Controllability and Stabilization of water waves

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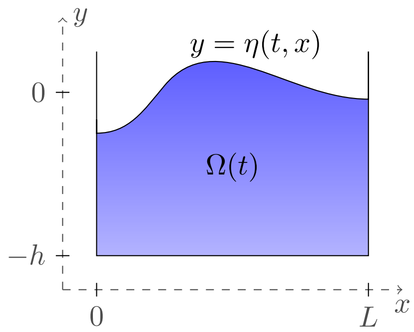
Partly based on a joint work with Pietro Baldi and Daniel Han-Kwan.

1. Introduction

Consider the incompressible Euler equations for a potential flow in a fluid domain located between with a free surface, two vertical walls and a flat bottom, which is at time t of the form

$$\Omega(t) = \{ (x, y) \in [0, L] \times \mathbb{R} : -h < y < \eta(t, x) \},$$

where L is the length of the basin, h is its depth and η, the free surface elevation, is an unknown function.



The Eulerian velocity field v is assumed to be irrotational. It follows that $v = \nabla_{x,y}\phi$ for some time-dependent potential ϕ satisfying

$$\Delta_{x,y}\phi = 0, \quad \partial_t\phi + \frac{1}{2} |\nabla_{x,y}\phi|^2 + P + gy = 0, \tag{1.1}$$

where $g > 0$ is the gravity acceleration, P is the pressure, $\nabla_{x,y} = (\partial_x, \partial_y)$ and $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. In addition we assume that the velocity satisfies the solid wall boundary condition on the bottom and the lateral walls:

$$\begin{aligned} &\partial_y\phi = 0 \quad \text{on } y = -h, \\ &\partial_x\phi = 0 \quad \text{on } x = 0 \text{ or } x = L, \end{aligned} \tag{1.2}$$

The water waves equations are then given by two boundary conditions on the free surface: the classical kinematic boundary condition, describing the deformations of the domain,

$$\partial_t\eta = \sqrt{1 + (\partial_x\eta)^2}\, \partial_n\phi|_{y=\eta},$$

where ∂_n is the outward normal derivative, so $\sqrt{1 + (\partial_x\eta)^2}\, \partial_n\phi = \partial_y\phi - (\partial_x\eta)\partial_x\phi$. Secondly, the balance of forces across the free surface reads

$$P|_{y=\eta(t,x)} = P_{\text{ext}}(t, x) - \kappa H(\eta),$$

where κ is a positive constant, P_{ext} is an external source term and $H(\eta)$ is the curvature of the free surface:

$$H(\eta) := \partial_x\left(\frac{\partial_x\eta}{\sqrt{1 + (\partial_x\eta)^2}} \right) = \frac{\partial_x^2\eta}{(1 + (\partial_x\eta)^2)^{3/2}}.$$

In this paper, we present two results which are concerned with the generation and the absorption of water waves by means of pressure disturbances applied above a small portion of the free surface.

Controllability. The first question is the following : given

- a time $T > 0$,
- an initial state (η_{in}, ϕ_{in}) and a final state $(\eta_{final}, \phi_{final})$ in some space,
- a small subset $\omega = (a, b) \subset [0, L]$,

is-it possible to find a function $P_{ext}(t, x)$ supported in $[0, T] \times \omega$ such that the solution to the water-wave equations with initial data (η_{in}, ϕ_{in}) satisfies $(\eta, \phi)|_{t=T} = (\eta_{final}, \phi_{final})$?

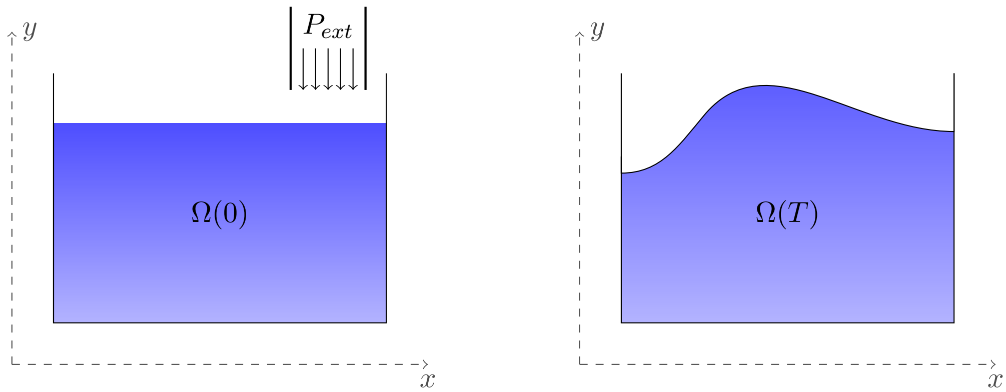


FIGURE 1. Generation of a water wave in time T starting from a fluid at rest, which means that $(\eta_{in}, \phi_{in}) = (0, 0)$.

There are many results about the controllability or the stabilization of linear or nonlinear equations describing water waves in some asymptotic regimes like Benjamin-Ono, KdV or nonlinear Schrödinger equations (see the Coron's book [18]). However, one cannot easily adapt these studies to the water-wave system since it is quasi-linear (instead of semi-linear) and since it is a pseudo-differential system, involving the Dirichlet-Neumann operator which is nonlocal and also depends nonlinearly on the unknown. The first results about the possible applications of control theory to the water-wave equations are due to Reid and Russell [41] and Reid [39, 40], who studied the linearized equations at the origin (see also Miller [38], Lissy [34] and Biccari [13] for other control results about dispersive equations involving a fractional Laplacian). In this survey paper, we present a result from Alazard, Baldi and Han-Kwan ([5]) about the controllability of the *nonlinear* water-wave problem.

Stabilization. Think of a rectangular wave basin, having vertical walls, equipped with a wave-maker at one extremity. The waves generated by the wave-maker will be reflected at the opposite side and then will interact with the wave produced by the wave-maker. Consequently, to simulate experimentally the open sea propagation, one has to introduce wave absorbers to minimize wave reflection. The same problem appears for the numerical analysis of the water-wave equations. Indeed, for computational reasons, one has to work in a bounded domain. To simulate propagation in an unbounded domain like the open sea, one can use either artificial boundary conditions (cf [27, 43, 29, 28]) or one can damp outgoing waves in an absorbing zone surrounding the computational boundary (see [27, 43]). For the water-wave equations, the idea of using the latter method goes back to Le Méhauté [36] in 1972. This approach is used in many numerical studies ([26, 17, 24, 14, 16, 25]). The mathematical study of the damping properties of these absorbers corresponds to the question of the stabilization of the water-wave equations.

To state this problem, let us introduce the energy $\mathcal{E} = \mathcal{E}(t)$, defined by

$$\mathcal{E} = \frac{g}{2}\int_0^L \eta^2 \,\mathrm{d}x + \kappa \int_0^L \left(\sqrt{1+\eta_x^2} - 1\right) \mathrm{d}x + \frac{1}{2}\int_0^L \int_{-h}^{\eta(t,x)} |\nabla_{x,y}\phi|^2 \,\mathrm{d}y \,\mathrm{d}x. \tag{1.3}$$

This is the sum of the gravitational potential energy, a surface energy due to stretching of the surface and the kinetic energy. Recall that the energy is conserved when there is no external pressure, which means that if $P_{ext} = 0$ then $\mathcal{E}(t) = \mathcal{E}(0)$ for all time. The stabilization problem for the water-wave equations consists in finding a pressure law, relating P_{ext} to the unknown (η, ψ), such that:

i) $\mathcal{E}$ is decreasing and converges to zero;
ii) $\partial_x P_{ext}$ is supported inside a small subset of $[0, L]$.

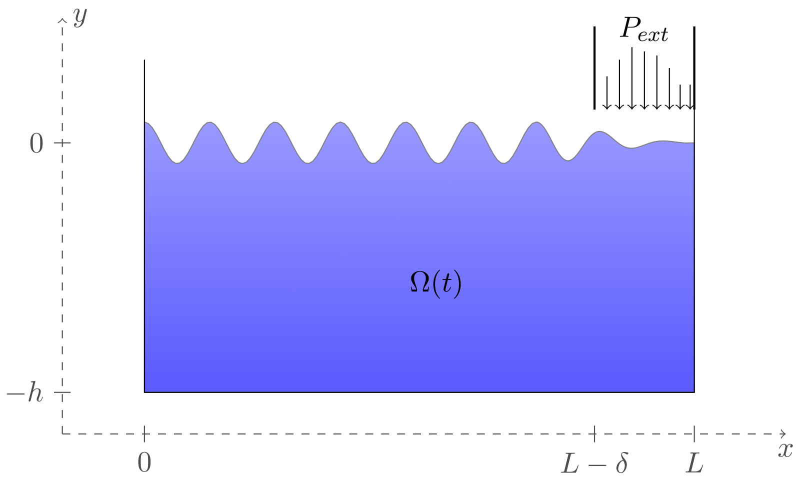


FIGURE 2. Absorption of water waves in the neighborhood of $x = L$ by means of an external counteracting pressure produced by blowing above the free surface.

2. Controllability

We begin by discussing the controllability problem.

Instead of working in a domain with lateral boundaries, we can reduce the study to a periodic problem in x. To do so, following a classical idea, we use a reflection/periodization procedure with respect to the normal variable to the boundary of the tank, as illustrated below (this raises some difficult questions about the regularity of the solutions that are studied in [7] for the case without surface tension).

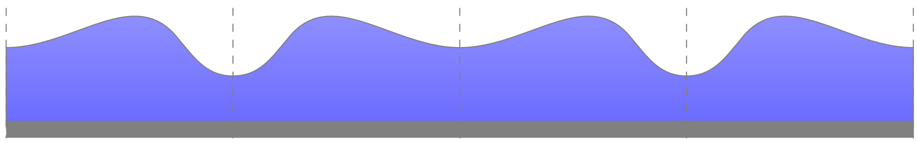


FIGURE 3. Periodization of the domain (see Boussinesq [11]).

Secondly, we use the so-called Craig-Sulem-Zakharov formulation (see [44, 21] and also [8]). This is a popular formulation of the water-wave problem seen as a system on the free surface. To obtain this system, one first notices that the velocity potential ϕ is harmonic and satisfies a Neumann boundary condition on the walls and the bottom. Consequently ϕ is fully determined by its evaluation at the free surface. Set

$$\psi(t, x) := \phi(t, x, \eta(t, x)).$$

Then one introduces the Dirichlet to Neumann operator, denoted by $G(\eta)$, relating ψ to the normal derivative of the potential by

$$(G(\eta)\psi)(t, x) = \sqrt{1 + (\partial_x \eta)^2}\, \partial_n \phi|_{y=\eta(t,x)}.$$

Then (η, ψ) solves

$$\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g\eta + \dfrac{1}{2}(\partial_x \psi)^2 - \dfrac{1}{2}\dfrac{\big(G(\eta)\psi + (\partial_x \eta)(\partial_x \psi)\big)^2}{1 + (\partial_x \eta)^2} = \kappa H(\eta) - P_{\text{ext}}. \end{cases} \tag{2.1}$$

This system is augmented with initial data

$$\eta|_{t=0} = \eta_{in}, \quad \psi|_{t=0} = \psi_{in}. \tag{2.2}$$

We consider the case when η and ψ are 2π-periodic in the space variable x and we set $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. Recall that the mean value of η is conserved in time and can be taken to be 0 without loss of generality. We thus introduce the Sobolev spaces $H^\sigma_0(\mathbb{T})$ of functions with mean value 0. Our main result asserts that, given any control domain ω and any arbitrary control time $T > 0$, the equation (2.1) is controllable in time T for small enough data.

The following result asserts that, in arbitrarily small time, one can generate any small amplitude, two-dimensional, gravity-capillary water waves.

Theorem 2.1 (Alazard, Baldi, Han-Kwan, from [5]). *Assume that $\kappa > 0$. Let $T > 0$ and consider a non-empty open subset $\omega \subset \mathbb{T}$. There exist σ large enough and a positive constant M_0 small enough such that, for any two pairs of functions (η_{in}, ψ_{in}), $(\eta_{final}, \psi_{final})$ in $H_0^{\sigma+\frac{1}{2}}(\mathbb{T}) \times H^\sigma(\mathbb{T})$ satisfying*

$$\|\eta_{in}\|_{H^{\sigma+\frac{1}{2}}} + \|\psi_{in}\|_{H^\sigma} < M_0, \quad \|\eta_{final}\|_{H^{\sigma+\frac{1}{2}}} + \|\psi_{final}\|_{H^\sigma} < M_0,$$

there exists P_{ext} in $C^0([0,T]; H^\sigma(\mathbb{T}))$, supported in $[0,T] \times \omega$, that is

$$\operatorname{supp} P_{ext}(t, \cdot) \subset \omega, \quad \forall t \in [0,T],$$

such that the Cauchy problem (2.1)-(2.2) has a unique solution

$$(\eta, \psi) \in C^0([0,T]; H_0^{\sigma+\frac{1}{2}}(\mathbb{T}) \times H^\sigma(\mathbb{T})),$$

and the solution (η, ψ) satisfies $(\eta|_{t=T}, \psi|_{t=T}) = (\eta_{final}, \psi_{final})$.

Remark 2.2. *i*) This result holds for any $T > 0$ and not only for T large enough. Compared to the Cauchy problem, for the control problem it is more difficult to work on short time intervals than on large time intervals.

ii) This result holds also in the infinite depth case.

In the rest of this section, our goal is to sketch the proof of the controllability result.

Step 1: Reduction to a dispersive equation. The proof is based on the fact that the water waves equation is a dispersive equation. To explain this, we begin by considering the linearized equation around the null solution. Recall that $G(0)$ is the Fourier multiplier $|D_x| \tanh(h\,|D_x|)$. Removing quadratic and higher order terms in the equations, System (2.1) becomes

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi + g\eta - \kappa \partial_x^2 \eta = P_{\text{ext}}. \end{cases}$$

Introduce the Fourier multiplier (of order 3/2)

$$L := \big((g - \kappa\partial_x^2)G(0)\big)^{\frac{1}{2}}.$$

The operator $G(0)^{-1}$ is well-defined on periodic functions with mean value zero. Then $u = \psi - iLG(0)^{-1}\eta$ satisfies the dispersive equation

$$\partial_t u + iLu = P_{ext}.$$

The first step of the proof is to obtain a similar equation starting with the nonlinear water-wave equations. To do so, we use the Eulerian formulation of the water waves equations, following [31, 9, 6]. More precisely, we use a paradifferential approach in order to paralinearize the water waves equations and then to symmetrize the obtained equations. We denote below by T_a the paradifferential operator with symbol $a = a(x,\xi)$.

It is proved in [6] that there are symbols $p = p(t,x,\xi)$ and $q = q(t,x,\xi)$ with p of order 0 in ξ and q of order 1/2, such that $u = T_p\psi + iT_q\eta$ satisfies an equation of the form

$$P(u)u = P_{ext} \quad \text{with} \quad P(u) := \partial_t + T_{V(u)}\partial_x + iL^{\frac{1}{2}}\big(T_{c(u)}L^{\frac{1}{2}}\,\cdot\,\big),$$

where $L^{\frac{1}{2}} = \big((g-\kappa\partial_x^2)G(0)\big)^{\frac{1}{4}}$, $T_{V(u)}$ and $T_{c(u)}$ are paraproducts. Here V, c depend on the unknown u with $V(0) = 0$ and $c(0) = 1$, and hence $P(0) = \partial_t + iL$ is the linearized operator around the null solution. Furthermore, the mapping $(\eta,\psi) \mapsto u$ is invertible (under a smallness assumption) and, up to modifying the sub-principal symbols of p and q, one can further require that

$$\int_{\mathbb{T}} \operatorname{Im} u(t,x)\,\mathrm{d}t = 0. \tag{2.3}$$

Here we have simplified the result (neglecting remainder terms and simplifying the dependence of V, c on u) and we refer the original article [5] for the full statement.

Step 2: Quasi-linear scheme. Since the water waves system (2.1) is quasi-linear, one cannot deduce the controllability of the nonlinear equation from the one of $P(0)$. Instead of using a fixed point argument, we use a quasi-linear scheme and seek P_{ext} as the limit of *real-valued* functions P_n determined by means of approximate control problems. To guarantee that P_{ext} will be real-valued we seek P_n as the real part of some function. To insure that $\operatorname{supp} P_n \subset \omega$ we seek P_n under the form

$$P_n = \chi_\omega \operatorname{Re} f_n.$$

Hereafter, we fix ω, a non-empty open subset of $\mathbb{T}$, and a C^∞ cut-off function χ_ω, supported on ω, such that $\chi_\omega(x) = 1$ for all x in some open interval $\omega_1 \subset \omega$.

The approximate control problems are defined by induction as follows: we choose f_{n+1} by requiring that the unique solution u_{n+1} of the Cauchy problem

$$P(u_n)u_{n+1} = \chi_\omega \operatorname{Re} f_{n+1}, \quad u_{n+1}|_{t=0} = u_{in}$$

satisfies $u(T) = u_{final}$. Our goal is to prove that

- this scheme is well-defined (that is one has to prove a controllability result for $P(u_n)$);
- the sequences (f_n) and (u_n) are bounded in $C^0([0,T];H^\sigma(\mathbb{T}))$;
- the series $\sum(f_{n+1} - f_n)$ and $\sum(u_{n+1} - u_n)$ converge in $C^0([0,T];H^{\sigma-\frac{3}{2}}(\mathbb{T}))$.

It follows that (f_n) and (u_n) are Cauchy sequences in $C^0([0,T];H^{\sigma-\frac{3}{2}}(\mathbb{T}))$ (and in fact, by interpolation, in $C^0([0,T];H^{\sigma'}(\mathbb{T}))$ for any $\sigma' < \sigma$).

To use the quasi-linear scheme, we need to study a sequence of linear approximate control problems. The key point is to study the control problem for the linear operator $P(\underline{u})$ for some given function $\underline{u}$. Our goal is to prove the following result.

Proposition 2.3. *Let $T > 0$. There exists s_0 such that, if $\|\underline{u}\|_{C^0([0,T];H^{s_0})}$ is small enough, depending on T, then the following properties hold.*

i) (Controllability) For all $\sigma \geq s_0$ and all

$$u_{in}, u_{final} \in \tilde{H}^\sigma(\mathbb{T}) := \left\{ w \in H^\sigma(\mathbb{T})\,;\, \operatorname{Im}\int_{\mathbb{T}} w(x)\,\mathrm{d}x = 0 \right\},$$

there exists f satisfying $\|f\|_{C^0([0,T];H^\sigma)} \leq K(T)(\|u_{in}\|_{H^\sigma} + \|u_{final}\|_{H^\sigma})$ such that the unique solution u to

$$P(\underline{u})u = \chi_\omega \operatorname{Re} f \quad ; \; u_{|t=0} = u_{in},$$

satisfies $u(T) = u_{final}$.

ii) (Stability) Consider another state $\underline{u}'$ *with* $\|\underline{u}'\|_{C^0([0,T];H^{s_0})}$ *small enough and denote by* f' *the control associated to* $\underline{u}'$*. Then*

$$\left\|f - f'\right\|_{C^0([0,T];H^{\sigma-\frac{3}{2}})} \le K'(T)(\|u_{in}\|_{H^\sigma} + \|u_{final}\|_{H^\sigma}) \left\|\underline{u} - \underline{u}'\right\|_{C^0([0,T];H^{s_0})}.$$

Again, we simplified the assumptions and refer the reader to the original article for the full statement.

Step 3: Reduction to a regularized problem. We next reduce the analysis by proving that it is sufficient

- to consider a *classical* equation instead of a *paradifferential* equation;
- to prove a L^2-result instead of a Sobolev-result.

This is obtained by commuting $P(\underline{u})$ with some well-chosen elliptic operator $\Lambda_{h,s}$ of order s with

$$s = \sigma - \frac{3}{2}$$

and depending on a small parameter denoted by $\hbar$ to avoid confusion with the depth h (the reason to introduce $\hbar$ is explained below). In particular $\Lambda_{\hbar,s}$ is chosen so that the operator

$$\widetilde{P}(\underline{u}) := \Lambda_{\hbar,s} P(\underline{u}) \Lambda_{\hbar,s}^{-1}$$

satisfies

$$\widetilde{P}(\underline{u}) = P(\underline{u}) + R(\underline{u}) \tag{2.4}$$

where $R(\underline{u})$ is a remainder term of order 0. For instance, if $s = 3m$ with $m \in \mathbb{N}$, set

$$\Lambda_{\hbar,s} = I + \hbar^s \mathcal{L}^{\frac{2s}{3}} \quad \text{where } \mathcal{L} := L^{\frac{1}{2}}\big(T_c L^{\frac{1}{2}}\,\cdot\big).$$

With this choice one has $[\Lambda_{\hbar,s}, \mathcal{L}] = 0$ so (2.4) holds with $R(\underline{u}) = [\Lambda_{\hbar,s}, T_{V(\underline{u})}]\Lambda_{\hbar,s}^{-1}$. It follows from symbolic calculus that $\|R(\underline{u})\|_{\mathcal{L}(L^2)} \lesssim \|V\|_{W^{1,\infty}}$ uniformly in $\hbar$.

Moreover, since $V(\underline{u})$ and $c(\underline{u})$ are continuous in time with values in $H^{s_0}(\mathbb{T})$ with s_0 large, one can replace paraproducts by usual products, up to remainder terms in $C^0([0,T];\mathcal{L}(L^2))$. We have

$$\widetilde{P}(\underline{u}) = \partial_t + V(\underline{u})\partial_x + iL^{\frac{1}{2}}\big(c(\underline{u})L^{\frac{1}{2}}\,\cdot\big) + R_2(\underline{u})$$

where

$$R_2(\underline{u}) := R(\underline{u}) + \big(T_{V(\underline{u})} - V(\underline{u})\big)\partial_x + iL^{\frac{1}{2}}\big(\big(T_{c(\underline{u})} - c(\underline{u})\big)L^{\frac{1}{2}}\,\cdot\big).$$

The remainder $R_2(\underline{u})$ belongs to $C^0([0,T];\mathcal{L}(L^2))$ uniformly in $\hbar$. On the other hand,

$$\left\|[\Lambda_{\hbar,s}, \chi_\omega]\Lambda_{\hbar,s}^{-1}\right\|_{\mathcal{L}(L^2)} = O(h), \tag{2.5}$$

which is the reason to introduce the parameter $\hbar$. The key point is that one can reduce the proof of Proposition 2.3 to the proof of the following result.

Proposition 2.4. *Let* $T > 0$*. There exists* s_0 *such that, if* $\|\underline{u}\|_{C^0([0,T];H^{s_0})}$ *is small enough, then the following properties hold.*

i) (Controllability) For all $v_{in} \in L^2(\mathbb{T})$ *there exists* f *with* $\|f\|_{C^0([0,T];L^2)} \le K(T)\|v_{in}\|_{L^2}$ *such that the unique solution* v *to* $\widetilde{P}(\underline{u})v = \chi_\omega \operatorname{Re} f$, $\quad v_{|t=0} = v_{in}$ *is such that* $v(T)$ *is an imaginary constant:*

$$\exists b \in \mathbb{R} \,/\, \forall x \in \mathbb{T}, \quad v(T,x) = ib.$$

ii) (Regularity) Moreover $\|f\|_{C^0([0,T];H^{\frac{3}{2}})} \le K(T)\|v_{in}\|_{H^{\frac{3}{2}}}$.

iii) (Stability) Consider another state $\underline{u}'$ *with* $\|\underline{u}'\|_{C^0([0,T];H^{s_0})}$ *small enough and denote by* f' *the control associated to* $\underline{u}'$*. Then*

$$\left\|f - f'\right\|_{C^0([0,T];L^2)} \le K'(T)\|v_{in}\|_{H^{\frac{3}{2}}} \left\|\underline{u} - \underline{u}'\right\|_{C^0([0,T];H^{s_0})}.$$

Let us explain how to deduce Proposition 2.3 from the latter proposition. Consider u_{in}, u_{final} in $\tilde{H}^\sigma(\mathbb{T})$ and seek $f \in C^0([0,T]; H^\sigma(\mathbb{T}))$ such that

$$P(\underline{u})u = \chi_\omega \operatorname{Re} f, \quad u(0) = u_{in} \quad \Longrightarrow \quad u(T) = u_{final}.$$

Since the equation is reversible in time, it is sufficient to consider the case where $u_{final} = 0$. Now, to deduce this result from Proposition 2.4, the main difficulty is that the conjugation with $\Lambda_{\hbar,s}$ introduces a nonlocal term: indeed, $\Lambda_{\hbar,s}^{-1}(\chi_\omega f)$ is not compactly supported in general. This is a possible source of difficulty since we seek a localized control term. We overcome this problem by considering the control problem for $\widetilde{P}(\underline{u})$ associated to some well-chosen initial data v_{in}. Proposition 2.4 asserts that for all $v_{in} \in H^{\frac{3}{2}}(\mathbb{T})$ there is $\widetilde{f} \in C^0([0,T]; H^{\frac{3}{2}}(\mathbb{T}))$ such that

$$\widetilde{P}(\underline{u})v_1 = \chi_\omega \operatorname{Re} \widetilde{f}, \quad v_1|_{t=0} = v_{in} \Longrightarrow v_1(T,x) = ib, \;\; b \in \mathbb{R}.$$

Define $\mathcal{K} v_{in} = v_2(0)$ where v_2 is the solution to

$$\widetilde{P}(\underline{u})v_2 = \big[\Lambda_{\hbar,s}, \chi_\omega\big]\Lambda_{\hbar,s}^{-1} \operatorname{Re} \widetilde{f}, \quad v_2|_{t=T} = 0.$$

Using (2.5) one can prove that the $\mathcal{L}(H^{\frac{3}{2}})$-norm of $\mathcal{K}$ is $O(\hbar)$ and hence $I + \mathcal{K}$ is invertible for $\hbar$ small. So, v_{in} can be so chosen that $v_{in} + \mathcal{K}v_{in} = \Lambda_{\hbar,s} u_{in}$. Then, setting $f := \Lambda_{\hbar,s}^{-1}\widetilde{f}$ and $u := \Lambda_{\hbar,s}^{-1}(v_1 + v_2)$, one checks that

$$P(\underline{u})u = \chi_\omega \operatorname{Re} f, \quad u(0) = u_{in}, \quad u(T,x) = ib, \;\; b \in \mathbb{R}.$$

It remains to prove that $u(T)$ is not only an imaginary constant, but it is 0. This follows from the property (2.3). Indeed, P can be so defined that if $P(\underline{u})u$ is a real-valued function, then $\frac{d}{dt}\int_{\mathbb{T}} \operatorname{Im} u(t,x)\, dx = 0$. Since $\int_{\mathbb{T}} \operatorname{Im} u(0,x)\, dx = 0$ by assumption, one deduces that $\int_{\mathbb{T}} \operatorname{Im} u(T,x)\, dx = 0$ and hence $u(T) = 0$.

Step 4: Reduction to a constant coefficient equation. The controllability of $\widetilde{P}(\underline{u})$ will be deduced from the classical HUM method. A key step in the HUM method consists in proving that some bilinear mapping is coercive. To determine the appropriate bilinear mapping, we follow an idea introduced in [4] and conjugate $\widetilde{P}(\underline{u})$ to a constant coefficient operator modulo a remainder term of order 0.

To do so, we use a change of variables and a pseudo-differential change of unknowns to find an operator $M(\underline{u})$ such that

$$M(\underline{u})\widetilde{P}(\underline{u})M(\underline{u})^{-1} = \partial_t + iL + \mathcal{R}(\underline{u}),$$

where $\|\mathcal{R}(\underline{u})\|_{\mathcal{L}(L^2)} \lesssim \|\underline{u}\|_{H^{s_0}}$ (and hence $\mathcal{R}(\underline{u})$ is a *small* perturbation of order 0).

To find $M(\underline{u})$, we begin by considering three changes of variables of the form

$$(1 + \partial_x\kappa(t,x))^{\frac{1}{2}}\, h(t, x + \kappa(t,x)), \quad h(a(t), x), \quad h(t, x - b(t)),$$

to replace $\widetilde{P}(\underline{u})$ with

$$Q(\underline{u}) = \partial_t + W\partial_x + iL + R_3, \tag{2.6}$$

where $W = W(t,x)$ satisfies $\int_{\mathbb{T}} W(t,x)\, dx = 0$, $\|W\|_{C^0([0,T];H^{s_0-d})} \lesssim \|\underline{u}\|_{C^0([0,T];H^{s_0})}$ where $d > 0$ is a universal constant, and R_3 is of order zero. This is not trivial since the equation is nonlocal and also because this exhibits a cancellation of a term of order $1/2$. Indeed, in general the conjugation of $L^{\frac{1}{2}}\big(c(\underline{u})L^{\frac{1}{2}}\,\cdot\big)$ and a change of variables generates also a term of order $3/2 - 1$. This term disappears here since we consider transformations which preserve the $L^2(dx)$ scalar product.

We next seek an operator A such that $i\big[A, |D_x|^{\frac{3}{2}}\big] + W\partial_x A$ is a zero order operator. This leads to consider, following [4], a pseudo-differential operator $A = \operatorname{Op}(a)$ for some symbol $a = a(x,\xi)$ in the Hörmander class $S^0_{\rho,\rho}$ with $\rho = \frac{1}{2}$, namely $a = \exp(i|\xi|^{\frac{1}{2}}\beta(t,x))$ for some function β depending on W.

Step 5: Observability. Then, we establish an observability inequality. That is, we prove that there exists $\varepsilon > 0$ such that for any initial data v_0 whose mean value $\langle v_0 \rangle = \frac{1}{2\pi} \int_{\mathbb{T}} v_0(x)\, \mathrm{d}x$ satisfies

$$|\mathrm{Re}\langle v_0 \rangle| \geq \frac{1}{2} |\langle v_0 \rangle| - \varepsilon \|v_0\|_{L^2}, \tag{2.7}$$

the solution v of

$$\partial_t v + iLv = 0, \quad v(0) = v_0$$

satisfies

$$\int_0^T \int_\omega |\mathrm{Re}(Av)(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \geq K \int_{\mathbb{T}} |v_0(x)|^2 \,\mathrm{d}x. \tag{2.8}$$

To prove this inequality with the real-part in the left-hand side allows to prove the existence of a real-valued control function.

The observability inequality is deduced using a variant of Ingham's inequality. Recall that Ingham's inequality is an inequality for the L^2-norm of a sum of oscillatory functions which generalizes Parseval's inequality (it applies to pseudo-periodic functions and not only to periodic functions). For example, one such result asserts that for any $T > 0$ there exist two positive constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that

$$C_1 \sum_{n\in\mathbb{Z}} |w_n|^2 \leq \int_0^T \left| \sum_{n\in\mathbb{Z}} w_n e^{in|n|^{\frac{1}{2}}t} \right|^2 \mathrm{d}t \leq C_2 \sum_{n\in\mathbb{Z}} |w_n|^2 \tag{2.9}$$

for all sequences (w_n) in $\ell^2(\mathbb{C})$. The fact that this result holds for any $T > 0$ (and not only for T large enough) is a consequence of a general result due to Kahane on lacunary series (see [30]).

Note that, since the original problem is quasi-linear, we are forced to prove an Ingham type inequality for sums of oscillatory functions whose phases differ from the phase of the linearized equation. For our purposes, we need to consider phases that do not depend linearly on t, of the form

$$\mathrm{sign}(n)\big[\ell(n)^{\frac{3}{2}}t + \beta(t,x)|n|^{\frac{1}{2}}\big], \qquad \ell(n) := \Big((g + |n|^2)\,|n| \tanh(h\,|n|)\Big)^{\frac{1}{2}},$$

where x plays the role of a parameter. Though it is a sub-principal term, to take into account the perturbation $\beta(t,x)|n|^{\frac{1}{2}}$ requires some care since $e^{i\beta(t,x)|n|^{1/2}} - 1$ is not small. In particular we need to prove upper bounds for expressions in which we allow some amplitude depending on time (and whose derivatives in time of order k can grow as $|n|^{k/2}$).

Step 6: HUM method. Inverting A, we deduce from (2.8) an observability result for the adjoint operator $Q(\underline{u})^*$ ($Q(\underline{u})$ is as given by (2.6)). Then the controllability will be deduced from the classical HUM method (we need in fact a version that makes it possible to consider a real-valued control). The idea is that the observability property implies that some bilinear form is coercive and hence the existence of the control follows from the Riesz's theorem and a duality argument. A possible difficulty is that the control P_{ext} is acting only on the equation for ψ. To explain this, consider the case where $(\eta_{final}, \psi_{final}) = (0,0)$. Since the HUM method is based on orthogonality arguments, the fact that the control is not acting on both equations means for our problem that the final state is orthogonal to a co-dimension 1 space. The fact that this final state can be chosen to be 0 will be obtained by choosing this co-dimension 1 space in an appropriate way, introducing an auxiliary function $M = M(x)$ which is chosen later on.

Consider any real function $M = M(x)$ with $M - 1$ small enough, and introduce

$$L^2_M := \left\{ \varphi \in L^2(\mathbb{T};\mathbb{C})\,;\, \mathrm{Im} \int_{\mathbb{T}} M(x)\varphi(x)\,\mathrm{d}x = 0 \right\}.$$

Notice that L^2_M is an $\mathbb{R}$-Hilbert space. Also, for any $v_0 \in L^2_M$, the condition (2.7) holds. Then, using a variant of the HUM method in this space, one deduces that for all $v_{in} \in L^2$

(not necessarily in L^2_M) there is $f \in C^0([0,T];L^2)$ such that, if

$$Q(\underline{u})w = \partial_t w + W\partial_x w + iLw + R_3 w = \chi_\omega \operatorname{Re} f, \quad w(0) = w_{in},$$

then

$$w(T,x) = ibM(x)$$

for some constant $b \in \mathbb{R}$. Now

$$Q(\underline{u}) = \Phi(\underline{u})^{-1}\widetilde{P}(\underline{u})\Phi(\underline{u}),$$

where $\Phi(\underline{u})$ is the composition of the transformations in (2.5). Since $\Phi(\underline{u})$ and $\Phi(\underline{u})^{-1}$ are local operators, one easily deduce a controllability result for $\widetilde{P}(\underline{u})$ from the one proved for $Q(\underline{u})$. Now, choosing $M = \Phi(\underline{u}(T,\cdot))(1)$ where 1 is the constant function 1, we deduce from $w(T,x) = ibM(x)$ that $u(T,x)$ is an imaginary constant, as asserted in statement $i)$ of Proposition 2.3. Concerning M, notice that $M \neq 1$ because of the factor $(1+\partial_x\kappa(t,x))^{\frac{1}{2}}$ multiplying $h(t,x+\kappa(t,x))$ in (2.5).

Step 7: Convergence of the scheme. To prove that the scheme converges, we prove that (f_n) and (u_n) are Cauchy sequences. This is where we need statement $ii)$ in Proposition 2.3, to estimate the difference of two controls associated with different coefficients. To prove this stability estimate we introduce an auxiliary control problem which, loosely speaking, interpolates the two control problems. Since the original nonlinear problem is quasi-linear, there is a loss of derivative (this reflects the fact that the flow map is expected to be merely continuous and not Lipschitz on Sobolev spaces). We overcome this loss by proving and using a regularity property of the control, see statement $ii)$ in Proposition 2.4. This regularity result is proved by adapting an argument used by Dehman-Lebeau [22] and Laurent [32]. We also need to study how the control depends on T or on the function M.

We mention that another scheme is used in [10], based on the Nash-Moser method, to study the controllability problem for similar quasilinear equations. With this scheme, it is not necessary to prove stability estimates (but then the final result does not include a stability property for the control).

3. Stabilization of the water-wave equations

3.1. **The multiplier method for the water-wave equations.** We now discuss the multiplier method for the water-wave problem, as introduced in [2] for the nonlinear water-wave equations. The motivation was to prove an observability inequality. Namely, the question studied in [2] is the following: is-it possible to estimate the energy of gravity water waves by looking only at the motion of some of the curves of contact between the free surface and the vertical walls? From the point of view of control theory, this is the question of boundary observability of gravity water waves.

For the sake of readability, we begin by recalling some well-known results for the linear wave equation

$$\partial_t^2 u - \Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad u|_{\partial\Omega} = 0. \tag{3.1}$$

The multiplier method, introduced by Morawetz, consists in multiplying the equations by $m(x)\cdot\nabla u(t,x)$, for some well-chosen function m, and to integrate by parts in space and time. For instance, by considering a smooth extension $m\colon \Omega \to \mathbb{R}^n$ of the normal $\nu(x)$ to the boundary $\partial\Omega$, one obtains

$$\int_0^T \int_{\partial\Omega} (\partial_n u)^2 \,\mathrm{d}\sigma\,\mathrm{d}t \le K(T)\mathcal{E}(u) \text{ where } \mathcal{E}(u) := \|u(0,\cdot)\|^2_{H^1_0(\Omega)} + \|\partial_t u(0,\cdot)\|^2_{L^2(\Omega)}. \tag{3.2}$$

This is the so-called *hidden regularity* property. The name comes from the fact that, using energy estimates, one controls only the square of the $C^0([0,T];L^2(\Omega))$-norm of $\nabla_x u$ by means of the right-hand side of (3.2), which is insufficient to control the left-hand side of (3.2) by means of classical trace theorems.

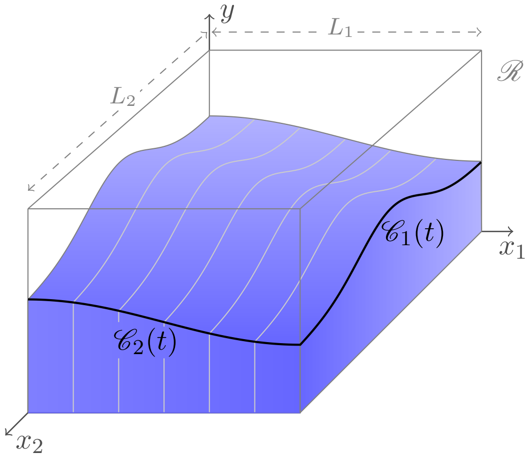


FIGURE 4. The boundary observability problem consists in bounding from below the energy by looking only at the motion of the curves $\mathscr{C}_1$ and $\mathscr{C}_2$.

Another key estimate is the so-called *boundary observability inequality*, which is, compared to (3.2), a reverse inequality where one can bound the norms of the initial data by the integral of $\partial_n u$ restricted to a domain $\Gamma_0 \subset \partial\Omega$. Let us recall the proof of such an inequality in the simplest case. Consider the one dimensional linear wave equation with Dirichlet boundary condition:

$$\partial_t^2 u - \partial_x^2 u = 0, \quad u(t,0) = u(t,1) = 0. \tag{3.3}$$

Multiply the equation by $x\partial_x u$ and integrate by parts, to obtain

$$\frac{1}{2}\int_0^T (\partial_x u(t,1))^2 \,\mathrm{d}t = \int_0^1 (\partial_t u)(x\partial_x u)\,\mathrm{d}x \,\Big|_0^T + \frac{1}{2}\iint_S \left[(\partial_t u)^2 + (\partial_x u)^2\right] \mathrm{d}x\,\mathrm{d}t \tag{3.4}$$

where $S = (0,T)\times(0,1)$. Since

$$\left|\int_0^1 (\partial_t u)(x\partial_x u)\,\mathrm{d}x\right| \le \mathcal{E} := \frac{1}{2}\int_0^1 \left[(\partial_t u)^2 + (\partial_x u)^2\right] dx, \tag{3.5}$$

by using the conservation of energy ($d\mathcal{E}/dt = 0$), we deduce

$$\int_0^T (\partial_x u(t,1))^2 \,\mathrm{d}t \ge (T-2)\int_0^1 \left[(\partial_t u)^2 + (\partial_x u)^2\right](0,x)\,dx. \tag{3.6}$$

This inequality implies that, for $T > 2$, one can bound the energy by means of an observation at the boundary. For more details, generalizations and extensions to other equations, we refer the reader to the SIAM Review article by Lions [33]. Similar results are known for many other wave equations and we only mention the paper by Machtyngier and Zuazua [35] for the Schrödinger equation $i\partial_t u + \Delta u = 0$. Biccari [13] used recently the multiplier method to analyze the interior controllability problem for the fractional Schrödinger equation $i\partial_t u + (-\Delta)^s u = 0$ with $s \ge 1/2$ in a $C^{1,1}$ bounded domain with Dirichlet boundary condition.

In [2] we proved the following exact identity analogous to (3.4) for the *nonlinear* water-wave equations without surface tension. (The result in [2] holds for 3D waves but here we give a statement for 2D waves to simplify notations.)

Theorem 3.1. *Consider the equations without surface tension (that is assume that $\kappa = 0$) and without external pressure ($P_{ext} = 0$). Consider a smooth enough solution defined on the time interval $[0,T]$ and introduce*

$$m(t) = \eta(t,L).$$

Then

$$\begin{aligned}\frac{L}{2}\int_0^T \left[gm(t)^2 - m(t)m'(t)^2\right] \mathrm{d}t = \frac{T}{2}\mathcal{H} \\ + \frac{L}{2}\int_0^T\int_{-h}^{m(t)} (\partial_y\phi)^2(t,L,y)\,\mathrm{d}y\,\mathrm{d}t \\ + \frac{1}{2}\int_0^T\int_0^L \left(h+\frac{7}{4}\eta\right)(\partial_x\phi)^2(t,x,-h)\,\mathrm{d}x\,\mathrm{d}t \\ - \frac{1}{4}\int_0^L \eta\psi\,dx\,\Big|_{t=0}^{t=T} - \int_0^L x\eta\partial_x\psi\,\mathrm{d}x\,\Big|_{t=0}^{t=T} \\ - \frac{7}{4}\int_0^T\iint_{\Omega(t)} (\partial_x\eta)(\partial_x\phi)(\partial_y\phi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,\end{aligned} \tag{3.7}$$

where $\int f\,dx|_{t=0}^{t=T}$ *stands for* $\int f(T,x)\,dx - \int f(0,x)\,dx$ *and*

$$\mathcal{H} = \frac{g}{2}\int_{-L}^{L}\eta^2(t,x)\,\mathrm{d}x + \frac{1}{2}\iint_{\Omega(t)} |\nabla_{x,y}\phi(t,x,y)|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.8}$$

The proof uses the Zakharov's formulation of the water-wave problem as a Hamiltonian system (see [44]) and the observation by Craig and Sulem [21] that the equations and the hamiltonian are most naturally expressed in terms of the Dirichlet to Neumann operator $G(\eta)$. The main ingredients of the proof of Theorem 2.1 are then: *i*) a Pohozaev identity for the Dirichlet to Neumann operator (that is a computation of $\int(G(\eta)\psi)x\partial_x\psi\,\mathrm{d}x$) which shows that the contributions due to the boundary conditions are positive and *ii*) some computations inspired by the analysis of Benjamin and Olver [12] of the conservation laws for water waves. In the appendix of [2], we give another proof of (3.7) which exploits the hamiltonian structure of the water-wave equations.

3.2. **Stabilization.** We now consider the stabilization problem.

Recall that the energy $\mathcal{E} = \mathcal{E}(t)$ is defined by

$$\mathcal{E} = \frac{g}{2}\int_0^L \eta^2\,\mathrm{d}x + \kappa\int_0^L\left(\sqrt{1+\eta_x^2}-1\right)\mathrm{d}x + \frac{1}{2}\int_0^L\int_{-h}^{\eta(t,x)} |\nabla_{x,y}\phi|^2\,\mathrm{d}y\,\mathrm{d}x.$$

Our goal is to prove that, for some pressure law relating P_{ext} to the unknown (η,ψ), there holds:

i) $\mathcal{E}$ is decreasing and converges to zero exponentially in time;
ii) $\partial_x P_{ext}$ is supported inside a small subset of $[0,L]$.

In the numerical literature, a popular choice is to assume that $P_{ext} = \chi\partial_t\eta$ for some cut-off function $\chi \geq 0$ supported[1] in $[L-\delta, L]$ for some $\delta > 0$. To explain this choice, we start by recalling that, as observed by Zakharov [44], the equations have the hamiltonian form

$$\frac{\partial\eta}{\partial t} = \frac{\delta\mathcal{E}}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta\mathcal{E}}{\delta\eta} - P_{ext}.$$

Consequently,

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \int\left(\frac{\delta\mathcal{E}}{\delta\eta}\frac{\partial\eta}{\partial t} + \frac{\delta\mathcal{E}}{\delta\psi}\frac{\partial\psi}{\partial t}\right)\mathrm{d}x = -\int\frac{\partial\eta}{\partial t}P_{ext}\,\mathrm{d}x, \tag{3.9}$$

and hence, if $P_{ext} = \chi\partial_t\eta$ with $\chi \geq 0$, we deduce that $\mathrm{d}\mathcal{E}/\,\mathrm{d}t \leq 0$. It is thus easily seen that the energy decays. However, it is much more complicated to prove that the energy converges exponentially to zero. To study this problem, we first need to pause to clarify the question since, in general, solutions of the water-wave equations do not exist globally

[1] We consider a damping located near $x = L$ only, and not also near $x = 0$, since we imagine that water waves are generated near $x = 0$. To do so, as explained above, one could use also the variations of an external pressure.

in time (they might blow-up in finite time, see [15, 19]). Our goal is in fact to prove that there exists a constant C such that, if a regular solution exists on the time interval $[0,T]$, then

$$\mathcal{E}(T) \leq \frac{C}{T}\mathcal{E}(0). \tag{3.10}$$

Since the equation is invariant by translation in time, one can iterate this inequality. Consequently, if the solution exists on time intervals of size nT_0 with $T_0 \geq 2C$, then $\mathcal{E}(nT_0) \leq 2^{-n}\mathcal{E}(0)$, which is the desired exponential decay.

We now define the pressure law.

Definition 3.2. *We assume that*

$$P_{ext}(t,x) = \chi(x)\partial_t\eta(t,x) - \frac{1}{2L}\int_{-L}^{L}\chi(x)\partial_t\eta(t,x)\,\mathrm{d}x,$$

where the cut-off function χ is defined as follows. Fix $\delta > 0$ and consider a $2L$-periodic C^∞ function κ, satisfying $0 \leq \kappa \leq 1$ and such that (see Figure 5)

$$\kappa(x) = \kappa(-x), \quad x\kappa'(x) \leq 0 \text{ for } x \in [-L,L], \quad \kappa(x) = \begin{cases} 1 \text{ if } x \in [0, L-\delta], \\ 0 \text{ if } x \in \left[L - \frac{\delta}{2}, L\right]. \end{cases}$$

We successively set

$$m(x) = x\kappa(x),$$

and

$$\chi(x) = 1 - m_x(x).$$

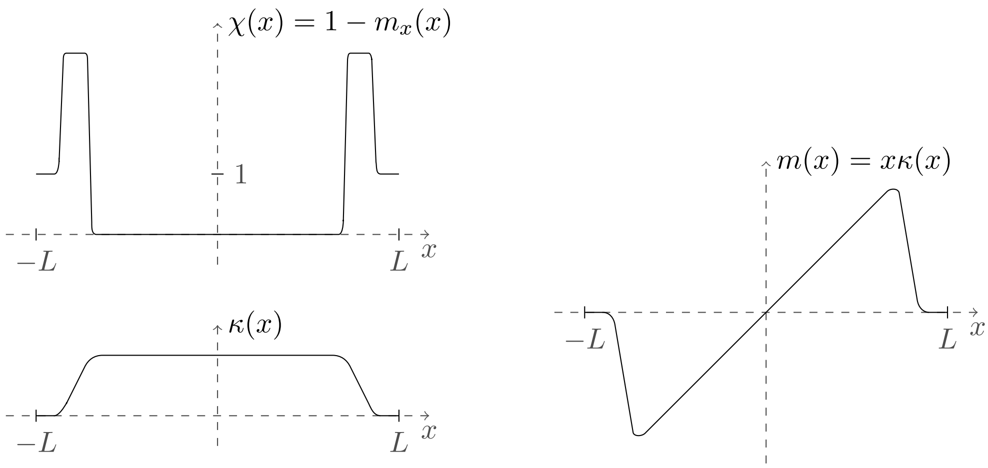


FIGURE 5. The cutoff function χ and the multiplier m.

Here is our main result.

Theorem 3.3. *Assume that $\kappa > 0$ and that*

$$\kappa \sup_{[-L,L]} m_{xx}(x)^2 \leq g.$$

Then there exists a positive constant C, depending only on the physical parameters κ, g, h, L, such that the following result holds. Let $T > 0$ and consider a regular solution (η, ψ, P_{ext}) defined on the time interval $[0,T]$ with P_{ext} as given by Definition 3.2 and set

$$\rho(t,x) = (m(x) - x)\eta_x(t,x) + \frac{9}{4}\eta(t,x) - \frac{1}{2}m_x(x)\eta(t,x).$$

If, for all $(t,x)\in[0,T]\times[-L,L]$,

$$i)\qquad \rho(t,x)\geq -\frac{h}{4},\qquad |\rho_x(t,x)|<\frac{1}{4},$$

$$ii)\qquad \int_0^L (1-m_x(x))\eta(t,x)\,\mathrm{d}x\leq \frac{h}{12},\qquad |m_x(x)|\,|\eta_x(t,x)|^2\leq 2,\quad |\eta_x(t,x)|\leq\frac{1}{2},$$

then one has the estimate

$$\mathcal{E}(T)\leq \frac{C}{T}\mathcal{E}(0).$$

Remark 3.4. (i) An important remark is that the constant C can be given by an explicit formula in terms of κ, g, h, L.

(ii) By combining this result with the local controllability result for small data proved by Alazard, Baldi and Han-Kwan in [5], this in turn implies that one can deduce controllability for larger data, but in large time (following an argument due to Dehman-Lebeau-Zuazua [23]).

(iii) We studied a similar problem in [3] for the case without surface tension. Assuming that P_{ext} is given by

$$\partial_x P_{ext}=\chi(x)\int_{-h}^{\eta(t,x)}\phi_x(t,x,y)\,\mathrm{d}y, \tag{3.11}$$

where $\chi\geq 0$ is a cut-off function, we proved in [3] an inequality of the form

$$\mathcal{E}(T)\leq \frac{C(N)}{\sqrt{T}}\mathcal{E}(0), \tag{3.12}$$

where the constant $C(N)$ depends on the frequency localization[2] of the solution (η,ψ). The fact that this constant must depend on the frequency localization can be easily understood by considering the linearized equations (see [3]).

To prove Theorem 3.3, it is sufficient to prove that there exists a positive constant C such that

$$\int_0^T \mathcal{E}(t)\,\mathrm{d}t\leq C\mathcal{E}(0). \tag{3.13}$$

Indeed, since $\mathcal{E}$ is a decreasing function, this will imply the wanted inequality

$$\mathcal{E}(T)\leq \frac{1}{T}\int_0^T \mathcal{E}(t)\,\mathrm{d}t\leq \frac{C}{T}\mathcal{E}(0).$$

The proof of (3.13) is in two steps.

First step. The first step consists in deriving an exact *identity* which involves the integral in time of the energy. Here one can explain one of the main difficulties one has to cope with to stabilize the water-wave equations: one cannot decouple the problem of the observability and the question of the stabilization. Compared to what is done for the usual wave equation for instance, since the water-wave system is quasi-linear, one cannot write the solution as the sum of the two different problems. Another difficulty is that we do not know how to deduce an internal observability inequality from a boundary observability inequality (for the wave equation or the Schrödinger equation, this is possible thanks to a hidden regularity result). To overcome these two problems, we prove directly an internal observability result for the water-wave system with an external source term, by considering a multiplier $m(x)\partial_x$ with $m(x)=x\kappa(x)$ where κ is a cut-off function satisfying $\kappa(x)=1$ for $0\leq x\leq L-\delta$ and $\kappa(x)=0$ for $L-\delta/2\leq x\leq L$.

We proceed by establishing an exact identity similar to the one given above for the case without surface tension. With surface tension, one has to handle many remainder terms.

[2]The quantity N measuring the frequency localization is of the ratio of two Sobolev norms. One can think of the ratio $\|u\|_{H^1}/\|u\|_{L^2}$, which is proportional to N for a typical function oscillating at frequency N, like $u(x)=\cos(2\pi Nx/L)$.

In the end, instead of stating an identity (as we did in Theorem 3.1 for the case without surface tension), we will obtain an inequality.

Consider a smooth function $m = m(x)$ with $m(0) = m(L) = 0$, and set

$$\zeta = \partial_x(m\eta) + \frac{3}{2}(1 - m_x)\eta - \frac{1}{4}\eta,$$

our first main task will be to derive the following inequality

$$\begin{aligned}
&\frac{1}{4}\int_0^T \mathcal{E}(t)\,\mathrm{d}t \le O + W + B - I \qquad \text{where} \\
&I := \frac{h}{4}\int_0^T\int_{-L}^{L} \phi_x(t,x,-h)^2\,\mathrm{d}x\,\mathrm{d}t, \\
&O := \int_0^T\int_{-L}^{L}\left(\frac{3}{2}(1-m_x)\psi + (x-m)\psi_x\right)G(\eta)\psi\,\mathrm{d}x\,\mathrm{d}t, \\
&W := -\int_0^T\int_{-L}^{L} P_{ext}\,\zeta\,\mathrm{d}x\,\mathrm{d}t, \\
&B := \int_{-L}^{L}\zeta(0,x)\psi(0,x)\,\mathrm{d}x - \int_{-L}^{L}\zeta(T,x)\psi(T,x)\,\mathrm{d}x.
\end{aligned} \tag{3.14}$$

These four quantities play different roles. Their key properties are the following:

- $I \ge 0$ and hence (3.14) gives a bound for the horizontal component of the velocity at the bottom (this plays a key role to control the velocity in terms of the pressure, see (3.16)).
- W is the only term which involves the pressure.
- O corresponds to an *observation*, this means that this term depends only on the behavior of the solutions near $\{x = -L\}$ or $\{x = L\}$ when m is as given by Definition 3.2. Indeed, $x - m$ and $1 - m_x$ vanish when $x \in [-L + \delta, L - \delta]$ by definition of m.
- B is not an integral in time, by contrast with the other terms and, in addition, it is easily estimated by $K\mathcal{E}(0)$.

Second step. The goal of the second step is to deduce the wanted result (3.13) from the inequality (3.14). To do so, it is sufficient to prove that there exists a constant K depending only on g, κ, h, L such that

$$O + W + B - I \le K\mathcal{E}(0) + a\int_0^T \mathcal{E}(t)\,\mathrm{d}t \quad \text{for some} \quad a < \frac{1}{4}. \tag{3.15}$$

Indeed, by combining (3.14) and (3.15) one obtains that

$$\int_0^T \mathcal{E}(t)\,\mathrm{d}t \le \frac{K}{1/4 - a}\mathcal{E}(0),$$

which is the wanted result (3.13). To prove (3.15), recalling that $I \ge 0$, we need to estimate the terms B, W, O.

The estimate of the term B is easy so we begin by explaining how to estimate W. Recall that the hamiltonian structure of the equation implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = -\int_{-L}^{L} P_{ext}\partial_t\eta\,\mathrm{d}x.$$

Since $\partial_t\eta = G(\eta)\psi$, one deduces the inequality

$$\int_0^T\int_{-L}^{L}\chi(\partial_t\eta)^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{-L}^{L}\chi(G(\eta)\psi)^2\,\mathrm{d}x\,\mathrm{d}t \le \mathcal{E}(0).$$

This gives an estimate for the L^2-norm of $\chi\partial_t\eta$, which is the main contribution to the definition of P_{ext}. A more tricky inequality, relying on the special choice $\chi(x) = 1 - m_x(x)$, is that

$$-\int_0^T \left[\left(\int_{-L}^L \chi \partial_t \eta \,\mathrm{d}x\right) \int_{-L}^L \zeta \,\mathrm{d}x\right] \mathrm{d}t \leq \frac{L}{g} \left\| (1-m_x)^2 \right\|_{L^\infty} \mathcal{E}(0).$$

By combining the above inequalities, we will be able to estimate the term W. The main difficulty is to bound the observation term O, and in particular to estimate

$$\iint (x-m)\psi_x G(\eta)\psi \,\mathrm{d}x \,\mathrm{d}t.$$

Using the Cauchy-Schwarz inequality, the key point is to estimate the L^2-norm of ψ_x in terms of the two quantities which are under control, that is the integral of $\chi(G(\eta)\psi)^2$ and the positive term I. In this direction, we will prove the following inequality, which is of independent interest: there exists a constant A, depending only on $\|\eta_x\|_{L^\infty_x}$, such that

$$\begin{aligned}\int_{-L}^L \chi \psi_x^2 \,\mathrm{d}x \leq A \int_{-L}^L \chi (G(\eta)\psi)^2 \,\mathrm{d}x + A \int_{-L}^L \phi_x^2|_{y=-h} \,\mathrm{d}x \\ - A \int_{-L}^L \int_{-h}^{\eta(t,x)} \chi_x \phi_x \phi_y \,\mathrm{d}y \,\mathrm{d}x.\end{aligned} \tag{3.16}$$

Notice that this result holds in fact for any smooth cut-off function χ and one can also take $\chi = 1$. In this case this inequality simplifies since the last term in the right-hand side vanishes. This gives a way to control the L^2-norm of ψ_x by the L^2-norm of $G(\eta)\psi$. When η is smooth, this can be obtained by delicate commutator estimates. However, it is possible to give a simple proof which applies for any Lipschitz domain.

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