard-Delort [17, 18], Ionescu-Pusateri [222], Hunter-Ifrim-Tataru [214, 215] to the critical problem, that is for $d = 1$, following earlier work by Sijue Wu [377] about almost global well-posedness. For the case with surface tension, the first result for $d = 2$ was obtained by Germain, Masmoudi and Shatah [185]. In the critical case ($d = 1$), the result was first obtained by Ifrim and Tataru [216] and then by Ionescu and Pusateri [223]. See also the paper by Ifrim-Tataru [217] for the case with vorticity and Harrop-Griffiths, Ifrim and Tataru [205] and Wang [370] for the finite depth case. See also the review article [159].
The well-posedness of the Cauchy problem is also true for the compressible equations, whether it is the isentropic system, or the complete entropic system. We refer the reader to articles by Tanaka and Tani [344], Lindblad [268], Coutand–Lindblad–Skholler [132], Coutand–Shkoller [134], Jang–Masmoudi [234, 233]. One can also couple the fluid mechanics equations to other equations; such systems have been studied by Trakhinin [358, 359]. Another very interesting extension consists in considering the case where several fluids are superimposed. This profoundly changes the nature of the equations and raises some very difficult questions; see the review text of Saut [323], recent articles by Shatah-Zeng [329] and Lannes [259] for the study of the Cauchy problem, and that of Iooss, Lombardi and Sun [225] for the study of travelling waves. In this direction, let us also mention the article by Bresch and Renardy [83]. Let us also mention the links with the Hele-Shaw problem studied by Günther and Prokert [198, 197].

In the study of the Cauchy problem, another hypothesis, hidden and much more problematic for applications, is that we consider a fixed initial data. A whole part of the theory consists precisely in deriving asymptotic equations in regimes where the solutions (and thus the initial data) depend on a small parameter that tends towards 0. One can think of estimates which are uniform with respect to the surface tension coefficient (see [37, 327, 292]). Interesting asymptotics are those that make appear new equations. In this the subject is magnificent because many classical equations (KdV,NLS,KP,BBM,...) are thus obtained from the free surface Euler equation (see [153, 342, 383]). Let us mention only the pioneering work of Craig [136], Kano and Nishida [238] as well as the work of David Lannes and his collaborators [34, 73, 72, 140, 168, 167, 166, 258, 274]. Several new models were found through mathematical analysis, I am thinking in particular of the article by Bona, Lannes and Saut [73].

As far as the Cauchy theory is concerned, we can show that the bottom has very little importance and that very irregular funds are appropriate. Other studies are interested in the case where the topography changes the dynamics (what happens in coastal oceanography, when near the shore the depth is shallow; which is fundamental in the study of refraction). This is the case, for example, of the work of Craig, Guyenne and Sulem [138], de Bouard, Craig, Díaz-Espinosa, Guyenne and Sulem [155], Alvarez-Samaniego and Lannes [33], Lannes [259], Chazel [104] and Israwi [232]. This raises a host of new questions and requires a wide variety of techniques. For example, in [140], Craig, Lannes and Sulem are interested in the so-called ”shallow water” limit in case
the bottom is strongly oscillating. The authors derive a new system, which corrects previous formal work, thanks to tools resulting from the analysis of homogenization problems.

1.5.3 Dispersive estimates

There is a vast literature concerning dispersive estimates. Here we mention a few results related to the smoothing effect mentioned previously. Let us quote the original article by Kato [240] for the Korteweg-de-Vries equation (KdV), the articles by Constantin and Saut [123], Sjölin [335] and Vega [366] for a general dispersive equation with constant coefficients, and for the Schrödinger equation with variable coefficients include the work of Craig, Kappeler and Strauss [139], Doi [162, 163, 164], Kenig-Ponce-Vega [242, 243], Robbiano and Zuily [316], Burq and Planchon [87]. Note that the regularizing effect of Kato for the KdV equation, intervenes in the demonstration by Rosier [317] of inequality observability for this equation (see also Coron’s book [129, section 2.2]). There are links between the free surface Euler equation with surface tension and the Euler-Korteweg system (see Benzoni [58]); Audiard [44] has recently shown an analog of the theorem 1.2.8 for this system. Sharp versions of the smoothing effect were obtained recently by Zhu [388] and Alazard-Ifrim-Tataru [19]. Dispersive properties can also be used to prove controllability properties of the waves [11, 387].

In addition to the smoothing effect, the water-wave equations enjoy Strichartz-type inequalities. For the water-wave equations with surface tension, in the special case of dimension $d = 1$, Strichartz estimates were proved by Christianson, Hur and Staffilani in [112] for smooth enough data and in [14] for the low regularity solutions constructed in [13].

One natural question is to combine these Strichartz estimates with the standard energy estimates to improve the threshold for the well-posedness theory. The main difficulty is that this requires to establish Strichartz inequalities at a lower level of regularity than the threshold where one is able to prove the existence of the solutions. This was first achieved in [15] for the gravity water-wave equations. This allowed to improve the Cauchy theory and in particular, for $d = 1$, to consider solutions such that the curvature of the initial free surface does not belong to $L^2$. Such Strichartz estimates are obtained by constructing parametrices on small time intervals tailored to the size of the frequencies
considered (in the spirit of the works by Lebeau [263], Bahouri-Chemin [46], Tataru [349], Staffilani-Tataru [337], and Burq-Gérard-Tzvetkov [86]). These semi-classical Strichartz estimates have been extended in several directions: 

i) to the case with surface tension by de Poyferré and Nguyen [157], 

ii) Ai obtained ([5]) optimal lossless Strichartz estimates. This provides the sharp regularity threshold with respect to the approach of combining Strichartz estimates with energy estimates.

\subsection{1.5.4 References about progressive waves}

In the rest of these lectures, we will focus on the study of the Cauchy problem and we will not study the Stokes waves. Since the study of those special solutions lies at the heart of the water-wave theory, we discuss in this section many recent research directions about them.

We know today that the existence of 2D waves simply comes from a bifurcation from a simple eigenvalue. This corresponds to a local theory of bifurcations which studies all the solutions for \( \varepsilon \) close to 0. There is also a global theory that studies the possibility of find a branch of solutions parameterized by \( \varepsilon \in \mathbb{R} \). Using this global theory we can prove the existence of what we call extreme Stokes waves (see [357, 85] for a complete introduction to the study of extremal waves and also the original articles of Toland [356], Amick, Fraenkel and Toland [40], Plotnikov [307]). For the problems we have studied, the difficulty is that the inversion of the linearized operator around a non-trivial state (in the orthogonal of its kernel) is possible, but at the price of a loss of derivatives (which comes from the presence of small divisors). The theory of bifurcations no longer applies because the implicit function theorem no longer applies. However, as has been observed in another context by Rabinowitz [312] (see also [135]), one can use the implicit function theorem of Nash-Moser.

The study of Stokes waves is a classical subject in physics: see the reference books of Whitham [372] and Wehausen and Laitone [371]. The question of reflection of a Stokes wave on a wall is illustrated in the book by Barber [51]. This is a classic subject for experimenters, and we refer here to the very complete review articles of Dias and Kharif [161] and Hammack, Henderson and Segur [204]. The more general question of the Stokes wave interaction is discussed in the books of Holthuijsen [206] and Kartashova [239]. The numerical study of surface waves is of course a huge subject and we refer the reader to
References on the problem of standing gravity waves (periodic in time and space) are given in the introduction of the article by Iooss, Plotnikov and Toland [231]. See also the note by Iooss [224] who cites Boussinesq as the first to consider this question in 1877. The question of the existence of these waves has remained open for a very long time. The first result is due to Plotnikov and Toland [308]. This work was extended by Iooss, Plotnikov and Toland ([231] then supplemented by the articles of Iooss and Plotnikov [226, 227] which show the existence of solutions unimodal ([231]) and multimodal ([226, 227]) in the completely degenerate case where the kernel is of infinite dimension (which comes from the fact that in [231, 226, 227] the domain is of infinite depth, a more difficult case for this problem than the case of a finite depth domain [308]). The stability of standing waves with respect to harmonic perturbations are studied by Wilkening in [373]. A new phenomenon concerning a directional drift for 3D waves has been demonstrated by Iooss and Plotnikov [228] for small amplitude 3D waves non-symmetrical; whose existence was demonstrated by Iooss and Plotnikov [230] which was a very difficult question. The small divisor problems for 3D water waves was further studied by Alazard and M´etivier [22]. The first result about the existence of gravity capillary standing water waves was obtained in [10]. By using the transformations introduced by Iooss-Plotnikov [229] and Alazard and Baldi [10] with methods to study quasi-periodic solutions, KAM results were then obtained, first for gravity-capillary water waves by Berti and Montalto [64] and then for the gravity water waves equations (see [50]).

The study of periodic travelling waves 3D can be traced back to the work of Reeder and Shinbrot [315]. It has been completed by Groves and Haragus [192] and Craig and Nicholls [141]. In the case where the surface tension is very high, the existence of localized capillary waves has been demonstrated by Groves and Sun [195] and Buffoni, Groves, Sun and Wahlén [84]. Let us also mention the previous papers by Groves and Mielke [194] (which we refer to for a discussion of Kirchgässner’s spatial dynamics approach [246]). The linear instability of capillary waves is studied by Groves and Haragus [193]. The case non-linear, very difficult, was solved by Rousset and Tzvetkov [318]. There are very many results in the case where the rotational is non-zero, see for example [369] and the references of this article.

Solitary water waves have been proved to exist in many interesting situations.
Their existence relies on a subtle balance of different factors, such as gravity, surface tension or the fluid bottom (see [179, 52, 41]). For this question, we refer to the article by Ifrim and Tataru [218] who prove that there are no solitary waves in 2D for purely gravity or capillary waves in infinite depth.

We have discussed the interaction of two progressive periodic waves. The interaction of two solitary waves is studied numerically and experimentally by Craig, Guyenne, Hammack, Henderson and C. Sulem in the article [137]. The interaction of solitons for the Korteweg de Vries equation has been studied, including in non-integrable situations, by Martel and Merle (see the survey [275]).

1.5.5 Historical References

The linear theory of gravity waves was much studied in the eighteenth and nineteenth centuries, notably by Laplace, Lagrange, Cauchy and Poisson. We refer to the articles of Craik [146, 147], to his book [145] as well as to that of Darrigol [151]. And note especially that the statues of Laplace, Lagrange, Cauchy and Poisson are located in the courtyard of the École normale supérieure rue d’Ulm in Paris, in the corner which is closest to the Pantheon.
The question by the French Academy question we cited can be found in Cauchy’s memoir [?]. Darrigol explains in his book [151, page 37] that it is due to Laplace.

Many microlocal analysis tools have their origin in the study of surface waves or in connection with questions that arise in this field. One can think of the introduction of the stationary phase method (in order to study the pattern of waves in the wake of a boat). Note that the notions of stationary phase, phase velocity and speed group appear in some calculations of Cauchy [?]. He had seen that harmonics of different wavelengths propagate at different speeds and thus they tend to disperse; they are said to disperse.

Let us also note that Cauchy and Poisson would have rediscovered Fourier’s integral formula (which appears in Fourier’s memoir, published in 1822, for which he won the prize for his answer (given in 1812) to a question asked by the Academy in 1811). This is explained in the article by Annaratone [42].

Part of the techniques we will use comes from the study of shock waves and rarefaction waves. A shock is a weak solution of a system of conservation laws, discontinuous through a regular hypersurface Σ, regular up to the boundary on either side of Σ. One can read in Tartar’s book on Sobolev’s spaces ([348, page 183]) that Stokes was the first to be interested in discontinuous solutions of the acoustic wave equation, in an article published one year after the publication of his 1847 article on the theory of oscillating waves [340].

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6See the minutes of the sessions of the Academy, which can also be consulted at http://gallica.bnf.fr/ark:/12148/cb32746437k/; more generally, the original articles of Cauchy, Stokes, Russell and also those of Boussinesq, that we will mention later, can be easily found on the internet, in free access.
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THE

Public Fountains

of the City of Dijon

EXPLANATION AND APPLICATION OF

Principles to be Followed and Formulas to be used

IN QUESTIONS ABOUT

Water Distribution Issues

COMPLETED WITH

an appendix relating to water supplies,

water filtering

AND TO

THE MANUFACTURE OF CAST IRON, SHEET METAL AND BITUMEN PIPES

BY

Henry Darcy

GENERAL MANAGER OF BRIDGES AND ROADWAYS

Since the good quality of water is one of the things that contribute most to the health of the citizens of a town, there is nothing in which we can spend more for the welfare of the population.

The water used for the domestic needs of men and animals, and to remedy the accidents by which this water could be obtained, either to the soils of the houses or to the streets, should be clean and pure. The water should not have been a dirt, or deadly in use. Since the springs are clean.

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1856

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Chapter 2

Darcy’s law and the Hele-Shaw equations

Consider a time-dependent surface \( \Sigma \) of the form

\[
\Sigma(t) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}; \ y = h(t, x)\} \quad (t \in \mathbb{R}_+).
\]

In this chapter, we consider several free boundary problems that are described by evolution equations that express the velocity of \( \Sigma \) at each point in terms of some nonlinear expressions depending on \( h \).

The most popular example is the mean-curvature equation, which stipulates that the normal component of the velocity of \( \Sigma \) is equal to the mean curvature at each point. It follows that:

\[
(2.0.1) \quad \partial_t h + \sqrt{1 + |\nabla h|^2} \kappa = 0 \quad \text{where} \quad \kappa = -\text{div}\left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right).
\]

The previous equation plays a fundamental role in differential geometry. Many other free boundary problems appear in fluid dynamics. Among these, we are chiefly concerned by the equations modeling the dynamics of a free surface transported by the flow of an incompressible fluid evolving according to Darcy’s law.
2.1 The Hele-Shaw equations

2.1.1 Darcy’s law

Consider the fluid domain

\[ \Omega(t) = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} ; y < h(t, x)\}. \]

One assumes that the velocity \( v : \Omega \to \mathbb{R}^{d+1} \) and the pressure \( P : \Omega \to \mathbb{R} \) solve the Darcy’s equations. This means that

\[ (2.1.1) \quad \text{div}_{x,y} v = 0, \quad \text{and} \quad v = -\nabla_{x,y}(P + gy) \quad \text{in } \Omega, \]

where \( g > 0 \) is the acceleration of gravity.

2.1.2 Boundary conditions

One imposes that

\[ \lim_{y \to -\infty} v = 0. \]

As for water waves, on the free surface, it is assumed that the normal component of \( v \) coincides with the normal component of the velocity of the free surface:

\[ (2.1.2) \quad \partial_t h = \sqrt{1 + \left| \nabla h \right|^2} v \cdot n \quad \text{on} \quad y = h, \]

where recall that \( \nabla = \nabla_x \) and \( n \) is the outward unit normal to \( \Sigma \), given by

\[ n = \frac{1}{\sqrt{1 + \left| \nabla h \right|^2}} \left( -\nabla h \begin{pmatrix} 1 \\ -\nabla h \end{pmatrix} \right). \]

The final equation states that the restriction of the pressure to the free surface is proportional to the mean curvature:

\[ P = \lambda \kappa \quad \text{on} \quad \Sigma, \]

where the parameter \( \lambda \) belongs to \([0, 1]\) and \( \kappa \) is given by \((2.0.1)\).
2.1.3 Formulation with the DN

Recall that the Dirichlet-to-Neumann operator $G(h)$ is defined by

$$G(h) \psi(x) = \sqrt{1 + |\nabla h|^2} \partial_n \varphi |_{y=h(x)} = \partial_y \varphi(x, h(x)) - \nabla h(x) \cdot \nabla \varphi(x, h(x)),$$

where

$$\Delta_{x,y} \varphi = 0 \quad \text{in} \quad \Omega, \quad \varphi |_{y=h} = \psi.$$

Since

$$\Delta_{x,y}(P + gy) = \text{div}_{x,y} v = 0; \quad P + gy |_{y=h} = gh + \lambda \kappa,$$

it follows that the Hele-Shaw problem is equivalent to

$$(2.1.3) \quad \partial_t h + G(h)(gh + \lambda \kappa) = 0.$$  

When $g = 1$ and $\lambda = 0$, the equation (2.1.3) is called the Hele-Shaw equation without surface tension. Hereafter, we will refer to this equation simply as the Hele-Shaw equation. If $g = 0$ and $\lambda = 1$, the equation is known as the Hele-Shaw equation with surface tension, also known as the Mullins-Sekerka equation. Let us record the terminology:

$$(2.1.4) \quad \partial_t h + G(h) h = 0 \quad \text{(Hele-Shaw)},$$  

$$(2.1.5) \quad \partial_t h + G(h) \kappa = 0 \quad \text{(Mullins-Sekerka)}.$$  

Recalling that $G(0) = |D|$, we see that the linearized equations read

$$(2.1.6) \quad \partial_t h + |D| h = 0 \quad \text{(Linearized Hele-Shaw)},$$  

$$(2.1.7) \quad \partial_t h + |D|^3 h = 0 \quad \text{(Linearized Mullins-Sekerka)}.$$  

They are parabolic evolution equations.

2.1.4 Lubrication approximation

Other parabolic equations appear naturally to describe asymptotic regime in the thin-film approximation. They are

$$(2.1.8) \quad \partial_t h - \text{div}(h \nabla h) = 0 \quad \text{(Boussinesq)},$$  

$$(2.1.9) \quad \partial_t h + \text{div}(h \nabla \Delta h) = 0 \quad \text{(thin-film)}.$$  

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2.1.5 The Cauchy problem

The Cauchy problems for the Hele-Shaw and Mullins-Sekerka equations have been studied by different techniques, for weak solutions, viscosity solutions or also classical solutions (see the numerous references in §2.3.3 at the end of this chapter). In this book, we will restrict ourselves to the study of classical solutions. We consider initial data in Sobolev spaces.

Definition 2.1.1. Given a real number \( s \geq 0 \), the Sobolev space \( H^s(\mathbb{R}^d) \) consists of those functions \( u \in L^2(\mathbb{R}^d) \) such that the following norm is finite:

\[
\|u\|_{H^s}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi,
\]

where \( \hat{u} \) is the Fourier transform of \( u \):

\[
\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} u(x) \, dx.
\]

The homogeneous Sobolev norms are defined by

\[
\|u\|_{\dot{H}^s}^2 := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi.
\]

Theorem 2.1.2. Let \( d \geq 1 \) and consider a real number \( s > \frac{d}{2} + 1 \). For any initial data \( h_0 \) in \( H^s(\mathbb{R}^d) \), there exists a time \( T > 0 \) such that the Cauchy problem

\[
\frac{\partial}{\partial t} h + G(h) h = 0, \quad h|_{t=0} = h_0,
\]

has a unique solution satisfying

\[
h \in C^0([0,T]; H^s(\mathbb{R}^d)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^d)) \cap L^2([0,T]; H^{s+\frac{1}{2}}(\mathbb{R}^d)).
\]

Moreover, \( h \) belongs to \( C^\infty((0,T] \times \mathbb{R}^d) \).

Proof. See Chapter ??.

Definition 2.1.3. i) Recall that

\[
\sigma > \frac{d}{2} \implies H^\sigma(\mathbb{R}^d) \hookrightarrow C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).
\]

So \( s > 1 + d/2 \) implies that \( H^s(\mathbb{R}^d) \hookrightarrow C^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d) \).

ii) We say that \( h \) is a regular solution to (2.1.10) defined on \([0,T]\) if \( h \) satisfies the conclusions of the above result.
2.1.6 Maximum principles and Lyapunov functionals

The Hele-Shaw equation is a nonlinear parabolic equation, so a natural question is to find maximum principles. We will consider maximum principle for \( h \) itself or for its derivatives.

**Proposition 2.1.4.**

i) Let \( d \geq 1 \). For any regular solution \( h \) of the Hele-Shaw equation \( \partial_t h + G(h)h = 0 \), there holds
\[
\sup_{x \in \mathbb{R}^d} |h(t,x)| \leq \sup_{x \in \mathbb{R}^d} |h(0,x)|.
\]

ii) Let \( h_1, h_2 \) be two regular solutions of the Hele-Shaw equation defined on the same time interval \([0,T]\), such that, initially,
\[
h_1(0,\cdot) \leq h_2(0,\cdot).
\]
Then
\[
h_1(t,\cdot) \leq h_2(t,\cdot)
\]
for all \( t \in [0,T] \).

*Proof.* See Chapter ??.

**Proposition 2.1.5.** Let \( d \geq 1 \) and consider a regular solution \( h \) to the Hele-Shaw equation \( \partial_t h + G(h)h = 0 \) defined on \([0,T]\). Then, whenever \( \omega \) is a modulus of continuity for \( h(0,\cdot) \), \( \omega \) is also a modulus of continuity for \( h(t,\cdot) \), for any \( t \in [0,T] \). In particular, we have
\[
\sup_{x \in \mathbb{R}^d} |\nabla h(t,x)| \leq \sup_{x \in \mathbb{R}^d} |\nabla h(0,x)|.
\]

*Proof.* See Chapter ??.

We will also study the monotonicity of some coercive quantities. Consider the \( L^2 \)-norm and the area functional
\[
\left( \int_{\mathbb{R}^d} h^2 \, dx \right)^{\frac{1}{2}}, \quad A(\Sigma) = \int_{\mathbb{R}^d} \left( \sqrt{1 + |\nabla h|^2} - 1 \right) \, dx.
\]

Recall that \( G(h) \) is a non-negative operator. Indeed, consider a function \( \psi = \psi(x) \) and denote by \( \varphi = \varphi(x,y) \) its harmonic extension given by
\[
\Delta_{x,y} \varphi = 0 \quad \text{in } \Omega = \{ y < h(t,x) \}, \quad \varphi |_{y=h} = \psi.
\]
It follows from Stokes’ theorem that

\[(2.1.11) \quad \int_{\mathbb{R}^d} \psi G(h) \psi \, dx = \int_{\partial \Omega} \varphi \partial_n \varphi \, d\sigma = \iint_{\Omega} |\nabla_{x,y} \varphi|^2 \, dy \, dx \geq 0.\]

1. This immediately implies that, for any regular solution to \( \partial_t h + G(h)h = 0 \), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} h^2 \, dx = - \int_{\mathbb{R}^d} hG(h)h \, dx \leq 0.
\]

We say that \( \int_{\mathbb{R}^d} h^2 \, dx \) is a Lyapunov functional.

2. On the other hand, if \( h \) is a regular solution to \( \partial_t h + G(h)\kappa = 0 \), then

\[
\frac{d}{dt} A(\Sigma) = \frac{d}{dt} \int_{\mathbb{R}^d} \left( \sqrt{1 + |\nabla h|^2} - 1 \right) \, dx
= \int_{\mathbb{R}^d} \frac{\nabla h \cdot \nabla \partial_t h}{\sqrt{1 + |\nabla h|^2}} \, dx
= \int_{\mathbb{R}^d} (\partial_t h)\kappa \, dx \quad (\text{integration by parts})
= - \int_{\mathbb{R}^d} \kappa G(h)\kappa \, dx \leq 0 \quad (\text{using (2.1.11)}).
\]

This proves that \( A(\Sigma) \) is a Lyapunov functional for the Mullins-Sekerka equation.

We refer to Section 2.3.5 for references and additional results related to these two examples. The next result generalizes the previous observations: the \( L^2 \)-norm and the area functional are Lyapunov functionals for the Hele-Shaw and Mullins-Sekerka equations, in any dimension, uniformly in \( g \) and \( \lambda \).

**Proposition 2.1.6.** Let \( d \geq 1 \), \( (g, \lambda) \in [0, +\infty)^2 \) and assume that \( h \) is a smooth solution to

\[
\partial_t h + G(h)(gh + \lambda \kappa) = 0.
\]

Then,

\[
\frac{d}{dt} \int_{\mathbb{R}^d} h^2 \, dx \leq 0 \quad \text{and} \quad \frac{d}{dt} A(\Sigma) \leq 0.
\]

**Proof.** See Chapter ??.

In addition, for the Hele-Shaw equation, the square of the \( L^2 \)-norm decays in a convex manner.
Proposition 2.1.7. Let $d \geq 1$. For any regular solution $h$ of the Hele-Shaw equation $\partial_t h + G(h)h = 0$, there holds

$$\frac{d}{dt} \int_{\mathbb{R}^d} h(t, x)^2 \, dx \leq 0 \quad \text{and} \quad \frac{d^2}{dt^2} \int_{\mathbb{R}^d} h(t, x)^2 \, dx \geq 0.$$ 

Proof. See Chapter ??.

2.2 The Muskat equation

2.2.1 The Muskat problem

The Muskat equation is a two-fluids in porous media, analogue of the Hele-Shaw equation. Here we consider a time-dependent free surface $\Sigma(t)$ separating two fluid domains $\Omega_1(t)$ and $\Omega_2(t)$. To simplify, we consider only two-dimensional fluids so that

$$\Omega_1(t) = \{(x, y) \in \mathbb{R} \times \mathbb{R} ; y > f(t, x)\},$$
$$\Omega_2(t) = \{(x, y) \in \mathbb{R} \times \mathbb{R} ; y < f(t, x)\},$$
$$\Sigma(t) = \partial \Omega_1(t) = \partial \Omega_2(t) = \{y = f(t, x)\}.$$ 

Introduce the density $\rho_i$, the velocity $v_i$ and the pressure $P_i$ in the domain $\Omega_i$ ($i = 1, 2$). One assumes that the velocities $v_1$ and $v_2$ obey Darcy’s law. Then, the equations by which the motion is to be determined are

$$v_i = -\nabla(P_i + \rho_i g y) \quad \text{in} \quad \Omega_i,$$
$$\text{div} \, v_i = 0 \quad \text{in} \quad \Omega_i,$$
$$P_1 = P_2 \quad \text{on} \quad \Sigma,$$
$$v_1 \cdot n = v_2 \cdot n \quad \text{on} \quad \Sigma,$$
$$\partial_t f = \sqrt{1 + (\partial_x f)^2} v_2 \cdot n$$

where $g$ is the gravity and $n$ is the outward unit normal to $\Omega_2$ on $\Sigma$,

$$n = \frac{1}{\sqrt{1 + (\partial_x f)^2}} \begin{pmatrix} -\partial_x f \\ 1 \end{pmatrix}.$$
2.2.2 The Córdoba-Gancedo formulation

Changes of unknowns, reducing the Muskat problem to an evolution equation for the free surface parametrization, have been known for quite a time (see the references at the end of this chapter). This approach was further developed by Córdoba and Gancedo who obtained the following beautiful compact formulation of the Muskat equation:

\[
\partial_t f = \frac{\rho}{2\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \, d\alpha,
\]

(2.2.1)

where \( \rho = \rho_2 - \rho_1 \) is the difference of the densities of the two fluids and \( \Delta_\alpha f \) is the slope

\[
\Delta_\alpha f(t, x) = \frac{f(t, x) - f(t, x - \alpha)}{\alpha}.
\]

(2.2.2)

We assume that \( \rho_2 > \rho_1 \) (heavier fluid below the lighter one) and then we may set \( \rho = 2 \) without loss of generality. It is not obvious that the right-hand side is well defined, and we will address this question later in this section.

Besides its esthetic aspect, this formulation allows to apply tools at interface of harmonic analysis and nonlinear partial differential equations. One can think of the circle of methods centering around the study of the Hilbert transform \( \mathcal{H} \) and Riesz potentials, or Besov and Triebel-Lizorkin spaces.

Recall that the Hilbert transform \( \mathcal{H} \) is defined by

\[
\mathcal{H}u(\xi) = -i\frac{\xi}{|\xi|} \hat{u}(\xi).
\]

It follows that the fractional Laplacian \( |D| = (-\partial_{xx})^{1/2} \) satisfies

\[
|D| = \partial_x \mathcal{H}.
\]

Alternatively, it can be defined by a singular integral:

\[
\mathcal{H}f(x) = \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy,
\]

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where the integral is understood in the sense of principal values:

\[ \mathcal{H} f(x) = \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy = \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{f(x - y)}{y} \, dy \]

\[ = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon < |y| < \frac{1}{2}} \frac{f(x - y)}{y} \, dy. \]

Now, observe that

\[ \mathcal{H} f(x) = -\frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{f(x) - f(x - \alpha)}{\alpha} \, d\alpha = -\frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \Delta_{\alpha} f \, d\alpha. \]

It follows that the fractional Laplacian \(|D| = \partial_x \mathcal{H}\) satisfies

\[ |D| f = -\frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \partial_x \Delta_{\alpha} f \, d\alpha. \]

Consequently, by writing

\[ \frac{\partial_x \Delta_{\alpha} f}{1 + (\Delta_{\alpha} f)^2} = \partial_x \Delta_{\alpha} f - (\partial_x \Delta_{\alpha} f) \frac{(\Delta_{\alpha} f)^2}{1 + (\Delta_{\alpha} f)^2}, \]

we see that the Muskat equation can be written under the form

\[ \partial_t f + |D| f = T(f) f \quad \text{where} \quad T(f) f = -\frac{1}{\pi} \int_{\mathbb{R}} (\partial_x \Delta_{\alpha} f) \frac{(\Delta_{\alpha} f)^2}{1 + (\Delta_{\alpha} f)^2} \, d\alpha. \]

To give a rigorous meaning to the Muskat equation, we will show in a moment that the map \( f \mapsto T(f) f \) is locally Lipschitz from \( H^1(\mathbb{R}) \cap H^2(\mathbb{R}) \) to \( L^2(\mathbb{R}) \).

To do so, it is convenient to recall a characterization of Sobolev spaces in terms of finite differences. Given a function \( f: \mathbb{R} \to \mathbb{R} \), an integer \( m \in \mathbb{N} \setminus \{0\} \) and a real number \( h \in \mathbb{R} \), we define the finite differences \( \delta_h^m f \) as follows:

\[ \delta_h f(x) = f(x) - f(x - h), \quad \delta_h^{m+1} f = \delta_h (\delta_h^m f). \]

Notice that, by notations, we have

\[ (\Delta_{\alpha} f)(x) = \frac{\delta_{\alpha} f}{\alpha}. \]

**Remark 2.2.1.** The operators \( \partial_x, \mathcal{H}, |D|^s \) and \( \delta_{\alpha} \) are Fourier multipliers:

\[ \hat{\partial_x} u(\xi) = i\xi \hat{u}(\xi), \quad \hat{\delta_{\alpha}} u(\xi) = (1 - e^{-i\alpha \xi}) \hat{u}(\xi), \]

\[ \hat{\mathcal{H}} u(\xi) = -i \frac{\xi}{|\xi|} \hat{u}(\xi), \quad \hat{|D|^s} u(\xi) = |\xi|^s \hat{u}(\xi). \]

Consequently, they commute with each other.
2.2.3 Gagliardo semi-norms

Given \( s \in (0, 1) \), we define the following Gagliardo semi-norm

\[
\| u \|_{\dot{F}^s_{2,2}}^2 := \iint_{\mathbb{R} \times \mathbb{R}} \frac{|u(x) - u(y)|^2}{|x-y|^{2s}} \, dx \, dy.
\]

Then, for \( s \in (0, 1) \),

\[
\| u \|_{\dot{H}^s}^2 = \frac{1}{4\pi c(s)} \| u \|_{\dot{F}^s_{2,2}}^2 \quad \text{with} \quad c(s) = \int_{\mathbb{R}} \frac{1 - \cos(h)}{|h|^{1+2s}} \, dh.
\]

Notice that this is equivalent to

\[
\| u \|_{\dot{H}^s}^2 = \frac{1}{4\pi c(s)} \iint_{\mathbb{R} \times \mathbb{R}} \frac{|u(x) - u(x-\alpha)|^2}{|\alpha|^{2s}} \, dx \, d\alpha \quad \text{with} \quad c(s) = \int_{\mathbb{R}} \frac{1 - \cos(h)}{|h|^{1+2s}} \, dh.
\]

This immediately implies the following

Lemma 2.2.2. For all \( a \in [0, +\infty) \) and \( b \in (0, 1) \), there exists a positive constant \( C > 0 \) such that

\[
\int_{\mathbb{R}} \| \delta_\alpha f \|_{\dot{H}^a}^2 \, d\alpha \|_{\dot{H}^{a+b}} = C \| f \|_{\dot{H}^{a+b}}^2.
\]

Proof. Since \( \| \delta_\alpha f \|_{\dot{H}^a} = \| \delta_\alpha (|D|^a f) \|_{L^2} \), (2.2.6) follows at once from (2.2.5) applied with \( u = |D|^a f \). \( \square \)

As an example of properties that is very simple to prove using the definition of Sobolev spaces in terms of finite differences, let us prove the following proposition (from Alazard and Lazare [21]).

Proposition 2.2.3. Consider the operator

\[
\mathcal{T}(f)g = -\frac{1}{\pi} \int_{\mathbb{R}} \Delta_\alpha g_x \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \, d\alpha,
\]

where \( g_x := \partial_x g \).
i) For all $f$ in $H^1(\mathbb{R})$ and all $g$ in $H^{3/2}(\mathbb{R})$, the function

$$\alpha \mapsto \nabla_\alpha g x \left( \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \right)$$

belongs to $L^1_\alpha(\mathbb{R}; L^2_x(\mathbb{R}))$. Consequently, $T(f)g$ belongs to $L^2(\mathbb{R})$. Moreover, there is a constant $C$ such that

$$\|T(f)g\|_{L^2} \leq C \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^{3/2}}.$$  

(2.2.7)

ii) There exists a constant $C > 0$ such that, for all functions $f_1, f_2$ in $H^1(\mathbb{R})$ and for all $g$ in $H^{3/2}(\mathbb{R})$,

$$\|(T(f_1) - T(f_2))g\|_{L^2} \leq C \|f_1 - f_2\|_{\dot{H}^1} \|g\|_{\dot{H}^{3/2}}.$$

iii) The map $f \mapsto T(f)f$ is locally Lipschitz from $H^{3/2}(\mathbb{R})$ to $L^2(\mathbb{R})$.

**Proof.** i) The proof will use the following Sobolev embedding (see Theorem 4.3.1 as well as the remark which follows the latter)

$$\dot{H}^t(\mathbb{R}) \hookrightarrow L^{\frac{2}{1-t}}(\mathbb{R}) \quad \text{for} \quad 0 \leq t < \frac{1}{2}. \quad (2.2.8)$$

In particular, for $t = 1/4$, this gives that $\dot{H}^{1/4}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$.

Recall that, by notation, we have

$$\Delta_\alpha f = \frac{\delta_\alpha f}{\alpha}.$$

By combining the previous Sobolev embedding with Hölder’s inequality, we obtain

$$\left\| \nabla_\alpha g x \left( \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \right) \right\|_{L^2} \leq \| (\Delta_\alpha g x)(\Delta_\alpha f) \|_{L^2} \leq \| \Delta_\alpha g x \|_{L^4} \| \Delta_\alpha f \|_{L^4} \leq \| \delta_\alpha g x \|_{\dot{H}^{3/2}} \| \delta_\alpha f \|_{\dot{H}^{3/2}} \frac{1}{|\alpha|} \| \delta_\alpha g x \|_{\dot{H}^{3/2}} \frac{1}{|\alpha|}.$$
Therefore,

\[
\|T(f)g\|_{L^2} \lesssim \int \frac{\|\delta_\alpha g_x\|_{\dot{H}^{1/2}}}{|\alpha|^{1/4}} \frac{\|\delta_\alpha f\|_{\dot{H}^{1/2}}}{|\alpha|^{1/4}} d\alpha
\]

\[
\lesssim \left( \frac{\|\delta_\alpha g_x\|_{\dot{H}^{1/2}}^2}{|\alpha|^{1/4}} \frac{\|\delta_\alpha f\|_{\dot{H}^{1/2}}^2}{|\alpha|^{1/4}} d\alpha \right)^{1/2} \left( \frac{\|\delta_\alpha f\|_{\dot{H}^{1/2}}^2}{|\alpha|^{3/4}} d\alpha \right)^{1/2}
\]

\[
\lesssim \|g_x\|_{\dot{H}^{1/2}} \|f\|_{H^1} = \|g\|_{\dot{H}^{1/2}} \|f\|_{H^1},
\]

hence the wanted inequality (2.2.7).

**ii)** Write that

\[(T(f_1) - T(f_2))g = -\frac{1}{\pi} \int \Delta_\alpha g_x \Delta_\alpha (f_1 - f_2) M(\alpha, x) \, d\alpha\]

where

\[M(\alpha, x) = \frac{(\Delta_\alpha f_1) + \Delta_\alpha f_2}{(1 + (\Delta_\alpha f_1)^2)(1 + (\Delta_\alpha f_2)^2)}.\]

Since \(|M(\alpha, x)| \leq 1\), by repeating similar arguments to those used in the first part, we get

\[
\|(T(f_1) - T(f_2))g\|_{L^2} \lesssim \frac{1}{\pi} \int \frac{\|\delta_\alpha g_x\|_{L^4}}{|\alpha|} \frac{\|\delta_\alpha (f_1 - f_2)\|_{L^4}}{|\alpha|} d\alpha
\]

\[
\lesssim \frac{1}{\pi} \int \frac{\|\delta_\alpha g_x\|_{\dot{H}^{1/2}}}{|\alpha|^{1/4}} \frac{\|\delta_\alpha (f_1 - f_2)\|_{\dot{H}^{1/2}}}{|\alpha|^{3/4}} d\alpha
\]

\[
\lesssim \|g_x\|_{\dot{H}^{1/2}} \|f_1 - f_2\|_{H^{1/2}}.
\]

which implies

\[
\|(T(f_1) - T(f_2))g\|_{L^2} \lesssim \|f_1 - f_2\|_{H^1} \|g\|_{\dot{H}^{1/2}}.
\]

**iii)** Consider \(f_1\) and \(f_2\) in \(H^{3/2}(\mathbb{R})\). Then

\[T(f_1)f_1 - T(f_2)f_2 = T(f_1)(f_1 - f_2) + (T(f_1) - T(f_2))f_2.\]

Then (2.2.7) implies that the \(L^2\)-norm of the first term is bounded by

\[C \|f_1\|_{H^1} \|f_1 - f_2\|_{\dot{H}^{3/2}}.\]
We estimate the second term by using statement \( ii \). It follows that
\[
\| T(f_1)f_1 - T(f_2)f_2 \|_{L^2} \lesssim (\| f_1 \|_{H^{3/2}} + \| f_2 \|_{H^{3/2}}) \| f_1 - f_2 \|_{H^{3/2}},
\]
which completes the proof. \( \square \)

### 2.2.4 The critical Cauchy problem

In section, I discuss the main result from a series of papers with Quoc-Hung Nguyen (see [26, 25, 24]) devoted to the study of solutions with critical regularity for the two-dimensional Muskat equation. We prove that the Cauchy problem is well-posed on the endpoint Sobolev space \( H^{3/2}(\mathbb{R}) \) of \( L^2 \) functions with three-half derivative in \( L^2 \). This result is optimal with respect to the scaling of the equation.

A key feature of the Muskat equation is that (2.2.1) is preserved by the change of unknowns:
\[
f(t, x) \mapsto f_\lambda(t, x) := \frac{1}{\lambda} f(\lambda t, \lambda x).
\]

Now, by a direct calculation, one verifies that
\[
\| f_\lambda \|_{t=0} \|_{\dot{H}^{3/2}} = \| f_0 \|_{\dot{H}^{3/2}}.
\]

This means that the space \( \dot{H}^{3/2} (\mathbb{R}) \) is a critical space for the study of the Cauchy problem.

**Exercise 2.2.4.** Check the previous claims.

**Theorem 2.2.5** (from [24]). \( i \) For any \( f_0 \in H^{3/2}(\mathbb{R}) \), there is \( T > 0 \) such that the Cauchy problem has a unique solution \( f \) in
\[
X^{3/2}(T) = \left\{ f \in C^0([0, T]; H^{3/2}(\mathbb{R})); \int_0^T \int_{\mathbb{R}} \frac{(\partial_{xx}f)^2}{1 + (\partial_x f)^2} \, dx \, dt < \infty \right\}.
\]

\( ii \) There exists \( \varepsilon_0 \) such that if \( \| f_0 \|_{\dot{H}^{3/2}} \leq \varepsilon_0 \) then the solution exists for all time.

The proof requires to

- uncover a certain null-type structure because of a degenerate parabolic behavior;
- estimate the solutions for a norm which depends on the initial data themselves.
2.2.5 Weighted fractional Laplacians

To prove Theorem 2.2.5, one well-known difficulty is that one cannot define a flow map such that the lifespan is bounded from below on bounded subsets of this critical Sobolev space. To overcome this, in [24] we estimate the solutions for a norm which depends on the initial data themselves, using the weighted fractional Laplacians introduced in our previous works [26, 25].

As the name indicates, a weighted fractional Laplacian attempts to be a slight modulation of the usual fractional Laplacian $|D|^s = (-\partial_{xx})^{s/2}$.

**Notation 2.2.6.** Consider $s \in [0, +\infty)$ and a function $\phi: [0, +\infty) \to [1, \infty)$. The weighted fractional Laplacian $|D|^{s,\phi}$ denotes the Fourier multiplier with symbol $|\xi|^s \phi(|\xi|)$, such that

$$\mathcal{F}(|D|^{s,\phi} f)(\xi) = |\xi|^s \phi(|\xi|) \mathcal{F}(f)(\xi).$$

Moreover, we define the space

$$(2.2.9) \quad \mathcal{H}^{s,\phi}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : |D|^{s,\phi} f \in L^2(\mathbb{R}) \}.$$

We will consider special weight functions $\phi$ depending on some extra functions $\kappa: [0, \infty) \to [1, \infty)$, of the form

$$(2.2.10) \quad \phi(\lambda) = \int_0^\infty \frac{1 - \cos(h)}{h^2} \kappa \left( \frac{\lambda}{h} \right) \, dh, \quad \text{for } \lambda \geq 0.$$

In addition we will always assume that $\kappa$ is an admissible weight, in the sense of the following definition.

**Definition 2.2.7.** An admissible weight is a function $\kappa: [0, \infty) \to [1, \infty)$ satisfying the following three conditions:

(H1) $\kappa$ is increasing;

(H2) there exists a positive constant $c_0$ such that $\kappa(2r) \leq c_0 \kappa(r)$ for any $r \geq 0$;

(H3) the function $r \mapsto \kappa(r) / \log(1+r)$ is decreasing on $[r_0, \infty)$, for some $r_0 > 0$ large enough.

The next results contain the main results about these operators.
Lemma 2.2.8. For all $\sigma > 0$, there exists $C_\sigma > 0$ such that, for all $0 < r \leq \mu$,

\begin{align}
(2.2.11) & \quad r^\sigma \kappa \left( \frac{1}{r} \right) \leq C_\sigma \mu^\sigma \kappa \left( \frac{1}{\mu} \right), \\
(2.2.12) & \quad r^\sigma \kappa^2 \left( \frac{1}{r} \right) \leq C_\sigma \mu^\sigma \kappa^2 \left( \frac{1}{\mu} \right).
\end{align}

Proof. i) Use the decomposition:

$$r^\sigma \kappa \left( \frac{1}{r} \right) = \frac{\kappa \left( \frac{1}{r} \right)}{\log \left( 4 + \frac{1}{r} \right)} \times \left[ r^\sigma \log \left( e^{\frac{1}{\sigma}} + \frac{1}{r} \right) \right] \times \frac{\log \left( 4 + \frac{1}{r} \right)}{\log \left( e^{\frac{1}{\sigma}} + \frac{1}{r} \right)}.$$  

By assumption, the first factor is an increasing function of $r$ for $r$ large enough. By computing the derivative, it is easily verified that the second factor is also an increasing function of $r$. Eventually, the third factor is a bounded function on $(0, +\infty)$.

ii) We proceed as above, using this time the decomposition

$$r^\sigma \kappa^2 \left( \frac{1}{r} \right) = \frac{\kappa^2 \left( \frac{1}{r} \right)}{\log^2 \left( 4 + \frac{1}{r} \right)} \times \left[ r^\sigma \log^2 \left( e^{\frac{1}{\sigma}} + \frac{1}{r} \right) \right] \times \frac{\log^2 \left( 4 + \frac{1}{r} \right)}{\log^2 \left( e^{\frac{1}{\sigma}} + \frac{1}{r} \right)}.$$  

This completes the proof. \qed

Remark 2.2.9. These inequalities have the following interpretation: even if the function $r \to \kappa(1/r)$ and $r \to \kappa^2(1/r)$ are decreasing, since the function $\kappa(r)/\log(2+r)$ is decreasing for $r$ large enough, one expects that $r^\sigma \kappa(1/r)$ and $r^\sigma \kappa^2(1/r)$ behave as increasing functions of $r$.

We will see that $\kappa$ and $\phi$ are equivalent. The reason for introducing two different functions to code a single operator is that we will use them for different purposes. We use $\phi$ when we prefer to work with the frequency variable, whereas we will use $\kappa$ when the physical variable is more practical. The next proposition will be used later to switch calculations between frequency and physical variables.

Proposition 2.2.10. Assume that $\phi$ is as defined in (2.2.10) for some function $\kappa$ satisfying Assumption 2.2.7. Then, for all $g \in \mathcal{S}(\mathbb{R})$, there holds

$$|D|^{1/\phi} g(x) = \frac{1}{4} \int_{\mathbb{R}} \frac{2g(x) - g(x+h) - g(x-h)}{h^2} \kappa \left( \frac{1}{|h|} \right) \, dh.$$  

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Proof. Notice that the Fourier transform of the function

$$
\int_{\mathbb{R}} \frac{2g(x) - g(x + h) - g(x - h)}{h^2} \kappa \left( \frac{1}{|h|} \right) \, dh,
$$

is given by

$$
\left( \int_{\mathbb{R}} \frac{2 - 2 \cos(h \xi)}{h^2} \kappa \left( \frac{1}{|h|} \right) \, dh \right) \hat{g}(\xi).
$$

Therefore

$$
\left( 4|\xi| \int_{0}^{\infty} \frac{1 - \cos(h)}{h^2} \kappa \left( \frac{|\xi|}{|h|} \right) \, dh \right) \hat{g}(\xi) = 4\phi(\xi) |\xi| \hat{g}(\xi) = 4|D|^{1,\phi} g(\xi),
$$

equivalent to the wanted result. □

Eventually, we will need the following link between $|D|^{s, \phi}$ and the function $\kappa$.

**Proposition 2.2.11.** i) There exist $c, C > 0$ such that, for all $\lambda \geq 0$,

\begin{equation}
(2.2.13) \quad c\kappa(\lambda) \leq \phi(\lambda) \leq C\kappa(\lambda).
\end{equation}

ii) Given $g \in \mathcal{S}(\mathbb{R})$, define the semi-norm

$$
\|g\|_{s, \kappa} = \left( \int_{\mathbb{R}^2} |2g(x) - g(x + h) - g(x - h)|^2 \kappa \left( \frac{1}{|h|} \right)^2 \frac{dx \, dh}{|h|} \right)^{\frac{1}{2}}.
$$

Then, for all $0 < s < 2$, there exist $c, C > 0$ such that, for all $g \in \mathcal{S}(\mathbb{R})$,

$$
c \int_{\mathbb{R}} |D|^{s, \phi} g(x) |^2 \, dx \leq \|g\|_{s, \kappa}^2 \leq C \int_{\mathbb{R}} |D|^{s, \phi} g(x) |^2 \, dx.
$$

Proof. We prove statement ii) only, the proof of statement i) is similar.

ii) For $h \in \mathbb{R}$, the Fourier transform of $x \mapsto 2g(x) - g(x + h) - g(x - h)$ is given by $(2 - 2 \cos(\xi h)) \hat{g}(\xi)$. So, Plancherel’s identity implies that

$$
\|g\|_{s, \kappa}^2 = \int_{\mathbb{R}^2} |2g(x) - g(x + h) - g(x - h)|^2 \kappa^2 \left( \frac{1}{|h|} \right) \frac{1}{|h|^{1+2s}} \, dx \, dh
$$

$$
= \int_{\mathbb{R}} |I(\xi)| \hat{g}(\xi) |^2 \, d\xi,
$$

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The proof of the lower bound is straightforward:

We claim that $I(\xi) \sim |\xi|^{2s} \phi(|\xi|)^2$.

Since $|1 - \cos(\theta)| \leq \min\{2, \theta^2\}$ for all $\theta \in \mathbb{R}$, we have

$$I(\xi) \leq \frac{4}{\pi^2} \int_{|h\xi| \geq 1} \kappa^2 \left( \frac{1}{|h|} \right) \frac{1}{|h|^{1+2s}} \, dh + \frac{1}{\pi^2} \int_{|h\xi| \leq 1} |\xi|^4 |h|^{4s} \kappa^2 \left( \frac{1}{|h|} \right) \frac{1}{|h|^{1+2s}} \, dh.$$ 

Since $\kappa$ is increasing (by assumption), the first integral is estimated by

$$\int_{|h\xi| \geq 1} \kappa^2 \left( \frac{1}{|h|} \right) \frac{1}{|h|^{1+2s}} \, dh \leq (\kappa(|\xi|))^2 \int_{|h\xi| \geq 1} \frac{dh}{|h|^{1+2s}} \lesssim \kappa^2(|\xi|) |\xi|^{2s}.$$ 

To estimate the second integral, we use Lemma 2.2.8 to infer

$$|h| \leq \frac{1}{|\xi|} \quad \Rightarrow \quad |h|^{2-2s} \kappa^2 \left( \frac{1}{|h|} \right) \lesssim \frac{1}{|\xi|^{2-2s} \kappa^2(|\xi|)}.$$

It follows that

$$\int_{|h\xi| \leq 1} |\xi|^4 |h|^{4s} \kappa^2 \left( \frac{1}{|h|} \right) \frac{1}{|h|^{1+2s}} \, dh = \int_{|h\xi| \leq 1} |\xi|^4 |h|^{4s} \kappa^2 \left( \frac{1}{|h|} \right) \frac{1}{|h|^{1+2s}} \, dh$$

$$= |\xi|^4 \int_{|h\xi| \leq 1} |h|^{2-2s} \kappa^2 \left( \frac{1}{|h|} \right) \frac{1}{|h|^{s-1}} \, dh$$

$$\lesssim \kappa(|\xi|)^2 |\xi|^{2s} \int_{|h| \leq 1/|\xi|} \frac{dh}{|h|^{s-1}} \lesssim \kappa(|\xi|)^2 |\xi|^{2s},$$

where we have used the assumption $s < 2$ to obtain the last inequality.

Now, by combining the previous estimates, we deduce that $I(\xi) \lesssim |\xi|^{2s} \kappa(|\xi|)^2$.

Since $\phi \sim \kappa$, this proves that $I(\xi) \lesssim |\xi|^{2s} \phi(|\xi|)^2$.

The proof of the lower bound is straightforward:

$$I(\xi) \gtrsim \int_{\frac{\pi}{2\pi} \leq \xi h \leq \frac{3\pi}{2\pi}} \kappa^2 \left( \frac{1}{|h|} \right) \frac{1}{|h|^{1+2s}} \, dh \gtrsim \left( \int_{\frac{2\pi}{2\pi} \leq \xi h \leq \frac{3\pi}{2\pi}} \frac{dh}{|h|^{1+2s}} \right) \kappa^2(|\xi|)|\xi|^{1+2s}$$

$$\gtrsim (\kappa(|\xi|))^2 |\xi|^{2s}.$$ 

Using again the equivalence $\phi \sim \kappa$, this concludes the proof of the claim $I(\xi) \sim |\xi|^{2s} \phi(|\xi|)^2$, which in turn concludes the proof of statement ii).
For large data, the time of existence must depend on the initial data themselves and not only on its critical Sobolev norm. To overcome this problem, our strategy is to estimate the solution for a stronger norm whose definition involves the initial data themselves. To do so, we use the following elementary result.

**Proposition 2.2.12.** For all \( f_0 \) in \( H^{3/2}(\mathbb{R}) \), there exists an admissible weight \( \kappa \) satisfying \( \lim_{r \to +\infty} \kappa(r) = \infty \) and such that \( f_0 \) belongs to \( \mathcal{H}^{-3/2,\phi}(\mathbb{R}) \) where \( \phi \) is given by (2.2.10).

**Lemma 2.2.13.** For any nonnegative integrable function \( \omega \in L^1(\mathbb{R}) \), there exists a function \( \eta: [0, \infty) \to [1, \infty) \) satisfying the following properties:

1. \( \eta \) is increasing and \( \lim_{r \to \infty} \eta(r) = \infty \),
2. \( \eta(2r) \leq 2\eta(r) \) for any \( r \geq 0 \),
3. \( \omega \) satisfies the enhanced integrability condition:

\[
(2.2.14) \quad \int_{\mathbb{R}} \eta(|r|)\omega(r) \, dr < \infty,
\]
4. moreover, the function \( r \mapsto \frac{\eta(r)}{\log(4 + r)} \) is decreasing on \([0, \infty)\).

**Proof.** Consider a sequence \((\alpha_k)_{k\geq 1}\) such that \( \alpha_1 \geq e^5 \) and \( \alpha_k \geq \alpha_{k-1}^{10} \) and in addition

\[
(2.2.15) \quad \forall k \geq 1, \quad \int_{|r| \geq \alpha_k} \omega(r) \, dr \leq 2^{-k}.
\]

We set

\[
(2.2.16) \quad \eta(r) = \begin{cases} 
2 & \text{if } 0 \leq r < \alpha_1, \\
2 + 1 + \frac{\log(\frac{4+r}{4+\alpha_k})}{\log(\frac{4+\alpha_{k+1}}{4+\alpha_k})} & \text{if } \alpha_k \leq r < \alpha_{k+1}.
\end{cases}
\]

It is easy to check that \( \eta: [0, \infty) \to [1, \infty) \) is an increasing function converging to \( +\infty \) when \( r \) goes to \( +\infty \). Moreover, \( \eta \) satisfies \( \eta(2r) \leq 2\eta(r) \) for any \( r \geq 0 \).
In addition,

\[
\int \eta(|r|) \omega(r) \, dr \leq \int_{|r| \leq \alpha_1} 2\omega(r) \, dr + \sum_{k=1}^{\infty} (k + 2) \int_{\alpha_k \leq |r| \leq \alpha_{k+1}} \omega(r) \, dr
\]

\[
\leq 2\|\omega\|_{L^1} + \sum_{k=1}^{\infty} (k + 2)2^{-k}
\]

\[
\leq 2\|\omega\|_{L^1} + C.
\]

It remains to prove that \( r \mapsto \eta(r)/\log(4 + r) \) is decreasing. To do so, write

\[
(2.2.17) \quad \frac{d}{dr} \left( \frac{\eta(r)}{\log(4 + r)} \right) = \frac{1}{\log(4 + r)} \left( \eta'(r) - \frac{1}{4 + r \log(4 + r)} \right).
\]

So, for \( 0 \leq r < \alpha_1 \),

\[
(2.2.18) \quad \frac{d}{dr} \left( \frac{\eta(r)}{\log(4 + r)} \right) < 0,
\]

while for \( \alpha_k \leq r < \alpha_{k+1} \) with \( k \geq 1 \), we have

\[
\frac{d}{dr} \left( \frac{\eta(r)}{\log(4 + r)} \right) \leq \frac{1}{(4 + r) \log(4 + r)^2} \left( \frac{\log(4 + r)}{\log(4 + \alpha_{k+1})} - k - 1 \right)
\]

\[
\leq \frac{1}{(4 + r) \log(4 + r)^2} \left( \frac{\log(4 + \alpha_{k+1})}{\log(4 + \alpha_{k+1})} - 2 \right) < 0,
\]

where we have used \( \alpha_{k+1} \geq e^{5 \times 10^k} \).

This proves that \( r \mapsto \eta(r)/\log(4 + r) \) is decreasing on \([0, \infty)\). The proof is complete. \( \Box \)

Then, we are in position to state an improvement of statement \( i) \) in Theorem 2.2.5 which asserts that, whenever one controls a bigger norm than the critical one, the time of existence depends only on the norm. Recall that \( \mathcal{H}^{3, \phi}(\mathbb{R}) \) is defined by (2.2.9).

**Theorem 2.2.14.** Consider a real number \( M_0 > 0 \) and a function \( \phi \) given by (2.2.10) for some admissible weight \( \kappa \) satisfying \( \lim_{r \to +\infty} \kappa(r) = \infty \). Then there exists a time \( T_0 > 0 \) depending on \( M_0, \kappa \) such that, for any initial data
In \( H^{\frac{3}{2},\phi}(\mathbb{R}) \) satisfying \( \|f_0\|_{H^{\frac{3}{2},\phi}} \leq M_0 \), the Cauchy problem for (2.2.1) has a unique solution in the space

\[
\mathcal{X}^{\frac{3}{2},\phi}(T_0) := \left\{ f \in C^0([0,T_0]; H^{\frac{3}{2},\phi}(\mathbb{R})) ; \int_0^{T_0} \int_{\mathbb{R}} \frac{|D^{2,\phi} f|^2}{1 + (\partial_x f)^2} \, dx \, dt < \infty \right\}.
\]

### 2.3 References (work in progress)

#### 2.3.1 References about the equations

The first rigorous derivation of Darcys law was done by Tartar [347] in the appendix of the Book [322]. Further extensions can be found in [31, 270, 290]. A review of the literature in terms of both mathematical results and physical motivation can be found in the book by Hornung [211]. The latter reference discuss many extensions of Darcy law, including Darcy law with memory, nonlinear Darcy law, or models of one-phase Newtonian flows as well as of non-Newtonian fluids, to name a few. The analysis of filtration in porous media plays a key role in many situations, see for instance [291, 364, 365] and the references there in.

The Hele-Shaw equation plays a key role in Mathematical Biology (see [281, 304, 306, 305, 389, 261]). Muskat introduced the equations that bear his name in [293], to model problems in petroleum engineering (see [294, 297] for many historical comments).

It is often convenient to observe that there is a gradient flow structure. The fact Mullins-Sekerka equation is a gradient flow for the area functional \( H^d(\Sigma) \) was first observed by Almgren [32] and Giacomelli and Otto [186].

The Boussinesq equation (2.1.8) was derived from the Hele-Shaw equation (2.1.6) by Boussinesq [79] to study groundwater infiltration; it also models the flow of gas in porous media (see the monograph of Vázquez [362]). The thin-film equation (2.1.9) was derived from the Mullins-Sekerka equation (2.1.7) by Constantin, Dupont, Goldstein, Kadanoff, Shelley and Zhou in [120] as a lubrication approximation model of the interface between two immiscible fluids in a Hele-Shaw cell.

It has long been understood that the Muskat problem can be reduced to a
parabolic evolution equation for the unknown function $f$ (see [89, 174, 309, 333]). The formulation (2.2.1) that we are using arises from the work by Córdoba and Gancedo [127] who have studied this problem using contour integrals, and obtained this beautiful formulation of the Muskat equation in terms of finite differences.

2.3.2 The Cauchy problem for the Hele-Shaw equation

The Cauchy problems for the Hele-Shaw and Mullins-Sekerka equations have been studied by different techniques, for weak solutions, viscosity solutions or also classical solutions (see [20, 23, 103, 108, 109, 125, 175, 180, 199, 203, 244, 250, 299, 310]). In this book, we will restrict ourselves to the study of classical solutions. There are many possible ways to study this problem: to mention a few approaches we quote various PDE methods based on $L^2$-energy estimates (see the works of Chen [108], Córdoba, Córdoba and Gancedo [125], Knüpfer and Masmoudi [250], Günther and Prokert [199], Cheng, Granero-Belinchón and Shkoller [109]), there are also methods based on functional analysis tools and maximal estimates (see Escher and Simonett [175], the results reviewed in the book by Prüss and Simonett [310]).

2.3.3 The Cauchy problem for the Muskat equation

The study of the Cauchy problem for the Muskat equation begun two decades ago. Inspired by the analysis of free boundary flows, several different approaches succeeded to establish local well-posedness results for smooth enough initial data starting with the works of Yi [381], Ambrose [35, 36], Caflisch, Howison and Siegel [333]. In the last several years, this problem was extensively studied. There are now many different proofs that the Cauchy problem is well-posed, locally in time. The well-posedness of the Cauchy problem was proved in [127] by Córdoba and Gancedo for initial data in $H^3(\mathbb{R})$ in the stable regime $\rho_2 > \rho_1$ (they also proved that the problem is ill-posed in Sobolev spaces when $\rho_2 < \rho_1$). In [109], Cheng, Granero-Belinchón, Shkoller proved the well-posedness of the Cauchy problem in $H^2(\mathbb{R})$ (introducing a Lagrangian point of view which can be used in a broad setting, see [189]). The Cauchy problem was then studied in various sub-critical spaces. Firstly, by Constantin, Gancedo, Shvydkoy and Vicol [121] in the Sobolev space $W^{2,p}(\mathbb{R})$ for some $p > 1$, by
Deng, Lei and Lin in Hölder spaces [160], and by Matioc [277, 278] for initial data in $H^s(\mathbb{R})$ with $s > 3/2$ (see also Alazard-Lazar [21] and Nguyen and Pausader [299]). Regularity criteria were obtained in [121, 187] in terms of a control of some critical quantities.

Since the Muskat equation is parabolic, the proof of the local well-posedness results also gives global well-posedness results under a smallness assumption, see Yi [381]. The first global well-posedness results under mild smallness assumptions, namely assuming that the Lipschitz semi-norm is smaller than 1, was obtained by Constantin, Córdoba, Gancedo, Rodríguez-Piazza and Strain [119] (see also [121, 303]).

On the other hand, there are blow-up results for some large enough data by Castro, Córdoba, Fefferman, Gancedo and López-Fernández ([96, 97, 99]). They prove the existence of solutions such that at time $t = 0$ the interface is a graph, at a later time $t_1 > 0$ the interface is no longer a graph and then at a subsequent time $t_2 > t_1$, the interface is $C^3$ but not $C^4$.

The previous discussion raises a question about the possible existence of a criteria on the slopes of the solutions which would force/prevent them to enter the unstable regime where the slope is infinite. Surprisingly, it was shown that it is possible to solve the Cauchy problem for initial data whose slope can be arbitrarily large. Deng, Lei and Lin in [160] obtained the first result in this direction, under the assumption that the initial data are monotone. Cameron [91] proved the existence of a modulus of continuity for the derivative, and hence a global existence result assuming only that the product of the maximal and minimal slopes is bounded by 1; thereby allowing arbitrarily large slopes too (recently, Abedin and Schwab also obtained the existence of a modulus of continuity in [1] via Krylov-Safonov estimates). Then, by using a new formulation of the Muskat equation involving oscillatory integrals, Córdoba and Lazar established in [128] that the Muskat equation is globally well-posed in time, assuming only that the initial data is sufficiently smooth and that the $\dot{H}^{3/2}(\mathbb{R})$-norm is small enough. This result was extended to the 3D case by Gancedo and Lazar [181]. Let us also quote papers by Vazquez [363], Granero-Belinchón and Scrobogna [188] for related global existence results for different equations.

Eventually, let us mention that many recent results focus on the existence and possible non-uniqueness of weak-solutions (we refer the reader to [80, 126, 343, 95, 178, 300]). These problems arise for instance in the unstable regime.
\( \rho_1 > \rho_2 \), to study the existence mixing zones or the dynamic between the two different regimes.

### 2.3.4 References about critical Cauchy problems

The study of the well-posedness of the Cauchy problem for various partial differential equations in critical spaces has attracted a lot of attention in the last decades.

The Muskat equation is parabolic, but it is interesting to also discuss other type of equations. We begin with the Schrödinger equation, which is the prototypical example of a semi-linear dispersive equation. For this equation, the study of Cauchy problem in the energy critical case goes back to the works of Cazenave and Weissler [101, 102] and culminates with the global existence results of Bourgain [78], Grillakis [191] and Colliander, Keel, Staffilani, Takaoka and Tao [116]. In sharp contrast with sub-critical problems, the time of existence given by the local theory in [101, 102] depends on the profile of the data and not only on its norm. As a result, the conservation of the energy is insufficient to obtain a global existence result. Detailed historical accounts of the subject can be found in the book by Tao [346]. We also refer to the recent paper by Merle, Raphaël, Rodnianski and Szeftel [282] which establishes an unexpected blow-up result for supercritical defocusing nonlinear Schrödinger equations. If instead of a semi-linear equation, one considers a quasi-linear problem, then the scaling is not necessarily the only relevant criteria. One key result in this direction is about a hyperbolic equation in general relativity, namely the resolution of the bounded \( L^2 \) curvature conjecture by Klainerman, Rodnianski and Szeftel [249].

Many papers have been devoted to the study of critical problems for parabolic equations. Consider for instance the equation

\[
\partial_t \theta + u \cdot \nabla \theta + (-\Delta)^{\frac{\alpha}{2}} \theta = 0 \quad \text{with} \quad u = \nabla^\perp (-\Delta)^{-\frac{\alpha}{2}} \theta.
\]

This equation arises as a dissipative version of the surface quasi-geostrophic equation introduced by Constantin-Majda-Tabak [122]. Here the critical case corresponds to \( \alpha = 1 \). In this case, the global in time well-posedness has been proved by Kiselev-Nazarov-Volberg [248], Caffarelli-Vasseur [88] and Constantin-Vicol [124] (see also [247, 334, 361]).
2.3.5 Entropies and maximum principles

In the literature, there are many definitions of entropies for various evolution equations. The common idea is that entropy dissipation methods allow to study the large time behavior or to prove functional inequalities (see [65, 94, 43, 176, 368, 71, 165, 70, 390, 236]). For a thorough discussion of the role of entropy methods in information theory, we refer to Villani’s lecture notes [367] and his book [368, Chapters 20, 21, 22].

The study of entropies plays a key role in the study of the thin-film equation (and its variant) since the works of Bernis and Friedman [62] and Bertozzi and Pugh [67]. The simplest observation is that, if \( h \) is a non-negative solution to \( \partial_t h + \partial_x (h \partial_x^3 h) = 0 \), then
\[
\frac{d}{dt} \int_T h^2 \, dx \leq 0, \quad \frac{d}{dt} \int_T (\partial_x h)^2 \, dx \leq 0.
\]
(This can be verified by elementary integrations by parts.) To give an example of hidden Lyapunov functionals, consider, for \( p \geq 0 \) and a function \( h > 0 \), the functionals
\[
H_p(h) = \int_T \frac{h^2}{h^p} \, dx.
\]
Laugesen discovered ([260]) that, for \( 0 \leq p \leq 1/2 \), \( H_p(h) \) is a Lyapunov functional. This result was complemented by Carlen and Ulusoy ([92]) who showed that \( H_p(f) \) is an entropy when \( 0 < p < (9 + 4\sqrt{15})/53 \). We also refer to [60, 149, 68, 237] for the study of entropies of the form \( \int h^p \, dx \) with \( 1/2 \leq p \leq 2 \).

Still for the thin-film equation, the study of the decay of Lebesgue norms was initiated by Bernis and Friedman [62] and continued by Beretta-Bertsch-Dal Passo [60], Dal Passo–Garcke–Grün [149] and more recently by Jüngel and Matthes [237], who performed a systematic study of entropies for the thin-film equation, by means of a computer assisted proof. The study of these decay estimates is related to the study of functional inequalities: we refer to Bernis [61], Dal Passo–Garcke–Grün [149] and [12].

In addition to Lyapunov functionals, maximum principles also play a key role in the study of these parabolic equations. One can think of the maximum principles for the mean-curvature equation obtained by Huisken [213] and Ecker and Huisken (see [172, 171]), used to obtain a very sharp global existence result
of smooth solutions. Many maximum principles exist also for the Hele-Shaw equations (see [244, 103]). In particular, we will use the maximum principle for space-time derivatives proved in [23]. For the thin-film equations of the form
\[ \partial_t h + \partial_x (f(h)\partial_x^3 h) = 0 \]
with \( f(h) = h^m \) with \( m \geq 3.5 \), in one space dimension, if the initial data \( h_0 \) is positive, then the solution \( h(x,t) \) is guaranteed to stay positive (see [62, 66] and [149, 68, 386, 82]).
Part II

Tools
Chapter 3

Introduction to paradifferential calculus

This chapter is a short self-contained introduction to the application of paradifferential calculus to the study of the Dirichlet-to-Neumann operator.

3.1 Introduction

3.1.1 Bony’s paradifferential operators

Bony’s theory of paradifferential operators allows to study the regularity of the solutions of nonlinear partial differential equations. This theory lies at the interface between harmonic analysis and microlocal analysis. It has a long history that owes a lot to Calderón and Zygmund, Coifman and Meyer, Kohn and Nirenberg, as well as Hörmander.

Since the work of Kohn–Nirenberg and Hörmander it is said that that $T$ is a pseudo-differential operator if we can define it from a function $a = a(x, \xi)$ by the relation

$$(3.1.1) \quad T(e^{ix\cdot\xi}) = a(x, \xi)e^{ix\cdot\xi}. $$

We then say that $a$ is the symbol for $T$ and we denote $T = \text{Op}(a)$. For instance, the operator associated with the symbol $a = \sum_{\alpha} a_{\alpha}(x)(i\xi)^{\alpha}$ is simply the differential operator $T = \sum_{\alpha} a_{\alpha}(x)\partial_{\xi}^{\alpha}$ (with classical notations). Another
fundamental example is the case of the operator $|D_x|$ that we have already defined by

$$|D_x| e^{ix \cdot \xi} = |\xi| e^{ix \cdot \xi},$$

so that $|D_x| = \text{Op}(|\xi|)$.

The pseudo-differential calculus is a process that associates to a symbol $a = a(x, \xi)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ an operator $\text{Op}(a)$ such that one can understand the properties of these operators (product, adjoint, boundedness on the usual spaces of functions...) simply by looking at the properties of the symbols.

The application that associates an operator $\text{Op}(a)$ to the symbol $a$ is called a quantification. There are very many quantizations that are known to be useful, which are variants of (3.1.1). Bony’s quantization is perfectly suited for non-linear problems. Its specificity is to quantize symbols that have a limited regularity in $x$. It will allow us to quantize, among others, the symbol

$$\sqrt{(1 + |\nabla \eta(x)|^2) |\xi|^2 - (\nabla \eta(x) \cdot \xi)^2},$$

in situations where $\eta$ is not very regular (for our subject matter, we are interested in the case where $\eta$ is lipschitzian, so that this symbol is barely bounded in $x$).

On the other hand, Calderón and his school favoured the point of view of the study of an operator by the study of its kernel. This distinction is also found in the study of free boundary problems. This is how the operator $|D_x|$ appears much more often written in the form $\partial_x \mathcal{H}$ where $\mathcal{H}$ is the Hilbert transform, which is defined on the functions of a single variable by

$$\mathcal{H} f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t| > \varepsilon} \frac{f(t)}{x-t} \, dt.$$

A central result in the theory of singular integrals concerns the commutator $A \circ |D_x| - |D_x| \circ A$ where $A$ is the operator of multiplication by one function $a(x)$. Calderón proved that this commutator is bounded to $L^2$ if and only if the function $a(x)$ is lipschitzian (see the introduction of Meyer’s book [288]). This estimate will play an important role in the problems we will consider.
3.1.2 The Dirichlet-to-Neumann operator

A notable part of the analysis of free boundary problems consists in studying the Dirichlet-to-Neumann operator. We present here some of the results on this topic which are proved later in Chapter 5.

Let \( \eta : \mathbb{R}^d \to \mathbb{R} \) be a smooth enough function and consider the open set

\[ \Omega := \{ (x, y) \in \mathbb{R}^d \times \mathbb{R} ; y < \eta(x) \} . \]

If \( \psi : \mathbb{R}^d \to \mathbb{R} \) is another function, and if we call \( \phi : \Omega \to \mathbb{R} \) the unique solution of \( \Delta_{x,y} \phi = 0 \) in \( \Omega \) satisfying \( \phi|_{y=\eta(x)} = \psi \) and a convenient vanishing condition at \( y \to -\infty \), one defines the Dirichlet-to-Neumann operator \( G(\eta) \) by

\[ G(\eta)\psi = \sqrt{1 + |\nabla \eta|^2} \partial_n \phi|_{y=\eta}, \]

where \( \partial_n \) is the outward normal derivative on \( \partial \Omega \). In Chapter 5 we make precise the above definition and study the action of \( G(\eta) \) on different spaces.

By standard variational arguments, one can prove that \( G(\eta)\psi \) is well-defined for any function \( \psi \) that belongs to the Sobolev space \( H^{1/2}(\mathbb{R}^d) \).

**Proposition 3.1.1.** Let \( \eta \) be a Lipschitz function in \( W^{1,\infty}(\mathbb{R}^d) \). Then \( G(\eta) \) is well-defined and bounded from \( H^{1/2}(\mathbb{R}^d) \) to \( H^{-1/2}(\mathbb{R}^d) \) and satisfies an estimate

\[ \| G(\eta)\psi \|_{H^{-1/2}} \leq C(\| \nabla \eta \|_{L^{\infty}}) \| |D|^{1/2} \psi \|_{L^2}, \]

where \( C(\cdot) \) is a non decreasing continuous function of its argument.

The previous proposition will be proven using classical variational arguments. On the other hand, when the domain is smoother, this problem can be studied by means of microlocal analysis. In particular, if \( \eta \in C_0^\infty(\mathbb{R}^d) \), it is known since Calderón that \( G(\eta) \) is a pseudo-differential operator of order 1. This is true in any dimension \( d \geq 1 \). If \( d = 1 \), one can give a very simple rigorous meaning to the previous statement. Indeed, the latter simplifies to the following

**Proposition 3.1.2.** Assume that \( d = 1 \) and \( \eta \in C_0^\infty(\mathbb{R}) \). Then \( G(\eta) \) can be written under the form

\[ G(\eta)\psi = |D_x| \psi + R(\eta)\psi, \]

(3.1.2)
where \( R(\eta)_f \) is a smoothing operator, bounded from \( H^\mu(\mathbb{R}) \) to \( H^{\mu+m}(\mathbb{R}) \) for any integer \( m \). Namely, for any \( m \in \mathbb{N} \), there exists a constant \( K \geq 1 \) such that

\[
\forall \mu \geq \frac{1}{2}, \quad \|R_0(\eta)_\psi\|_{H^{\mu+m}} \leq C(\|\eta\|_{H^{\mu+K}}) \|\psi\|_{H^\mu},
\]

where \( C(\cdot) \) is a non decreasing continuous function of its argument.

This result is not satisfactory for the analysis of the free boundary problems we want to study. For our subject matters, \( \eta \) and \( \psi \) are expected to have essentially the same regularity so that the constant \( K \) that appears in (3.1.3) corresponds to a loss of derivatives. We need estimates without loss of derivatives. In this direction, let us state the following result about the boundedness of \( G(\eta)_f \) on Sobolev spaces.

**Proposition 3.1.3.** Let \( d \geq 1 \), \( s > 1 + \frac{d}{2} \) and \( \frac{1}{2} \leq \sigma \leq s \). Then, for all \( \eta \in H^s(\mathbb{R}^d) \) and all \( f \in H^\sigma(\mathbb{R}^d) \), \( G(\eta)_f \) belongs to \( H^{\sigma-1}(\mathbb{R}^d) \), together with the estimate

\[
\|G(\eta)_f\|_{H^{\sigma-1}} \leq C(\|\eta\|_{H^s}) \|f\|_{H^\sigma},
\]

where \( C(\cdot) \) is a non decreasing continuous function of its argument.

Ideally, we would like a result which includes the two previous propositions. To do so, we need to pause to introduce paraproducts.

### 3.1.3 Paraproducts

To make this introductory chapter self-contained, we recall here the definition of a paraproduct, which is the simplest example of a paradifferal operator.

One can define a paraproduct very simply, using the Fourier inversion formula:

\[
a(x)b(x) = \frac{1}{(2\pi)^{2d}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{ix \cdot (\xi_1 + \xi_2)} \hat{a}(\xi_1) \hat{b}(\xi_2) \, d\xi_1 \, d\xi_2.
\]

Let us decompose the integral in three terms

\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} = \iint_{|\xi_1 + \xi_2| < |\xi_2|} + \iint_{|\xi_1 + \xi_2| < |\xi_1|} + \iint_{|\xi_1| < |\xi_2|}
\]

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to obtain the following decomposition of the product

\[ ab = T_a b + T_b a + R(a, b). \]

To define these operators more precisely, let us consider a cut-off function \( \theta \) in \( C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) such that

\[ \theta(\xi_1, \xi_2) = 1 \quad \text{if} \quad |\xi_1| \leq \varepsilon_1 |\xi_2|, \quad \theta(\xi_1, \xi_2) = 0 \quad \text{if} \quad |\xi_1| \geq \varepsilon_2 |\xi_2|, \]

with \( 0 < \varepsilon_1 < \varepsilon_2 < 1 \). Given two functions \( a = a(x) \) and \( b = b(x) \) one defines

\[
T_a b = \frac{1}{(2\pi)^{2d}} \int e^{ix \cdot (\xi_1 + \xi_2)} \theta(\xi_1, \xi_2) \hat{a}(\xi_1) \hat{b}(\xi_2) \, d\xi_1 \, d\xi_2,
\]

\[
T_b a = \frac{1}{(2\pi)^{2d}} \int e^{ix \cdot (\xi_1 + \xi_2)} \theta(\xi_2, \xi_1) \hat{a}(\xi_1) \hat{b}(\xi_2) \, d\xi_1 \, d\xi_2,
\]

\[
R_B(a, b) = \frac{1}{(2\pi)^{2d}} \int e^{ix \cdot (\xi_1 + \xi_2)} (1 - \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1)) \hat{a}(\xi_1) \hat{b}(\xi_2) \, d\xi_1 \, d\xi_2.
\]

This is Bony’s decomposition of the product of two functions. One says that \( T_a b \) and \( T_b a \) are paraproducts, while \( R_B(a, b) \) is a remainder. The key property is that a paraproduct by an \( L^\infty \) function acts on any Sobolev spaces \( H^s \) with \( s \) in \( \mathbb{R} \). The remainder term \( R_B(a, b) \) is smoother than the paraproducts \( T_a b \) and \( T_b a \). We have the following results:

\[
\forall \sigma \in \mathbb{R}, \quad a \in L^\infty(\mathbb{R}^d), \quad b \in H^\sigma(\mathbb{R}^d) \quad \Rightarrow \quad T_a b \in H^\sigma(\mathbb{R}^d),
\]

\[
\forall \sigma \in (0, +\infty), \quad a \in H^\sigma(\mathbb{R}^d), \quad b \in H^\sigma(\mathbb{R}^d) \quad \Rightarrow \quad R_B(a, b) \in H^{2\sigma-d/2}(\mathbb{R}^d).
\]

### 3.1.4 Paralinearization of the Dirichlet–Neumann operator

Once paraproducts have been defined, we are in position to introduce a paralinearization formula for the Dirichlet-to-Neumann operator. The next result, proved in collaboration with Guy M´etivier ([22]) allows to study the microlocal properties of the Dirichlet-to-Neumann operator \( G(\eta) \psi \) in the case where \( \eta \) and \( \psi \) have the same regularity. Again, for the sake of simplicity, we state the result in the particular case where \( d = 1 \) and refer to Chapter 5 for the general case.
Proposition 3.1.4 (from [22]). There exists $K > 0$ such that the following property holds: For all functions $\eta$ and $\psi$ belonging to the Sobolev space $H^s(\mathbb{R})$, with $s \geq K$, we have

$$G(\eta)\psi = |D_x| (\psi - T\partial_y\phi|_{y=\eta}\eta) - \partial_x (T\partial_x\phi|_{y=\eta}\eta) + F(\eta, \psi),$$

where $\phi$ denotes the harmonic extension of $\psi$,

$$\Delta_{x,y}\phi = 0 \quad \text{in} \quad \Omega = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < \eta(x)\}, \quad \phi|_{y=\eta} = \psi,$$

and where

$$\|F(\eta, \psi)\|_{H^{2s-K}} \leq C\left(\|\eta\|_{H^s}\right) \|\psi\|_{H^s},$$

where $C$ is a non-decreasing function of its argument.

For $\eta \in C^\infty$, this gives the Calderón result alluded to earlier. This result makes it easy to prove estimates that would otherwise only be obtained using (difficult) estimates of solutions to elliptic problems. For instance, the commutator properties between the Dirichlet-to-Neumann operator and a Fourier multiplier can be deduced directly from the properties of symbolic calculus.

The novelty is that we understand perfectly how the operator $G(\eta)$ depends on $\eta$, except for a regular remainder. The fact that the remainder $R$ is twice as regular as $\eta$ and $\psi$ plays an important role for applications to the study of small divisors problems (see [22, 10]) or to the study of exact controllability (see [11, 387]) in the study of 3D travelling waves.

On the other hand, in the study of Cauchy’s problem with irregular data, it is enough to have a remainder which has the same regularity as $\eta$. The difficulty is then to know how to deal with indices $s$ which are small. In this direction, we have proven with Nicolas Burq and Claude Zuily various results. For instance, in the article [13] we have shown that the previous theorem is true with $s > 2 + d/2$ and $K = (3 + d)/2$. In addition we gave an estimate of the differential of $R(\eta, \psi)$ with respect to $\eta$. Another useful result is the following

Proposition 3.1.5 (from [15]). Consider real numbers $s, \sigma, \varepsilon$ such that

$$s > 1 + \frac{1}{2}, \quad \frac{1}{2} \leq \sigma \leq s - \frac{1}{2}, \quad 0 < \varepsilon \leq \frac{1}{2}, \quad \varepsilon < s - 1 - \frac{1}{2}.$$

Then there exists a non-decreasing function $C: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|G(\eta)f - |D_x| f\|_{H^{s-1+\varepsilon}(\mathbb{R})} \leq C\left(\|\eta\|_{H^s(\mathbb{R})}\right) \|f\|_{H^s(\mathbb{R})}.$$
3.2 The Calderón-Vaillancourt theorem

Given a function \( a = a(x, \xi) \) and a function \( u = u(x) \), we want to study operators \( A \) defined by expression of the form

\[
Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) \, d\xi.
\]

Notice that, if \( a = 1 \), then the Fourier inversion theorem implies that \( A = \text{Id} \).

More generally, if \( a(x, \xi) = \sum a_\alpha(x) (i\xi)^\alpha \), then \( A \) is the differential operator \( \sum a_\alpha \partial_\alpha x \).

A pseudo-differential operator is an operator of the previous form, but where the function \( a(x, \xi) \) is not necessarily a polynomial function. In this chapter, we propose to study two elementary results about these operators. We will assume that \( a \) is a smooth bounded function and prove that \( A \) is bounded on \( L^2(\mathbb{R}^d) \).

3.2.1 Continuity on the Schwartz class

Consider a function \( a \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d) \). By definition, this means that, for all multi-indices \( \alpha \) and \( \beta \) in \( \mathbb{N}^d \), we have

\[
\sup_{(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d} \left| \partial_\alpha x \partial_\beta \xi a(x, \xi) \right| < +\infty.
\]

Given any function \( u \in \mathcal{S}(\mathbb{R}^d) \) in the Schwartz class and a fixed \( x \in \mathbb{R}^d \), the function \( \xi \mapsto a(x, \xi)\hat{u}(\xi) \) belongs to \( \mathcal{S}(\mathbb{R}^d) \). In particular, it is integrable and we may define the function \( \text{Op}(a)u \) by

\[
\text{Op}(a)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) \, d\xi.
\]

We say that \( \text{Op}(a) \) is a pseudo-differential operator and we call \( a \) its symbol.

**Proposition 3.2.1.** For any \( a \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) and any \( u \in \mathcal{S}(\mathbb{R}^d) \), the function \( \text{Op}(a)u \) is well-defined and belongs to \( \mathcal{S}(\mathbb{R}^d) \). Moreover \( \text{Op}(a) \) is continuous from \( \mathcal{S}(\mathbb{R}^d) \) into \( \mathcal{S}(\mathbb{R}^d) \).

**Proof.** Since \( \hat{u} \in \mathcal{S}(\mathbb{R}^d) \), we can apply Lebesgue’s differentiation theorem to check easily that \( \text{Op}(a)u \in C^\infty(\mathbb{R}^d) \). So it will suffice to prove estimates.
Using $\|a\|_{L^\infty} < +\infty$ and $\|\langle \xi \rangle^{2d} \hat{u}\|_{L^\infty} < +\infty$, we get the inequality

$$|\operatorname{Op}(a)u(x)| \leq (2\pi)^{-n} \int \|a\|_{L^\infty} \|\langle \xi \rangle^{2d} \hat{u}\|_{L^\infty} \langle \xi \rangle^{-2d} \, d\xi,$$

which implies that $\operatorname{Op}(a)u$ is bounded together with the estimate

$$\|\operatorname{Op}(a)u\|_{L^\infty} \leq CN_{2d}(\hat{u})$$

where we used the notation $N_p(\varphi) = \sum_{|\alpha| \leq p, |\beta| \leq p} \|x^\alpha \partial^\beta_x \varphi\|_{L^\infty}$ to denote the canonical semi-norms on the Schwartz space; let us recall that the Fourier transform is continuous from $S(\mathbb{R}^d)$ into $S(\mathbb{R}^d)$ and that

$$N_{2d}(\hat{u}) \leq C_{2d}N_{3d+1}(u).$$

To estimate the other semi-norms in $S(\mathbb{R}^d)$ of $\operatorname{Op}(a)u$, we use the following formulas (to be checked as an exercise)

$$\partial_{x_j} \operatorname{Op}(a)u = \operatorname{Op}(a)(\partial_{x_j} u) + \operatorname{Op}(\partial_{x_j} a)u,$$

$$x_j \operatorname{Op}(a)u = \operatorname{Op}(a)(x_j u) + i \operatorname{Op}(\partial_{\xi_j} a)u.$$

Thus, $x^\alpha \partial^\beta_x \operatorname{Op}(a)u$ can written as a linear combination of terms of the form

$$\operatorname{Op}(\partial^\alpha_x \partial^\beta_\xi a)(x^{\alpha-\delta} \partial^{\beta-\gamma}_x u).$$

Since $\partial^\alpha_x \partial^\beta_\xi a$ belongs to $C^\infty_b(\mathbb{R}^d \times \mathbb{R}^d)$ and since $x^{\alpha-\delta} \partial^{\beta-\gamma}_x u$ belongs to $S(\mathbb{R}^d)$, we are back to the previous case. This shows that $\operatorname{Op}(a)u$ belongs to $S(\mathbb{R}^d)$ and that we have estimates for the semi-norms of $\operatorname{Op}(a)u$ in terms of the semi-norms of $u$, which in turn proves that $\operatorname{Op}(a)$ is continuous from $S(\mathbb{R}^d)$ to itself. \qed

### 3.2.2 Boundedness on $L^2$

We can now state the main result, which asserts that one can extend $\operatorname{Op}(a)$ as a bounded operator from $L^2(\mathbb{R}^d)$ into itself.

**Theorem 3.2.2.** If $a \in C^\infty_b(\mathbb{R}^{2d})$, the operator $\operatorname{Op}(a)$ can be uniquely extended as a bounded linear operator in $\mathcal{L}(L^2(\mathbb{R}^d))$.

We will demonstrate this result by assuming, to simplify the notations, that the space dimension $d$ is less than or equal to 3 (otherwise just replace the polynomial $P(\zeta)$ below by $(1 + |\zeta|^2)^k$ where $k$ is an integer such that $4k > d$).
Let us introduce the polynomial

\[ P(\zeta) = 1 + |\zeta|^2 \quad (\zeta \in \mathbb{R}^d, \ d = 1, 2, 3). \]

**Lemma 3.2.3.** Given a function \( u \in \mathcal{S}(\mathbb{R}^d) \), we introduce the function

\[
Wu(x, \xi) = \int_{\mathbb{R}^d} e^{-iy \cdot \xi} P(x - y)^{-1} u(y) \, dy \quad ((x, \xi) \in \mathbb{R}^{2d}).
\]

i) Then \( Wu \) is a function \( C^\infty_b(\mathbb{R}^{2d}) \) and moreover for any multi-indices \( \alpha, \beta, \gamma \),

\[
\sup_{\mathbb{R}^{2d}} |\xi|^{\gamma} \left| (\partial_\alpha x \partial_\beta \xi Wu)(x, \xi) \right| < +\infty.
\]

ii) There is a constant \( A \) such that

\[
(3.2.1) \quad \|Wu\|_{L^2(\mathbb{R}^{2d})} = A \|u\|_{L^2(\mathbb{R}^d)}
\]

for any \( u \) in \( \mathcal{S}(\mathbb{R}^d) \).

iii) For any \( \gamma \in \mathbb{N}^d \), there are \( A_\gamma \) such as

\[
\|\partial_\gamma^2 Wu\|_{L^2(\mathbb{R}^{2d})} \leq A_\gamma \|u\|_{L^2(\mathbb{R}^d)}.
\]

**Proof.** i) We verify that

\[
\xi^\gamma (\partial_\alpha^a \partial_\xi^b Wu)(x, \xi) = \int e^{-iy \cdot \xi} (\partial_\gamma^a (\partial_\gamma^b (P(x - y)^{-1} u(y)) dy,
\]

so, by integrating by parts

\[
\xi^\gamma (\partial_\alpha^a \partial_\xi^b Wu)(x, \xi)
\]

\[
= \sum_{\gamma' + \gamma'' = \gamma} \frac{\gamma!}{\gamma'! \gamma''!} \int (-i)^{\gamma'} \partial_\gamma^a (u(y) (-iy)^\beta \partial_\gamma^b (P(x - y)^{-1} u(y) dy.
\]

We next use the elementary estimates

\[
|\partial_\gamma^a (\zeta)^{-2} | \leq C_\alpha \langle \zeta \rangle^{-2-|\alpha|} \leq C_\alpha \langle \zeta \rangle^{-2}
\]

to deduce that

\[
|\partial^a (1/P)(x - y)| \leq C_\alpha (1 + |x - y|^2)^{-1} \leq 2C_\alpha (1 + |x|^2)^{-1} (1 + |y|^2),
\]
where the last inequality comes from the fact that
\[ 1 + |x|^2 = 1 + |x - y + y|^2 \leq 1 + 2|x - y|^2 + 2|y|^2 \leq 2(1 + |x - y|^2)(1 + |y|^2). \]

ii) For any \( x \in \mathbb{R}^d \), \( Wu(x, \cdot) \) is the Fourier transform of \( y \mapsto u(y)P(x - y)^{-1} \). So
\[
\int |W(x, \xi)|^2 \, d\xi = (2\pi)^d \int |u(y)P(x - y)^{-1}|^2 \, dy
\]
according to Plancherel’s formula. So
\[
\iint |W(x, \xi)|^2 \, d\xi \, dx = (2\pi)^d \iint |u(y)P(x - y)^{-1}|^2 \, dy \, dx = A^2 \|u\|_{L^2(\mathbb{R}^d)}^2.
\]

iii) By combining the above observations. \( \square \)

**Lemma 3.2.4.** We have
\[
\hat{u}(\xi) = e^{-ix\cdot\xi}(I - \Delta_\xi)(e^{ix\cdot\xi}Wu(x, \xi))
\]
and
\[
\overline{v}(x) = \frac{1}{(2\pi)^d} e^{-ix\cdot\xi}(I - \Delta_x)(e^{ix\cdot\xi}\overline{W\hat{v}}(\xi, x)).
\]

**Proof.** Like \((I - \Delta_\xi)e^{iX\cdot\xi} = P(X)\) we have
\[
e^{ix\cdot\xi}\hat{u}(\xi) = \int e^{i(x-y)\cdot\xi}u(y) \, dy = (I - \Delta_\xi) \int e^{i(x-y)\cdot\xi}P(x - y)^{-1}u(y) \, dy.
\]
In a dual way, using the inverse Fourier transform, we have
\[
e^{ix\cdot\xi}\overline{v}(x) = \frac{1}{(2\pi)^d} \int e^{i(\xi - \eta)\cdot x}\overline{v}(\eta) \, d\eta
\]
\[
= \frac{1}{(2\pi)^d} (I - \Delta_x) \int e^{i(\xi - \eta)\cdot x}P(\xi - \eta)^{-1}\overline{v}(\eta) \, d\eta,
\]
which implies the second identity. \( \square \)

**Proof of Theorem 3.2.2.** Given the density of \( \mathcal{S}(\mathbb{R}^d) \) in \( L^2(\mathbb{R}^d) \), it is enough to demonstrate the inequality
\[
\|\text{Op}(a)u\|_{L^2} \leq C \|u\|_{L^2}
\]
for any $u$ in $\mathcal{S}(\mathbb{R}^d)$. Let us consider two functions $u, v$ in $\mathcal{S}(\mathbb{R}^d)$ and let us set
\[
I := \int \int e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) \overline{v}(x) \, d\xi \, dx.
\]
We want to show that $|I| \leq C \|u\|_{L^2} \|v\|_{L^2}$. For this we will rewrite $I$ as a scalar product in $L^2(\mathbb{R}^{2d})$ of functions involving $W u$ and $\hat{W} v$.

Let us start by writing $I$ in the form
\[
I = \int \int a(x, \xi) \left( (I - \Delta_\xi) \left( e^{ix \cdot \xi} W u(x, \xi) \right) \overline{v}(x) \right) \, d\xi \, dx.
\]
Since $(I - \Delta_\xi) \left( e^{ix \cdot \xi} W u(x, \xi) \overline{v}(x) \right)$ belongs to $\mathcal{S}(\mathbb{R}^{2d})$, we can integrate by parts in $\xi$ and deduce that
\[
I = \int \int \left( (I - \Delta_\xi) a(x, \xi) \right) W u(x, \xi) e^{ix \cdot \xi} \overline{v}(x) \, d\xi \, dx.
\]
Using the identity for $v$ it comes
\[
I = \int \int \left( (I - \Delta_\xi) a(x, \xi) \right) W u(x, \xi) (I - \Delta_x) \left( e^{ix \cdot \xi} \hat{W} v(\xi, x) \right) \, dx
\]
and integrating by parts in $x$,
\[
I = \int \int (I - \Delta_x) \left( ((I - \Delta_\xi) a(x, \xi)) W u(x, \xi) \right) e^{ix \cdot \xi} \hat{W} v(\xi, x) \, d\xi \, dx
\]
so
\[
I = \sum_{|\beta| \leq 2, |\alpha| + |\gamma| \leq 2} C_{\alpha \beta \gamma} \int \int (\partial^\alpha_x \partial^{\beta_\xi} a(x, \xi)) \partial^\gamma_x W u(x, \xi) \hat{W} v(\xi, x) e^{ix \cdot \xi} \, dx \, d\xi.
\]
We conclude the proof with the Cauchy-Schwarz inequality and the previous results:
\[
\|\partial^\gamma_x W u\|_{L^2(\mathbb{R}^{2d})} \leq A_\gamma \|u\|_{L^2(\mathbb{R}^d)},
\]
\[
\|\hat{W} v(\xi, x)\|_{L^2(\mathbb{R}^{2d})} = A \|\overline{v}\|_{L^2} = A(2\pi)^{\frac{d}{2}} \|v\|_{L^2(\mathbb{R}^d)},
\]
where the Plancherel formula was used in the last inequality. □
3.3 Littlewood-Paley Decomposition

We begin by introducing a dyadic decomposition of the unity. This decomposition allows to introduce a parameter (large or small) in a problem which does not have any. It is a simple and extremely fruitful idea.

Lemma 3.3.1. Let $d \geq 1$. There exist $\psi \in C^\infty_0(\mathbb{R}^d)$ and $\varphi \in C^\infty_0(\mathbb{R}^d)$ such that the following properties hold:

(i) (Support conditions) We have $0 \leq \psi \leq 1$, $0 \leq \varphi \leq 1$ and
\[
\text{supp } \psi \subset \{|\xi| \leq 1\}, \quad \text{supp } \varphi \subset \left\{\frac{3}{4} \leq |\xi| \leq 2\right\}.
\]

(ii) (Decomposition of the unity) For any $\xi \in \mathbb{R}^d$,
\[
1 = \psi(\xi) + \sum_{p=0}^{\infty} \varphi(2^{-p}\xi).
\]

(iii) (Almost orthogonality) For any $\xi \in \mathbb{R}^d$,
\[
\frac{1}{3} \leq \psi^2(\xi) + \sum_{p=0}^{+\infty} \varphi^2(2^{-p}\xi) \leq 1.
\]

Proof. Let $\psi \in C^\infty_0(\mathbb{R}^d, \mathbb{R})$ be a radial function verifying $\psi(\xi) = 1$ for $|\xi| \leq 3/4$, and $\psi(\xi) = 0$ for $|\xi| \geq 1$, and decreasing (if $|\xi| \geq |\eta|$ then $\psi(\eta) \leq \psi(\eta)$). Then, we set $\varphi(\xi) = \psi(\xi/2) - \psi(\xi)$ and notice that $\varphi$ is supported in the annulus $\{3/4 \leq |\xi| \leq 2\}$. For any integer $N$ and any $\xi \in \mathbb{R}^d$, we have
\[
\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi) = \psi(2^{-N-1}\xi),
\]
which immediately implies (3.4.2) by letting $N$ goes to $+\infty$.

It remains to prove (3.3.2). For any integer $N$ we have
\[
\psi^2(\xi) + \sum_{p=0}^{N} \varphi^2(2^{-p}\xi) \leq \left(\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi)\right)^2.
\]
On the other hand, notice that, for all \( \xi \in \mathbb{R}^d \), there are never more than three non-zero terms in the set \( \{ \psi(\xi), \varphi(\xi), \ldots, \varphi(2^{-p}\xi), \ldots \} \). Consequently, using the elementary inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\), we easily get

\[
(\psi(\xi) + \sum_{p=0}^{N} \varphi(2^{-p}\xi))^2 \leq \sum_{p=0}^{N} \varphi^2(2^{-p}\xi))^2.
\]

Then we obtain (3.3.2) by letting \( N \) goes to \( +\infty \) in the previous inequalities.

Let us define, for \( p \geq -1 \), the Fourier multipliers \( \Delta_p \) as follows:

\[
\Delta_{-1} := \psi(D_x) \quad \text{and} \quad \Delta_p := \varphi(2^{-p}D_x) \quad (p \geq 0).
\]

Let us also introduce, for \( p \geq 0 \), the Fourier multipliers \( S_p \):

\[
S_p := \psi(2^{-p}D_x) = \sum_{k=1}^{p-1} \Delta_k.
\]

The partition of the unit also implies a partition of the identity.

**Proposition 3.3.2.** We have

\[
I = \sum_{p \geq -1} \Delta_p,
\]

in the sense of distributions: For any \( u \in S'(\mathbb{R}^d) \), the series \( \sum u_p \) converges to \( u \) in \( S'(\mathbb{R}^d) \), which means that \( \sum_p (\Delta_p u, \varphi)_{S' \times S} \) converges to \( \langle u, \varphi \rangle_{S' \times S} \) for any \( \varphi \in S(\mathbb{R}^n) \).

**Proof.** Let \( u \in S'(\mathbb{R}^d) \) and \( \theta \in S(\mathbb{R}^d) \). The partial sums \( S_Nu = \sum_{p \geq 0} \Delta_p u \) are well defined and

\[
\langle \mathcal{F}(S_Nu), \theta \rangle = \langle \psi(2^{-N}\xi) \mathcal{F}(u), \theta \rangle = \langle \mathcal{F}(u), \psi(2^{-N}\xi) \theta \rangle.
\]

Now \( \lim_{N \to +\infty} \psi(2^{-N}\xi) \theta = \theta \) in \( S(\mathbb{R}^d) \), so

\[
\mathcal{F}(S_Nu) \xrightarrow[p \to +\infty]{}\mathcal{F}(u) \quad \text{in} \quad S'(\mathbb{R}^d).
\]

By continuity of \( \mathcal{F}^{-1} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) we have \( u = \sum_{p \geq -1} \Delta_p u \). \( \square \)
Proposition 3.3.3. (i) For all $u \in L^2(\mathbb{R}^d)$,
\begin{equation}
\sum_{p \geq -1} \|\Delta_p u\|_{L^2}^2 \leq \|u\|_{L^2}^2 \leq 3 \sum_{p \geq -1} \|\Delta_p u\|_{L^2}^2 .
\end{equation}

(ii) Consider $s \in \mathbb{R}$. A temperate distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the Sobolev space $H^s(\mathbb{R}^d)$ if and only if
(a) $\Delta_{-1} u \in L^2(\mathbb{R}^d)$ and for all $p \geq 0$, $\Delta_p u \in L^2(\mathbb{R}^d)$;
(b) the sequence $\delta_p = 2^{ps} \|\Delta_p u\|_{L^2} \in \ell^2(\mathbb{N} \cup \{-1\})$.

Moreover, there exists a constant $C$ such that
\begin{equation}
\frac{1}{C} \|u\|_{H^s} \leq \left(\sum_{p=-1}^{+\infty} \delta_p^2\right)^{\frac{1}{2}} \leq C \|u\|_{H^s} .
\end{equation}

Proof. The first point follows immediately from (3.3.2) and Plancherel’s identity.

Since $\|u\|_{H^s} = \|\langle D_x \rangle^s u\|_{L^2}$, by applying (3.3.3) with $u$ replaced by $\langle D_x \rangle^s u$, we obtain
\begin{equation}
\sum_{p \geq -1} \|\Delta_p \langle D_x \rangle^s u\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq 3 \sum_{p \geq -1} \|\Delta_p \langle D_x \rangle^s u\|_{L^2}^2 .
\end{equation}

Consider $p \geq 0$ and write that
\[ \|\Delta_p \langle D_x \rangle^s u\|_{L^2}^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \varphi(2^{-p}\xi) |\hat{u}(\xi)|^2 \, d\xi. \]

Since $(1 + |\xi|^2)^s \varphi(2^{-p}\xi) \sim 2^{2ps}$ on the support of $\varphi(2^{-p}\xi)$, we see that
\begin{equation}
\frac{1}{C} 2^{2ps} \|\Delta_p u\|_{L^2}^2 \leq \|\Delta_p \langle D_x \rangle^s u\|_{L^2}^2 \leq C 2^{2ps} \|\Delta_p u\|_{L^2}^2 ,
\end{equation}

for some constant $C$ depending only on $s$. We have a similar estimate for $\Delta_{-1} u$ and the wanted result easily follows. \qed

Proposition 3.3.4. i) Consider $s \in \mathbb{R}$ and $R \geq 1$. Assume that $(u_j)_{j \geq -1}$ is a sequence of functions in $L^2(\mathbb{R}^d)$ such that
\[ \text{supp } \hat{u}_{-1} \subset \{|\xi| \leq R\}, \quad \text{supp } \hat{u}_j \subset \left\{ \frac{1}{R} 2^j \leq |\xi| \leq R 2^j \right\}, \]

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and, in addition,

\[(3.3.6) \quad \sum_{j \geq -1} 2^{2js} \|u_j\|_{L^2}^2 < +\infty.\]

Then the series \(\sum u_j\) converges to a function \(u \in H^s(\mathbb{R}^d)\) and moreover,

\[\|u\|_{H^s}^2 \leq C \sum_{j \geq -1} 2^{2js} \|u_j\|_{L^2}^2,
\]

for some constant \(C\) depending only on \(s\) and \(R\).

\(\text{ii)}\) If \(s > 0\), then the previous result holds under the weaker assumption that \(\text{supp} \hat{u}_j\) is included in the ball \(B(0, R2^j)\).

**Proof.** \(\text{i)}\) We begin by proving that the series \(\sum u_j\) is normally convergent in \(H^r(\mathbb{R}^d)\) for any \(0 < r < s\). Assuming that \(\text{supp} \hat{u}_j\) is included in an annulus \(\{\frac{1}{R}2^j \leq |\xi| \leq R2^j\}\), parallel to (3.3.5), we see that \(2^{jr} \|u_j\|_{L^2} \sim \|u_j\|_{H^r}\). So, the Cauchy-Schwarz inequality implies that

\[\sum_{j \geq -1} \|u_j\|_{H^r}^2 \leq \left( \sum_{j \geq -1} 2^{2js} \|u_j\|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sum_{j \geq -1} 2^{2j(r-s)} \right)^{\frac{1}{2}} \lesssim \left( \sum_{j \geq -1} 2^{2js} \|u_j\|_{L^2}^2 \right)^{\frac{1}{2}} < +\infty.\]

This shows that the series \(\sum u_j\) is normally convergent and hence convergent in \(H^s(\mathbb{R}^d)\). Now we can set \(u = \sum_{j \geq -1} u_j\). Our goal is then to prove that \(u\) belongs to \(H^s(\mathbb{R}^d)\).

The assumption that \(\text{supp} \hat{u}_j\) is included in an annulus \(\{\frac{1}{R}2^j \leq |\xi| \leq R2^j\}\) implies that there exists some integer \(N\) depending only on \(R\) such that \(\Delta_p u_j = 0\) if \(|j - p| > N\). Therefore

\[\|\Delta_p u\|_{L^2} \leq \sum_{|j-p| \leq N} \|\Delta_p u_j\|_{L^2} \leq \sum_{|j-p| \leq N} \|u_j\|_{L^2},\]

whence the result.

\(\text{ii)}\) If one only assumes that \(\text{supp} \hat{u}_j\) is included in a ball \(|\xi| \leq R2^j\), then we just have for some integer \(N\),

\[\Delta_p u = \sum_{j \geq p-N} \Delta_p u_j.\]
It follows from the triangle inequality that
\[ 2^{ps} \| \Delta_p u \|_{L^2} \leq \sum_{j \geq p-N} 2^{(p-j)s} 2^{js} \| u_j \|_{L^2}. \]

Now, since \( s > 0 \), the sequence \( (2^{(p-j)s}) \) belongs to \( \ell^1 \) and the convolution inequality \( \ell^1 \ast \ell^2 \hookrightarrow \ell^2 \) gives the result. \( \square \)

### 3.4 Operators of type \((1, 1)\)

We continue the study of pseudo-differential operators of the form

\[(3.4.1) \quad \text{Op}(a)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x, \xi) \hat{u}(\xi) \, d\xi. \]

In Section \S 3.2, we have introduced the study of these operators, assuming that \( a \in C_b^\infty(\mathbb{R}^{2d}; \mathbb{C}) \). Here we consider more general symbols, belonging to the following spaces.

**Definition 3.4.1.** For \( m \in \mathbb{R} \) and \( 0 \leq \delta \leq \rho \leq 1 \), the symbol class \( S_{\rho, \delta}^m(\mathbb{R}^d) \) is the space of functions \( a \in C^\infty(\mathbb{R}^{2d}; \mathbb{C}) \) such that, for all multi-indices \( \alpha \in \mathbb{N}^d \) and \( \beta \in \mathbb{N}^d \), there exists a constant \( C \) such that

\[ \left| \partial^\alpha_x \partial^\beta_\xi a(x, \xi) \right| \leq C(1 + |\xi|)^{m+\delta|\alpha|-\rho|\beta|}. \]

We say that \( a \) is a symbol of order \( m \) and type \( (\rho, \delta) \).

**Remark 3.4.2.** Notice that \( C_b^\infty(\mathbb{R}^{2d}; \mathbb{C}) = S_{0,0}^0(\mathbb{R}^d) \).

For any real numbers \( m \in \mathbb{R} \) and \( 0 \leq \delta \leq \rho \leq 1 \), and for any symbol \( a \in S_{\rho, \delta}^m(\mathbb{R}^d) \), by using similar arguments to those used to prove Proposition 3.2.1, one can prove that the relation (3.4.1) defines \( \text{Op}(a) \) as a continuous operator from \( S(\mathbb{R}^d) \) to \( S(\mathbb{R}^d) \).

Let us state a generalization of Theorem 3.2.2 to the case of general symbol.

**Theorem 3.4.3** (Calderón-Vaillancourt). Let \( a \in S_{\rho, \delta}^m(\mathbb{R}^d) \) with \( 0 \leq \delta \leq \rho \leq 1 \) and \( \delta < 1 \). Then the operator \( \text{Op}(a) \) can be extended as a bounded operator from \( L^2(\mathbb{R}^d) \) to itself. Moreover,

\[ \| \text{Op}(a) \|_{L(2)} \leq C \sup_{|\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1} \sup_{|\beta| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1} \sup_{(x, \xi) \in \mathbb{R}^{2d}} \left| (1 + |\xi|)^{\delta|\alpha|-\rho|\beta|} \partial^\alpha_x \partial^\beta_\xi p(x, \xi) \right|, \]

for some absolute constant \( C \) depending only on \( d, \rho, \delta \).
Proof. We will not use this result and refer to [90] for the proof. The precise bound in terms of the semi-norms of \( p \) is proved for instance by Coifman and Meyer [115]. □

It is proved in Exercise 3.8.3 that the statement of Theorem 3.4.3 does not hold for \((\rho, \delta) = (1, 1)\). This means that an operator of 0 and type \((1, 1)\) is not bounded in general from \( L^2(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^d) \). However, the following result, due to Stein, states that such an operator is bounded from \( H^s(\mathbb{R}^d) \) to \( H^s(\mathbb{R}^d) \) for any \( s > 0 \).

**Theorem 3.4.4** (Stein). Assume that \( a \in S_{1,1}^0(\mathbb{R}^d) \). Then the operator \( \text{Op}(a) \) is bounded from \( H^s(\mathbb{R}^d) \) to \( H^s(\mathbb{R}^d) \) for all \( s > 0 \).

Proof. We will not use this result and refer to [285] for the proof. □

**Theorem 3.4.5.** Let \( \varepsilon \in [0, 1) \) and consider a function \( a \in C^\infty(\mathbb{R}^{2d}; \mathbb{C}) \) such that

\[
M := \sup_{|\beta| \leq \lfloor \frac{d}{2} \rfloor + 1} \sup_{(x, \xi) \in \mathbb{R}^{2d}} |(1 + |\xi|)^{\lfloor \frac{d}{2} \rfloor} \partial_\xi^\beta a(x, \xi)| < +\infty.
\]

Assume in addition that, for all \( \xi \in \mathbb{R}^d \), the partial Fourier transform

\[
\hat{a}(\eta, \xi) = \int_{\mathbb{R}^d} e^{-iy \cdot \eta} a(y, \xi) \, dy
\]

is supported in the ball \( \{ \eta \in \mathbb{R}^d \mid |\eta| \leq \varepsilon |\xi| \} \). Then \( \text{Op}(a) \in \mathcal{L}(L^2(\mathbb{R}^d)) \) and

\[
\|\text{Op}(a)\|_{L^2 \to L^2} \leq CM,
\]

for some constant \( C \) depending only on \( \varepsilon \).

**Remark 3.4.6.** We will see in the proof that \( a \) belongs to \( S_{1,1}^0(\mathbb{R}^d) \).

Proof. Set \( N = 1 + \lceil d/2 \rceil \).

**Step 1: Littlewood-Paley decomposition.** We use the decomposition of the unity introduced in Lemma 3.3.1. Write

\[
a(x, \xi) = a(x, \xi)\psi(\xi) + \sum_{p=0}^{\infty} a(x, \xi)\varphi(2^{-p}\xi),
\]

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and then set
\[ a_{-1}(x, \xi) = a(x, \xi)\psi(\xi) \quad ; \quad a_p(x, \xi) = a(x, \xi)\varphi(2^{-p}\xi) \quad \text{for} \quad p \geq 0. \]

**Step 2: Bernstein Lemma.** We claim that, for any multi-indices \( \alpha \in \mathbb{N}^d \) and \( \beta \in \mathbb{N}^d \) with \( |\beta| \leq N \), there exists a positive constant \( C \) such that,
\[
\left| \partial_\alpha^a \partial_\xi^\beta a_{-1}(x, \xi) \right| \leq CM, \tag{3.4.3}
\]
and, for \( p \geq 0 \),
\[
\left| \partial_\alpha^a \partial_\xi^\beta a_p(x, \xi) \right| \leq CM2^{|\alpha|-|\beta|}. \]

Since \( |\xi| \sim 2^p \) on the support of \( \varphi(2^{-p}\xi) \) (resp. \( \psi(\xi) \) for \( p = 0 \)), this follows from the assumption that the partial Fourier transform \( \hat{a}(\eta, \xi) \) is supported in the ball \( \{|\eta| \leq \varepsilon |\xi|\} \), by using the following

**Lemma 3.4.7.** Consider a function \( f \in L^\infty(\mathbb{R}^d) \) whose Fourier transform is included in the ball \( \{|\xi| \leq \lambda\} \). Then \( f \in C^\infty(\mathbb{R}^d) \) and, for all \( \alpha \in \mathbb{N}^d \), there exists a constant \( C = C(d, \alpha) \) such that
\[
\|\partial_\alpha^a f\|_{L^\infty} \leq C\lambda^{|\alpha|} \|f\|_{L^\infty}. \]

**Proof.** Introduce \( \theta \in C^\infty_0(\mathbb{R}^d) \) such that \( \theta(\xi) = 1 \) for \( |\xi| \leq 1 \) and set \( \theta_\lambda(\xi) = \theta(\xi/\lambda) \). Then \( \theta_\lambda \hat{f} = \hat{f} \), which implies that
\[
f = \kappa_\lambda \ast f, \quad \text{where} \quad \kappa_\lambda = \mathcal{F}^{-1}(\theta_\lambda). \]

We are now in position to estimate the derivatives of \( f \) by exploiting the relation
\[
\partial_\alpha^a f = (\partial_\alpha^a \kappa_\lambda) \ast f.
\]
Observing that \( \kappa_\lambda(x) = \lambda^d \kappa(\lambda x) \) with \( \kappa = \mathcal{F}^{-1}(\theta) \), we obtain that
\[
\|\partial_\alpha^a \kappa_\lambda\|_{L^1(\mathbb{R}^d)} = \lambda^{|\alpha|} \|\partial_\alpha^a \kappa\|_{L^1(\mathbb{R}^d)},
\]
and the result follows. \( \square \)

**Step 3: low frequency component.** In view of (3.4.3), it follows directly from the Calderón-Vaillancourt theorem (see Theorem 3.2.2), implies that \( \text{Op}(a_{-1}) \) is bounded from \( L^2(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^d) \), and satisfies the estimate
\[
\|\text{Op}(a_{-1})\|_{L^2 \to L^2} \leq CM.
\]
Step 4: rescaling. We want to prove that the operators $\text{Op}(a_p)$ are also bounded from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. To do so, we use a rescaling argument. More precisely, given a positive real-number $\lambda > 0$, introduce the operator $H_\lambda$ defined by

$$(H_\lambda u)(x) = \lambda^\frac{d}{2} u(\lambda x).$$

Then

$$\|H_\lambda u\|_{L^2} = \|u\|_{L^2}.$$ 

In addition, for any symbol $p = p(x, \xi)$, we have

$$\text{Op}(p)(H_\lambda u) = H_\lambda(\text{Op}(p_\lambda)u) \quad \text{where} \quad p_\lambda(x, \xi) = p\left(\frac{x}{\lambda}, \lambda \xi\right).$$

This implies that $\text{Op}(a_p) \in \mathcal{L}(L^2(\mathbb{R}^d))$ if and only if $\text{Op}(b_p) \in \mathcal{L}(L^2(\mathbb{R}^d))$ where

$$b_p(x, \xi) = a_p(2^{-p}x, 2^p\xi),$$

and then $\|\text{Op}(a_p)\|_{L^2 \to L^2} = \|\text{Op}(b_p)\|_{L^2 \to L^2}$.

Step 5: boundedness of the rescaled operators. Notice that, for any multi-indices $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^d$ with $|\beta| \leq N$, there holds

$$\left| \partial_\xi^\alpha \partial_\xi^\beta b_p(x, \xi) \right| \leq CM.$$

Then, as already explained above, it follows from the Calderón-Vaillancourt theorem (see Theorem 3.2.2) that

$$\|\text{Op}(a_p)\|_{L^2 \to L^2} = \|\text{Op}(b_p)\|_{L^2 \to L^2} \leq CM.$$ 

Step 6: spectral localization. Notice that

$$\widehat{\text{Op}(a_p)}u(\eta) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{a}(\eta - \xi, \xi) \varphi(2^{-p}\xi) \hat{u}(\xi) \, d\xi.$$

Introduce the function $u_p$ defined by

$$\hat{u}_p(\xi) = \hat{u}(\xi) \quad \text{if} \quad \xi \in \Gamma_p := \{ 3^{-1} \cdot 2^p \leq |\xi| \leq 3 \cdot 2^p \},$$

and $\hat{u}_p(\xi) = 0$ whenever $\xi \notin \Gamma_p$. Then $\varphi(2^{-p}\xi) \hat{u}(\xi) = \varphi(2^{-p}\xi) \hat{u}_p(\xi)$, which in turn implies that

$$\text{Op}(a_p)u = \text{Op}(a_p)u_p.$$
Exploiting again that the partial Fourier transform $\hat{a}(\eta, \xi)$ is supported in the ball $\{|\eta| \leq \varepsilon |\xi|\}$, we verify that the support of $\mathcal{F}(\text{Op}(a)u_p)$ is included in the larger shell

$$\Gamma'_p = \left\{ \xi \in \mathbb{R}^d; \frac{1}{3} - \frac{\varepsilon}{2} \leq |\xi| \leq 3 \cdot (1 + \varepsilon)^2 \right\}.$$

Now, since any $\eta$ is included in at most $2 \log(3/(1 - \varepsilon)) / \log(2)$ dyadic shells $\Gamma'_p$, we deduce from the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ that

$$\left| \sum_p \hat{\text{Op}(a_p)}u(\eta) \right|^2 \leq C(\varepsilon) \sum_p \left| \hat{\text{Op}(a_p)}u(\eta) \right|^2.$$

It follows from Plancherel’s theorem that

$$\|\text{Op}(a)u\|^2_{L^2} \sim \left| \sum_p \hat{\text{Op}(a_p)}u \right|^2_{L^2} \leq \sum_p \left| \hat{\text{Op}(a_p)}u \right|^2_{L^2} \sim \sum_p \|\text{Op}(a_p)u\|^2_{L^2}.$$

Remembering that $\text{Op}(a_p)u = \text{Op}(a_p)u_p$ and using the fact that $\text{Op}(a_p)$ is bounded in $\mathcal{L}(L^2(\mathbb{R}^d))$, we get

$$\sum_p \|\text{Op}(a_p)u\|^2_{L^2} = \|\text{Op}(a_p)u_p\|^2_{L^2} \lesssim M^2 \|u_p\|^2_{L^2},$$

we conclude that

$$\|\text{Op}(a_p)u\|^2_{L^2} \lesssim M^2 \sum_p \|u_p\|^2_{L^2}.$$

Eventually, since each $\xi$ is contained in at most a fix number of dyadic shells $\Gamma_p$, we have

$$\sum_p \|u_p\|^2_{L^2} \lesssim \|u\|^2_{L^2}.$$

This concludes the proof. \qed

### 3.5 Symbolic calculus

In this paragraph we review notations and results about Bony’s paradifferential calculus. We refer to [75, 208, 285, 286, 352] for the general theory. Here we follow the presentation by Mézivier in [285]. We refer also to the recent book of Benzoni-Gavage and Serre [59] for applications of paradifferential calculus to hyperbolic systems.
We denote by \( \hat{u} \) or \( F u \) the Fourier transform acting on temperate distributions \( u \in \mathcal{S}'(\mathbb{R}^d) \), and in particular on periodic distributions. The spectrum of \( u \) is the support of \( F u \). Fourier multipliers are defined by the formula
\[
p(D_x)u = F^{-1} (p F u),
\]
provided that the multiplication by \( p \) is defined at least from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \); \( p(D_x) \) is the operator associated to the symbol \( p(\xi) \).

We denote by \( C^0(\mathbb{R}) \) the space of bounded continuous functions. For any \( \rho \in \mathbb{N} \), we denote by \( C^\rho(\mathbb{R}) \) the space of \( C^0(\mathbb{R}) \) functions whose derivatives of order less or equal to \( \rho \) are in \( C^0(\mathbb{R}) \). For any \( \rho \in \mathbb{N} \), we denote by \( C^\rho(\mathbb{R}) \) the space of bounded functions whose derivatives of order \( \rho \) are uniformly Hölder continuous with exponent \( \rho \).

**Definition 3.5.1.** Consider \( \rho \) in \( [0, +\infty) \) and \( m \in \mathbb{R} \). One denotes by \( \Gamma^\rho_m(\mathbb{R}^d) \) the space of locally bounded functions \( a(x, \xi) \) on \( \mathbb{R}^d \times (\mathbb{R}^d \setminus 0) \), which are \( C^\infty \) functions of \( \xi \) outside the origin and such that, for any \( \alpha \in \mathbb{N}^d \) and any \( \xi \neq 0 \), the function \( x \mapsto \partial_\xi^\alpha a(x, \xi) \) belongs to \( C^\rho(\mathbb{R}^d) \) and there exists a constant \( C_{\alpha} \) such that,
\[
\forall |\xi| \geq 2, \quad \| \partial_\xi^\alpha a(\cdot, \xi) \|_{C^\rho} \leq C_{\alpha} (1 + |\xi|)^{m-|\alpha|}.
\]

**Remark 3.5.2.** Note that we consider symbols \( a(x, \xi) \) that need not be smooth for \( \xi = 0 \), for instance \( a(x, \xi) = |\xi|^m \) with \( m \in \mathbb{R}^+ \). The main motivation for considering such symbols comes from the principal symbol of the Dirichlet to Neumann operator. As we shall see this symbol is given by
\[
\sqrt{1 + |\nabla \eta(x)|^2} |\xi|^2 - (\nabla \eta(x) \cdot \xi)^2.
\]
If \( \eta \in C^s(\mathbb{R}^d) \) then this symbol belongs to \( \Gamma^1_{s-1}(\mathbb{R}^d) \). Of course, this symbol is not \( C^\infty \) with respect to \( \xi \in \mathbb{R}^d \).

Given a symbol \( a \), to define the paradifferential operator \( T_a \) we need to introduce a cutoff function \( \theta \).

**Definition 3.5.3.** Fix \( \theta \in C^\infty(\mathbb{R} \times \mathbb{R}) \) satisfying the three following properties.

(i) There exists \( \varepsilon_1, \varepsilon_2 \) satisfying \( 0 < 2\varepsilon_1 < \varepsilon_2 < 1/2 \) such that
\[
\theta(\xi_1, \xi_2) = 1 \quad \text{if} \quad |\xi_1| \leq \varepsilon_1 |\xi_2| \quad \text{and} \quad |\xi_2| \geq 2,
\]
\[
\theta(\xi_1, \xi_2) = 0 \quad \text{if} \quad |\xi_1| \geq \varepsilon_2 |\xi_2| \quad \text{or} \quad |\xi_2| \leq 1.
\]
(ii) For all $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is $C_{\alpha, \beta}$ such that
\[
\forall (\xi_1, \xi_2) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \left| \partial^\alpha_{\xi_1} \partial^\beta_{\xi_2} \theta(\xi_1, \xi_2) \right| \leq C_{\alpha, \beta}(1 + |\xi_2|)^{-|\alpha| - |\beta|}.
\]

(iii) $\theta$ satisfies the following symmetry conditions:
\[
\theta(\xi_1, \xi_2) = \theta(-\xi_1, -\xi_2) = \theta(-\xi_1, \xi_2).
\]

Remark 3.5.4. Notice that $\theta(\xi_1, \xi_2) = 0$ for $|\xi_2|$ small enough. This choice is important in our analysis since we have to handle symbols which are homogeneous in $\xi_2$ and hence not regular for $\xi_2 = 0$.

Example 3.5.5. As an example, fix $d = 1$ and $\varepsilon_1, \varepsilon_2$ such that $0 < 2\varepsilon_1 < \varepsilon_2 < 1/2$ and a function $f$ in $C^\infty_0(\mathbb{R})$ satisfying $f(t) = f(-t)$, $f(t) = 1$ for $|t| \leq 2\varepsilon_1$ and $f(t) = 0$ for $|t| \geq \varepsilon_2$. Then set
\[
\theta(\xi_1, \xi_2) = (1 - f(\xi_2))f\left(\frac{\xi_1}{\xi_2}\right).
\]

Properties (i), (ii) and (iii) are clearly satisfied.

Figure 3.1: The support of the cut-off function $\theta(\xi_1, \xi_2)$ is in grey. The set of points $(\xi_1, \xi_2)$ where $\theta(\xi_1, \xi_2) = 1$ is in darker grey.

The paradifferential operator $T_a$ with symbol $a$ is defined by
\[
(3.5.3) \quad \widehat{T_a u}(\xi) = (2\pi)^{-d} \int \theta(\xi - \eta, \eta)\widehat{a}(\xi - \eta, \eta) \hat{u}(\eta) \, d\eta,
\]
where $\widehat{a}(\theta, \xi) = \int e^{-ix\cdot\theta}a(x, \xi) \, dx$ is the Fourier transform of $a$ with respect to $x$. 
Remark 3.5.6. It follows from (3.5.2) that, if $a$ and $u$ are real-valued functions, so is $T_a u$.

Remark 3.5.7. One says that $\Theta = \Theta(\xi_1, \xi_2)$ is an admissible cut-off function if $\Theta$ satisfies the properties (i) and (ii) in Definition 3.5.3. All the results given in this appendix remain true for any admissible cut-off function (except Remark 3.5.6).

Remark 3.5.8. Given a symbol $a = a(x, \xi)$ which is homogeneous in $\xi$ and smooth in $x$, we define the pseudo-differential operator $\text{Op}(a)$ by

\begin{equation}
\hat{\text{Op}(a)}u(\xi) = (2\pi)^{-d} \int \hat{a}(\xi - \eta, \eta) \psi(\eta) \hat{u}(\eta) \, d\eta,
\end{equation}

where again $\hat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi) \, dx$ is the Fourier transform of $a$ with respect to the first variable; and where $\psi$ is as in (3.5.3). Note that the only difference between (3.5.3) and (3.5.4) is the cut-off function $\chi$; this cut-off allows to define operators for non smooth symbols by means of symbol smoothing.

We shall use quantitative results from [285]. To do so, introduce the following semi-norms.

**Definition 3.5.9.** For $m \in \mathbb{R}$, $\rho \geq 0$ and $a \in \Gamma^m_\rho(\mathbb{R}^d)$, we set

\begin{equation}
M^m_\rho(a) = \sup_{|\alpha| \leq \frac{d}{2} + 1 + \rho} \sup_{|\xi| \geq 1/2} \| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(\cdot, \xi) \|_{C^0(\mathbb{R}^d)}.
\end{equation}

We shall use quantitative results from [285] about operator norms estimates in symbolic calculus. To do so, introduce the following semi-norms.

**Definition 3.5.10.** For $m \in \mathbb{R}$, $\rho \geq 0$ and $a \in \Gamma^m_\rho(\mathbb{R}^d)$, we set

\begin{equation}
M^m_\rho(a) = \sup_{|\alpha| \leq \frac{d}{2} + 1 + \rho} \sup_{|\xi| \geq 1/2} \| (1 + |\xi|)^{|\alpha| - m} \partial_\xi^\alpha a(\cdot, \xi) \|_{W^{\rho, \infty}(\mathbb{R}^d)}.
\end{equation}

The main features of symbolic calculus for paradifferential operators are given by the following theorems.

**Definition 3.5.11.** Let $m \in \mathbb{R}$. An operator $T$ is said of order $m$ if, for all $\mu \in \mathbb{R}$, it is bounded from $H^\mu$ to $H^{\mu - m}$.
**Theorem 3.5.12.** Let \( m \in \mathbb{R} \). If \( a \in \Gamma^m_0(\mathbb{R}^d) \), then \( T_a \) is of order \( m \). Moreover, for all \( \mu \in \mathbb{R} \) there exists a constant \( K \) such that
\[
\| T_a \|_{H^\mu \to H^{\mu-m}} \leq K M^m_0(a).
\]

**Theorem 3.5.13 (Composition).** Let \( m \in \mathbb{R} \) and \( \rho > 0 \). If \( a \in \Gamma^m_\rho(\mathbb{R}^d) \), \( b \in \Gamma^{m'}_\rho(\mathbb{R}^d) \) then \( T_a T_b - T_{a\#b} \) is of order \( m + m' - \rho \) where
\[
a\#b = \sum_{|\alpha| < \rho} \frac{1}{i|\alpha|!} \partial_\xi^\alpha a \partial_x^\alpha b.
\]
Moreover, for all \( \mu \in \mathbb{R} \) there exists a constant \( K \) such that
\[
\| T_a T_b - T_{a\#b} \|_{H^\mu \to H^{\mu-m-\rho}} \leq K M^m_\rho(a) M^{m'}_\rho(b).
\]

**Theorem 3.5.14 (Adjoint).** Let \( m \in \mathbb{R} \), \( \rho > 0 \) and \( a \in \Gamma^m_\rho(\mathbb{R}^d) \). Denote by \( (T_a)^* \) the adjoint operator of \( T_a \) and by \( \bar{a} \) the complex-conjugated of \( a \). Then \( (T_a)^* - T_{a^*} \) is of order \( m - \rho \) where
\[
a^* = \sum_{|\alpha| < \rho} \frac{1}{i|\alpha|!} \partial_\xi^\alpha \partial_x^\alpha \bar{a}.
\]
Moreover, for all \( \mu \) there exists a constant \( K \) such that
\[
\|(T_a)^* - T_{a^*}\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M^m_\rho(a).
\]

### 3.6 Paraproducts

If \( a = a(x) \) is a function of \( x \) only, then \( T_a \) is a called a paraproduct.

If \( a \in L^\infty(\mathbb{R}) \) then \( T_a \) is an operator of order 0, together with the estimate
\[
\forall \sigma \in \mathbb{R}, \quad \| T_a u \|_{H^\sigma} \lesssim \| a \|_{L^\infty} \| u \|_{H^\sigma}.
\]
A paraproduct with an \( L^\infty \)-function acts on any Hölder space \( C^\rho(\mathbb{R}) \) with \( \rho \) in \( \mathbb{R}^*_+ \setminus \mathbb{N} \),
\[
\forall \rho \in \mathbb{R}^*_+ \setminus \mathbb{N}, \quad \| T_a u \|_{C^\rho} \lesssim \| a \|_{L^\infty} \| u \|_{C^\rho}.
\]
If \( a = a(x) \) and \( b = b(x) \) are two functions then (??) simplifies to \( a\#b = ab \) and hence (??) implies that, for any \( \rho > 0 \),
\[
\| T_a T_b - T_{ab} \|_{H^\mu \to H^{\mu+\rho}} \leq K \| a \|_{C^\rho} \| b \|_{C^\rho},
\]
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provided that \( a \) and \( b \) belong to \( C^\rho(\mathbb{R}) \).

A key feature of paraproducts is that one can replace nonlinear expressions by paradifferential expressions, to the price of error terms which are smoother than the main terms. As an illustration, we give the following result of Bony [75].

**Definition 3.6.1.** Given two functions \( a \) and \( b \), we define the remainder

\[
R_B(a, u) = au - T_a u - T_u a.
\]

We record here two estimates about the remainder \( R_B(a, b) \) (see chapter 2 in [47]).

**Theorem 3.6.2.** Let \( \alpha \in \mathbb{R}_+ \) and \( \beta \in \mathbb{R} \) be such that \( \alpha + \beta > 0 \). Then

\[
\| R_B(a, u) \|_{H^{\alpha+\beta}(\mathbb{R})} \leq K \| a \|_{H^\alpha(\mathbb{R})} \| u \|_{H^\beta(\mathbb{R})},
\]

\[
\| R_B(a, u) \|_{C^{\alpha+\beta}(\mathbb{R})} \leq K \| a \|_{C^\alpha(\mathbb{R})} \| u \|_{H^\beta(\mathbb{R})}.
\]

We next recall a well-known property of products of functions in Sobolev spaces (see chapter 8 in [208]) that can be obtained from (3.6.1) and (3.6.6): If \( u_1, u_2 \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and \( s > 0 \) then

\[
\| u_1 u_2 \|_{H^s} \leq K \| u_1 \|_{L^\infty} \| u_2 \|_{H^s} + K \| u_2 \|_{L^\infty} \| u_1 \|_{H^s}.
\]

Similarly, recall that, for \( s > 0 \) and \( F \in C^\infty(\mathbb{C}^N) \) such that \( F(0) = 0 \), there exists a non-decreasing function \( C : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\| F(U) \|_{H^s} \leq C(\| U \|_{L^\infty}) \| U \|_{H^s},
\]

for any \( U \in (H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N \).

### 3.7 References

#### 3.7.1 Pseudo-differential operators

The study of pseudo-differential operators is a very vast subject of which we have barely touched the surface here. We refer for instance to Alinhac and Gérard [30], Grigis and Sjöstrand [190], Hörmander [209], Lerner [266], Saint-Raymond [321], Taylor [351] or Zworski [391].
We have studied the boundedness of pseudo-differential operators on $L^2(\mathbb{R}^d)$. Let us briefly discuss the boundedness on other functions spaces.

(i) Firstly, a pseudo-differential operator of order 0 and type $(\rho, \delta)$ is not bounded in general on Lebesgue spaces $L^p(\mathbb{R}^d)$ with $p \neq 2$. Nevertheless, Fefferman proved in [177] that, for any $0 \leq \delta \leq \rho \leq 1$ with $\delta < 1$, and any symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^d)$, the operator $\text{Op}(a)$ belong to $L^p(\mathbb{R}^d)$ provided that

$$m \leq -d(1 - \rho) \left| \frac{1}{2} - \frac{1}{p} \right|^\cdot$$

We also refer to David and Journé (see [154]) for the boundedness of pseudo-differential operators on $L^p(\mathbb{R}^d)$ when $p = 1 = \delta$.

(ii) Assume that $a \in S^0_{0,1}(\mathbb{R}^d)$. Then $\text{Op}(a)$ is also bounded from $C^{k,\rho}(\mathbb{R}^d)$ to $C^{k,\rho}(\mathbb{R}^d)$ for all $k \in \mathbb{N}$ and all $\rho \in (0, 1)$.

(iii) One can also consider the case where $\delta > \rho$, see Hörmander [207].

3.7.2 Littlewood-Paley decomposition

For an introduction to this topic, we refer the reader to Bahouri [45] or Danchin [150]. There are many books which develop a systematic study of this tool. In addition to the book by Coifman and Meyer [115, 289] and Métivier [285] that we have already mentioned, we refer to Alinhac and Gérard [30], Bahouri, Danchin and Chemin [47], Tao [346] or Taylor [353, 355].

The decomposition of a product into two paraproducts and a remainder has been introduced by Bony in [75]. However, the discrete version of it using Littlewood-Paley decomposition is due to Gérard and Rauch (see [182]).

3.7.3 Variants of paradifferential calculus

Bony’s paradifferential calculus has been completed in particular by Chemin who introduced a bilinear symbolic computation and it has been enriched by the introduction of the operators of paracomposition by Alinhac and also by Fourier para-integral operators (the interested reader is referred to the original articles [27, 28, 106, 105] as well as to Taylor’s book [353]). Let us mention that we can use versions much more simple of this calculus in particular
situations (see the articles by Shnirelman [331, 332] for what is perhaps a first appearance of the ideas of decomposition underlying the paradifferential calculus). Paraproducts play a key role in the study of multi-linear Fourier multipliers. The reference works are the books by Stein [338], Coifman and Meyer [115, 289], Meyer [287, 288], Christ [110] and now that of Bahouri, Chemin and Danchin [47]. To return to the initial question, the application of paradifferential calculus to the non-linear interaction of singularities and to their propagation, we refer the reader to to the notes of Bony [76].

If $a \in \Gamma_m^\rho(\mathbb{R}^d)$ then the symbol $\sigma$ defined by (??) belongs to the class $S^{m_{1,1}}(\mathbb{R}^d)$ and satisfy some additional properties (see [77, 76, 208, 285]). Other frequency regularization lead to symbols $S^{m_{1,\delta}}(\mathbb{R}^d)$ with $\delta < 1$. As noted Lebeau [264], the case $\delta < 1$ is very useful, for instance to construct parametrixes and prove Strichartz’s estimates ([14, 16]).

Calderón’s projectors have been studied in very general frameworks; see the books by Egorov, Komech and Shubin [173], Hörmander [210], Trèves [360], Grubb [196]. The $\Psi$DO operators with irregular coefficients have been extensively studied, see Nagase [295], Kumano-go and Nagase [254], Beals and Reed [55] and the books of Taylor [352, 354, 355, 353]. Lerner’s book contains a lot of additional results [266].

### 3.7.4 About the Dirichlet-to-Neumann operator

The idea of studying the incompressible Euler equation at free surface using tools derived from the analysis of singular integrals goes back to an article by Craig–Schanz–Sulem [142]; this idea has been pursued by, among others, Craig–Schanz–Sulem [142], Lannes [257], Ming-Zhang [292] and Iooss–Plotnikov [229]. The paradifferential analysis of the Dirichlet-to-Neumann operator is introduced by Alazard-Métivier in [22].

The paradifferential approach is inspired by another context of free boundary problems: the study of shock waves. and rarefaction waves for conservation law systems by Majda [273, 272], Alinhac [29, 28] and Métivier [283, 284]. The good unknown of Alinhac plays a key role in the study of surface waves. For the water-wave problems, it was used by Lannes [257] and Trakhinin [358] to study the linearized equations.

There are close links between Alinhac’s good unknown and the geometric anal-
ysis of Ambrose and Masmoudi [37, 38, 39], Lindblad [269, 268] and Shatah and Zeng [327]. There are also links with the work of Coulombel and Secchi [130] on the study of the problem of vortex pockets for the equation compressible Euler in dimension two (see also [131]).

We now discuss the boundedness of the Dirichlet-to-Neumann operator on Sobolev spaces. Expressing $G(\eta)$ as a singular integral operator, it was proved by Craig, Schanz and Sulem [142] that if $\eta$ is in $C^{k+1}$ and $\psi$ is in $H^{k+1}$ for some integer $k$, then $G(\eta)\psi$ belongs to $H^k$. Moreover, it was proved by Lannes [257] that when $\eta$ is a function with limited smoothness, then $G(\eta)$ is a pseudodifferential operator with symbol of limited regularity. This implies that if $\eta$ is in $H^s$ and $\psi$ is in $H^s$ for some $s$ large enough, then $G(\eta)\psi$ belongs to $H^{s-1}$ (which was first established by Craig and Nicholls [141] and Wu [375, 376] by different methods). We refer to [15, 13, 327, 328] for results in rough domains.

Numerous works have been devoted to the study of elliptic equations in domains whose boundary is only Lipschitz. One could cite many works by Dahlberg, Dauge, Jerison, Kenig, Maz’ya and many others (see for instance [142, 152, 148, 235, 241] for many references). However, we are exclusively interested in situations where the boundary has a Sobolev regularity, the threshold of lipschitzian regularity being seen only by the Sobolev injection; this allows to show results that are impossible to show for boundaries that are only $C^1$. One can nevertheless cite works closer to those of interest to us we refer, for instance, to chapter 14 of Maz’ya and Shaposhnikova [280] (see also [279]) and the article by Gérard-Varet andillairet [183].

An important extension of the estimates for Dirichlet-Neumann is given by Alvarez-Samniego and Lannes in [33] which prove estimates that are uniform with respect to the various physical parameters encountered in the study of surface waves. Let us also mention that these estimates are true for any bottom (as shown in [13]) under the assumption that the distance $h$ between the free surface and the bottom is strictly positive. See also Lannes [259] for an analysis of the case where the constant $h$ tends towards 0.

There are many extensions of the Theorem ?? for higher order derivatives (see [259]). That is, one can give the Taylor development of $G(\eta)$. This intervenes in the computation of formal solutions ([229]), to derive the equations in order to study the problem of Cauchy ([259, 69, 318]) and in the normal form methods ([184, 18]).
3.7.5 Propagation of singularities

Let us conclude this chapter by discussing the applications of paradifferential calculus to the study of singularities for nonlinear equations.

An important question in PDE is to determine the wavefront of the distribution solutions of the equation \( Pf = 0 \) where \( P \) is a differential operator of order \( m \) with coefficients \( C^\infty \),

\[
P = \sum p_\alpha(x)D_x^\alpha, \quad (|\alpha| \leq m, \ D_x = -i\partial_x).
\]

An important geometrical object is constituted by the characteristic variety of \( P \) that we denote \( \text{Car}(P) \) and which is the closed (homogeneous in \( \xi \)) defined by

\[
\text{Car}(P) = \{ (x, \xi) \in T^*\mathbb{R}^d; \ p_m(x, \xi) = 0 \} \quad \text{where} \quad p_m(x, \xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha.
\]

The first important result of the theory is the following: the singularities are contained in the characteristic variety. What means that in any point \((x_0, \xi_0) \in \mathbb{R}^d \setminus \{0\}\) such that \( p_m(x_0, \xi_0) \neq 0 \), there exists a symbol \( \varphi = \varphi(x, \xi) \) homogeneous of order 0 in \( \xi \), satisfying \( \varphi(x_0, \xi_0) \neq 0 \), such that \( \text{Op}(\varphi)u \in C^\infty \). See the text of Bony [74] and that of Lebeau [265] for two excellent introductions to these issues.

The paradiferential calculus has been introduced by Jean-Michel Bony to study the singularities of the equations to the non-linear partial derivatives of the form

\[
(3.7.1) \quad F((\partial_x^\alpha f)_{|\alpha| \leq m}) = 0,
\]

where \( f \in H^s(\mathbb{R}^d) \) with \( s > s_0 = d/2 + m \) so that \( \partial_x^\alpha f \) is continuous and bounded for all multi-index \( \alpha \) of length less than \( m \).

We cannot describe for such a general equation the set of points where the function is not microlocally \( C^\infty \). On the other hand, thanks to the paradiirectional calculus, we can say things if we replace \( C^\infty \) by a space of functions with limited regularity. Let us introduce

\[
p_m(f; x, \xi) = \sum_{|\alpha|=m} \frac{\partial F}{\partial f_\alpha}((\partial_x^\alpha f(x))_{|\alpha| \leq m})(i\xi)^\alpha.
\]
The big difference between the linear and the non-linear case is that here the characteristic variety depends on the unknown of the problem. Bony then shows two results. The simplest one says that, in any point \((x_0, \xi_0) \in \mathbb{R}^d \setminus \{0\}\) such that \(p_m(x_0, \xi_0) \neq 0\), \(f\) is microlocally twice as regular: it is microlocally of class \(H^t\) for all \(t < 2s - s_0\), which means that there exists a symbol \(\varphi = \varphi(x, \xi)\) homogeneous of order 0 in \(\xi\), satisfying \(\varphi(x_0, \xi_0) \neq 0\), such that \(\text{Op}(\varphi)u \in H^t\).

The central point in the proof is to show that if \(f\) satisfies the equation (3.7.1) then \(T_p f \in H^{t-m}(\mathbb{R}^d)\), where

\[
p(x, \xi) = \sum_{|\beta| > 2m - (s-d/2)} \frac{\partial F}{\partial f^\beta} ((\partial_x^\alpha f(x))_{|\alpha| \leq m})(i\xi)^\beta.
\]

It is said that we have parallelized (3.7.1) (we have replaced a non-linear equation by a linear paradifferential equation).

### 3.8 Exercises

**Exercise 3.8.1** (Semi-classical operators). Consider a real number \(h \in (0, 1]\) and a symbol \(a = a(x, \xi)\) which belongs to \(C^\infty_b(\mathbb{R}^{2n})\). We define

\[
\text{Op}_h(a)(x) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} a(x, h\xi) \hat{u}(\xi) \, d\xi.
\]

We want to show that

\[
\|\text{Op}_h(a)\|_{L^2(\mathbb{R}^d)} \leq C \sup_{\mathbb{R}^{2d}} |a| + O(h^{\frac{1}{2}}).
\]

1. Show that

\[
\text{Op}_h(a)u(x) = \left( \text{Op}(a_h)(x) \right) (h^{-\frac{1}{2}} x)
\]

where

\[
a_h(x, \xi) = a(h^{\frac{1}{2}} x, h^{\frac{1}{2}} \xi), \quad u_h(y) = u(h^{\frac{1}{2}} y).
\]

2. Deduce that there is a constant \(C\) and an integer \(M\) such that for all \(a \in C^\infty_b(\mathbb{R}^{2d})\) and all \(h \in (0, 1]\),

\[
\|\text{Op}_h(a)\|_{L^2(\mathbb{R}^d)} \leq C \sup_{(x, \xi) \in \mathbb{R}^{2n}} |a(x, \xi)|
\]

\[
+ C \sup_{1 \leq |\alpha| + |\beta| \leq M} \sup_{(x, \xi) \in \mathbb{R}^{2n}} h^{\frac{1}{2} (|\alpha| + |\beta|)} \left| \partial_x^\alpha \partial_\xi^\beta a \right|.
\]

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Exercise 3.8.2 (Wave packet transformation). Let \( u : \mathbb{R} \to \mathbb{C} \) in the class of Schwartz \( S(\mathbb{R}) \). The wave packet transform of \( u \) is the function \( W u : \mathbb{R} \to \mathbb{C} \) defined by

\[
W u(x, \xi) = \int_{\mathbb{R}} e^{i(x-y)\xi - \frac{1}{2}(x-y)^2} u(y) \, dy.
\]

1. Show that \((x, \xi) \mapsto xW u(x, \xi)\) and \((x, \xi) \mapsto \xi W u(x, \xi)\) are bounded on \( \mathbb{R}^2 \).

Show more generally that \( W u \) belongs to the Schwartz class \( S(\mathbb{R}^2) \).

2. Show that, for any \( x \in \mathbb{R} \),

\[
\int |W u(x, \xi)|^2 \, d\xi = 2\pi \int e^{-(x-y)^2/2} |u(y)|^2 \, dy.
\]

Deduce that there is a constant \( A > 0 \) such that, for every \( u \) in \( S(\mathbb{R}) \), we have

\[
\iint |W u(x, \xi)|^2 \, dx \, d\xi = A \int |u(y)|^2 \, dy.
\]

(It is not required to calculate \( A \).)

3. Show that for any function \( u \) in the Schwartz class \( S(\mathbb{R}) \),

\[
W u(x, \xi) = ce^{ix\xi} (\hat{W}u)(\xi, -x),
\]

for a certain constant \( c \) (it is not required to calculate \( c \)).

4. Let \( \varepsilon \in (0, 1] \) and \( u \) in Schwartz’s class \( S(\mathbb{R}^2) \). We introduce

\[
W^{\varepsilon} u(x, \xi) = \varepsilon^{-3/4} \int_{\mathbb{R}} e^{i(x-y)\xi/\varepsilon - (x-y)^2/2\varepsilon} u(y) \, dy.
\]

Check that \( A^{-1/2}W^{\varepsilon} \) is an isometry and then show that there is \( K \) such that for all \( \varepsilon \in (0, 1] \) and all functions \( u \) and \( v \) in the Schwartz class \( S(\mathbb{R}) \),

\[
\left\| \varepsilon W^{\varepsilon} u - W^{\varepsilon}(\varepsilon u) \right\|_{L^2(\mathbb{R}^2)} \leq K \varepsilon^{1/2} \| \partial_x v \|_{L^\infty(\mathbb{R})} \| u \|_{L^2(\mathbb{R})}.
\]

5. Show that there is \( K' \) such that, for all \( \varepsilon \in (0, 1] \) and for all function \( u \) in the Schwartz class \( S(\mathbb{R}) \),

\[
\| i\xi W^{\varepsilon} u - W^{\varepsilon}(\varepsilon \partial_x u) \|_{L^2(\mathbb{R}^2)} \leq K' \varepsilon^{1/2} \| u \|_{L^2(\mathbb{R})}.
\]
Exercise 3.8.3 (An unbounded operator on $L^2$). Let $\chi \in C^\infty_0(\mathbb{R})$ be such that 
$$\text{supp} \chi \subset \{ \xi \in \mathbb{R}, 2^{-1/2} \leq |\xi| \leq 2^{1/2} \}, \quad \chi(\xi) = 1 \quad \text{si} \quad 2^{-1/4} \leq |\xi| \leq 2^{1/4}.$$ 
Set 
$$a(x, \xi) = \sum_{j=1}^{+\infty} \exp(-i2^j x) \chi(2^{-j} \xi).$$

1. Show that $a \in C^\infty(\mathbb{R}^2)$ verifies 
$$|\partial_\alpha \partial_\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{||\alpha|-|\beta||} \quad \forall \alpha, \beta \in \mathbb{N}^2, \forall (x, \xi) \in \mathbb{R}^2.$$ 

2. Let $f_0$ be a function of the Schwartz class whose Fourier transform $\hat{f}_0$ is to support in the range $[-1/2, 1/2]$. For $N \in \mathbb{N}$ we set 
$$f_N(x) = \sum_{j=2}^{N} \frac{1}{j} \exp(i2^j x) f_0(x).$$ 
Using Plancherel’s formula, show that 
$$\|f_N\|^2_{L^2} = \left( \sum_{j=2}^{N} j^{-2} \right) \|f_0\|^2_{L^2} \leq c.$$ 

3. Show that 
$$\text{Op}(a)f_N = \left( \sum_{j=2}^{N} j^{-1} \right) f_0.$$ 

Chapter 4

Sobolev estimates

4.1 Inequalities in Lebesgue spaces

Proposition 4.1.1 (Hölder). Consider three real numbers $p, q, r$ in $[1, +\infty]$, satisfying

$$
\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.
$$

Then, for any couple of functions $(f, g) \in L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$, the product $fg$ belongs to $L^r(\mathbb{R}^d)$. Moreover, the following estimate holds

\begin{equation}
\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.
\end{equation}

Proposition 4.1.2 (Minkowski). Suppose that $(S_1, \mu_1)$ and $(S_2, \mu_2)$ are two $\sigma$-finite measure spaces and $F: S_1 \times S_2 \to \mathbb{R}$ is measurable. Then for all $p \in [1, +\infty)$,

\begin{equation}
\left( \int_{S_2} \left( \int_{S_1} |F(x, y)| \mu_1(dx) \right)^p \mu_2(dy) \right)^{\frac{1}{p}} \leq \left( \int_{S_1} \left( \int_{S_2} |F(x, y)|^p \mu_2(dy) \right)^{\frac{1}{p}} \mu_1(dx) \right).
\end{equation}

Proposition 4.1.3 (Hardy). For any real number $p \in [1, +\infty]$ and for all function $f$ in $L^p([0, +\infty[)$, there holds

\begin{equation}
\int_0^{+\infty} \left( \frac{1}{x} \int_0^x f(y) \, dy \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^{+\infty} |f(x)|^p \, dx.
\end{equation}
Proposition 4.1.4 (Young). Consider three real numbers $p, q, r$ such that

\begin{equation}
1 \leq p, q, r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.
\end{equation}

Then, for any $f$ in $L^p(\mathbb{R}^d)$ and any $g$ in $L^q(\mathbb{R}^d)$, the integral

$$
f * g(x) = \int_{\mathbb{R}^d} f(x - y)g(y) \, dy
$$

converges for almost all $x \in \mathbb{R}^d$. In addition, $f * g$ belongs to $L^r(\mathbb{R}^d)$ and

$$
\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.
$$

We also recall the definition of Riesz potentials as well as the classical Hardy-Littlewood-Sobolev inequality.

Given a real number $\alpha > 0$, the Riesz potential $I_\alpha$ is the operator defined by

$$
I_\alpha(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-\alpha}} \, dy.
$$

Notice that $I_\alpha f(x)$ is well-defined for any $\alpha > 0$ and any function $f$ with compact support. To see this, decompose $\mathbb{R}^d$ into two parts: the ball $B(x, 1)$ with center $x$ and radius 1, and its complementary. Since $f$ is bounded, the fact that the integral over $B(x, 1)$ is well-defined is straightforward since $\alpha > 0$. On the complementary, the integral is well-defined since we are integrating a bounded function with compact support.

Theorem 4.1.5 (Hardy-Littlewood-Sobolev). Let $d \in \mathbb{N}^*$ and consider three real positive numbers $(p, q, \alpha)$ such that

$$
\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d}, \quad 1 < p < \frac{d}{\alpha}.
$$

Then there exists a constant $C = C(p, q, d, \alpha)$ such that, for any function $f$ in $C^1_0(\mathbb{R}^d)$, there holds

$$
\|I_\alpha f\|_{L^q} \leq C \|f\|_{L^p}.
$$
4.2 Fourier analysis and Sobolev spaces

4.2.1 Definitions and first properties

Recall the notation
\[ \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}. \]

Given a real number \( s \in [0, +\infty) \), we say that a function \( u \in L^2(\mathbb{R}^d) \) belongs to the Sobolev space \( H^s(\mathbb{R}^d) \) if
\[
\int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 \, d\xi < +\infty.
\]

**Proposition 4.2.1.** Let \( s \in [0, +\infty) \). Equipped with the scalar product
\[
(u, v)_{H^s} = (2\pi)^{-d} \int (1 + |\xi|^2)^s \hat{u}(\xi) \overline{\hat{v}(\xi)} \, d\xi,
\]
and therefore the norm
\[
\|u\|_{H^s} = (2\pi)^{-n/2} \| (1 + |\xi|^2)^{s/2} \hat{u} \|_{L^2},
\]
the Sobolev space \( H^s(\mathbb{R}^d) \) is a Hilbert space.

**Proof.** The application \( u \mapsto (2\pi)^{-d/2}(1 + |\xi|^2)^{s/2} \hat{u} \) is by definition an isometric bijection of \( H^s(\mathbb{R}^d) \) on \( L^2(\mathbb{R}^d) \). This last space being a Banach space, it is the same for \( H^s(\mathbb{R}^d) \) with the norm defined above. \( \square \)

**Proposition 4.2.2.** The Schwartz space \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( H^s(\mathbb{R}^d) \) for all \( s \geq 0 \).

**Proof.** Let us consider the isometry \( u \mapsto (2\pi)^{-d/2}(1 + |\xi|^2)^{s/2} \hat{u} \) from \( H^s(\mathbb{R}^d) \) onto \( L^2(\mathbb{R}^d) \). The inverse isometry transforms the dense subspace \( \mathcal{S}(\mathbb{R}^d) \) of \( L^2(\mathbb{R}^d) \) into a dense subspace of \( H^s(\mathbb{R}^d) \). Now this application is a bijection of \( \mathcal{S}(\mathbb{R}^d) \) onto itself. We deduce that \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( H^s(\mathbb{R}^d) \). \( \square \)

**Proposition 4.2.3.** For any real number \( s > d/2 \),
\[
H^s(\mathbb{R}^d) \subset C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),
\]
with continuous injection.
Proof. According to Cauchy-Schwarz inequality, for any \( f \in \mathcal{S}(\mathbb{R}^d) \),
\[
\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1} \leq \|\langle \xi \rangle^s \hat{f}\|_{L^2} \|\langle \xi \rangle^s \hat{f}\|_{L^2},
\]
and we deduce the result by density of \( \mathcal{S}(\mathbb{R}^d) \) in \( H^s(\mathbb{R}^d) \). \( \square \)

Theorem 4.2.4. For any real number \( s > d/2 \), the product of two elements of \( H^s(\mathbb{R}^d) \) belongs to \( H^s(\mathbb{R}^d) \). In addition, there is a constant \( C \) such that for any \( u, v \in H^s(\mathbb{R}^d) \),
\[
\|uv\|_{H^s} \leq C \|u\|_{H^s} \|v\|_{H^s}.
\]

Proof. The proof rests on the following inequality: for every \( \xi, \eta \in \mathbb{R}^d \) we have
\[
\forall s \geq 0, \quad (1 + |\xi|^2)^{s/2} \leq 2^s \left\{ (1 + |\xi - \eta|^2)^{s/2} + (1 + |\eta|^2)^{s/2} \right\},
\]
which is deduced from the triangular inequality and the bound \((a + b)^r \leq 2^r(a^r + b^r)\) for any triplet \((a, b, r)\) of positive numbers. Let us write then that for every \( u, v \in \mathcal{S}(\mathbb{R}^d) \), we have (check the following formula in exercise)
\[
\hat{uv} (\xi) = (2\pi)^{-d} \int \hat{u}(\xi - \eta) \hat{v}(\eta) \, d\eta.
\]
Multiplying the two members by \( \langle \xi \rangle^s \) and using the previous inequality, we find
\[
\langle \xi \rangle^s |\hat{uv}(\xi)| \leq C \int \langle \xi - \eta \rangle^s |\hat{u}(\xi - \eta)| |\hat{v}(\eta)| \, d\eta
\]
\[
+ C \int |\hat{u}(\xi - \eta)| \langle \eta \rangle^s |\hat{v}(\eta)| \, d\eta.
\]
If \( s > d/2 \) then \( \mathcal{F}(H^s(\mathbb{R}^d)) \subset L^1(\mathbb{R}^d) \) as we have already seen (cf (4.2.1)). We then recognize above two products of convolution between a function of \( L^1(\mathbb{R}^d) \) and another of \( L^2(\mathbb{R}^d) \), that belong to \( L^2(\mathbb{R}^d) \). This implies that \( \langle \xi \rangle^s \hat{uv} \in L^2(\mathbb{R}^d) \), hence the desired result \( uv \in H^s(\mathbb{R}^d) \). \( \square \)

We have seen that, any real number \( s > d/2 \), the product of two elements of \( H^s(\mathbb{R}^d) \) is still in \( H^s(\mathbb{R}^d) \). The following proposition shows that we can also define the product \( \varphi u \) for everything \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) and any \( u \in H^s(\mathbb{R}^d) \) with \( s \in [0, +\infty[ \).

Proposition 4.2.5. For any \( s \in \mathbb{R} \), if \( u \in H^s(\mathbb{R}^d) \) and \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) then \( \varphi u \in H^s(\mathbb{R}^d) \).
Proof. The proof uses an inequality, called Peetre’s inequality that states that for every $\xi, \eta$ in $\mathbb{R}^d$, we have

$$\forall s \in \mathbb{R}, \quad (1 + |\xi|^2)^s \leq 2^{|s|} (1 + |\eta|^2)^s (1 + |\xi - \eta|^2)^{|s|}.$$ 

Let us assume that $s \geq 0$. To obtain this inequality, just use the triangular inequality

$$1 + |\xi|^2 \leq 1 + (|\eta| + |\xi - \eta|)^2 \leq 1 + 2|\eta|^2 + 2|\xi - \eta|^2 \leq 2(1 + |\eta|^2)(1 + |\xi - \eta|^2),$$

then raise both sides to the power $s \geq 0$. If $s < 0$, then $-s > 0$ and the previous inequality leads to

$$(1 + |\eta|^2)^{-s} \leq 2^{-s}(1 + |\xi|^2)^{-s}(1 + |\xi - \eta|^2)^{-s}.$$ 

The desired result is obtained by dividing by $(1 + |\eta|^2)^{-s}(1 + |\xi|^2)^{-s}$.

We then proceed as in the proof of the theorem 4.2.4. Indeed, one can still write for $u \in H^s(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\hat{\varphi}u(\xi)$ as a convolution product. As $\hat{\varphi}(\zeta)$ is in Schwartz’s class, the previous inequality allows the product of convolution of a function to appear. of $L^1$ and $\langle \eta \rangle^s|\hat{u}(\eta)|$ which is in $L^2$. □

**Proposition 4.2.6 (Interpolation in Sobolev spaces).** Let $s_1 < s_2$ be two real numbers and $s \in]s_1, s_2[$. Let us write $s$ in the form $s = \alpha s_1 + (1 - \alpha)s_2$ with $\alpha \in [0, 1]$. There is a constant $C(s_1, s_2)$ such as for all $u \in H^{s_2}(\mathbb{R}^d)$,

$$\|u\|_{H^s} \leq C(s_1, s_2) \|u\|_{H^{s_1}}^\alpha \|u\|_{H^{s_2}}^{1-\alpha}.$$ 

Proof. Let us write that

$$\|u\|_{H^s}^2 = (2\pi)^{-d} \int \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 \, d\xi$$

$$= (2\pi)^{-d} \int \langle \xi \rangle^{2\alpha s_1} |\hat{u}(\xi)|^{2\alpha} \langle \xi \rangle^{2(1-\alpha)s_2} |\hat{u}(\xi)|^{2(1-\alpha)} \, d\xi$$

so that the desired inequality is a consequence of Hölder’s inequality. □

### 4.3 Sobolev embeddings

We will now study the injection of Sobolev spaces $H^s(\mathbb{R}^d)$ into Lebesgue spaces $L^p(\mathbb{R}^d)$.
Theorem 4.3.1. Let \( d \geq 1 \) and \( s \) be a real such that \( 0 \leq s < d/2 \). Then the Sobolev space \( H^s(\mathbb{R}^d) \) is continuously embedded into \( L^p(\mathbb{R}^d) \) for any \( p \) such that
\[
2 \leq p \leq \frac{2d}{d-2s}.
\]

Remark 4.3.2. The previous theorem states that for any real number \( s \) in \([0,d/2]\), we have
\[
\|f\|_{L^p} \leq C_s \|f\|_{H^s}.
\]
In fact, we will show a stronger result (see (4.3.1)):
\[
\|f\|_{L^p} \leq C \|f\|_{\dot{H}^s} := \left( \int |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.
\]
In particular, for \( s = 1 \), this gives another proof of the fact that
\[
q = \frac{2d}{d-2} \Rightarrow \|f\|_{L^q} \leq C \|\nabla f\|_{L^2}.
\]

Proof. We will show that there is a constant \( C \) such as, for any \( f \in S(\mathbb{R}^d) \), we have
\[
(4.3.1) \quad p = \frac{2d}{d-2s} \Rightarrow \|f\|_{L^p} \leq C \|f\|_{H^s} := \left( \int |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.
\]
This is a stronger result than the one stated. Indeed, if \( p < 2d/(d-2s) \) then there is \( s' \in [0,s) \) such that \( p = 2d/(d-2s') \) and hence
\[
\|f\|_{L^p} \leq C \|f\|_{\dot{H}^{s'}} \leq C \|f\|_{H^s}.
\]
(A word of caution: one cannot bound \( \|f\|_{\dot{H}^{s'}} \) by \( \|f\|_{\dot{H}^s} \) because we do not have \( |\xi|^{2s'} \leq |\xi|^{2s} \) for \( |\xi| \leq 1 \).)

We use the proof of Chemin and Xu which is based on the estimate of level sets. We will denote by \( \{|f| > \lambda\} \) the set \( \{x \in \mathbb{R}^d : |f(x)| > \lambda\} \) and \( |\{|f| > \lambda\}| \) the Lebesgue measure of this set.

Let us consider a function \( f \in S(\mathbb{R}^d) \). We can assume without loss of generality that \( \|f\|_{H^s} = 1 \). We start from the classical identity
\[
\|f\|_{L^p}^p = p \int_0^{\infty} \lambda^{p-1} |\{|f| > \lambda\}| \, d\lambda.
\]

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To estimate \(|\{|f| > \lambda\}|\), we will use a decomposition in terms of low and high frequencies. For any \(\lambda > 0\), we will decompose \(f\) into the form

\[
f = g_\lambda + h_\lambda
\]

where, for a certain constant \(A_\lambda\) to be determined,

\[
\hat{g}_\lambda(\xi) = \hat{f}(\xi) \quad \text{if} \quad |\xi| \leq A_\lambda, \quad \hat{g}_\lambda(\xi) = 0 \quad \text{if} \quad |\xi| > A_\lambda
\]

\[
\hat{h}_\lambda(\xi) = 0 \quad \text{if} \quad |\xi| \leq A_\lambda, \quad \hat{h}_\lambda(\xi) = \hat{f}(\xi) \quad \text{if} \quad |\xi| > A_\lambda.
\]

So, according to the triangular inequality,

\[
\{|f| > \lambda\} \subset \{|g_\lambda| > \lambda/2\} \cup \{|h_\lambda| > \lambda/2\}.
\]

We will choose the constant \(A_\lambda\) so that \(\{|g_\lambda| > \lambda/2\} = \emptyset\). Then we will have

\[
\{|f| > \lambda\| \leq \{|h_\lambda| > \lambda/2\| \leq \frac{4}{\lambda^2} \|h_\lambda\|^2_{L^2},
\]

because

\[
\|h_\lambda\|^2_{L^2} \geq \int_{\{|h_\lambda| > \lambda/2\}} |h_\lambda|^2 \, dx \geq \frac{\lambda^2}{4} \{|h_\lambda| > \lambda/2\|.
\]

Combining the above observations, we conclude

\[
(4.3.2) \quad \|f\|_{L^p}^p \leq 4p \int_0^{+\infty} x^{p-3} \|h_\lambda\|^2_{L^2} \, d\lambda.
\]

**Choice of \(A_\lambda\).** According to the Fourier inversion theorem, we have

\[
|g_\lambda(x)| = \left| \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \hat{g}_\lambda(\xi) \, d\xi \right| = \left| \frac{1}{(2\pi)^n} \int_{|\xi| \leq A_\lambda} e^{ix\cdot\xi} \hat{f}(\xi) \, d\xi \right|.
\]

As \(2s < d\), we can use the Cauchy-Schwarz inequality and write that

\[
|g_\lambda(x)| \leq \frac{1}{(2\pi)^n} \left( \int_{|\xi| \leq A_\lambda} |\xi|^{-2s} \, d\xi \right)^{\frac{1}{2}} \left( \int |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.
\]

If we switch to polar coordinates, we obtain

\[
\int_{|\xi| \leq A_\lambda} |\xi|^{-2s} \, d\xi = \int_0^{A_\lambda} \int_{S^{d-1}} r^{d-1-2s} \, d\theta \, dr = \frac{|S^{d-1}| A_\lambda^{d-2s}}{d - 2s}.
\]

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As \( \|f\|_{H^s} = 1 \) by assumption, we finally get
\[
\|g_\lambda\|_{L^\infty} \leq C_1(s,d)A^\frac{d}{2} - s.
\]
We then define \( A_\lambda \) by
\[
C_1(s,d)A^\frac{d}{2} - s = \frac{\lambda}{2}.
\]
So \( \|g_\lambda\|_{L^\infty} \leq \lambda/2 \). Since \( g_\lambda \) is a continuous function (it is the Fourier transform of an integrable function), we deduce that \( \{|g_\lambda| > \lambda/2\} = \emptyset \), which is the desired result.

End of the proof. By definition of \( h_\lambda \), using the identity (4.3.2) and Plancherel’s formula, we find
\[
\|f\|_{L^p}^p \leq 4p(2\pi)^d \int_0^{+\infty} \int_{|\xi| \geq A_\lambda} \lambda^{p-3} |\hat{f}(\xi)|^2 d\xi d\lambda.
\]
By definition of \( A_\lambda \), if \( |\xi| \geq A_\lambda \) then
\[
\lambda \leq \Lambda(\xi) := 2C_1(s,d) |\xi|^{\frac{d}{2} - s},
\]
so, using Fubini’s theorem, it comes
\[
\|f\|_{L^p}^p \leq 4p(2\pi)^d \int_{\mathbb{R}^d} \left( \int_0^{\Lambda(\xi)} \lambda^{p-3} d\lambda \right) |\hat{f}(\xi)|^2 d\xi,
\]
from where
\[
\|f\|_{L^p}^p \leq C_2(s,d) \int_{\mathbb{R}^d} \Lambda(\xi)^{p-2} |\hat{f}(\xi)|^2 d\xi.
\]
As \( \frac{d}{2} - s = \frac{d}{p} \), we have
\[
\Lambda(\xi) \leq C_1(s,d) |\xi|^{\frac{d}{p}}.
\]
We finally get
\[
\|f\|_{L^p}^p \leq C_3(s,d) \int_{\mathbb{R}^d} |\xi|^{\frac{d(g-2)}{p}} |\hat{f}(\xi)|^2 d\xi,
\]
which is the desired result. \( \square \)

**Corollary 4.3.3** (Sobolev embeddings). Let \( d \geq 1 \) and \( p \in (1,d) \). Define \( p^* \) by
\[
\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}.
\]
Then there exists a constant $C$ such that, for any function $f \in C_0^\infty(\mathbb{R}^d)$,
\[ \|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^d)}. \]

Proof. Here we use the following identity
\[ f(x) = -\frac{1}{|S^{d-1}|} \int_{\mathbb{R}^d} (x-y) \cdot \nabla f(y) \frac{dy}{|x-y|^d}. \]
It follows that
\[ |f| \leq \frac{1}{|S^{d-1}|} f_1(|\nabla f|), \]
and hence the wanted inequality follows directly from the Hardy-Littlewood-Sobolev inequality. \qed

Theorem 4.3.4 (Inequality of Brué and Nguyen). Consider an integer $d \geq 1$ and two real numbers $s \in (0, 1)$ and $p \in (0, d/s)$, then set
\[ p^* = \frac{dp}{d - sp}. \]
There exists a constant $C$ such that for all $f$ in $C^1_b(\mathbb{R}^d)$ and all $x$ in $\mathbb{R}^d$ we have
\[ |f(x)|^{p^*} \leq C \|f\|^{p^* - p} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{d+ps}} \frac{dy}{d}. \]
It follows that
\[ \|f\|_{L^{p^*}} \leq C \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{d+ps}} \frac{dy \cdot dx}{d} \right)^\frac{1}{p}. \]

Proof. We denote by $C$ several constants that do not have to be depend on $d, p$ or $s$ and whose values can change from one line to another.

Step 1. We first check that the integral
\[ \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{d+ps}} \frac{dy}{d} \]
is well defined for all $f$ in $S(\mathbb{R}^d)$ and all $x$ in $\mathbb{R}^d$. To do so, we cut the integral on $\mathbb{R}^d$ in two parts: the integral on $B(x, 1)$ and that on $\mathbb{R}^d \setminus B(x, 1)$. On $B(x, 1)$, we use the estimate
\[ |f(x) - f(y)| \leq K |x-y| \quad \text{with} \quad K = \sup_{m \in \mathbb{R}^d} |\nabla f(m)|, \]
while on $\mathbb{R}^d \setminus B(x, 1)$ one writes $|f(x) - f(y)| \leq 2 \sup |f|$.

**Step 2.** Let us fix $x \in \mathbb{R}^d$ and a real $t > 0$. We denote by $C_t$ the annulus

$$C_t = B(0, 2t) - B(0, t) = \{ y \in \mathbb{R}^d ; t \leq |y| < 2t \},$$

and we denote by $|C_t|$ its Lebesgue measure.

Then

$$|f(x)|^p = \frac{1}{|C_t|} \int_{C_t} |f(x)|^p \, dh \leq \frac{1}{|C_t|} \int_{C_t} (|f(x + h) - f(h)| + |f(x + h)|)^p \, dh.$$ 

Since

$$|a + b|^p \leq (2 \max\{|a|, |b|\})^p \leq 2^p(|a|^p + |b|^p),$$

we deduce that

$$|f(x)|^p \leq \frac{2^p}{|C_t|} \int_{C_t} |f(x + h) - f(x)|^p \, dh + \frac{2^p}{|C_t|} \int_{C_t} |f(x + h)|^p \, dh.$$
We will show that

\[(4.3.3) \quad I(\lambda) \leq C \int_{\mathbb{R}^d} \frac{|f(x + h) - f(x)|^p}{|h|^{d+sp}} \, dh \]

which will imply the desired inequality.

Note that $|C_t| \sim t^d$. Moreover, on $C_t$ we have $t \sim |h|$. So there exists $C$ such that, if $t \in [2^{\ell}, 2^{\ell+1}]$ with $\ell \in \mathbb{Z}$ then

\[
\frac{1}{t^{1+sp}} \frac{1}{|C_t|} \int_{C_t} |f(x + h) - f(x)|^p \, dh \leq \frac{1}{\ell} \int_{C_t} \frac{|f(x + h) - f(x)|^p}{|h|^{n+sp}} \, dh \\
\leq \frac{1}{2^\ell} \int_{B(0,2^{\ell+2}) \setminus B(0,2^\ell)} \frac{|f(x + h) - f(x)|^p}{|h|^{n+sp}} \, dh.
\]

We cut the integral $I(\lambda)$ into integrals on $[2^\ell, 2^{\ell+1}]$ for $\ell \in \mathbb{Z}$. As one integrates on a size interval $2^{\ell+1} - 2^\ell = 2^\ell$, it comes

\[
I(\lambda) \leq C \sum_{\ell=-\infty}^{+\infty} \int_{B(0,2^{\ell+2}) \setminus B(0,2^\ell)} \frac{|f(x + h) - f(x)|^p}{|h|^{d+sp}} \, dh.
\]

This series is convergent, by telescopic summation, and we check that it satisfies (4.3.3).

**Step 5.** Multiplying the two members of the inequality in the previous question by $\lambda^{sp}$, we find

\[
|f(x)|^p \leq C \|f\|_{L^p}^p \lambda^{sp-d} + C \lambda^{sp} \int_{\mathbb{R}^d} \frac{|f(x + h) - f(x)|^p}{|x - y|^{d+ps}} \, dh.
\]

We then choose $\lambda$ such that

\[
\|f\|_{L^p}^p \lambda^{sp-d} = \lambda^{sp} \int_{\mathbb{R}^d} \frac{|f(x + h) - f(x)|^p}{|x - y|^{d+ps}} \, dh
\]

and the desired inequality is inferred.
Part III

Study of the Cauchy problem
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The Dirichlet-to-Neumann operator
Chapter 6

The water-wave problem
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The Cauchy problem for the Hele-Shaw and Muskat equations
Chapter 8

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