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Gravity Capillary Standing Water Waves

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Abstract

The paper deals with the 2D gravity-capillary water waves equations in their Hamiltonian formulation, addressing the question of the nonlinear interaction of a plane wave with its reflection off a vertical wall. The main result is the construction of small amplitude, *standing* (namely periodic in time and space, and not travelling) solutions of Sobolev regularity, for almost all values of the surface tension coefficient, and for a large set of time-frequencies. This is an existence result for a quasi-linear, Hamiltonian, reversible system of two autonomous pseudo-PDEs with small divisors. The proof is a combination of different techniques, such as a Nash–Moser scheme, microlocal analysis and bifurcation analysis.

Contents

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1. Introduction

This paper deals with the 2D gravity-capillary water waves equations in their Hamiltonian formulation [see Equation (2.1)]. The main result (Theorem 2.4) is the construction of small amplitude, *standing* (namely periodic in time and space, and not travelling) solutions of Sobolev regularity, for almost all values of the surface tension coefficient, and for a large set of time-frequencies. This is an existence result for a quasi-linear system of two autonomous pseudo-PDEs with small divisors.

Before stating precisely the result and describing the strategy of the proof, we introduce the problem within a more general framework.

A classical topic in the mathematical theory of hydrodynamics concerns the Euler equations for the irrotational flow of an incompressible fluid in a domain which, at time t, is of the form

$$\Omega(t) = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \mid y < \eta(t, x)\},$$

whose boundary is a free surface, which means that η is an unknown. The simplest type of nontrivial solution for the problem is a *progressive wave*, which is a profile of the form $\eta(t, x) = \sigma(k \cdot x - \omega t)$ for some periodic function $\sigma : \mathbb{R} \to \mathbb{R}$, together with a similar property for the velocity field. Despite intensive research on this old subject, many natural questions are far from being fully resolved. Among these, most questions about the boundary behavior of water waves are not understood. Also, the question of the nonlinear interactions of several progressive waves is mostly open.

This paper is concerned with these two problems. We shall study the reflection of a progressive wave off a wall. More precisely, we shall study the nonlinear interaction of a 2D gravity-capillary plane wave with its reflection off a vertical wall. To clarify matters, recall that 2D waves are waves such that the motion is the same in every vertical section, so that one can consider the two-dimensional motion in one such section (the free surface is then a 1D curve).

Interaction of two gravity-capillary waves. The problem consists in seeking solutions of the water waves equations, periodic in space and time, and such that

$$\eta(t, x) = \varepsilon \cos(k_1 \cdot x - \omega(k_1)t) + \varepsilon \cos(k_2 \cdot x - \omega(k_2)t) + O(\varepsilon^2), \qquad (1.1)$$

together with a similar property for the velocity field, where

- ε is a small parameter which measures the amplitude of the waves;
- k_1 and k_2 belong to $\mathbb{R}^2$ and are mirror images such that either $k_1 = -k_2$ or $k_1 = (1, \tau)$, $k_2 = (1, -\tau)$ for some τ. This is the assumption that there is one incident plane wave, say $\varepsilon \cos(k_1 \cdot x - \omega(k_1)t)$, and one reflected wave, $\varepsilon \cos(k_2 \cdot x - \omega(k_2)t)$;

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- there holds

$$\omega(k) := \sqrt{g|k| + \kappa|k|^3}$$

where $g > 0$ is the acceleration of gravity and $\kappa \in [0, 1]$ is the surface tension coefficient. Thus $k \to \omega(k)$ is the dispersion relation of the water waves equation linearized at the rest position, which (after transforming the system into a single equation) can be written in the form

$$Lu := \partial_t^2 u + g|D_x|u + \kappa|D_x|^3 u = 0. \tag{1.2}$$

Here $|D_x|$ is the Fourier multiplier defined by $|D_x|e^{ik\cdot x} = |k|e^{ik\cdot x}$, where $|k|$ is the Euclidean norm of k. Note that $\omega(k_1) = \omega(k_2)$.

This problem was initiated by Reeder and Shinbrot [46] and further developed by Craig and Nicholls [20,21] and Groves and Haragus [26]. They considered the superposition of two oblique 2D travelling waves, such that $k_1 = (1, \tau)$ and $k_2 = (1, -\tau)$ for some τ. This produces 3D short crested waves. Indeed, setting $\omega = \omega(k_1) = \omega(k_2)$ and writing $x = (x_1, x_2)$, one has

$$\varepsilon\cos(x_1 + \tau x_2 - \omega t) + \varepsilon\cos(x_1 - \tau x_2 - \omega t) = 2\varepsilon\cos(x_1 - \omega t)\cos(\tau x_2).$$

Since these waves are propagating in the direction $(1, 0)$ (the x_1 axis), one has to study solutions of the equation that are obtained by replacing ∂_t with $-\omega\partial_{x_1}$ in (1.2), which is

$$Ku := (\omega\partial_{x_1})^2 u + g|D_x|u + \kappa|D_x|^3 u = 0. \tag{1.3}$$

For $\kappa > 0$, K is an elliptic operator. Consequently, in this context, the existence of solutions for the nonlinear equation is a problem in bifurcation theory (without small divisors).

Standing waves. In this paper, we consider the case where the crests of the incident waves are parallel to the wall. This implies that $k_1 = -k_2$ and therefore

$$\varepsilon\cos(k_1\cdot x - \omega(k_1)t) + \varepsilon\cos(k_2\cdot x - \omega(k_2)t) = 2\varepsilon\cos(\omega(k_1)t)\cos(k_1\cdot x).$$

Hence the waves obtained by superimposing the incident and the reflected waves are *standing waves* (namely periodic in time and space, and not travelling). Since standing waves are not travelling, one cannot replace the time derivative by a space derivative. This changes dramatically the nature of the problem, as *small divisors* appear. In this paper we are interested in the case with surface tension $\kappa > 0$, while in the case without surface tension, namely $\kappa = 0$, a similar small divisors problem was studied in a series of papers of Iooss, Plotnikov and Toland [28–33,43], which are described below.

The standing waves we are interested in are 2D waves. Without loss of generality, we can assume that $k_1 = (\lambda, 0) = -k_2$ for some $\lambda \neq 0$. Thus we shall consider functions that are independent of x_2. In the rest of the paper, $x \in \mathbb{R}$.

Small divisors. Let us explain why small divisors enter into the analysis. Looking for solutions that are 2π-periodic in space and $(2\pi/\omega)$-periodic in time (where the time frequency ω is an unknown of the problem), (1.2) gives

$$Le^{i(jx+\omega\ell t)} = p(\omega\ell, j)e^{i(jx+\omega\ell t)}, \quad p(\omega\ell, j) := -\omega^2\ell^2 + g|j| + \kappa|j|^3.$$

In general, namely for almost all values of ω, κ, the eigenvalues $\{p(\omega\ell, j)\}_{\ell, j\in\mathbb{Z}}$ of L accumulate to zero. To invert L in the orthogonal of its kernel, one finds these small eigenvalues as denominators (in fact, small divisors), so that the inverse of L is not a bounded operator, in the sense that it does not map any function space (Sobolev or analytic or Hölder or others) into itself. This makes it impossible to apply the standard implicit function theory to solve the orthogonal component of the nonlinear problem (i.e. the range equation, in the language of bifurcation theory).
Main result. Our main result is stated in the next section, see Theorem 2.4. It asserts that, for almost all values κ of the surface tension coefficient, for ε_0 small enough there exists a set $\mathcal{G} \subset [0, \varepsilon_0]$ of Lebesgue measure greater than $\varepsilon_0(1 - C\varepsilon_0^{1/18})$, such that for ε in $\mathcal{G}$ there exists a standing wave whose free surface is of the form (1.1), or, more precisely, $\eta(t, x) = \varepsilon \cos(\omega t)\cos(x) + O(\varepsilon^2)$, with time-frequency $\omega = \sqrt{g+\kappa} + O(\varepsilon^2)$. (In Theorem 2.4 the result is stated precisely for the problem one obtains after normalizing the gravity constant g, rescaling time, and introducing an additional amplitude parameter ξ, see Section 2.1.).

Our proof is based on Nash-Moser methods for quasi-linear PDEs on the one side, and on techniques of microlocal analysis on the other.

Regarding Nash-Moser and KAM theory for quasi-linear PDEs, we remark that in general, as it was proved in the works of Lax, Klainerman and Majda on the formation of singularities (see e.g. [35]), the presence of *nonlinear unbounded* operators—as it is in our water waves problem—can compromise the existence of invariant structures of the dynamics like periodic or quasi-periodic solutions. In fact, the wide existing literature on KAM and Nash-Moser theory for PDEs mainly deals with problems where the perturbation is bounded (see e.g. Kuksin [36] and Wayne [48]; see [37], [19] for a survey). For unbounded perturbations where the nonlinear term contains *less* derivatives than the linear one, time-periodic solutions have been obtained by Craig [19] and Bourgain [16], while quasi-periodic solutions for PDEs of that type have been constructed via Nash–Moser or KAM methods by Bourgain [15], Kuksin [37], Kappeler-Pöschel [34] for KdV, and, more recently, by Liu-Yuan [40] and Zhang–Gao–Yuan [50] for NLS and Benjamin-Ono, and Berti--Biasco--Procesi [11,12] for NLW.

For *quasi-linear* PDEs, namely for equations where there are as many derivatives in the nonlinearity as in the linear part (sometimes called "strongly nonlinear" PDEs, e.g. in [37]), the extension of KAM and Nash–Moser theory is a very recent subject, which counts very few results. Time-periodic solutions for this class of equations have been constructed by Iooss, Plotnikov and Toland for gravity water waves [33,43] (which, even more, is a *fully nonlinear* system), and by Baldi for forced Kirchhoff [6] and autonomous Benjamin–Ono equation [7], all using Nash–Moser methods. We also mention the pioneering Nash–Moser results of Rabinowitz [44,45] for periodic solutions of fully nonlinear wave equations (where, however, small divisors are avoided by a dissipative term). The existence (and linear stability) of quasi-periodic solutions for a quasi-linear PDE has been only proved very recently by Baldi, Berti and Montalto for forced Airy [8] and autonomous KdV [9] equations, by Nash-Moser, linear KAM reducibility, and Birkhoff normal forms.

Regarding the water waves problem, in [43] Plotnikov and Toland proved the existence of pure gravity (i.e. $\kappa = 0$) standing waves, periodic in time and space.

This work has been extended by Iooss, Plotnikov and Toland [33] and then by Iooss and Plotnikov [29,30] who proved the existence of unimodal [33] and multimodal [29,30] solutions in Sobolev class via Nash–Moser theory, overcoming the difficulty of a complete resonance of the linearized operator at the origin. On the contrary, the gravity-capillary case has an additional parameter, the surface tension coefficient κ, whose arithmetic properties determine the bifurcation analysis of the linear theory. In particular, for all irrational values of κ (and therefore for almost all κ) the linearized operator at the origin has a one-dimensional kernel (see Section 4.1).

We also mention the recent proof by Iooss and Plotnikov [31,32] of the existence of *three-dimensional* periodic progressive gravity waves, obtained with Nash-Moser techniques related to [29,30]. The question of the existence of such waves was a well known problem in the theory of surface waves—we refer the reader to [10,17,20,24,25,28,31] for references and an historical survey of the background of this problem.

Apart from the dimension of the kernel of the linear problem, there are other important differences between the gravity water waves problem, as studied by Iooss–Plotnikov–Toland, and the gravity-capillary water waves problem studied here. The difference is clear at the level of the dispersion relation $\omega = \sqrt{g|k| + \kappa|k|^3}$. One could think that the dispersion is stronger for $\kappa > 0$ than for $\kappa = 0$ (and this is certainly true for the *linear* part of the problem), but this does not help the study of the nonlinear problem, because higher order derivatives also appear in the nonlinearity. In fact, this requires the introduction of new techniques. A more detailed explanation about which are the new problems emerging in presence of surface tension and why the techniques of [33] do not work for $\kappa > 0$ is given in the lines below (1.4).

To conclude this introduction, let us discuss the main ingredients in our proof. Applying a Nash–Moser scheme, the main difficulty regards the invertibility of the operator linearized at a nonzero point. As in [7–9,33], we seek a sufficiently accurate asymptotic expansion of the eigenvalues in terms of powers of ε and in terms of inverse powers of the spatial wavelength. To do so, we conjugate the linearized operator to a constant coefficient operator plus a smoothing remainder which can be handled as a perturbation term. (We remark that a similar eigenvalues expansion was obtained in [33] using inverse powers ∂_t^{-1} of the time-derivative, instead of space-derivative ∂_x^{-1}, destroying the structure of dynamical system. Such a structure is preserved, instead, by the transformations performed in this paper, as well as those in [7–9].) To obtain such a precise knowledge of the asymptotic behavior of the eigenvalues requires one to find the dispersion relationship associated to a variable coefficient equation, which in turn requires microlocal analysis. In this direction, we shall follow a now well developed approach in the analysis of water waves equations, which consists in working with the Craig–Sulem–Zakharov formulation of the equations, introducing the Dirichlet–Neumann operator. In particular, we shall use in a crucial way two facts proved by Lannes in [38]. Firstly, by introducing what is known as the good unknown of Alinhac (see [3,5]), one can overcome an apparent loss of derivative in the analysis of the linearized equation. (Another advantage is that, working with the Craig–Sulem–Zakharov formulation (2.1) of the problem as a dynamical system in two unknowns, and thanks to the good unknown of Alinhac, here we do not need to introduce any "approximate inverse" for the linearized operator, as it was done in [33].) Secondly, one can use pseudo-differential analysis to study the Dirichlet–Neumann operator in domains with limited regularity. In [3], this analysis is improved by showing that one can paralinearize the Dirichlet–Neumann operator, introducing the paradifferential version of the good unknown of Alinhac (see [4]). Notice that the analysis of the Cauchy problem for capillary waves requires an analysis of the sub-principal terms (see [1,42]). In this paper, we shall use in an essential way the fact that it is also possible to symmetrize *sub-sub-principal* terms (the method used below can be extended at any order). We underline, in particular, the use of a pseudo-differential operator with symbol in Hörmander class $S^0_{\rho,\delta}$, $\rho = \delta = 1/2$ (except for the fact that the symbol here has finite regularity), see (9.2) and also the discussion about the related model problem (1.4). Moreover, to apply a Nash–Moser scheme one needs tame estimates (these are estimates which are linear with respect to the highest-order norms). We shall use the estimates proved in [2] together with several estimates proved in Section 12 of this paper using a paradifferential decomposition of the frequencies.

Eventually, we refer the reader to [18,41] for recent results establishing the existence of progressive waves localized in space.

The content of the next sections is as follows. In Section 2 the water waves equations are written, some symmetries are shown (such symmetries play a role in the bifurcation analysis of Section 4.1, especially to deal with the space-average terms), the functional setting is introduced, and the main theorem is stated. Section 3 collects preliminary facts about the Dirichlet–Neumann operator, and fixes some notations.

In Section 4 we perform a bifurcation analysis. In particular, in Section 4.3 we construct an "approximate solution" $\bar{u}_\varepsilon$, which corresponds to the need of any quadratic Newton scheme to have a sufficiently good "initial guess". In Section 4.4 we exploit the (nonlinear) construction of Section 4.3 to deduce a restriction for the (linear) problem of the inversion of the linearized operator $F'(u)$ at a nonzero point u which is close to $\bar{u}_\varepsilon$. This restriction is given by a linearized version of the Lyapunov-Schmidt approach first (Lemma 4.1), and then by a further restriction with respect to the space frequencies only, closer to a dynamical system point of view (Lemma 4.4).

In Sections 5–9 we conjugate the linearized operator $F'(u)$ to the sum of a constant coefficient part $\tilde{\mathcal{D}}$ and a regularizing, small remainder $\tilde{\mathcal{R}}_6$, see (9.53). In Section 5 we use (a linear version of) the good unknown of Alinhac. In Section 6 we perform a time-dependent change of the space variable, a (space-independent) reparametrization of the time-variable, and a matrix multiplication, to obtain constant coefficients in front of the highest order terms, see (6.8). In Section 7 we symmetrize the highest order terms, keeping altogether the few terms that are not small in ε (this is visible in (7.7)). Section 8 completes the symmetrization procedure, obtaining a symmetric part (see the operator matrix $\mathcal{L}_5$ in (8.1)) plus a remainder of order $O(\varepsilon|D_x|^{-3/2})$, after solving a block-triangular system of 8 equations in 8 unknowns. At this point the 2×2 real linear system can be written as a single equation for a single complex-valued function $h : \mathbb{T}^2 \to \mathbb{C}$, see (8.13).

Here comes the most interesting part of our conjugation analysis, where it the reason for which the method of [33] does not work in the presence of surface tension becomes most evident. The point can be better explained when reformulated in terms of a modified model problem (for the full operator see (8.13)).

• **Model problem:** *conjugate the linear operator*

$$\omega\partial_t + ic|D_x|^{3/2} + a(t,x)\partial_x \tag{1.4}$$

to constant coefficients up to order $O(|D_x|^{-3/2})$, *knowing that c is a real constant, and the variable coefficient* $a(t,x)$ *is small in size, odd in* t, and odd in x. The technique used in [33] to eliminate the term $a(t,x)\partial_x$ was the one of conjugating the vector field $\omega\partial_t + a(t,x)\partial_x$ to $\omega\partial_t$ using a suitable change of variable, namely the composition map with a diffeomorphism of the torus (this can be done by the method of the characteristics, i.e. by solving an ODE). However, in (1.4), the change of variables that rectifies $\omega\partial_t + a(t,x)\partial_x$ produces a variable coefficient in front of $|D_x|^{3/2}$, which is even worse than (1.4) for our purposes. A similar effect is obtained by any other Fourier integral operator (FIO) with homogeneous phase function (recall that the changes of variables are special cases of FIOs). On the other hand, one cannot eliminate $a(t,x)\partial_x$ by commuting the equation with any multiplication operator, or any other "standard" pseudo-differential operator of order zero (with a symbol in Hörmander class $S^0_{\rho,\delta}$ with $\rho = 1$, $\delta = 0$, see Chapter 7.8 in [27]). Indeed, the commutator between such an operator and $|D_x|^{3/2}$ is an operator of order $1/2$, which leaves $a(t,x)\partial_x$ unchanged. Hence, the algebraic rigidity of the equation forces us to commute the equation with a pseudo-differential operator with symbol of type $S^0_{\rho,\delta}$ with $\rho = \delta = 1/2$, see (9.2). In Section 9 we calculate the right candidate to complete the reduction to constant coefficients up to $O(|D_x|^{-3/2})$. The study of this operator, namely the proof of its invertibility, its commutators expansion and tame estimates, is developed in Section 12, using pseudo-differential and also paradifferential calculus.

Once the linearized operator has been reduced to constant coefficients up to a domesticated remainder (end of Section 9), its invertibility is straightforward by imposing the first order Melnikov non-resonance conditions [see (10.2)], and is proved in Section 10, where the dependence of the eigenvalues on the parameters is also discussed. In Section 11 we construct a solution of the water waves problem as the limit of a Nash–Moser sequence converging in a Sobolev norm, for a large set of parameters, whose Lebesgue measure is estimated in Section 11.2. Finally, Section 13 collects some standard technical facts used in the previous sections.

2. The Equation and Main Result

We recall the Craig–Sulem–Zakharov formulation which allows one to reduce the analysis of the Euler equation to a problem on the boundary, by introducing the Dirichlet–Neumann operator.

We consider an incompressible liquid occupying a domain Ω with a free surface. Namely, at time $t \geq 0$, the fluid domain is

$$\Omega(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R} \,:\, y < \eta(t,x)\}$$

where η is an unknown function. We assume that the flow is incompressible and also irrotational, so that the velocity field v is given by $v = \nabla_{x,y}\phi$ for some harmonic velocity potential $\phi\colon \Omega \to \mathbb{R}$. Following ZAKHAROV [49], introduce the trace of the potential on the free surface:

$$\psi(t,x) = \phi(t,x,\eta(t,x)).$$

Since ϕ is harmonic, η and ψ fully determines ϕ. CRAIG and SULEM (see [23]) observed that one can form a system of two evolution equations for η and ψ by introducing the Dirichlet–Neumann operator $G(\eta)$ which relates ψ to the normal derivative $\partial_n\phi$.

Definition 2.1. Given any functions $\eta, \psi\colon \mathbb{R} \to \mathbb{R}$, set $\Omega := \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y < \eta(x)\}$ and define ϕ as the harmonic extension of ψ in Ω:

$$\Delta\phi = 0 \quad \text{in} \quad \Omega, \quad \phi|_{y=\eta} = \psi, \quad \nabla\phi \to 0 \quad \text{as } y \to -\infty.$$

The Dirichlet–Neumann operator is defined by

$$G(\eta)\psi(x) = \sqrt{1+\eta_x^2}\,\partial_n\phi|_{y=\eta(x)} = (\partial_y\phi)(x,\eta(x)) - \eta_x(x)\cdot(\partial_x\phi)(x,\eta(x)).$$

(We denote by η_x the derivative $\partial_x\eta$.)

The water waves equations are a system of two coupled equations: one equation describing the deformations of the domain and one equation coming from the assumption that the jump of pressure across the free surface is proportional to the mean curvature. Using the Dirichlet–Neumann operator, these equations are

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + g\eta + \dfrac{1}{2}\psi_x^2 - \dfrac{1}{2}\dfrac{\big(G(\eta)\psi + \eta_x\psi_x\big)^2}{1+\eta_x^2} = \kappa H(\eta), \end{cases} \tag{2.1}$$

where g and κ are positive constants and $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) := \partial_x\left(\frac{\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2}}\right) = \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}.$$

The gravity constant g can be normalized by jointly rescaling the time t and the amplitude of ψ. With no loss of generality, in this paper we assume that $g = 1$.

Periodic solutions. We seek solutions $u(t,x) = (\eta(t,x), \psi(t,x))$ of system (2.1) which are periodic, with period 2π in space and period $T = 2\pi/\omega$ in time, where the parameter $\omega > 0$ is an unknown of the problem. Rescaling the time $t \to \omega t$, the problem becomes

$$F(u,\omega) = 0,$$

where $F = (F_1, F_2)$, $u = (\eta, \psi)$ is 2π-periodic both in time and space, and

$$F_1(\eta,\psi) := \omega\partial_t\eta - G(\eta)\psi \tag{2.2}$$

$$F_2(\eta,\psi) := \omega\partial_t\psi + \eta + \frac{1}{2}\psi_x^2 - \frac{1}{2}\frac{1}{1+\eta_x^2}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 - \kappa\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}. \tag{2.3}$$

Functional setting. We use Sobolev spaces of functions with the same regularity both in time and in space: consider the exponential basis $\{e^{i(lt+jx)} : (l, j) \in \mathbb{Z}^2\}$ on $\mathbb{T}^2$, and the standard Sobolev space $H^s := H^s(\mathbb{T}^2, \mathbb{R})$ on $\mathbb{T}^2$ given by

$$H^s = \left\{ f = \sum_{(l,j)\in\mathbb{Z}^2} \hat{f}_{l,j}\, e^{i(lt+jx)} : \|f\|_s^2 := \sum_{(l,j)\in\mathbb{Z}^2} |\hat{f}_{l,j}|^2 \langle l, j\rangle^{2s} < \infty \right\}, \tag{2.4}$$

where $\langle l, j\rangle := \max\{1, |l| + |j|\}$. Also, we set in the natural way $H^s(\mathbb{T}^2, \mathbb{R}^2) := \{u = (\eta, \psi) : \eta, \psi \in H^s\}$ with norm $\|u\|_s^2 := \|\eta\|_s^2 + \|\psi\|_s^2$.

Remark 2.2. Regularity in time and in space could be handled separately, as it is natural when thinking of the Cauchy problem (see [13]). However, we shall consider changes of variables of the form $(t, x) \mapsto (t, x + \beta(t, x))$, which, in some sense, mix the regularity in time and the one in space. To work with regularity in the time-space pair is a convenient choice.

Symmetries. Because of reversibility in time and symmetry in space, the problem has an invariant subspace, where we look for solutions: $X \times Y = \{u = (\eta, \psi) : \eta \in X,\ \psi \in Y\}$,

$$X := \{\eta(t, x) : \eta \text{ even}(t),\ \text{even}(x)\}, \quad Y := \{\psi(t, x) : \psi \text{ odd}(t),\ \text{even}(x)\}. \tag{2.5}$$

The restriction to this subspace is used in Section 4.1 to deal with the space and time averages, and in Section 9 [see (9.15)].

2.1. Main Result

We assume three hypotheses on the surface tension coefficient $\kappa > 0$, which are discussed in the comments below Theorem 2.4. First, $\kappa \notin \mathbb{Q}$. Second, $\kappa \neq \rho_0$, where ρ_0 is the unique real root of the polynomial $p(x) := 136x^3 + 66x^2 + 3x - 8$ (by the rational root test, ρ_0 is an irrational number, and it is in the interval $0.265 < \rho_0 < 0.266$). Third, we assume that κ satisfies the Diophantine condition

$$|\sqrt{1+\kappa}\, l + \sqrt{j + \kappa j^3}| > \frac{\gamma_*}{j^{\tau_*}} \quad \forall l \in \mathbb{Z},\ j \geq 2, \tag{2.6}$$

for some $\gamma_* \in (0, \frac{1}{2})$, where $\tau_* > 1$. The next lemma says that (2.6) is a very mild restriction on κ.

Lemma 2.3. *Let $\kappa_0 > 0$, $\tau_* > 1$. The set*

$$\mathcal{K} = \{\kappa \in [0, \kappa_0] : \exists\, \gamma_* \in (0, 1/2) \text{ such that } \kappa \text{ satisfies } (2.6)\} \tag{2.7}$$

has full Lebesgue measure $|\mathcal{K}| = \kappa_0$.

The proof of Lemma 2.3 is at the end of Section 11.2. Note that almost all positive real numbers κ satisfy all these three hypotheses. The main result of the paper is in the following theorem.

Theorem 2.4. *Assume that $\kappa > 0$ is an irrational number, $\kappa \neq \rho_0$, and κ satisfies (2.6) for some $\gamma_* \in (0, 1/2)$, with $\tau_* = 3/2$. Then there exist constants $C > 0$, $s_0 > 12$, $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there exists a set $\mathcal{G}_\varepsilon \subset [1, 2]$ of parameters with the following properties.*

For every $\xi \in \mathcal{G}_\varepsilon$ there exists a solution $u = (\eta, \psi)$ of $F(u, \omega) = 0$ with time-frequency

$$\omega = \bar{\omega} + \bar{\omega}_2 \varepsilon^2 \xi + \bar{\omega}_3 \varepsilon^3 \xi^{3/2}, \tag{2.8}$$

where $\bar{\omega} := \sqrt{1+\kappa}$ and the coefficients $\bar{\omega}_2, \bar{\omega}_3$ depend only on κ, with $\bar{\omega}_2 \neq 0$. The solution has Sobolev regularity $u \in H^{s_0}(\mathbb{T}^2, \mathbb{R}^2)$, it has parity $u \in X \times Y$, and small amplitude $u = O(\varepsilon)$. More precisely, u has ε-expansion

$$\eta = \varepsilon \sqrt{\xi} \cos(t) \cos(x) + O(\varepsilon^2), \quad \psi = -\varepsilon \sqrt{\xi} \sqrt{1+\kappa}\, \sin(t) \cos(x) + O(\varepsilon^2),$$

where $O(\varepsilon^2)$ denotes a function f such that $\|f\|_{s_0} \leq C\varepsilon^2$.

The set $\mathcal{G}_\varepsilon \subset [1, 2]$ has positive Lebesgue measure $|\mathcal{G}_\varepsilon| \geq 1 - C\varepsilon^{1/18}$, asymptotically full $|\mathcal{G}_\varepsilon| \to 1$ as $\varepsilon \to 0$.

Some comments about the role of the three hypotheses on the surface tension coefficient κ:

1. The first assumption $\kappa \notin \mathbb{Q}$ implies that the linearized problem at the equilibrium $u = 0$ has a nontrivial one-dimensional kernel (see Section 4.1), from which the solution of the nonlinear water waves problem bifurcates. Rational values of κ would lead to a different bifurcation analysis.
2. The second assumption $\kappa \neq \rho_0$ implies that the coefficient $\bar{\omega}_2$ is nonzero (a "twist" condition). As a consequence, the map $\xi \mapsto \omega$ in (2.8) is a bijection for ε sufficiently small. Thus Theorem 2.4 gives the existence of a solution of the water waves system with time-frequency ω for many values ω in an ε^2-neighbourhood of the "unperturbed" frequency $\bar{\omega} := \sqrt{1+\kappa}$. Instead, for $\kappa = \rho_0$ one should push forward with the analysis of the frequency-amplitude relation, looking for a higher order twist condition.
3. The third assumption (2.6) on κ gives a Diophantine control on the small divisors of the *unperturbed* problem, and it is used in the measure estimates in Section 11.2, see in particular Remark 11.3.

Remark 2.5. One could rename $\tilde{\varepsilon} := \varepsilon\sqrt{\xi}$ and work with one parameter $\tilde{\varepsilon}$ instead of two (ε, ξ). However, it is convenient to work with the two parameters ε and ξ to split two different roles: $\varepsilon \ll 1$ merely gives the smallness, while $\xi \in [1, 2]$ allows to control the small divisors by imposing the Melnikov non-resonance conditions.

3. Preliminaries

Notations. The notation $a \leq_s b$ indicates that $a \leq C(s)b$ for some constant $C(s) > 0$ depending on s and possibly on the data of the problem, namely κ, γ_*, τ_* [κ is the surface tension coefficient and γ_*, τ_* are in (2.6)].

Given $\varepsilon > 0$, for functions $u \in H^s(\mathbb{T}^2)$ depending on a parameter $\xi \in \mathcal{G} \subset [1, 2]$, we define

$$\|u\|_s^{\mathrm{Lip}(\varepsilon)} := \|u\|_s^{\sup} + \varepsilon \|u\|_s^{\mathrm{lip}} \quad \text{with}$$
$$\|u\|_s^{\sup} := \sup_{\xi \in \mathcal{G}} \|u(\xi)\|_s, \qquad \|u\|_s^{\mathrm{lip}} := \sup_{\substack{\xi_1, \xi_2 \in \mathcal{G} \\ \xi_1 \neq \xi_2}} \frac{\|u(\xi_1) - u(\xi_2)\|_s}{|\xi_1 - \xi_2|}. \tag{3.1}$$

Properties of the Dirichlet–Neumann operator. We collect some fundamental properties of the Dirichlet–Neumann operator G that are used in the paper, referring to [2,3,31,38] for more details.

The mapping $(\eta, \psi) \to G(\eta)\psi$ is linear with respect to ψ and nonlinear with respect to η. The derivative with respect to η is called the "shape derivative", and it is given by Lannes' formula (see [38,39])

$$G'(\eta)[\tilde{\eta}]\psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{G(\eta + \varepsilon\tilde{\eta})\psi - G(\eta)\psi\} = -G(\eta)(B\tilde{\eta}) - \partial_x(V\tilde{\eta}) \tag{3.2}$$

where we introduced the notations

$$B := B(\eta, \psi) := \frac{\eta_x \psi_x + G(\eta)\psi}{1 + \eta_x^2}, \qquad V := V(\eta, \psi) := \psi_x - B\eta_x. \tag{3.3}$$

Craig, Schanz and Sulem (see [22] and [47, Chapter 11]) have shown that one can expand the Dirichlet–Neumann operator as a sum of pseudo-differential operators with precise estimates for the remainders. Using (3.2) repeatedly, we get the second order Taylor expansion

$$\begin{aligned} G(\eta)\psi &= G(0)\psi + G'(0)[\eta]\psi + \frac{1}{2}G''(0)[\eta, \eta]\psi + G_{\geq 4}(\eta)\psi \\ &= |D_x|\psi - |D_x|(\eta|D_x|\psi) - \partial_x(\eta\partial_x\psi) + \frac{1}{2}\partial_{xx}(\eta^2|D_x|\psi) \\ &\quad + |D_x|(\eta|D_x|(\eta|D_x|\psi)) + \frac{1}{2}|D_x|(\eta^2\psi_{xx}) + G_{\geq 4}(\eta)\psi, \end{aligned} \tag{3.4}$$

where $G(0) = |D_x|$ and $G_{\geq 4}(\eta)\psi$ is of order 4, such that $G_{\geq 4}(\eta)\psi = O(\eta^3\psi)$. Moreover, it follows from [2, Section 2.6] that it satisfies the following estimate: for $s_0 \geq 10$ and any $s \geq s_0$, if $\|\eta\|_{s_0}$ is small enough, then

$$\left\| G_{\geq 4}(\eta)\psi \right\|_{H^s(\mathbb{T})} \leq_s \|\eta\|^2_{H^{s_0}(\mathbb{T})} \left\{ \|\psi\|_{H^{s_0}(\mathbb{T})} \|\eta\|_{H^{s+4}(\mathbb{T})} + \|\eta\|_{H^{s_0}(\mathbb{T})} \|\psi\|_{H^{s+5}(\mathbb{T})} \right\}. \tag{3.5}$$

A key property is that one can use microlocal analysis to study G. In the present case the matter is easier than in a more general case, because the physical problem has space-dimension 2 (see Definition 2.1, where the space variables are $(x, y) \in \mathbb{R}^2$), it is periodic in the horizontal direction $x \in \mathbb{T}$, with infinite depth, so that

$$G(\eta) = |D_x| + \mathcal{R}_G(\eta), \tag{3.6}$$

where the remainder $\mathcal{R}_G = \mathcal{R}_G(\eta)$ is bounded in t and regularizing in x at expense of η. More precisely, there exists a positive constant δ such that, for $\eta(x), h(x)$ functions of x only, independent of time, if $\|\eta\|_{H^5(\mathbb{T})} \leq \delta$, then for any $s \geq 1$,

$$\|\mathcal{R}_G(\eta)\psi\|_{H^s(\mathbb{T})} \leq_s \|\eta\|_{H^{s+4}(\mathbb{T})}\|\psi\|_{H^{1/2}(\mathbb{T})}.$$

This estimate is proved in [2] (see Proposition 2.7.1 in Chapter 2, noticing that the smallness condition on $\|\eta\|_{C^\gamma}$ assumed in this proposition is satisfied provided that $\|\eta\|_{H^5(\mathbb{T})}$ is small enough). Moreover, if $\|\eta\|_{H^5(\mathbb{T})} \leq \delta$, then for all $s \geq 4$ and all $5 \leq \mu \leq s-1$

$$\|G(\eta)\psi\|_{H^\mu(\mathbb{T})} \leq_s \|\psi\|_{H^{\mu+1}(\mathbb{T})} + \|\eta\|_{H^{s+1}(\mathbb{T})}\|\psi\|_{H^5(\mathbb{T})}. \tag{3.7}$$

Similar estimates can be also proved for functions of $(t, x) \in \mathbb{T}^2$, where t plays the role of a parameter, using repeatedly the time-derivative formula

$$\partial_t\{G(\eta)\psi\} = G(\eta)\partial_t\psi + G'(\eta)[\partial_t\eta]\psi = G(\eta)\psi_t - G(\eta)(B\eta_t) - \partial_x(V\eta_t)$$

(see the argument of Section 12.4, where it is explained in details how to extend estimates for functions of $x \in \mathbb{T}$ to include the dependence on time $t \in \mathbb{T}$). Thus in $H^s(\mathbb{T}^2)$ equipped with the norm $\|\cdot\|_s$ defined by (2.4), we have the following (non-sharp) tame bounds: if $\|\eta\|_6 \leq \delta$, then for all $s \geq 2$, $m \geq 0$,

$$\|\mathcal{R}_G(\eta)|D_x|^m\psi\|_s \leq_s \|\psi\|_s\|\eta\|_{7+m} + \|\psi\|_2\|\eta\|_{s+5+m}; \tag{3.8}$$

if $\|\eta\|_6 \leq \delta$, then, for all $s \geq 6$,

$$\|G(\eta)\psi\|_s \leq_s \|\psi\|_{s+1} + \|\eta\|_{s+1}\|\psi\|_6. \tag{3.9}$$

Finally, regarding parities, we note that, if $\eta \in X$, then $G(\eta)$ preserves the parities, namely $G(\eta)\psi \in X$ for $\psi \in X$, and $G(\eta)\psi \in Y$ for $\psi \in Y$.

4. Bifurcation Analysis

Let us consider the linearized equations around the equilibrium $(\eta, \psi) = (0, 0)$. Directly from (2.2)-(2.3), one finds that the linearized operator is

$$L_\omega := F'(0,0) = \begin{pmatrix} \omega\partial_t & -G(0) \\ 1 - \kappa\partial_{xx} & \omega\partial_t \end{pmatrix}. \tag{4.1}$$

4.1. Kernel

We study the kernel of L_ω. Let $\eta \in X$, $\psi \in Y$, namely

$$\eta(t,x) = \sum_{l\geq 0,\, j\geq 0} \eta_{lj} \cos(lt)\cos(jx), \quad \psi(t,x) = \sum_{l\geq 1,\, j\geq 0} \psi_{lj} \sin(lt)\cos(jx), \tag{4.2}$$

where $\eta_{lj}, \psi_{lj} \in \mathbb{R}$ and, for convenience, we also define $\psi_{lj} := 0$ for $l = 0$, as it does not change anything in the sum. Recall that $G(0) = |D_x|$,

$$|D_x|\cos(jx) = |j|\cos(jx), \quad |D_x|\sin(jx) = |j|\sin(jx) \quad \forall j \in \mathbb{Z},$$

and $|D_x| = \partial_x \mathcal{H}$, where $\mathcal{H}$ is the Hilbert transform, with

$$\mathcal{H}\cos(jx) = \operatorname{sign}(j)\sin(jx), \quad \mathcal{H}\sin(jx) = -\operatorname{sign}(j)\cos(jx) \quad \forall j \in \mathbb{Z},$$

namely $|D_x|e^{ijx} = |j|e^{ijx}$, $\mathcal{H}e^{ijx} = -i\operatorname{sign}(j)e^{ijx}$ for all $j \in \mathbb{Z}$. Hence

$$L_\omega[\eta,\psi] = \sum_{l,j\geq 0} \begin{pmatrix} (-\omega l\eta_{lj} - j\psi_{lj})\sin(lt)\cos(jx) \\ [(1+\kappa j^2)\eta_{lj} + \omega l\psi_{lj}]\cos(lt)\cos(jx) \end{pmatrix}. \tag{4.3}$$

Assume that $L_\omega[\eta,\psi] = 0$, so that $\omega l\eta_{lj} + j\psi_{lj} = 0 = (1+\kappa j^2)\eta_{lj} + \omega l\psi_{lj}$ for all $l, j \geq 0$. Since $1+\kappa j^2 \geq 1 > 0$, we get

$$\eta_{lj} = -\frac{\omega l}{1+\kappa j^2}\psi_{lj}, \quad \{\omega^2 l^2 - j(1+\kappa j^2)\}\psi_{lj} = 0.$$

At $l = 0$, this implies $\eta_{0j} = \psi_{0j} = 0$ for all $j \geq 0$ (recall that $\psi_{0j} = 0$ by definition, see above). For $l \geq 1$, choosing $\omega \neq 0$, we deduce that at $j = 0$ one has $\eta_{l0} = \psi_{l0} = 0$ for all $l \geq 1$.

Hence η_{lj}, ψ_{lj} can be nonzero only for $l, j \geq 1$. If $(\eta_{lj}, \psi_{lj}) \neq (0,0)$, then $\psi_{lj} \neq 0$, and therefore (we assume $\omega > 0$)

$$\omega = \frac{\sqrt{j(1+\kappa j^2)}}{l}.$$

Suppose that there are two pairs (l_1, j_1), (l_2, j_2) that give the same ω, namely

$$\frac{\sqrt{j_1(1+\kappa j_1^2)}}{l_1} = \frac{\sqrt{j_2(1+\kappa j_2^2)}}{l_2}.$$

Taking the square, $\kappa(j_1^3 l_2^2 - j_2^3 l_1^2) + (j_1 l_2^2 - j_2 l_1^2) = 0$. Now, if $\kappa \notin \mathbb{Q}$, then both the integers $(j_1^3 l_2^2 - j_2^3 l_1^2)$ and $(j_1 l_2^2 - j_2 l_1^2)$ are zero, whence $(l_2, j_2) = (l_1, j_1)$. Thus irrational values of κ give a kernel of dimension one. We consider the simplest nontrivial case $(l, j) = (1, 1)$, and fix

$$\bar\omega := \sqrt{1+\kappa}, \qquad \kappa > 0,\ \kappa \notin \mathbb{Q}.$$

The factor

$$\{\bar\omega^2 l^2 - j(1+\kappa j^2)\} \neq 0 \quad \forall l, j \geq 0,\ (l, j) \notin \{(1,1),(0,0)\}. \tag{4.4}$$

The case $(l, j) = (0, 0)$, as already seen, does not give any contribution to the kernel. Thus the kernel of $L_{\bar{\omega}}$ is

$$V := \mathrm{Ker}(L_{\bar{\omega}}) = \{\lambda v_0 : \lambda \in \mathbb{R}\}, \quad v_0 := \begin{pmatrix} \cos(t)\cos(x) \\ -\bar{\omega}\sin(t)\cos(x) \end{pmatrix}. \tag{4.5}$$

There is some freedom in fixing another vector w_0 to span the subspace $(l, j) = (1, 1)$. It is convenient to define

$$\begin{aligned} W &:= \{(\eta, \psi) \in X \times Y : (4.2) \text{ holds, and } (\eta_{11}, \psi_{11}) = \psi_{11}(\bar{\omega}, 1)\} \\ &= W^{(1,1)} \oplus W^{(\neq)}, \end{aligned} \tag{4.6}$$

where

$$W^{(1,1)} := \{\lambda w_0 : \lambda \in \mathbb{R}\}, \quad w_0 := \begin{pmatrix} \bar{\omega}\cos(t)\cos(x) \\ \sin(t)\cos(x) \end{pmatrix} \tag{4.7}$$

and

$$W^{(\neq)} := \{(\eta, \psi) \in X \times Y : (4.2) \text{ holds, and } \eta_{11} = \psi_{11} = 0\}. \tag{4.8}$$

Thus $X \times Y = V \oplus W^{(1,1)} \oplus W^{(\neq)}$, namely every $u \in X \times Y$ can be written in a unique way as $u = av_0 + bw_0 + w$, where $a, b \in \mathbb{R}$ and $w \in W^{(\neq)}$.

4.2. Range

Like F, also $L_{\bar{\omega}}$ maps $X \times Y \to Y \times X$. Let $(f, g) \in Y \times X$, namely

$$f(t, x) = \sum_{l,j \geq 0} f_{lj} \sin(lt)\cos(jx), \quad g(t, x) = \sum_{l,j \geq 0} g_{lj} \cos(lt)\cos(jx), \tag{4.9}$$

with $f_{lj}, g_{lj} \in \mathbb{R}$, $f_{0j} = 0$ for $l = 0$, $j \geq 0$. By (4.3) (with $\omega = \bar{\omega}$), the equation $L_{\bar{\omega}}[\eta, \psi] = (f, g)$ is equivalent to

$$-\bar{\omega} l \eta_{lj} - j\psi_{lj} = f_{lj}, \qquad (1 + \kappa j^2)\eta_{lj} + \bar{\omega} l \psi_{lj} = g_{lj}. \tag{4.10}$$

By (4.4), if $(l, j) \notin \{(1, 1), (0, 0)\}$, then system (4.10) is invertible, with solution

$$\eta_{lj} = \frac{-\bar{\omega} l f_{lj} - j g_{lj}}{\bar{\omega}^2 l^2 - j(1 + \kappa j^2)}, \qquad \psi_{lj} = \frac{(1 + \kappa j^2) f_{lj} + \bar{\omega} l g_{lj}}{\bar{\omega}^2 l^2 - j(1 + \kappa j^2)}. \tag{4.11}$$

For $(l, j) = (0, 0)$, system (4.10) is also invertible, with solution $\eta_{00} = g_{00}$, $\psi_{00} = 0$ (remember that $\psi_{00} = 0 = f_{00}$ by assumption). For $(l, j) = (1, 1)$, system (4.10) has rank one, and it has solutions if and only if $g_{11} + \bar{\omega} f_{11} = 0$, in which case the solutions are $(\eta_{11}, \psi_{11}) = (0, -f_{11}) + \lambda(1, -\bar{\omega})$, $\lambda \in \mathbb{R}$ (clearly $\lambda(1, -\bar{\omega})$ corresponds to λv_0, namely an element of the kernel V). Thus

$$\begin{aligned} R &:= \mathrm{Range}(L_{\bar{\omega}}) = \{(f, g) \in Y \times X : (4.9) \text{ holds, and } g_{11} + \bar{\omega} f_{11} = 0\} \\ &= R^{(1,1)} \oplus R^{(\neq)}, \end{aligned}$$

where

$$R^{(1,1)} := \{\lambda r_0 : \lambda \in \mathbb{R}\}, \quad r_0 := \begin{pmatrix} -\sin(t)\cos(x) \\ \bar{\omega}\cos(t)\cos(x) \end{pmatrix} \tag{4.12}$$

and

$$R^{(\neq)} := \{(f, g) \in Y \times X : (4.9) \text{ holds, and } f_{11} = g_{11} = 0\}. \tag{4.13}$$

There is some freedom in fixing another vector z_0 to span the subspace $(l, j) = (1, 1)$. It is convenient to define

$$Z := \{\lambda z_0 : \lambda \in \mathbb{R}\} \subset Y \times X, \qquad z_0 := \begin{pmatrix} \sin(t)\cos(x) \\ \bar{\omega}\cos(t)\cos(x) \end{pmatrix}. \tag{4.14}$$

Note that $L_{\bar{\omega}}$ is an invertible map of $W^{(\neq)} \to R^{(\neq)}$, and (using the equality $\bar{\omega}^2 = 1 + \kappa$)

$$L_{\bar{\omega}}[w_0] = (\bar{\omega}^2 + 1)r_0 = (2 + \kappa)r_0, \qquad \partial_t v_0 = -z_0. \tag{4.15}$$

Thus $Y \times X = Z \oplus R^{(1,1)} \oplus R^{(\neq)}$, namely every $u \in Y \times X$ can be written in a unique way as $u = az_0 + br_0 + r$, where $a, b \in \mathbb{R}$ and $r \in R^{(\neq)}$. The formula for the projection on r_0, z_0 is

$$\begin{pmatrix} p\sin(t)\cos(x) \\ q\cos(t)\cos(x) \end{pmatrix} = \lambda_r r_0 + \lambda_z z_0, \quad \lambda_r = -\frac{p}{2} + \frac{q}{2\bar{\omega}}, \quad \lambda_z = \frac{p}{2} + \frac{q}{2\bar{\omega}}, \; p, q \in \mathbb{R}. \tag{4.16}$$

4.3. Construction of an Approximate Solution

We look for solutions of (2.1) with frequency ω close to the "unperturbed" frequency $\bar{\omega} = \sqrt{1 + \kappa}$. Write

$$\omega = \bar{\omega} + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots, \quad u = (\eta, \psi) = \varepsilon u_1 + \varepsilon^2 u_2 + \dots,$$

$$F(u) = (\omega - \bar{\omega})\partial_t u + L_{\bar{\omega}}u + \mathcal{N}_2(u) + \mathcal{N}_3(u) + \dots, \quad \mathcal{N}_k(u) = T_k[u, \dots, u],$$

where T_k is a symmetric k-linear map, so that $\mathcal{N}_2(u)$ denotes the quadratic part of F, $\mathcal{N}_3(u)$ the cubic one, etc.. We get

$$F(u) = \varepsilon\mathcal{F}_1 + \varepsilon^2\mathcal{F}_2 + \varepsilon^3\mathcal{F}_3 + \varepsilon^4\mathcal{F}_4 + O(\varepsilon^5),$$

where $\mathcal{F}_1 = L_{\bar{\omega}}u_1$,

$$\begin{aligned}
\mathcal{F}_2 &= L_{\bar{\omega}}u_2 + \omega_1\partial_t u_1 + T_2[u_1, u_1], \\
\mathcal{F}_3 &= L_{\bar{\omega}}u_3 + \omega_2\partial_t u_1 + \omega_1\partial_t u_2 + 2T_2[u_1, u_2] + T_3[u_1, u_1, u_1], \\
\mathcal{F}_4 &= L_{\bar{\omega}}u_4 + \omega_3\partial_t u_1 + \omega_2\partial_t u_2 + \omega_1\partial_t u_3 + 2T_2[u_1, u_3] + T_2[u_2, u_2] \\
&\quad + 3T_3[u_1, u_1, u_2] + T_4[u_1, u_1, u_1, u_1].
\end{aligned}$$

We prove that there exist $u_1, u_2, u_3, u_4, \omega_1, \omega_2, \omega_3$ such that $F(u) = O(\varepsilon^5)$.

Order ε. $\mathcal{F}_1 = 0$ if and only if $u_1 \in V$, namely $u_1 = a_1 v_0$, for some $a_1 \in \mathbb{R}$, where v_0 is defined in (4.5). We assume that $a_1 \neq 0$ (otherwise the construction of u becomes trivial).

Order ε^2. The quadratic part of F [see (2.2)–(2.3)] is

$$T_2[u, u] = \begin{pmatrix} \partial_x(\eta\psi_x) + G_0[\eta G_0\psi] \\ \frac{1}{2}[\psi_x^2 - (G_0\psi)^2] \end{pmatrix},$$

where, for brevity, we write $G_0 := G(0) = |D_x|$. More generally, if $u' = (\eta', \psi')$ and $u'' = (\eta'', \psi'')$, then

$$T_2[u', u''] = \frac{1}{2}\begin{pmatrix} \partial_x(\eta'\psi''_x + \eta''\psi'_x) + G_0[\eta' G_0\psi'' + \eta'' G_0\psi'] \\ \psi'_x\psi''_x - (G_0\psi')(G_0\psi'') \end{pmatrix}. \tag{4.17}$$

In general, for $n, j \geq 0$,

$$\begin{aligned} &\partial_x[\cos(nx)\partial_x \cos(jx)] + G_0[\cos(nx) G_0 \cos(jx)] \\ &\quad = \frac{1}{2} j(|j-n| + n - j) \cos((j-n)x), \end{aligned} \tag{4.18}$$

and $|j-n| + n - j = 2(n-j)$ for $j < n$, and it is zero for $j \geq n$. In particular, for $n = 1$

$$\partial_x[\cos(x)\partial_x \cos(jx)] + G_0[\cos(x) G_0 \cos(jx)] = 0 \quad \forall j \geq 0. \tag{4.19}$$

For $u_1 = a_1 v_0$ [where v_0 is defined in (4.5)], we calculate

$$T_2[u_1, u_1] = a_1^2 T_2[v_0, v_0], \quad T_2[v_0, v_0] = \frac{\bar\omega^2}{4}\begin{pmatrix} 0 \\ [\cos(2t) - 1]\cos(2x) \end{pmatrix}. \tag{4.20}$$

In particular, $T_2[u_1, u_1]$ has no component Fourier-supported on $(l, j) = (1, 1)$. On the contrary, $\partial_t u_1 = a_1 \partial_t v_0$ is Fourier-supported only on $(l, j) = (1, 1)$. Split

$$u_2 = a_2 v_0 + b_2 w_0 + a_1^2 w_2, \quad a_2, b_2 \in \mathbb{R}, \quad w_2 \in W^{(\neq)},$$

where w_0, $W^{(\neq)}$ are defined in (4.7), (4.8). The equation $\Pi_{R^{(\neq)}}\mathcal{F}_2 = 0$ (i.e. the projection on the Fourier modes $(l, j) \neq (1, 1)$) is

$$a_1^2(L_{\bar\omega} w_2 + T_2[v_0, v_0]) = 0. \tag{4.21}$$

Since $L_{\bar\omega} : W^{(\neq)} \to R^{(\neq)}$ is invertible and $a_1 \neq 0$, we solve $L_{\bar\omega} w_2 + T_2[v_0, v_0] = 0$ and, by (4.11) and (4.20), we calculate

$$w_2 = \begin{pmatrix} \alpha_{02}\cos(2x) + \alpha_{22}\cos(2t)\cos(2x) \\ \beta_{22}\sin(2t)\cos(2x) \end{pmatrix} \tag{4.22}$$

with

$$\alpha_{02} := \frac{1+\kappa}{4(1+4\kappa)}, \quad \alpha_{22} := \frac{1+\kappa}{4(1-2\kappa)}, \quad \beta_{22} := -\bar\omega\alpha_{22}$$

(the denominators $1 + 4\kappa$, $1 - 2\kappa$ are nonzero because $\kappa \notin \mathbb{Q}$).

It remains to solve the equation $\mathcal{F}_2 = 0$ on $(l, j) = (1, 1)$. By (4.15), the component of $L_{\bar\omega} u_2$ that is Fourier-supported on $(1, 1)$ is $L_{\bar\omega}[a_2 v_0 + b_2 w_0] = b_2 L_{\bar\omega}[w_0] = b_2(2+\kappa) r_0$. By (4.15), $\partial_t u_1 = a_1 \partial_t v_0 = -a_1 z_0$, while $T_2[u_1, u_1]$ gives no contribution on $(1, 1)$ according to (4.20). Thus the equation projected on $(1, 1)$ is

$$b_2(2+\kappa) r_0 - \omega_1 a_1 z_0 = 0,$$

where r_0 and z_0 are linearly independent. Since $a_1 \neq 0$, we have to choose $\omega_1 = 0$ and $b_2 = 0$. There is no constraint on a_2. It is convenient to fix $a_2 = 0$. With this choice we have $u_2 = a_1^2 w_2$.

Order ε^3. Since $\omega_1 = 0$, one has $\mathcal{F}_3 = L_{\bar\omega} u_3 + \omega_2 \partial_t u_1 + 2T_2[u_1, u_2] + T_3[u_1, u_1, u_1]$. Using the Taylor expansion (3.4) of $G(\eta)$ at $\eta = 0$, for a general $u = (\eta, \psi)$, the cubic part of F is given by

$$T_3[u,u,u] = \begin{pmatrix} -\frac{1}{2}\partial_{xx}(\eta^2 G_0\psi) - G_0(\eta G_0(\eta G_0\psi)) - \frac{1}{2}G_0(\eta^2\psi_{xx}) \\ (G_0\psi)\big(\eta\psi_{xx} + G_0(\eta G_0\psi)\big) + \frac{1}{2}\kappa\,\partial_x(\eta_x^3) \end{pmatrix}.$$

We calculate $T_3[u_1, u_1, u_1] = a_1^3\, T_3[v_0, v_0, v_0]$, where v_0 is in (4.5):

$$T_3[v_0, v_0, v_0] = \frac{-1}{32}\begin{pmatrix} 2\bar\omega[\sin(t) + \sin(3t)]\cos(x) \\ \{(2+11\kappa)\cos(t) + (\kappa-2)\cos(3t)\}[\cos(x) - \cos(3x)] \end{pmatrix}$$

(as usual, we have used that $\bar\omega^2 = 1 + \kappa$). We also calculate $2T_2[u_1, u_2] = 2a_1^3\, T_2[v_0, w_2]$. By (4.17),

$$T_2[v_0, w_2] = -\frac{\bar\omega}{4}\begin{pmatrix} \{(2\alpha_{02} - \alpha_{22})\sin(t) + \alpha_{22}\sin(3t)\}\cos(x) \\ 2\bar\omega\alpha_{22}[\cos(t) - \cos(3t)]\cos(3x) \end{pmatrix}.$$

Split $u_3 = a_3 v_0 + b_3 a_1^3 w_0 + a_1^3 w_3$, where $a_3, b_3 \in \mathbb{R}$, $w_3 \in W^{(\neq)}$, and w_0, $W^{(\neq)}$ are defined in (4.7), (4.8). The projection on $R^{(\neq)}$ of the equation $\mathcal{F}_3 = 0$ is

$$a_1^3\big(L_{\bar\omega} w_3 + \Pi_{R^{(\neq)}}\{2T_2[v_0, w_2] + T_3[v_0, v_0, v_0]\}\big) = 0 \tag{4.23}$$

because $u_1 = a_1 v_0$ and $u_2 = a_1^2 w_2$. Since $a_1 \neq 0$ and $L_{\bar\omega} : W^{(\neq)} \to R^{(\neq)}$ is invertible, the equation (4.23) determines w_3, which depends only on κ.

Let us study the projection of the equation $\mathcal{F}_3 = 0$ on $(l, j) = (1, 1)$. As above, $L_{\bar\omega}[a_3 v_0 + b_3 a_1^3 w_0] = b_3 a_1^3 (2+\kappa) r_0$ and $\partial_t u_1 = a_1 \partial_t v_0 = -a_1 z_0$. Using (4.16), we calculate the projection $\Pi_{(1,1)}$ on $R^{(1,1)} \oplus Z$ (namely on the Fourier mode $(l, j) = (1, 1)$ in $Y \times X$):

$$\begin{aligned} \Pi_{(1,1)}(2T_2[u_1, u_2] + T_3[u_1, u_1, u_1]) &= -\frac{a_1^3}{32}\begin{pmatrix} 2\bar\omega[16\alpha_{02} - 8\alpha_{22} + 1]\sin(t)\cos(x) \\ (2+11\kappa)\cos(t)\cos(x) \end{pmatrix} \\ &= -\frac{a_1^3}{32}\Big(-\bar\omega[16\alpha_{02} - 8\alpha_{22} + 1] + \frac{2+11\kappa}{2\bar\omega}\Big) r_0 \\ &\quad - \frac{a_1^3}{32}\Big(\bar\omega[16\alpha_{02} - 8\alpha_{22} + 1] + \frac{2+11\kappa}{2\bar\omega}\Big) z_0. \end{aligned}$$

Since v_0 and r_0 are linearly independent, $\Pi_{(1,1)}\mathcal{F}_3 = 0$ if and only if

$$\begin{aligned} b_3 a_1^3 (2+\kappa) - \frac{a_1^3}{32}\Big(-\bar\omega[16\alpha_{02} - 8\alpha_{22} + 1] + \frac{2+11\kappa}{2\bar\omega}\Big) &= 0, \\ -\omega_2 a_1 - \frac{a_1^3}{32}\Big(\bar\omega[16\alpha_{02} - 8\alpha_{22} + 1] + \frac{2+11\kappa}{2\bar\omega}\Big) &= 0. \end{aligned}$$

Since $a_1 \neq 0$, the second equation determines ω_2 as

$$\omega_2 = a_1^2 \bar{\omega}_2, \quad \bar{\omega}_2 := \bar{\omega}_2(\kappa) := -\frac{\bar{\omega}}{32}\left(\frac{4(1+\kappa)}{1+4\kappa} - \frac{2(1+\kappa)}{1-2\kappa} + 1 + \frac{2+11\kappa}{2(1+\kappa)}\right) \neq 0, \tag{4.24}$$

then the first equation determines b_3 depending only on κ. Note that $\bar{\omega}_2$ is nonzero for $\kappa \neq \rho_0$, where ρ_0 is the unique real root of the polynomial $p(x) = 136x^3 + 66x^2 + 3x - 8$ (after writing the common denominator in (4.24), one has $\bar{\omega}_2 = 0$ if and only if $p(\kappa) = 0$). There is no constraint on a_3. It is convenient to fix $a_3 = 0$. With this choice we have $u_3 = a_1^3(b_3 w_0 + w_3)$.

Order ε^4. In the previous steps we have found $\omega_1 = 0$, $\omega_2 = a_1^2\bar{\omega}_2$, and $u_1 = a_1 v_0$, $u_2 = a_1^2 w_2$, $u_3 = a_1^3(b_3 w_0 + w_3)$. Let $u_4 = a_4 v_0 + a_1^4(b_4 w_0 + w_4)$, with $a_4, b_4 \in \mathbb{R}$ and $w_4 \in W^{(\neq)}$. The equation $\mathcal{F}_4 = 0$ becomes

$$\begin{aligned} 0 = \omega_3 a_1 \partial_t v_0 + a_1^4\big\{&b_4 L_{\bar{\omega}}[w_0] + L_{\bar{\omega}}[w_4] + \bar{\omega}_2 \partial_t w_2 + 2b_3 T_2[v_0, w_0] + 2T_2[v_0, w_3] \\ &+ T_2[w_2, w_2] + 3T_3[v_0, v_0, w_2] + T_4[v_0, v_0, v_0, v_0]\big\}. \end{aligned} \tag{4.25}$$

Its projection on $R^{(\neq)}$, after eliminating the factor $a_1^4 \neq 0$, is

$$\begin{aligned} L_{\bar{\omega}}[w_4] + \Pi_{R^{(\neq)}}\big\{&\bar{\omega}_2 \partial_t w_2 + 2T_2[v_0, b_3 w_0 + w_3] + T_2[w_2, w_2] \\ &+ 3T_3[v_0, v_0, w_2] + T_4[v_0, v_0, v_0, v_0]\big\} = 0. \end{aligned}$$

Since $L_{\bar{\omega}} : W^{(\neq)} \to R^{(\neq)}$ is invertible, this equation determines w_4, depending only on κ.

By (4.15), the projection of (4.25) on the Fourier mode $(l, j) = (1, 1)$ is

$$a_1^4(b_4(2+\kappa) + \alpha) r_0 + (-\omega_3 a_1 + \beta a_1^4) z_0 = 0$$

for some real coefficients α, β depending only on κ. We choose $\omega_3 = \beta a_1^3$ and $b_4 = -\alpha/(2+\kappa)$, and the equation is satisfied. We also fix $a_4 = 0$, so that $u_4 = a_1^4(b_4 w_0 + w_4)$, and rename $a_1^2 := \xi > 0$, $\beta := \bar{\omega}_3 = \bar{\omega}_3(\kappa)$.

In conclusion, we have found the *frequency-amplitude relation*

$$\omega = \bar{\omega} + \varepsilon^2 \omega_2 + \varepsilon^3 \omega_3 = \bar{\omega} + \varepsilon^2 \bar{\omega}_2 \xi + \varepsilon^3 \bar{\omega}_3 \xi^{3/2} \tag{4.26}$$

where the coefficient $\bar{\omega}_2$ is nonzero and both $\bar{\omega}_2, \bar{\omega}_3$ depend only on κ, and an "approximate solution"

$$\begin{aligned} \bar{u}_\varepsilon = \bar{u}_\varepsilon(\xi) &= \varepsilon \bar{u}_1 + \varepsilon^2 \bar{u}_2 + \varepsilon^3 \bar{u}_3 + \varepsilon^4 \bar{u}_4 \\ &= \varepsilon \sqrt{\xi} v_0 + \varepsilon^2 \xi \bar{w}_2 + \varepsilon^3 \xi^{3/2} \bar{w}_3 + \varepsilon^4 \xi^2 \bar{w}_4, \end{aligned} \tag{4.27}$$

where v_0 is defined in (4.5), $\bar{w}_2 := w_2 \in W^{(\neq)}$, $\bar{w}_3 := b_3 w_0 + w_3 \in W$, $\bar{w}_4 := b_4 w_0 + w_4 \in W$, such that $F(\bar{u}_\varepsilon) = O(\varepsilon^5)$. All $v_0, \bar{w}_2, \bar{w}_3, \bar{w}_4$ depend only on κ. Moreover $\bar{u}_\varepsilon$ is a trigonometric polynomial, Fourier-supported on $\cos(lt)\cos(jx)$, $\sin(lt)\cos(jx)$, with both $l, j \in [0, 5]$.

4.4. Restriction of the Linear Inversion Problem

In view of the Nash–Moser scheme, we have to study the inversion problem for the linearized system: given f, find h such that $F'(u)[h] = f$. In this section we split this (linear) inversion problem in a way that takes advantage of the (nonlinear) calculations we have already done in Section 4.3 to construct the approximate solution $\bar{u}_\varepsilon$.

We assume that $u = \bar{u}_\varepsilon + \tilde{u}$ and $\|\tilde{u}\|_{s_0+\sigma} \le C\varepsilon^{2+\delta}$, where $s_0 \ge 2$, $\sigma \ge 4$, $\delta > 0$. The linearized operator is

$$F'(u)[h] = (\varepsilon^2\omega_2 + \varepsilon^3\omega_3)\partial_t h + L_{\bar{\omega}}h + 2T_2[u, h] + 3T_3[u, u, h] + \mathcal{N}'_{\ge 4}(u)[h]$$

where $\mathcal{N}_{\ge 4}(u)$ denotes the component of F of order at least quartic. In the direction $h = U_\varepsilon := \bar{u}_1 + 2\varepsilon\bar{u}_2 + 3\varepsilon^2\bar{u}_3$ (which is $\partial_\varepsilon\bar{u}_\varepsilon$ truncated at order ε^2) one has

$$\begin{aligned} F'(u)[U_\varepsilon] &= F'(u)[\bar{u}_1 + 2\varepsilon\bar{u}_2 + 3\varepsilon^2\bar{u}_3] \\ &= 2\varepsilon\{L_{\bar{\omega}}[\bar{u}_2] + T_2[\bar{u}_1, \bar{u}_1]\} \\ &\quad + \varepsilon^2\{\omega_2\partial_t\bar{u}_1 + 3L_{\bar{\omega}}[\bar{u}_3] + 6T_2[\bar{u}_1, \bar{u}_2] + 3T_3[\bar{u}_1, \bar{u}_1, \bar{u}_1]\} + \rho \\ &= -2\varepsilon^2\omega_2\partial_t\bar{u}_1 + \rho, \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} \rho := \varepsilon^3\Big\{&\omega_3\partial_t\bar{u}_1 + (\omega_2 + \varepsilon\omega_3)\partial_t\{2\bar{u}_2 + 3\varepsilon\bar{u}_3\} + 6T_2[\bar{u}_1, \bar{u}_3] \\ &+ 2T_2[\bar{u}_2, 2\bar{u}_2 + 3\varepsilon\bar{u}_3] + 2T_2[\bar{u}_3 + \varepsilon\bar{u}_4, U_\varepsilon] + 3T_3[\bar{u}_1, \bar{u}_1, 2\bar{u}_2 + 3\varepsilon\bar{u}_3] \\ &+ 3T_3[2\bar{u}_1 + \varepsilon\bar{u}_2 + \varepsilon^2\bar{u}_3 + \varepsilon^3\bar{u}_4, \bar{u}_2 + \varepsilon\bar{u}_3 + \varepsilon^2\bar{u}_4, U_\varepsilon]\Big\} \\ &+ 2T_2[\tilde{u}, U_\varepsilon] + 6\varepsilon T_3[\tilde{u}, \bar{u}_1 + \varepsilon\bar{u}_2 + \varepsilon^2\bar{u}_3 + \varepsilon^3\bar{u}_4, U_\varepsilon] + 3T_3[\tilde{u}, \tilde{u}, U_\varepsilon] \\ &+ \mathcal{N}'_{\ge 4}(\bar{u}_\varepsilon + \tilde{u})[U_\varepsilon]. \end{aligned}$$

To get (4.28) we have used the equalities $\mathcal{F}_i = 0$, $i = 1, 2, 3$, namely the equations at order ε, ε^2, ε^3 solved in Section 4.3. For $s \ge s_0 \ge 10$, we claim that the function ρ satisfies

$$\|\rho\|_s \le_s \varepsilon^3 + \|\tilde{u}\|_{s+4}. \tag{4.29}$$

Indeed, directly from the above definition of ρ, this estimate is clear for all the terms inside the brackets (recalling that $\bar{u}_k$ are trigonometric polynomials). To estimate the terms $T_2[\tilde{u}, U_\varepsilon]$, $T_3[\tilde{u}, \bar{u}_1 + \varepsilon\bar{u}_2 + \varepsilon^2\bar{u}_3 + \varepsilon^3\bar{u}_4, U_\varepsilon]$ and $T_3[\tilde{u}, \tilde{u}, U_\varepsilon]$, we use the usual nonlinear estimates in Sobolev spaces. The last term $\mathcal{N}'_{\ge 4}(\bar{u}_\varepsilon + \tilde{u})[U_\varepsilon]$ is estimated starting from the linearization formula recalled below in (5.1), using the estimates (3.5) and (3.7) for the Dirichlet–Neumann operator and its Taylor expansion, together with similar estimates for the Taylor expansion for the coefficients B and V which appear in (5.1), see [2, Section 2.6] for these estimates.

Remember that $-\partial_t v_0 = z_0$. Split the datum $f = bz_0 + \tilde{f}$, where $b \in \mathbb{R}$ and $\tilde{f} \in R$. We look for h of the form

$$h = aU_\varepsilon + \tilde{h} = a(\bar{u}_1 + 2\varepsilon\bar{u}_2 + 3\varepsilon^2\bar{u}_3) + \tilde{h},$$

where $a \in \mathbb{R}$, $\tilde{h} = \tilde{h}(t,x) \in W$ are unknowns. For h of this form,

$$F'(u)[h] = aF'(u)[U_\varepsilon] + F'(u)[\tilde{h}] = a(2\varepsilon^2\omega_2\sqrt{\xi}\, z_0 + \rho) + F'(u)[\tilde{h}].$$

Projecting onto Z and R, one has $F'(u)[h] = f$ if and only if

$$\begin{cases} a(2\varepsilon^2\omega_2\sqrt{\xi}\, z_0 + \Pi_Z\rho) + \Pi_Z F'(u)[\tilde{h}] = bz_0 \\ a\Pi_R\rho + \Pi_R F'(u)[\tilde{h}] = \tilde{f}. \end{cases} \tag{4.30}$$

Assume that the restricted operator $\mathcal{L}_R^W := \Pi_R F'(u)_{|W} : W \to R$ is invertible, with

$$\|(\mathcal{L}_R^W)^{-1}g\|_s \leq_s \gamma^{-1}\big(\|g\|_{s+2} + \gamma^{-1}\|\tilde{u}\|_{s+\sigma}\|g\|_{s_0+2}\big) \tag{4.31}$$

for $s \geq s_0$, where $\gamma := \varepsilon^p$, $p := 5/6$. Then we solve for $\tilde{h}$ in the second equation in (4.30) and find

$$\tilde{h} = (\mathcal{L}_R^W)^{-1}(\tilde{f} - a\Pi_R\rho), \tag{4.32}$$

with estimate

$$\|\tilde{h}\|_s \leq_s \gamma^{-1}\{\|\tilde{f}\|_{s+2} + \gamma^{-1}\|\tilde{u}\|_{s+\sigma}\|\tilde{f}\|_{s_0+2} + |a|(\varepsilon^3 + \|\tilde{u}\|_{s+\sigma})\}, \tag{4.33}$$

because $\varepsilon^{2+\delta}\gamma^{-1} < 1$ and $\sigma \geq 4$. Substituting (4.32) in the first equation of (4.30) gives

$$a\{2\varepsilon^2\omega_2\sqrt{\xi}\, z_0 + \Pi_Z\rho - \Pi_Z F'(u)(\mathcal{L}_R^W)^{-1}\Pi_R\rho\} = bz_0 - \Pi_Z F'(u)(\mathcal{L}_R^W)^{-1}\tilde{f}. \tag{4.34}$$

Since $\Pi_Z L_{\bar{\omega}} = 0$, the operator $\Pi_Z F'(u)$ starts quadratically, and it satisfies

$$\|\Pi_Z F'(u)g\|_0 \leq C\|u\|_{s_0+2}\|g\|_2 \leq C\varepsilon\|g\|_{s_0},$$
$$\|\Pi_Z F'(u)(\mathcal{L}_R^W)^{-1}g\|_0 \leq_{s_0} \varepsilon\gamma^{-1}\|g\|_{s_0+2}$$

for all g. As a consequence,

$$\|\Pi_Z F'(u)(\mathcal{L}_R^W)^{-1}\Pi_R\rho\|_0 \leq_{s_0} \varepsilon\gamma^{-1}\|\rho\|_{s_0+2} \leq_{s_0} \varepsilon^{2+\delta}.$$

Therefore the coefficient of az_0 in (4.34) is $2\varepsilon^2\omega_2 + o(\varepsilon^2)$, which is nonzero for ε sufficiently small because $\omega_2 \neq 0$. Hence from (4.34) we find a as a function of b, $\tilde{f}$, with estimate

$$|a| \leq_{s_0} \varepsilon^{-2}(|b| + \varepsilon\gamma^{-1}\|\tilde{f}\|_{s_0+2}) \leq_{s_0} \varepsilon^{-2}(|b| + \|\tilde{f}\|_{s_0+2}) \leq_{s_0} \varepsilon^{-2}\|f\|_{s_0+2} \tag{4.35}$$

(we have used that $\varepsilon\gamma^{-1} < 1$). Then, substituting the value of a in (4.32), we find a formula for $\tilde{h}$ as a function of b, $\tilde{f}$. We have solved the inversion problem $F'(u)[h] = f$. Since $\|h\|_s = \|\tilde{h} + aU_\varepsilon\|_s \leq \|\tilde{h}\|_s + C(s)|a|$, by (4.33), (4.35) we get

$$\begin{aligned} \|h\|_s = \|F'(u)^{-1}f\|_s &\leq_s \gamma^{-1}\|f\|_{s+2} + \varepsilon^{-2}\{1 + \gamma^{-1}\|\tilde{u}\|_{s+\sigma}\}\|f\|_{s_0+2} \\ &\leq_s \varepsilon^{-2}\big(\|f\|_{s+2} + \gamma^{-1}\|\tilde{u}\|_{s+\sigma}\|f\|_{s_0+2}\big). \end{aligned} \tag{4.36}$$

We have proved the following inversion result:

Lemma 4.1. *Let* $u = \bar{u}_\varepsilon + \tilde{u}$, *with* $\|\tilde{u}\|_{s_0+\sigma} \le C\varepsilon^{2+\delta}$ *and* $s_0 \ge 2$, $\sigma \ge 4$, $\delta > 0$. *Assume that the restricted operator* $\mathcal{L}_R^W := \Pi_R F'(u)_{|W} : W \to R$ *is invertible, and its inverse satisfies* (4.31), *where* $\gamma = \varepsilon^{5/6}$. *Then* $F'(u)$ *is invertible, and its inverse satisfies* (4.36) *for all* $s \ge s_0$.

The "loss of regularity" in (4.36) is due to the inverse $(\mathcal{L}_R^W)^{-1}$ on the component W, R, while the "loss of smallness" ε^{-2} ($\gg \gamma^{-1}$) in (4.36) is due to the inverse on the kernel component V, Z. The starting point $\bar{u}_\varepsilon$ of the Nash–Moser scheme is sufficiently accurate ($F(\bar{u}_\varepsilon) = O(\varepsilon^5)$) to overcome both the loss of derivative and the loss of smallness, and therefore we do not need to distinguish the components on W, R, V, Z in the Nash–Moser iteration of Section 11.

In conclusion, we have reduced the inversion problem for the linearized operator $F'(u)$ to the one of inverting $\mathcal{L}_R^W = \Pi_R F'(u)_{|W} : W \to R$.

In view of the transformations of the next sections, it is convenient (although this is not the only option) to split the inversion problem for $\mathcal{L}_R^W$ into its space-Fourier components $\cos(jx)$, with $j = 0, 1$ or $j \ge 2$, because these three cases lead to different situations: $j = 0$, the space average, is the only space-frequency for which $L_{\bar{\omega}}$ gives a triangular, not symmetrizable system; $j = 1$ is the space-frequency of the kernel V; all the other $j \ge 2$ can be studied together using non-trivial infinite-dimensional linear transformations and a symmetrization argument (Sections 5–9).

Thus, to solve the equation $\Pi_R F'(u)_{|W}[h] = f$, we split $R = R_0 \oplus R_1 \oplus R_2$, where the elements of R_0 depend only on time, those of R_1 are space-Fourier-supported on $\cos(x)$, and those of R_2 are space-Fourier-supported on $\cos(jx)$, $j \ge 2$. Denote $R_{01} := R_0 \oplus R_1$. Decompose $W = W_0 \oplus W_1 \oplus W_2$ in the same way, and denote $W_{01} := W_0 \oplus W_1$. Split $h = h_{01} + h_2 \in W$, $f = f_{01} + f_2 \in R$, where $h_{01} \in W_{01}$, $h_2 \in W_2$, and $f_{01} \in R_{01}$, $f_2 \in R_2$. The problem $\Pi_R F'(u)_{|W}[h] = f$ becomes

$$\begin{cases} \Pi_{R_{01}} F'(u)[h_{01} + h_2] = f_{01}, \\ \Pi_{R_2} F'(u)[h_{01} + h_2] = f_2, \end{cases} \quad \text{i.e.} \quad \begin{cases} \mathcal{L}_{01}^{01} h_{01} + \mathcal{L}_{01}^{2} h_2 = f_{01}, \\ \mathcal{L}_{2}^{01} h_{01} + \mathcal{L}_{2}^{2} h_2 = f_2, \end{cases} \tag{4.37}$$

where $\mathcal{L}_{01}^2 := \Pi_{R_{01}} F'(u)_{|W_2} : W_2 \to R_{01}$, etc..

Lemma 4.2. $\mathcal{L}_{01}^{01} : W_{01} \to R_{01}$ *is invertible, with*

$$\|(\mathcal{L}_{01}^{01})^{-1} f\|_s \le_s \|f\|_s + \|\tilde{u}\|_{s+2}\|f\|_{s_0}, \quad \|(\mathcal{L}_{01}^{01})^{-1} f\|_{s_0} \le_{s_0} \|f\|_{s_0}. \tag{4.38}$$

Proof. To invert $\mathcal{L}_{01}^{01}$, we write the linearized operator as $F'(u) = L_\omega + \mathcal{N}'(u)$, where $\mathcal{N} = \mathcal{N}_2 + \mathcal{N}_3 + \dots$ is the nonlinear component of F, and L_ω is its linear one, defined in (4.1). We begin with the invertibility of L_ω as a map of $W_{01} \to R_{01}$. Since L_ω maps $W_0 \to R_0$ and $W_1 \to R_1$, "off-diagonal" one simply has $\Pi_{R_0} L_{\omega|W_1} = 0$ and $\Pi_{R_1} L_{\omega|W_0} = 0$.

Step 1. The restricted linear part $\Pi_{R_0} L_{\omega|W_0} : W_0 \to R_0$ is invertible, because the equation $L_\omega h = f$ in the unknown $h = (\eta, \psi) \in W_0$ with datum $f = (\alpha, \beta) \in R_0$ is the triangular system $\omega\eta'(t) = \alpha(t)$, $\eta(t) + \omega\psi'(t) = \beta(t)$, where η, β are even and ψ, α are odd, and $\eta, \psi, \alpha, \beta$ are functions of t only (this calculation has been done in Section 4.2 with $\bar{\omega}$ instead of ω). Thus $\|(\Pi_{R_0} L_{\omega|W_0})^{-1} f\|_s \le C\|f\|_{s-1}$ for all $s \ge 1$.

Step 2. We prove that the restricted linear part $\Pi_{R_1} L_{\omega|W_1} : W_1 \to R_1$ is invertible. Consider

$$\eta = \sum_{l\geq 0} \eta_l \cos(lt)\cos(x), \quad \psi = \sum_{l\geq 0} \psi_l \sin(lt)\cos(x), \quad \psi_0 = 0, \quad \eta_1 = \bar\omega \psi_1,$$

so that $(\eta, \psi) \in W_1$ (recall (4.7)). Similarly, let $f = \sum_{l\geq 0} f_l \sin(lt)\cos(x)$, $g = \sum_{l\geq 0} g_l \cos(lt)\cos(x)$, with $f_0 = 0$ and $g_1 = -\bar\omega f_1$, so that $(f, g) \in R_1$ [recall (4.12)]. Using the projection (4.16) for $l = 1$, the definition (4.12) of r_0, the assumption $\eta_1 = \bar\omega\psi_1$ and the equality $1 + \kappa = \bar\omega^2$, we obtain

$$\Pi_{R_1} L_\omega(\eta, \psi) = \begin{pmatrix} 0 \\ \bar\omega^2 \eta_0 \cos(x) \end{pmatrix} + \alpha \psi_1 r_0 + \sum_{l\geq 2} \begin{pmatrix} -(\omega l \eta_l + \psi_l)\sin(lt)\cos(x) \\ (\bar\omega^2 \eta_l + \omega l \psi_l)\cos(lt)\cos(x) \end{pmatrix},$$

where $\alpha := (\omega + \bar\omega)(1 + \bar\omega^2)/(2\bar\omega)$. Thus $\Pi_{R_1} L_\omega(\eta, \psi) = (f, g)$ if and only if $\eta_0 = g_0/\bar\omega^2$, $\psi_1 = -f_1/\alpha$, and

$$\eta_l = \frac{-\omega l f_l - g_l}{\omega^2 l^2 - \bar\omega^2}, \quad \psi_l = \frac{\bar\omega^2 f_l + \omega l g_l}{\omega^2 l^2 - \bar\omega^2}, \quad l \geq 2. \tag{4.39}$$

For all $l \geq 2$ the denominator is $\omega^2 l^2 - \bar\omega^2 \geq C l^2$, therefore $|\eta_l| + |\psi_l| \leq C(|f_l| + |g_l|)/l$ for all $l \geq 2$. Hence $\Pi_{R_1} L_{\omega|W_1}$ is invertible, with

$$\|(\Pi_{R_1} L_{\omega|W_1})^{-1} f\|_s \leq C\|f\|_{s-1}$$

for all $f \in R_1$, all $s \geq 1$.

Collecting Steps 1-2 we deduce that $\Pi_{R_{01}} L_{\omega|W_{01}} : W_{01} \to R_{01}$ is invertible, with

$$\|(\Pi_{R_{01}} L_{\omega|W_{01}})^{-1} f\|_s \leq C\|f\|_{s-1} \quad \forall f \in R_{01},\ s \geq 1.$$

Step 3. The linear operator $\mathcal{N}'(u)$ does not contain derivatives with respect to time, and it is a pseudo-differential operator of order 2 with respect to the space variable x. Denoting N_{ij}, $i, j = 1, 2$, the operator-matrix entries of $\mathcal{N}'(u)$, for $h = (\eta(t)\cos(x), \psi(t)\cos(x)) \in W_1$ one simply has

$$\begin{aligned} \mathcal{N}'(u)[h] &= \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} \eta(t)\cos(x) \\ \psi(t)\cos(x) \end{pmatrix} = \eta(t) \begin{pmatrix} N_{11}[\cos(x)] \\ N_{21}[\cos(x)] \end{pmatrix} \\ &\quad + \psi(t) \begin{pmatrix} N_{12}[\cos(x)] \\ N_{22}[\cos(x)] \end{pmatrix}, \end{aligned}$$

and similarly for $h \in W_0$ (just replace $\cos(x)$ with 1). Therefore, for all $h \in W_{01}$,

$$\begin{aligned} &\|\mathcal{N}'(u)_{|W_{01}}[h]\|_s \leq C(s_0)\varepsilon\|h\|_s + C(s)\|u\|_{s+2}\|h\|_{s_0}, \\ &\|\mathcal{N}'(u)_{|W_{01}}[h]\|_{s_0} \leq C(s_0)\varepsilon\|h\|_{s_0}, \end{aligned} \tag{4.40}$$

because $\|u\|_{s_0+2} \leq C\varepsilon$. The conclusion follows by tame Neumann series. $\square$

By Lemma 4.2, we solve for h_{01} in the first line of (4.37), and the system becomes

$$(\mathcal{L}_2^2 + \mathcal{R})h_2 = f_2 - \mathcal{L}_2^{01}(\mathcal{L}_{01}^{01})^{-1} f_{01}, \tag{4.41}$$

where

$$\mathcal{R} := -\mathcal{L}_2^{01}(\mathcal{L}_{01}^{01})^{-1}\mathcal{L}_{01}^2 : W_2 \to R_2.$$

Lemma 4.3. *Let $u = \bar{u}_\varepsilon + \tilde{u}$, with $\|\tilde{u}\|_{s_0+2+m} \le C\varepsilon^{2+\delta}$ for some $s_0 \ge 2$, $m \ge 0$, $\delta > 0$. Then the operator $\mathcal{R}$ defined in (4.41) satisfies for all $s \ge s_0$*

$$\|\mathcal{R}|D_x|^m h\|_s \le_s \varepsilon^2 \|h\|_s + \varepsilon \|\tilde{u}\|_{s+2+m}\|h\|_{s_0}. \tag{4.42}$$

Proof. $(\mathcal{L}_{01}^{01})^{-1}$ is estimated in Lemma 4.2, and $\mathcal{L}_2^{01} = \Pi_{R_2}\mathcal{N}'(u)_{|W_{01}}$ satisfies (4.40) because $\|\Pi_{R_2} h\|_s \le \|h\|_s$. It remains to estimate $\mathcal{L}_{01}^2$. Let $h = (\eta, \psi)$. By the explicit formula (5.1), using integration by parts and the self-adjointness of the Dirichlet–Neumann operator, it follows that both the first and the second component of $\Pi_{01}\mathcal{N}'(u)[\partial_x^m h]$ have the form

$$\left(\int_{\mathbb{T}} (\eta a_0 + \psi b_0)\,\mathrm{d}x\right) + \left(\int_{\mathbb{T}} (\eta a_1 + \psi b_1)\,\mathrm{d}x\right)\cos(x)$$

for some coefficients $a_i(t, x)$, $b_i(t, x)$, $i = 0, 1$, depending on u and of size $O(u)$. Note that both the derivatives contained in $\mathcal{N}'(u)$ and the additional derivatives $|D_x|^m$ go to the coefficients a_i, b_i and do not affect η, ψ. Now, any function $f(t, x)$ of the form $f(t, x) = g(t)$ or $f(t, x) = g(t)\cos(x)$ satisfies $\|f\|_s \le_s \|g\|_{H_t^s}$ (as usual, H_t^s means $H^s(\mathbb{T})$ where the variable is $t \in \mathbb{T}$). Also

$$\begin{aligned}\left\| \int_{\mathbb{T}} \eta(t, x) a(t, x)\,\mathrm{d}x \right\|_{H_t^s} &\le_s \|\eta\|_{L_x^2 H_t^s}\|a\|_{L_x^2 H_t^1} + \|\eta\|_{L_x^2 H_t^1}\|a\|_{L_x^2 H_t^s} \\ &\le_s \|\eta\|_s \|a\|_1 + \|\eta\|_1 \|a\|_s\end{aligned}$$

and similarly for ψb. Therefore, if $\|u\|_{3+m} \le C$, we get $\|\Pi_{01}\mathcal{N}'(u)[\partial_x^m h]\|_s \le_s \|h\|_s \|u\|_{3+m} + \|h\|_1 \|u\|_{s+2+m}$, and the lemma follows by composition. $\square$

Lemma 4.3 will be used with $m = 3/2$ or $m = 2$.

Using (4.37), (4.41) and the estimates for $\mathcal{L}_{01}^{01}$, $\mathcal{L}_{01}^2$, $\mathcal{L}_2^{01}$, we deduce the following inversion result:

Lemma 4.4. *Let $u = \bar{u}_\varepsilon + \tilde{u}$, with $\|\tilde{u}\|_{s_0+\sigma} \le C\varepsilon^{2+\delta}$ and $s_0 \ge 2$, $\sigma \ge 4$, $\delta > 0$. Assume that $\mathcal{L}_2^2 + \mathcal{R} : W_2 \to R_2$ is invertible, and its inverse satisfies*

$$\|(\mathcal{L}_2^2 + \mathcal{R})^{-1} g\|_s \le_s \gamma^{-1}\big(\|g\|_{s+2} + \gamma^{-1}\|\tilde{u}\|_{s+\sigma}\|g\|_{s_0+2}\big) \tag{4.43}$$

for all $s \ge s_0$, where $\gamma = \varepsilon^{5/6}$. Then $\mathcal{L}_R^W := \Pi_R F'(u)_{|W} : W \to R$ is invertible, and its inverse satisfies (4.31) (with the same σ, γ).

Remark 4.5. Collecting Lemmas 4.1 and 4.4, we have reduced the inversion problem for $F'(u)$ to the one of inverting $\mathcal{L}_2^2 + \mathcal{R} : W_2 \to R_2$, where $\mathcal{L}_2^2 = \Pi_{R_2} F'(u)_{|W_2}$ and $\mathcal{R}$ satisfies (4.42). Therefore our goal now is to invert $\mathcal{L}_2^2 + \mathcal{R}$ and to prove (4.43).

We finish this section with the following lemma, which rests on both the result of Section 4.3 and property (4.28) of the linearized operator. Recall the definition (3.1) of the norm $\| \|_s^{\mathrm{Lip}(\varepsilon)}$.

Lemma 4.6. *Let $\omega, \bar{u}_\varepsilon$ be as defined in (4.26)–(4.27). Then*

$$\|F(\bar{u}_\varepsilon, \omega)\|_s^{\mathrm{Lip}(\varepsilon)} \leq_s \varepsilon^5$$

for all $s \geq 0$.

Proof. By the construction of Section 4.3 it follows immediately that $\|F(\bar{u}_\varepsilon, \omega)\|_s \leq_s \varepsilon^5$. It remains to estimate the derivative $\partial_\xi\{F(\bar{u}_\varepsilon, \omega)\} = (\partial_\xi \omega)\partial_t \bar{u}_\varepsilon + F'(\bar{u}_\varepsilon)[\partial_\xi \bar{u}_\varepsilon]$. Recalling (4.27) and the definition $U_\varepsilon := \bar{u}_1 + 2\varepsilon \bar{u}_2 + 3\varepsilon^2 \bar{u}_3$, we get

$$\partial_\xi \bar{u}_\varepsilon = \frac{\varepsilon}{2\xi} U_\varepsilon + 2\varepsilon^4 \xi \bar{w}_4.$$

Hence, using (4.28), and recalling that $\omega_2 = \bar{\omega}_2 \xi$, for $\xi \in [1, 2]$ one has

$$\begin{aligned}\partial_\xi\{F(\bar{u}_\varepsilon, \omega)\} &= \Big(\varepsilon^2 \bar{\omega}_2 + \varepsilon^3 \bar{\omega}_3 \frac{3}{2} \xi^{1/2}\Big)\partial_t \bar{u}_\varepsilon + \frac{\varepsilon}{2\xi} F'(\bar{u}_\varepsilon)[U_\varepsilon] + 2\varepsilon^4 \xi F'(\bar{u}_\varepsilon)[\bar{w}_4] \\ &= \varepsilon^3 \bar{\omega}_2 \partial_t \bar{u}_1 + O(\varepsilon^4) + \frac{\varepsilon}{2\xi}(-2\varepsilon^2 \omega_2 \partial_t \bar{u}_1 + \rho) + O(\varepsilon^4) = O(\varepsilon^4).\end{aligned}$$

Thus $\|\partial_\xi\{F(\bar{u}_\varepsilon, \omega)\}\|_s \leq_s \varepsilon^4$, and the lemma is proved. □

5. Linearized Equation

The computation of the linearized equations is based on formula (3.2) for the "shape derivative" of $G(\eta)\psi$. The derivative of the two components F_1, F_2 of F [see (2.2)–(2.3)] at the point $u = (\eta, \psi)$ in the direction $\tilde{u} = (\tilde{\eta}, \tilde{\psi})$ is

$$\begin{aligned} F_1'(u)[\tilde{u}] &= \omega \partial_t \tilde{\eta} + \partial_x(V\tilde{\eta}) - G(\eta)(\tilde{\psi} - B\tilde{\eta}) \\ F_2'(u)[\tilde{u}] &= \omega \partial_t \tilde{\psi} + V\partial_x \tilde{\psi} - BG(\eta)\tilde{\psi} + (1 + BV_x)\tilde{\eta} + BG(\eta)(B\tilde{\eta}) \\ &\quad - \kappa \partial_x((1 + \eta_x^2)^{-3/2}\partial_x \tilde{\eta}),\end{aligned}$$

as it can be checked by a direct computation, noticing that $B\partial_x(V\tilde{\eta}) - B\psi_x\tilde{\eta}_x + B^2\eta_x\tilde{\eta}_x = BV_x\tilde{\eta}$, where B, V are defined in (3.3).

Notation: any function a is identified with the corresponding multiplication operators $h \mapsto ah$, and, where there is no parenthesis, composition of operators is understood. For example, $\partial_x c \partial_x$ means: $h \mapsto \partial_x(c\partial_x h)$.

Using this notation, one can represent the linearized operator as a 2×2 operator matrix

$$\begin{aligned} F'(u)[\tilde{u}] &= F'(\eta, \psi)\begin{bmatrix}\tilde{\eta} \\ \tilde{\psi}\end{bmatrix} \\ &= \begin{pmatrix} \omega\partial_t + \partial_x V + G(\eta)B & -G(\eta) \\ (1 + BV_x) + BG(\eta)B - \kappa\partial_x c \partial_x & \omega\partial_t + V\partial_x - BG(\eta)\end{pmatrix}\begin{bmatrix}\tilde{\eta} \\ \tilde{\psi}\end{bmatrix}, \end{aligned} \tag{5.1}$$

where

$$c := (1 + \eta_x^2)^{-3/2}. \tag{5.2}$$

The linearized operator $F'(u)$ has the following conjugation structure:

$$F'(u) = \mathcal{Z}\mathcal{L}_0\mathcal{Z}^{-1}, \tag{5.3}$$

where

$$\mathcal{Z} := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \quad \mathcal{Z}^{-1} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}, \quad \mathcal{L}_0 := \begin{pmatrix} \omega\partial_t + \partial_x V & -G(\eta) \\ a - \kappa\partial_x c \partial_x & \omega\partial_t + V\partial_x \end{pmatrix}, \tag{5.4}$$

and a is the coefficient

$$a := 1 + \omega B_t + V B_x. \tag{5.5}$$

Here, a, B, V, c are periodic functions of (t, x), namely they are variable coefficients. This conjugation structure is now well-known (see [38]). Formula (5.3) is verified by a direct calculus, and it is a consequence of the following two facts:

(i) the pseudodifferential terms in F_1' and F_2' are equal except than for a factor B. Hence they cancel in the sum $F_2' + BF_1'$;

(ii) both in the sum $F_2'+BF_1'$ and in F_2' the quantity $\tilde{\phi} := \tilde{\psi} - B\tilde{\eta}$ arises naturally and replaces $\tilde{\psi}$ completely. This $\tilde{\phi}$ is the "good unknown of Alinhac" (see [3,5]).

Remark 5.1. *Parities.* Since $u = (\eta, \psi) \in X \times Y$, it follows that $B \in Y$, $V =$ odd(t), odd(x), $c, a =$ even(t), even(x), and $\mathcal{Z}$ maps $X \to X$ and $Y \to Y$.

We want to obtain a conjugation similar to (5.3) for $\mathcal{L}_2^2 + \mathcal{R}$, see Remark 4.5, where $\mathcal{R} : W_2 \to R_2$ satisfies (4.42) and $\mathcal{L}_2^2 = \Pi_{R_2} F'(u)_{|W_2} : W_2 \to R_2$ (remember that R_2, W_2 are subspaces of functions that are space-Fourier-supported on $\cos(jx)$, $j \geq 2$). Denote, in short, $\mathbb{P}, \mathbb{F}$ the projection

$$\mathbb{P}h(x) = \sum_{j\geq 2} h_j \cos(jx), \quad \mathbb{F}h(x) = h_0 + h_1\cos(x),$$

where $h(x) = \sum_{j\geq 0} h_j \cos(jx)$. Clearly for all $s \geq 0$, $m \geq 0$, all $h(t, x)$,

$$\|\mathbb{P}h\|_s \leq \|h\|_s, \quad \||D_x|^m \mathbb{F}h\|_s \leq \|\mathbb{F}h\|_s \leq \|h_0\|_{H_t^s} + \|h_1\|_{H_t^s} \leq \|h\|_s. \tag{5.6}$$

We want to conjugate $\mathcal{L}_2^2 + \mathcal{R} = \mathbb{P}F'(u)\mathbb{P} + \mathcal{R}$. Let $\tilde{\mathcal{Z}} := \mathbb{P}\mathcal{Z}\mathbb{P}$. By the parity of B, the operator $\tilde{\mathcal{Z}}$ maps $R_2 \to R_2$ and $W_2 \to W_2$. Moreover, by Neumann series, $\tilde{\mathcal{Z}}$ is invertible (see Lemma 5.2 below). Using the equalities $F'(u)\mathcal{Z} = \mathcal{Z}\mathcal{L}_0$ and $I = \mathbb{P} + \mathbb{F}$, we get

$$\begin{aligned}(\mathcal{L}_2^2 + \mathcal{R})\tilde{\mathcal{Z}} &= (\mathbb{P}F'(u)\mathbb{P} + \mathcal{R})\mathbb{P}\mathcal{Z}\mathbb{P} \\ &= \mathbb{P}\mathcal{Z}\mathbb{P}\mathcal{L}_0\mathbb{P} + \mathbb{P}\mathcal{Z}\mathbb{F}\mathcal{L}_0\mathbb{P} - \mathbb{P}F'(u)\mathbb{F}\mathcal{Z}\mathbb{P} + \mathcal{R}\mathbb{P}\mathcal{Z}\mathbb{P}\end{aligned}$$

whence

$$\begin{aligned}\mathcal{L}_2^2 + \mathcal{R} = \tilde{\mathcal{Z}}(\tilde{\mathcal{L}}_0 + \tilde{\mathcal{R}}_0)\tilde{\mathcal{Z}}^{-1}, \quad \tilde{\mathcal{L}}_0 := \mathbb{P}\mathcal{L}_0\mathbb{P}, \\ \tilde{\mathcal{R}}_0 := \tilde{\mathcal{Z}}^{-1}\{\mathbb{P}\mathcal{Z}\mathbb{F}\mathcal{L}_0\mathbb{P} - \mathbb{P}F'(u)\mathbb{F}\mathcal{Z}\mathbb{P} + \mathcal{R}\tilde{\mathcal{Z}}\}.\end{aligned} \tag{5.7}$$

The remainder $\tilde{\mathcal{R}}_0$ has size $O(u^2)$ and it is regularizing of any order in ∂_x. More precisely, we have:

Lemma 5.2. (i) *Let* $u = \bar{u}_\varepsilon + \tilde{u}$, *with* $\|\tilde{u}\|_{s_0+1} \le C\varepsilon^{2+\delta}$, $s_0 \ge 5$, $\delta > 0$. *Then for all* $s \ge s_0$ *the functions* B, V *satisfy* $\|B\|_s + \|V\|_s \le_s \|u\|_{s+1} \le_s \varepsilon + \|\tilde{u}\|_{s+1}$, *and*

$$\|(\mathcal{Z} - I)h\|_s \le C(s_0)\|u\|_{s_0+1}\|h\|_s + C(s)\|u\|_{s+1}\|h\|_{s_0}. \tag{5.8}$$

Also $\tilde{\mathcal{Z}}$ *and* $\tilde{\mathcal{Z}}^{-1}$ *satisfy* (5.8) (*with possibly larger constants* $C(s_0), C(s)$).

(ii) *There is* $\sigma \ge 2$ *such that, if* $\|\tilde{u}\|_{s_0+\sigma+m} \le C\varepsilon^{2+\delta}$, *then the operator* $\tilde{\mathcal{R}}_0 : W_2 \to R_2$ *defined in* (5.7) *satisfies for all* $s \ge s_0$

$$\|\tilde{\mathcal{R}}_0|D_x|^m h\|_s \le_s \varepsilon^2\|h\|_s + \varepsilon\|\tilde{u}\|_{s+\sigma+m}\|h\|_{s_0}. \tag{5.9}$$

Proof. (5.8) holds because $\|(\mathcal{Z} - I)h\|_s \le \|Bh\|_s$. For $\tilde{\mathcal{Z}}$ use that $\|\mathbb{P}h\|_s \le \|h\|_s$, for $\tilde{\mathcal{Z}}^{-1}$ use tame Neumann series. To get the estimate for $\tilde{\mathcal{R}}_0$, note that in $\mathbb{F}\mathcal{L}_0\mathbb{P}$ and in $\mathbb{P}F'(u)\mathbb{F}$ there is no derivative ∂_t, because $\mathbb{F}\partial_t\mathbb{P} = 0 = \mathbb{P}\partial_t\mathbb{F}$. Also use that $\mathbb{P}\mathcal{Z}\mathbb{F} = \mathbb{P}(\mathcal{Z} - I)\mathbb{F}$ and $\mathbb{F}\mathcal{Z}\mathbb{P} = \mathbb{F}(\mathcal{Z} - I)\mathbb{P}$. □

6. Changes of Variables

We have arrived at the inversion problem for $\tilde{\mathcal{L}}_0 + \tilde{\mathcal{R}}_0$ defined in (5.7), where

$$\mathcal{L}_0 = \begin{pmatrix} \omega\partial_t + V\partial_x + V_x & -|D_x| - \mathcal{R}_G \\ a - \kappa c\partial_{xx} - \kappa c_x\partial_x & \omega\partial_t + V\partial_x \end{pmatrix},$$

and $|D_x| + \mathcal{R}_G = G(\eta)$. Our first goal is to obtain a constant coefficient in the term of order ∂_{xx}. To do that, we use two changes of variables: a space-independent change of the time variable (i.e. a reparametrization of time), and a time-dependent change of the space variable.

We start with an elementary observation. Given b_1, b_2, b_3, b_4 functions of (t, x), the system

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

has solutions f, g, p, q,

$$f = \frac{b_1}{\lambda_1 p}, \quad g = \frac{b_3}{\lambda_3 p}, \quad q = \frac{\lambda_1 b_2 p}{b_1 \lambda_2}, \quad p = \text{any},$$

if and only if b_i, λ_i satisfy

$$\frac{b_1 b_4}{b_2 b_3} = \frac{\lambda_1 \lambda_4}{\lambda_2 \lambda_3}.$$

In particular, to have λ_i constant, it is necessary that $b_1 b_4 / b_2 b_3$ is a constant, and, at the leading order, this is the condition we want to obtain after changing the coefficients of $\mathcal{L}_0$ by the changes of variables.

First, consider the change of variable $y = x + \beta(t, x) \Leftrightarrow x = y + \tilde{\beta}(t, y)$, where $\beta(t, x)$ is a periodic function with $|\beta_x| \le 1/2$, and $\tilde{\beta}(t, y)$ is given by the inverse diffeomorphism. Denote

$$(\mathcal{B}h)(t, x) := h(t, x + \beta(t, x)).$$

Conjugation rules for $\mathcal{B}$ are these: $\mathcal{B}^{-1}a\mathcal{B} = (\mathcal{B}^{-1}a)$, namely the conjugate of the multiplication operator $h \mapsto ah$ is the multiplication operator $h \mapsto (\mathcal{B}^{-1}a)h$, and

$$\begin{aligned}
\mathcal{B}^{-1}\partial_x\mathcal{B} &= \{\mathcal{B}^{-1}(1+\beta_x)\}\partial_y, \\
\mathcal{B}^{-1}\partial_{xx}\mathcal{B} &= \{\mathcal{B}^{-1}(1+\beta_x)\}^2\partial_{yy} + (\mathcal{B}^{-1}\beta_{xx})\partial_y, \\
\mathcal{B}^{-1}\partial_t\mathcal{B} &= \partial_t + (\mathcal{B}^{-1}\beta_t)\partial_y, \\
\mathcal{B}^{-1}|D_x|\mathcal{B} &= \{\mathcal{B}^{-1}(1+\beta_x)\}|D_y| + \mathcal{R}_{\mathcal{B}},
\end{aligned}$$

where $\mathcal{R}_{\mathcal{B}} := \{\mathcal{B}^{-1}(1+\beta_x)\}\partial_y(\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H})$ is bounded in time, regularizing in space at expense of η, because

$$\begin{aligned}
\mathcal{B}^{-1}|D_x|\mathcal{B} &= \mathcal{B}^{-1}\partial_x\mathcal{H}\mathcal{B} = (\mathcal{B}^{-1}\partial_x\mathcal{B})(\mathcal{B}^{-1}\mathcal{H}\mathcal{B}) \\
&= \{\mathcal{B}^{-1}(1+\beta_x)\}\partial_y[\mathcal{H} + (\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H})],
\end{aligned}$$

and $(\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H})$ is bounded in time, regularizing in space at expense of η (see Lemma 13.4 in Section 13.1 of the Appendix). Thus

$$\mathcal{L}_1 := \mathcal{B}^{-1}\mathcal{L}_0\mathcal{B} = \begin{pmatrix} \omega\partial_t + a_1\partial_y + a_2 & -a_3|D_y| + \mathcal{R}_1 \\ -\kappa a_4\partial_{yy} - \kappa a_5\partial_y + a_6 & \omega\partial_t + a_1\partial_y \end{pmatrix},$$

where the variable coefficients $a_i = a_i(t, y)$ are

$$\begin{aligned}
a_1 &:= \mathcal{B}^{-1}[\omega\beta_t + V(1+\beta_x)], & a_2 &:= \mathcal{B}^{-1}(V_x), \\
a_3 &:= \mathcal{B}^{-1}(1+\beta_x), & a_4 &:= \mathcal{B}^{-1}[c(1+\beta_x)^2], \\
a_5 &:= \mathcal{B}^{-1}[c\beta_{xx} + c_x(1+\beta_x)], & a_6 &:= \mathcal{B}^{-1}a,
\end{aligned}$$

and $\mathcal{R}_1 := -\mathcal{R}_{\mathcal{B}} - \mathcal{B}^{-1}\mathcal{R}_G\mathcal{B}$.

We want to conjugate $\tilde{\mathcal{L}}_0 + \tilde{\mathcal{R}}_0 = \mathbb{P}\mathcal{L}_0\mathbb{P} + \tilde{\mathcal{R}}_0$. Let $\tilde{\mathcal{B}} := \mathbb{P}\mathcal{B}\mathbb{P}$. The operator $\tilde{\mathcal{B}}$ maps $R_2 \to R_2$ and $W_2 \to W_2$ and it is invertible (see Lemma 6.3 below). Using the equalities $\mathcal{L}_0\mathcal{B} = \mathcal{B}\mathcal{L}_1$ and $I = \mathbb{P} + \mathbb{F}$, we get

$$\tilde{\mathcal{B}}^{-1}(\tilde{\mathcal{L}}_0 + \tilde{\mathcal{R}}_0)\tilde{\mathcal{B}} = \tilde{\mathcal{L}}_1 + \tilde{\mathcal{R}}_1,$$

with

$$\tilde{\mathcal{L}}_1 := \mathbb{P}\mathcal{L}_1\mathbb{P}, \quad \tilde{\mathcal{R}}_1 := \tilde{\mathcal{B}}^{-1}\{\mathbb{P}\mathcal{B}\mathbb{F}\mathcal{L}_1\mathbb{P} - \mathbb{P}\mathcal{L}_0\mathbb{F}\mathcal{B}\mathbb{P} + \tilde{\mathcal{R}}_0\tilde{\mathcal{B}}\}.$$

Second, consider a reparametrization of time $\tau = t + \alpha(t) \Leftrightarrow t = \tau + \tilde{\alpha}(\tau)$, where $\alpha(t)$ is a periodic function with $|\alpha'| \leq 1/2$, and $\tilde{\alpha}(\tau)$ is given by the inverse diffeomorphism. Denote

$$(\mathcal{A}h)(t, y) := h(t + \alpha(t), y).$$

Conjugation rules for $\mathcal{A}$ are these: $\mathcal{A}^{-1}a\mathcal{A} = (\mathcal{A}^{-1}a)$, namely the conjugate of the multiplication operator $h \mapsto ah$ is the multiplication operator $h \mapsto (\mathcal{A}^{-1}a)h$, and

$$\mathcal{A}^{-1}\partial_y\mathcal{A} = \partial_y, \quad \mathcal{A}^{-1}|D_y|\mathcal{A} = |D_y|, \quad \mathcal{A}^{-1}\partial_t\mathcal{A} = \{\mathcal{A}^{-1}(1+\alpha')\}\partial_\tau.$$

Thus

$$\mathcal{L}_2 := \mathcal{A}^{-1}\mathcal{L}_1\mathcal{A} = \begin{pmatrix} \omega a_7\partial_\tau + a_8\partial_y + a_9 & -a_{10}|D_y| + \mathcal{R}_2 \\ -\kappa a_{11}\partial_{yy} - \kappa a_{12}\partial_y + a_{13} & \omega a_7\partial_\tau + a_8\partial_y \end{pmatrix},$$

where $\mathcal{R}_2 = \mathcal{A}^{-1}\mathcal{R}_1\mathcal{A}$ and where the coefficients $a_i = a_i(\tau, y)$ are

$$a_7 := \mathcal{A}^{-1}(1+\alpha'), \qquad a_k := \mathcal{A}^{-1}(a_{k-7}), \quad k = 8, \dots, 13.$$

Note that $a_7(\tau)$ does not depend on y.

We want to conjugate $\tilde{\mathcal{L}}_1 + \tilde{\mathcal{R}}_1 = \mathbb{P}\mathcal{L}_1\mathbb{P} + \tilde{\mathcal{R}}_1$. The transformation $\mathcal{A}$ maps $W_2 \to W_2$ and $R_2 \to R_2$, and commutes with $\mathbb{P}$. Hence $\tilde{\mathcal{A}} := \mathbb{P}\mathcal{A}\mathbb{P} = \mathcal{A}\mathbb{P}$ is the restriction of $\mathcal{A}$ to the subspace W_2 or R_2, and we get

$$\tilde{\mathcal{A}}^{-1}(\tilde{\mathcal{L}}_1 + \tilde{\mathcal{R}}_1)\tilde{\mathcal{A}} = \tilde{\mathcal{L}}_2 + \tilde{\mathcal{R}}_2, \qquad \tilde{\mathcal{L}}_2 := \mathbb{P}\mathcal{L}_2\mathbb{P}, \quad \tilde{\mathcal{R}}_2 := \tilde{\mathcal{A}}^{-1}\tilde{\mathcal{R}}_1\tilde{\mathcal{A}}.$$

Following the elementary observation above, we look for α, β such that

$$m a_7^2 = a_{10}a_{11} \tag{6.1}$$

for some constant $m \in \mathbb{R}$. Since $a_7 = \mathcal{A}^{-1}(1+\alpha')$, $a_{10} = \mathcal{A}^{-1}a_3$, $a_{11} = \mathcal{A}^{-1}a_4$, and $\mathcal{A}^{-1}$ is bijective, (6.1) is equivalent to

$$m(1+\alpha')^2 = a_3 a_4. \tag{6.2}$$

Since $a_3 = \mathcal{B}^{-1}(1+\beta_x)$, $a_4 = \mathcal{B}^{-1}[c(1+\beta_x)^2]$, and $\alpha' = \mathcal{B}^{-1}(\alpha')$, (6.2) is equivalent to

$$m(1+\alpha')^2 = c(1+\beta_x)^3, \tag{6.3}$$

namely $m^{1/3}\,(1+\alpha'(t))^{2/3}\,c(t,x)^{-1/3} = 1 + \beta_x(t,x)$. Integrating this equality in $\mathrm{d}x$, the term β_x disappears because it has zero mean. Therefore

$$1 + \alpha'(t) = m^{-1/2}\left(\frac{1}{2\pi}\int_0^{2\pi} c(t,x)^{-1/3}\,\mathrm{d}x\right)^{-3/2}.$$

Integrating the last equality in $\mathrm{d}t$ determines the constant m:

$$m = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} c(t,x)^{-1/3}\,\mathrm{d}x\right)^{-3/2}\mathrm{d}t\right\}^2 \tag{6.4}$$

and, by (5.2), $c^{-1/3} = (1+\eta_x^2)^{1/2}$.

By construction, $[m^{-1/2}\,(\frac{1}{2\pi}\int c^{-1/3}\,\mathrm{d}x)^{-3/2} - 1]$ has zero average in t, therefore we can fix $\alpha(t)$ as

$$\alpha = \partial_t^{-1}\left[m^{-1/2}\left(\frac{1}{2\pi}\int_0^{2\pi} c(t,x)^{-1/3}\,\mathrm{d}x\right)^{-3/2} - 1\right], \tag{6.5}$$

where ∂_t^{-1} is the Fourier multiplier

$$\partial_t^{-1}\,\mathrm{e}^{ilt} = \frac{1}{il}\,\mathrm{e}^{ilt} \quad \forall l \in \mathbb{Z}\backslash\{0\}, \qquad \partial_t^{-1}\,1 = 0.$$

By construction, $[m^{1/3}(1+\alpha')^{2/3}c^{-1/3}-1]$ has zero average in x, therefore we can fix $\beta(t,x)$ as

$$\beta = \partial_x^{-1}\Big[m^{1/3}(1+\alpha'(t))^{2/3}c(t,x)^{-1/3}-1\Big], \tag{6.6}$$

where ∂_x^{-1} is defined in the same way as ∂_t^{-1}. With these choices of α, β, (6.1) holds, with m given in (6.4). We have found formulas

$$\begin{aligned} 1+\beta_x &= \sqrt{1+\eta_x^2}\,\left(\frac{1}{2\pi}\int_{\mathbb{T}}\sqrt{1+\eta_x^2}\,\mathrm{d}x\right)^{-1}, \\ 1+\alpha'(t) &= \frac{1}{\sqrt{m}}\left(\frac{1}{2\pi}\int_0^{2\pi}\sqrt{1+\eta_x^2}\,\mathrm{d}x\right)^{-3/2}. \end{aligned} \tag{6.7}$$

Remark 6.1. Since $c \in X$, it follows that $\alpha = \text{odd}(t), \text{even}(x)$ and $\beta = \text{even}(t), \text{odd}(x)$. As a consequence, both the transformations $\mathcal{A}$ and $\mathcal{B}$ preserve parities, namely they map $X \to X$ and $Y \to Y$, and $\mathcal{A}^{-1}, \mathcal{B}^{-1}$ do the same. Therefore

$$\begin{aligned} &a_1, a_8 = \text{odd}(t), \text{odd}(x); \quad a_2, a_9 \in Y; \quad a_3, a_4, a_6, a_7, a_{10}, a_{11}, a_{13} \in X; \\ &a_5, a_{12} = \text{even}(t), \text{odd}(x). \end{aligned}$$

□

We follow the elementary observation above, with $\lambda_1 = \lambda_2 = \lambda_4 = 1, \lambda_3 = m$, $p = 1$. Let

$$\begin{aligned} P &:= \begin{pmatrix} a_7 & 0 \\ 0 & a_{11}m^{-1} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} a_7^{-1} & 0 \\ 0 & a_{11}^{-1}m \end{pmatrix}, \\ Q &:= \begin{pmatrix} 1 & 0 \\ 0 & a_{10}^{-1}a_7 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & a_{10}a_7^{-1} \end{pmatrix}, \end{aligned}$$

and calculate

$$\mathcal{L}_3 := P^{-1}\mathcal{L}_2 Q = \begin{pmatrix} \omega\partial_\tau + a_{14}\partial_y + a_{15} & -|D_y| + a_{16}\mathcal{H} + \mathcal{R}_3 \\ -m\kappa\partial_{yy} + m\kappa a_{17}\partial_y + m a_{18} & \omega\partial_\tau + a_{14}\partial_y + a_{19} \end{pmatrix}, \tag{6.8}$$

where, using (6.1),

$$\begin{aligned} &a_{14} := \tfrac{a_8}{a_7}, \qquad a_{15} := \tfrac{a_9}{a_7}, \qquad a_{16} := -\tfrac{a_{10}}{a_7}\left(\tfrac{a_7}{a_{10}}\right)_y, \\ &a_{17} := -\tfrac{a_{12}}{a_{11}}, \qquad a_{18} := \tfrac{a_{13}}{a_{11}}, \qquad a_{19} := m\omega\tfrac{a_7}{a_{11}}\left(\tfrac{a_7}{a_{10}}\right)_\tau + m\tfrac{a_8}{a_{11}}\left(\tfrac{a_7}{a_{10}}\right)_y \end{aligned}$$

and

$$\mathcal{R}_3 := -\frac{a_{10}}{a_7}\,\partial_y\Big[\mathcal{H}, \frac{a_7}{a_{10}}\Big] + \frac{1}{a_7}\,\mathcal{R}_2\frac{a_7}{a_{10}}. \tag{6.9}$$

The commutator $[\mathcal{H}, f]$ of the Hilbert transform $\mathcal{H}$ and the multiplication by any function f is bounded in τ and regularizing in y at expense of f (see Lemma 13.4 in Section 13.1 of the Appendix). To calculate (6.8) we have used (6.1).

We want to conjugate $\tilde{\mathcal{L}}_2+\tilde{\mathcal{R}}_2=\mathbb{P}\mathcal{L}_2\mathbb{P}+\tilde{\mathcal{R}}_2$. Let $\tilde{P}:=\mathbb{P}P\mathbb{P}$ and $\tilde{Q}:=\mathbb{P}Q\mathbb{P}$. Using the equalities $P\mathcal{L}_3=\mathcal{L}_2Q$ and $I=\mathbb{P}+\mathbb{F}$, we get

$$\tilde{P}^{-1}(\tilde{\mathcal{L}}_2+\tilde{\mathcal{R}}_2)\tilde{Q}=\tilde{\mathcal{L}}_3+\tilde{\mathcal{R}}_3,$$

with

$$\tilde{\mathcal{L}}_3:=\mathbb{P}\mathcal{L}_3\mathbb{P},\quad \tilde{\mathcal{R}}_3:=\tilde{P}^{-1}\{\mathbb{P}P\mathbb{F}\mathcal{L}_3\mathbb{P}-\mathbb{P}\mathcal{L}_2\mathbb{F}Q\mathbb{P}+\tilde{\mathcal{R}}_2\tilde{Q}\}. \tag{6.10}$$

Thus we have conjugate

$$\tilde{\mathcal{L}}_0+\tilde{\mathcal{R}}_0=\tilde{\mathcal{B}}\tilde{\mathcal{A}}\tilde{P}(\tilde{\mathcal{L}}_3+\tilde{\mathcal{R}}_3)\tilde{Q}^{-1}\tilde{\mathcal{A}}^{-1}\tilde{\mathcal{B}}^{-1},$$

and the coefficients of ∂_τ, ∂_{yy}, $|D_y|$ in $\mathcal{L}_3$ are constants.

Remark 6.2. Using the parities of a_i, $i\le 13$, it follows that

$$a_{14}=\text{odd}(\tau),\text{odd}(y);\quad a_{15},a_{19}\in Y;\quad a_{16},a_{17}=\text{even}(\tau),\text{odd}(y);\quad a_{18}\in X.$$

□

Lemma 6.3. *There is $\sigma\ge 2$ with the following properties.* (i) *Let $u=\bar{u}_\varepsilon+\tilde{u}$, with $\|\tilde{u}\|_{s_0+\sigma}\le C\varepsilon^{2+\delta}$, $s_0\ge 5$, $\delta>0$. Then all the operators $\tilde{\mathcal{B}},\tilde{\mathcal{A}},\tilde{P},\tilde{Q}$ map $W_2\to W_2$ and $R_2\to R_2$, and they are all invertible. The inverse operators $\tilde{\mathcal{B}}^{-1},\tilde{\mathcal{A}}^{-1},\tilde{P}^{-1},\tilde{Q}^{-1}$ also map $W_2\to W_2$ and $R_2\to R_2$. All theses operators satisfy, for all $s\ge s_0$,*

$$\|Ah\|_s\le_s\|h\|_s+\|\tilde{u}\|_{s+\sigma}\|h\|_{s_0},\qquad A\in\left\{\tilde{\mathcal{B}},\tilde{\mathcal{A}},\tilde{P},\tilde{Q},\tilde{\mathcal{B}}^{-1},\tilde{\mathcal{A}}^{-1},\tilde{P}^{-1},\tilde{Q}^{-1}\right\}. \tag{6.11}$$

(ii) *The functions $a_i(\tau,y)$, $i=14,\dots,19$ satisfy*

$$\|a_{14}\|_s+\|a_{15}\|_s+\|a_{16}\|_s+\|a_{17}\|_s+\|a_{18}-1\|_s+\|a_{19}\|_s\le_s\varepsilon+\|\tilde{u}\|_{s+\sigma}.$$

(iii) *If $\|\tilde{u}\|_{s_0+\sigma+m}\le C\varepsilon^{2+\delta}$ for some $m\ge 0$, then for all $s\ge s_0$ the operator $\tilde{\mathcal{R}}_3:W_2\to R_2$ defined in (6.10) satisfies the same estimate (5.9) as $\tilde{\mathcal{R}}_0$, and the operator $\mathcal{R}_3$ defined in (6.9) satisfies*

$$\|\mathcal{R}_3|D_x|^m h\|_s\le_s\varepsilon\|h\|_s+\|\tilde{u}\|_{s+\sigma+m}\|h\|_{s_0}. \tag{6.12}$$

Proof. (i) The proof of the invertibility of $\tilde{\mathcal{B}}$ is based on these arguments: $\mathcal{B}$ is invertible of $X\to X$ and $Y\to Y$; $\mathcal{B}-I$ is of order $O(\beta)=O(\varepsilon^2)$ in size and of order 1 in ∂_x, therefore $\mathbb{F}(\mathcal{B}-I)\mathbb{F}$ is small and bounded (because $\mathbb{F}\partial_x$ is bounded). As a consequence, $\mathbb{F}\mathcal{B}\mathbb{F}$ is invertible by the Neumann series. Then $\mathbb{P}\mathcal{B}\mathbb{P}$ is invertible by a standard argument of linear systems. (See also the proof of Lemma 9.3, where the same argument is used to prove the invertibility of another operator, and it is described in more detail.) The invertibility of $\tilde{\mathcal{A}}$ is trivial, because $\tilde{\mathcal{A}}h=\mathcal{A}h$ for all $h\in W_2$ or $h\in R_2$, and $\mathcal{A}$ is invertible. The invertibility of $P,Q,\tilde{P},\tilde{Q}$ follows by a Neumann series.

(ii) Composition estimates for all $a_1,\dots,a_{19}$.

(iii) The estimate for $\tilde{\mathcal{R}}_3$ is proved similarly as for $\tilde{\mathcal{R}}_0$, see Lemma 5.2. The estimate for the term $\mathcal{R}_G$ in $\mathcal{R}_3$ comes from (3.8). □

7. Symmetrization of Top Order

From now on, we will use Fourier multipliers $|D_x|^\alpha$ also for negative powers α. Even if our operators act on periodic functions, it is convenient to define their symbols on all $\mathbb{R}$, and to fix the following definition.

Definition 7.1. Given $\alpha \in \mathbb{R}$, the operator $|D_x|^\alpha$ is the Fourier multiplier with symbol $g_\alpha(\xi)$, where $g_\alpha : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $g_\alpha(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, $g_\alpha(\xi) = |\xi|^\alpha$ for $|\xi| \geq 2/3$, and $g_\alpha(\xi) = 0$ for $|\xi| \leq 1/3$.

Hence for any periodic function $h(x) = \sum_{j\in\mathbb{Z}} h_j e^{ijx}$, from the definition one has $|D_x|^\alpha h(x) = \sum_{j\neq 0} h_j |j|^\alpha e^{ijx}$, for any $\alpha \in \mathbb{R}$ (and note that the frequency $j = 0$ becomes irrelevant after applying the projection $\mathbb{P}$).

Now let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $g(\xi) > 0$ for all $\xi \in \mathbb{R}$, and

$$g(\xi) := \left(\frac{1+\kappa\xi^2}{|\xi|}\right)^{\frac{1}{2}} \quad \forall|\xi| \geq 2/3; \qquad g(\xi) := 1 \quad \forall|\xi| \leq 1/3. \tag{7.1}$$

Let Λ be the Fourier multiplier of symbol g. Let

$$S = \begin{pmatrix} 1 & 0 \\ 0 & m^{1/2}\Lambda \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & m^{-1/2}\Lambda^{-1} \end{pmatrix},$$

where m is the real constant in (6.8), so that

$$\begin{aligned} \mathcal{L}_3^+ &:= S^{-1}\mathcal{L}_3 S = \begin{pmatrix} A_3 & m^{1/2}B_3\Lambda \\ m^{-1/2}\Lambda^{-1}C_3 & \Lambda^{-1}D_3\Lambda \end{pmatrix} =: \begin{pmatrix} A_3^+ & B_3^+ \\ C_3^+ & D_3^+ \end{pmatrix}, \\ \mathcal{L}_3 &= \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix}, \end{aligned} \tag{7.2}$$

where, in short, A_3, B_3, C_3, D_3 are the entries of $\mathcal{L}_3$ [see (6.8)]. Recall the asymptotic formula for the composition of the Fourier multiplier Λ and any multiplication operator $h \mapsto ah$:

$$\Lambda(au) \sim \sum_{n=0}^{\infty} \frac{1}{i^n n!} (\partial_x^n a)(x)\ \mathrm{Op}(\partial_\xi^n g)u, \tag{7.3}$$

where $\mathrm{Op}(\partial_\xi^n g)$ is the Fourier multiplier with symbol $\partial_\xi^n g(\xi)$. Thus $A_3^+ = A_3 = \omega\partial_\tau + a_{14}\partial_y + a_{15}$,

$$B_3^+ = -T + \sqrt{m}a_{16}\mathcal{H}|D_y|^{-1/2}(1-\kappa\partial_{yy})^{1/2} + \sqrt{m}\,\mathcal{R}_3\Lambda \tag{7.4}$$

$$\begin{aligned} C_3^+ &= T + \sqrt{m}\,\kappa a_{17}\Lambda^{-1}\partial_y + \sqrt{m}\,\kappa(a_{17})_y(\Lambda^{-1})_1\partial_y + \sqrt{m}(a_{18}-1)\Lambda^{-1} \\ &\quad + \sqrt{m}\,\pi_0 + \mathcal{R}_{3,C}^+ \end{aligned} \tag{7.5}$$

$$\begin{aligned} D_3^+ &= \omega\partial_\tau + a_{14}\partial_y + (a_{14})_y(\Lambda^{-1})_1\Lambda\partial_y + (a_{14})_{yy}(\Lambda^{-1})_2\Lambda\partial_y + a_{19} \\ &\quad + (a_{19})_y(\Lambda^{-1})_1\Lambda + \mathcal{R}_{3,D}^+ \end{aligned} \tag{7.6}$$

where

$$T := \sqrt{m}|D_y|^{1/2}(1-\kappa\partial_{yy})^{1/2}, \tag{7.7}$$

- π_0 is the space average, $\pi_0(h) := \frac{1}{2\pi}\int_{\mathbb{T}} h \,\mathrm{d}x$;
- $(\Lambda^{-1})_1 := \mathrm{Op}(-i\partial_\xi(1/g))$ and $(\Lambda^{-1})_2 := \mathrm{Op}(-\partial_\xi^2(1/g))$ are the terms corresponding to $n = 1$ and $n = 2$ respectively in the expansion (7.3) of Λ^{-1};
- the remainder $\mathcal{R}^+_{3,C}$ satisfies (6.12) with $m = 3/2$ (use Lemma 12.10 with $\beta = 0$ to estimate the remainders in the composition formula for $\Lambda^{-1}a_{17}$, $\Lambda^{-1}(a_{18} - 1)$, and use the bounds in Lemma 6.3 to estimate a_{17} and $(a_{18})_y = (a_{18} - 1)_y$);
- the remainder $\mathcal{R}^+_{3,D}$ satisfies (6.12) with $m = 2$ (use Lemma 12.10 with $\beta = 0$, and the bounds in Lemma 6.3 for a_{14}, a_{19}).

Note that, regarding size,

$$\mathbb{P}\mathcal{L}_3^+\mathbb{P} = \begin{pmatrix} \omega\partial_t & -T \\ T & \omega\partial_t \end{pmatrix}\mathbb{P} + O(\varepsilon). \tag{7.8}$$

For $|j| \geq 1$, let

$$r_j := \sqrt{1 + \kappa j^2} - \sqrt{\kappa}|j| - \frac{1}{2\sqrt{\kappa}|j|}. \tag{7.9}$$

Then $|r_j| \leq C_\kappa |j|^{-3}$ for all $j \neq 0$, for some constant $C_\kappa > 0$ depending only on κ. Therefore

$$(1 - \kappa\partial_{yy})^{1/2} = \sqrt{\kappa}|D_y| + \frac{1}{2\sqrt{\kappa}}|D_y|^{-1} + \mathcal{R}_\kappa + \pi_0 \tag{7.10}$$

where $\mathcal{R}_\kappa$ is the Fourier multiplier of symbol r_j in (7.9), which satisfies $\|\mathcal{R}_\kappa |D_x|^3 h\|_s \leq C_\kappa \|h\|_s$ for all s. Similarly, we expand

$$\begin{aligned}
\Lambda^{-1} &= \frac{1}{\sqrt{\kappa}}|D_y|^{-1/2} - \frac{1}{2\kappa^{3/2}}|D_y|^{-5/2} + O(|D_y|^{-9/2}),\\
(\Lambda^{-1})_1 &= -\frac{1}{2\sqrt{\kappa}}|D_y|^{-3/2}\mathcal{H} + O(|D_y|^{-7/2}),\\
(\Lambda^{-1})_2 &= -\frac{3}{4\sqrt{\kappa}}|D_y|^{-5/2} + O(|D_y|^{-9/2}).
\end{aligned}$$

Each remainder denoted by $O(|D_y|^\alpha)$ in the last three equalities is a Fourier multiplier whose symbol $g(\xi)$ satisfies $|g(\xi)| \leq C(1 + |\xi|)^\alpha$ for all $\xi \in \mathbb{R}$, for some $C > 0$ depending only on κ. Using the equality $\partial_y = -|D_y|\mathcal{H}$, we get

$$B_3^+ = -T + \sqrt{m\kappa}\, a_{16}|D_y|^{1/2}\mathcal{H} + O(|D_y|^{-3/2}) \tag{7.11}$$

$$C_3^+ = T - \sqrt{m\kappa}\, a_{17}|D_y|^{1/2}\mathcal{H} + a_{25}|D_y|^{-1/2} + \sqrt{m}\,\pi_0 + O(|D_y|^{-3/2}) \tag{7.12}$$

$$D_3^+ = \omega\partial_\tau + a_{14}\partial_y + a_{27} + a_{28}|D_y|^{-1}\mathcal{H} + O(|D_y|^{-3/2}) \tag{7.13}$$

where

$$a_{25} := \frac{\sqrt{m}}{\sqrt{\kappa}}\,(a_{18} - 1) - \frac{1}{2}\sqrt{m\kappa}\,(a_{17})_y, \qquad a_{27} := a_{19} - \frac{1}{2}(a_{14})_y,$$

$$a_{28} := \frac{3}{4}(a_{14})_{yy} - \frac{1}{2}(a_{19})_y.$$

The three remainders denoted by $O(|D_y|^{-3/2})$ in (7.11), (7.12), (7.13) all satisfy (6.12) with $m = 3/2$.

Let $\tilde{S} := \mathbb{P}S\mathbb{P} = \mathbb{P}S = S\mathbb{P}$, and note that $\tilde{S}^{-1} = S^{-1}\mathbb{P}$. Define

$$\mathcal{L}_4 := \begin{pmatrix} A_4 & B_4 \\ C_4 & D_4 \end{pmatrix}, \tag{7.14}$$

$$\begin{aligned} A_4 &:= \omega\partial_\tau + a_{14}\partial_y + a_{15}, \\ B_4 &:= -T + \sqrt{m\kappa}\, a_{16}|D_y|^{1/2}\mathcal{H}, \\ C_4 &:= T - \sqrt{m\kappa}\, a_{17}|D_y|^{1/2}\mathcal{H} + a_{25}|D_y|^{1/2}, \\ D_4 &:= \omega\partial_\tau + a_{14}\partial_y + a_{27} + a_{28}|D_y|^{-1}\mathcal{H}. \end{aligned}$$

Thus

$$\begin{aligned} &\tilde{S}^{-1}(\tilde{\mathcal{L}}_3 + \tilde{\mathcal{R}}_3)\tilde{S} = \tilde{\mathcal{L}}_4 + \tilde{\mathcal{R}}_4, \\ &\tilde{\mathcal{L}}_4 := \mathbb{P}\mathcal{L}_4\mathbb{P}, \quad \tilde{\mathcal{R}}_4 := \mathbb{P}(\mathcal{L}_3^+ - \mathcal{L}_4)\mathbb{P} + \tilde{S}^{-1}\tilde{\mathcal{R}}_3\tilde{S}. \end{aligned} \tag{7.15}$$

Unlike in C_3^+, the average term π_0 is not present in C_4 because $\pi_0\mathbb{P} = 0$.

Remark 7.2. By the parities of a_i, $i \le 19$, one has $a_{25} \in X$, $a_{27} \in Y$, $a_{28} = \text{odd}(t), \text{odd}(x)$.

Lemma 7.3. (i) *The Fourier multiplier S is an operator of order $1/2$, with $\|Sh\|_s \le C\|h\|_{s+(1/2)}$, for all $s \in \mathbb{R}$. S is invertible, and $\|S^{-1}h\|_s \le C\|h\|_s$, for all s. Moreover $\tilde{S}$, $\tilde{S}^{-1}$ satisfy the same estimates as S, S^{-1} respectively.*

(ii) *There is $\sigma \ge 2$ such that, if $u = \bar{u}_\varepsilon + \tilde{u}$, with $\|\tilde{u}\|_{s_0+\sigma} \le C\varepsilon^{2+\delta}$, $s_0 \ge 5$, $\delta > 0$, then for all $s \ge s_0$*

$$\|a_{25}\|_s + \|a_{27}\|_s + \|a_{28}\|_s \le_s \varepsilon + \|\tilde{u}\|_{s+\sigma}.$$

The operator $\tilde{\mathcal{R}}_4 : W_2 \to R_2$ defined in (7.15) satisfies the same estimate (6.12) as $\mathcal{R}_3$, with $m = 3/2$.

8. Symmetrization of Lower Orders

Let

$$\mathcal{L}_5 := \begin{pmatrix} A_5 & -C_5 \\ C_5 & A_5 \end{pmatrix}, \qquad \begin{aligned} A_5 &:= \omega\partial_\tau + a_{14}\partial_y + a_{29} + a_{30}\mathcal{H}|D_y|^{-1}, \\ C_5 &:= T + a_{31}\mathcal{H}|D_y|^{1/2} + a_{32}|D_y|^{-1/2}, \end{aligned} \tag{8.1}$$

so that $\mathcal{L}_5$ is a "symmetrized" version of $\mathcal{L}_4$ in (7.14). The coefficients $a_{29}, a_{30}, a_{31}, a_{32}$ are unknown real-valued periodic functions of (τ, y). We prove that there is a transformation

$$M = \begin{pmatrix} 1 & g \\ 0 & v \end{pmatrix}, \qquad \begin{aligned} v &= 1 + v_2\mathcal{H}|D_y|^{-1} + v_4|D_y|^{-2}, \\ g &= g_3|D_y|^{-3/2} + g_5\mathcal{H}|D_y|^{-5/2}, \end{aligned}$$

where v_2, v_4, g_3, g_5 are real-valued, periodic functions of (τ, y), such that $\mathcal{L}_4 M - M\mathcal{L}_5 = O(|D_y|^{-3/2})$. Using formula (7.3) to commute $|D_y|^s$ with multiplication operators, namely

$$|D_y|^s a = a|D_y|^s + s a_y |D_y|^{s-1}\mathcal{H} - \frac{s(s-1)}{2} a_{yy}|D_y|^{s-2} + O(|D_y|^{s-3})$$

and also the fact that $\mathcal{H}$ commutes with multiplication operators up to a regularizing rest that enters in the remainder of order $O(|D_y|^{-3/2})$, we calculate the entries of the matrix

$$\mathcal{L}_4 M - M\mathcal{L}_5 = \begin{pmatrix} A_4 - A_5 - gC_5 & B_4 v + C_5 + A_4 g - gA_5 \\ C_4 - vC_5 & D_4 v - vA_5 + C_4 g \end{pmatrix}.$$

To make the notation uniform, we write $\partial_y = -\mathcal{H}|D_y|$, $\partial_y \mathcal{H} = |D_y|$, and $\mathcal{H}\mathcal{H} = -I + \pi_0$. In the following calculations, remember that m, κ are constants, i.e. they do not depend on (τ, y). Denote, in short,

$$\lambda := \sqrt{m\kappa}.$$

Position (1,1). Calculate $gC_5 = \lambda g_3 + (g_3 a_{31} + \lambda g_5)\mathcal{H}|D_y|^{-1} + O(|D_y|^{-2})$. As a consequence, $A_4 - A_5 - gC_5 = O(|D_y|^{-2})$ if

$$a_{15} - a_{29} - \lambda g_3 = 0, \tag{8.2}$$

$$a_{30} + \lambda g_5 + g_3 a_{31} = 0. \tag{8.3}$$

Position (1,2). Calculate

$$\begin{aligned} B_4 v &= -T + \lambda(a_{16} - v_2)\mathcal{H}|D_y|^{1/2} + \lambda\big\{\tfrac{3}{2}(v_2)_y - v_4 - a_{16}v_2\big\}|D_y|^{-1/2} \\ &\quad + O(|D_y|^{-3/2}), \\ A_4 g &= g\omega\partial_\tau - a_{14}g_3\mathcal{H}|D_y|^{-1/2} + O(|D_y|^{-3/2}), \\ gA_5 &= g\omega\partial_\tau - g_3 a_{14}\mathcal{H}|D_y|^{-1/2} + O(|D_y|^{-3/2}). \end{aligned}$$

Therefore $B_4 v + C_5 + A_4 g - gA_5 = O(|D_y|^{-3/2})$ if

$$\lambda(a_{16} - v_2) + a_{31} = 0, \tag{8.4}$$

$$\lambda\{\tfrac{3}{2}(v_2)_y - v_4 - a_{16}v_2\} + a_{32} = 0. \tag{8.5}$$

Position (2,1). Calculate

$$vC_5 = T + (a_{31} + \lambda v_2)\mathcal{H}|D_y|^{1/2} + (a_{32} - v_2 a_{31} + \lambda v_4)|D_y|^{-1/2} + O(|D_y|^{-3/2}).$$

Therefore $C_4 - vC_5 = O(|D_y|^{-3/2})$ if

$$\lambda a_{17} + a_{31} + \lambda v_2 = 0, \tag{8.6}$$

$$a_{25} - a_{32} + v_2 a_{31} - \lambda v_4 = 0. \tag{8.7}$$

Position (2,2). Calculate

$$
\begin{aligned}
D_4 v &= v\omega\partial_\tau - a_{14}\mathcal{H}|D_y| + (a_{27} + a_{14}v_2) \\
&\quad + \big\{\omega(v_2)_\tau + a_{14}(v_2)_y - a_{14}v_4 + a_{27}v_2 + a_{28}\big\}\mathcal{H}|D_y|^{-1} \\
&\quad + O(|D_y|^{-2}), \\
vA_5 &= v\omega\partial_\tau - a_{14}\mathcal{H}|D_y| + (a_{29} + v_2 a_{14}) \\
&\quad + (a_{30} - v_2(a_{14})_y + v_2 a_{29} - v_4 a_{14})\mathcal{H}|D_y|^{-1} + O(|D_y|^{-2}), \\
C_4 g &= \lambda g_3 + \lambda\{\tfrac{3}{2}\,(g_3)_y + g_5 - a_{17}g_3\}\mathcal{H}|D_y|^{-1} + O(|D_y|^{-2}).
\end{aligned}
$$

Therefore $D_4 v - vA_5 + C_4 g = O(|D_y|^{-3/2})$ if

$$
a_{27} - a_{29} + \lambda g_3 = 0, \tag{8.8}
$$

$$
\begin{aligned}
&\omega(v_2)_\tau + (a_{14}v_2)_y + v_2(a_{27} - a_{29}) + a_{28} - a_{30} + \lambda\tfrac{3}{2}\,(g_3)_y \\
&\quad + \lambda g_5 - \lambda a_{17}g_3 = 0.
\end{aligned} \tag{8.9}
$$

Solution of the symmetrization system. (8.2)-(8.9) is a system of 8 equations in the 8 unknowns v_2, v_4, g_3, g_5, a_{29}, a_{30}, a_{31}, a_{32}. First, we solve (8.4) and (8.6), which give

$$
v_2 := \frac{1}{2}\,(a_{16} - a_{17}), \qquad a_{31} := -\frac{\lambda}{2}\,(a_{16} + a_{17}). \tag{8.10}
$$

Next, we solve (8.2) and (8.8), which give

$$
a_{29} := \frac{1}{2}\,(a_{15} + a_{27}), \qquad g_3 := \frac{1}{2\lambda}\,(a_{15} - a_{27}).
$$

Then we solve (8.5) and (8.7), which give

$$
\begin{aligned}
v_4 &:= \frac{1}{2\lambda}\left(\frac{3\lambda}{2}\,(v_2)_y + v_2(a_{31} - \lambda a_{16}) + a_{25}\right), \\
a_{32} &:= \frac{1}{2}\left((\lambda a_{16} + a_{31})v_2 - \frac{3\lambda}{2}\,(v_2)_y + a_{25}\right),
\end{aligned}
$$

and then (8.3) and (8.9), which give

$$
\begin{aligned}
g_5 \; &:= -\frac{1}{2\lambda}\,\big\{\omega(v_2)_\tau + (a_{14}v_2)_y + v_2(a_{27} - a_{29}) + a_{28} + \lambda\tfrac{3}{2}\,(g_3)_y \\
&\qquad + g_3(a_{31} - \lambda a_{17})\big\}\,, \\
a_{30} \; &:= \frac{1}{2}\,\big\{\omega(v_2)_\tau + (a_{14}v_2)_y + v_2(a_{27} - a_{29}) + a_{28} + \lambda\tfrac{3}{2}\,(g_3)_y - g_3(a_{31} \\
&\qquad + \lambda a_{17})\big\}\,.
\end{aligned}
$$

System (8.2)–(8.9) is solved. To be more precise: the system is solved up to a remainder, say $\mathcal{R}_c$, which is arbitrarily regularizing and is the sum of a fixed, finite number of commutators, all of the type $[a, \mathcal{H}]$, where a is a multiplication operator $h \mapsto ah$ by a real-valued function $a(\tau, y)$.

We have found M, $\mathcal{L}_5$ such that $\mathcal{R}_5 := \mathcal{L}_4 M - M\mathcal{L}_5 = O(|D_y|^{-3/2})$ and, more precisely, $\mathcal{R}_5$ satisfies (6.12) with $m = 3/2$. Let $\tilde{M} := \mathbb{P}M\mathbb{P}$, which is invertible by the Neumann series. By the equalities $\mathcal{L}_4 M = M\mathcal{L}_5 + \mathcal{R}_5$ and $I = \mathbb{P} + \mathbb{F}$ we get

$$\tilde{M}^{-1}(\tilde{\mathcal{L}}_4 + \tilde{\mathcal{R}}_4)\tilde{M} = \tilde{\mathcal{L}}_5 + \tilde{\mathcal{R}}_5, \qquad \tilde{\mathcal{L}}_5 := \mathbb{P}\mathcal{L}_5\mathbb{P}, \tag{8.11}$$

$$\tilde{\mathcal{R}}_5 := \tilde{M}^{-1}\{\tilde{\mathcal{R}}_4\tilde{M} + \mathbb{P}\mathcal{R}_5\mathbb{P} + \mathbb{P}M\mathbb{F}\mathcal{L}_5\mathbb{P} - \mathbb{P}\mathcal{L}_4\mathbb{F}M\mathbb{P}\}. \tag{8.12}$$

Both η and ψ are real-valued. Therefore, using the complex representation $h := \eta + i\psi \in \mathbb{C}$ of the pair $(\eta, \psi) \in \mathbb{R}^2$, $\eta = \mathrm{Re}\,(h)$, $\psi = \mathrm{Im}\,(h)$,

$$\mathcal{L}_5 = \omega\partial_\tau + iT + a_{14}\partial_y + ia_{31}\mathcal{H}|D_y|^{1/2} + a_{29} + ia_{32}|D_y|^{-1/2} + a_{30}\mathcal{H}|D_y|^{-1}. \tag{8.13}$$

Remark 8.1. By the parity of a_i, $i \le 28$, it follows that

$$a_{29}, g_3 \in Y; \quad a_{30}, g_5 = \mathrm{odd}(t), \mathrm{odd}(x); \quad a_{32}, v_4 \in X; \quad a_{31}, v_2 = \mathrm{even}(t), \mathrm{odd}(x).$$

Hence v maps $X \to X$ and $Y \to Y$ (it preserves the parity), g maps $X \to Y$ and $Y \to X$ (it changes the parity), and M maps the product space $X \times Y = \{(\eta, \psi) : \eta \in X, \psi \in Y\}$ into itself. In complex notation, $M : (X + iY) \to (X + iY)$. The operator $\mathcal{L}_5$ maps $(X + iY) \to (Y + iX)$ (and this is obvious, because $\mathcal{L}_5 = M^{-1}\mathcal{L}_4 M$, and $\mathcal{L}_4 : X \times Y \to Y \times X$).

Lemma 8.2. *There is $\sigma \ge 2$ such that, if $u = \bar{u}_\varepsilon + \tilde{u}$, with $\|\tilde{u}\|_{s_0+\sigma} \le C\varepsilon^{2+\delta}$, $s_0 \ge 5$, $\delta > 0$, then*

$$\begin{aligned} \|a_{29}\|_s &+ \|a_{30}\|_s + \|a_{31}\|_s + \|a_{32}\|_s + \|v_2\|_s + \|v_4\|_s + \|g_3\|_s + \|g_5\|_s \\ &\le_s \varepsilon + \|\tilde{u}\|_{s+\sigma}, \end{aligned}$$

and $\tilde{\mathcal{R}}_5 : W_2 \to R_2$ defined in (8.12) satisfies the same estimate (6.12) as $\mathcal{R}_3$, with $m = 3/2$.

9. Reduction to Constant Coefficients

We have arrived at $\tilde{\mathcal{L}}_5 + \tilde{\mathcal{R}}_5$, where $\tilde{\mathcal{L}}_5 = \mathbb{P}\mathcal{L}_5\mathbb{P}$, $\mathcal{L}_5$ is defined in (8.11) (and T in (7.7)). Rename the variables $y = x$, $\tau = t$. Consider a transformation A of the form

$$h(t, x) = \sum_{j \in \mathbb{Z}} h_j(t)\, \mathrm{e}^{ijx} \quad \mapsto \quad Ah(t, x) = \sum_{j \in \mathbb{Z}} h_j(t)\, p(t, x, j)\, \mathrm{e}^{i\phi(t,x,j)},$$

where the amplitude $p(t, x, j)$ is a symbol of order zero, periodic in (t, x), the phase function $\phi(t, x, j)$ is of the form

$$\phi(t, x, j) = jx + |j|^{1/2}\beta(t, x), \tag{9.1}$$

and $\beta(t, x)$ is a periodic function, with $|\beta_x(t, x)| < 1/2$. A is a periodically-t-dependent x-Fourier integral operator with non-homogeneous phase function. Moreover, A is also the pseudo-differential operator $\mathrm{Op}(a)$ of symbol

$$a(t, x, j) := p(t, x, j)\mathrm{e}^{i|j|^{1/2}\beta(t,x)} \tag{9.2}$$

in Hörmander class $S^0_{\frac{1}{2},\frac{1}{2}}$ (except for the fact that a in (9.2) has finite regularity). Let

$$\mathcal{D} := \omega\partial_t + iT + i\lambda_1|D_x|^{1/2} + i\lambda_{-1}|D_x|^{-1/2}, \tag{9.3}$$

with $\lambda_1, \lambda_{-1} \in \mathbb{R}$. In this section we prove that there exist real constants λ_1, λ_{-1} and functions $p(t,x,j)$, $\beta(t,x)$ such that $\mathcal{L}_5 A - A\mathcal{D} = O(|D_x|^{-3/2})$.

Let $\tau : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that

$$\tau(\xi) = \{m|\xi|(1+\kappa\xi^2)\}^{1/2} \quad \forall|\xi| \geq 2/3; \qquad \tau(\xi) = 0 \quad \forall|\xi| \leq 1/3, \tag{9.4}$$

so that $\mathrm{Op}(\tau) = T$ on the periodic functions. By Lemma 12.10, the commutator $[T, A]$ is given by

$$[T, A] = \sum_{n=1}^{5} \frac{1}{n!}\,\mathrm{Op}(\partial_x^n a) \circ \mathrm{Op}(i^{-n}\partial_\xi^n \tau) + \mathcal{R}_{A1}, \quad \mathcal{R}_{A1} = O(|D_x|^{-3/2}),$$

where a is defined in (9.2) and the remainder $\mathcal{R}_{A1}$ is estimated in the proof of Lemma 9.5. Here only terms with $n \leq 5$ are relevant for our purpose, because $\mathrm{Op}(\partial_x^n a) = O(|D_x|^{n/2})$, $\mathrm{Op}(i^{-n}\partial_\xi^n\tau) = O(|D_x|^{\frac{3}{2}-n})$, and $\frac{n}{2} + \frac{3}{2} - n \leq -\frac{3}{2}$ for all $n \geq 6$. Now, using (7.9) and proceeding like in (7.10),

$$\begin{aligned}
\mathrm{Op}\big(i^{-1}\partial_\xi\tau\big) &= \frac{3\lambda}{2}\,|D_x|^{1/2}\mathcal{H} - \frac{\sqrt{m}}{4\sqrt{\kappa}}\,|D_x|^{-3/2}\mathcal{H} + O\big(|D_x|^{-7/2}\big),\\
\mathrm{Op}\big(i^{-2}\partial_\xi^2\tau\big) &= -\frac{3\lambda}{4}\,|D_x|^{-1/2} + O\big(|D_x|^{-5/2}\big),\\
\mathrm{Op}\big(i^{-3}\partial_\xi^3\tau\big) &= \frac{3\lambda}{8}\,|D_x|^{-3/2}\mathcal{H} + O\big(|D_x|^{-7/2}\big),\\
\mathrm{Op}\big(i^{-4}\partial_\xi^4\tau\big) &= \frac{9\lambda}{16}\,|D_x|^{-5/2} + O\big(|D_x|^{-9/2}\big),\\
\mathrm{Op}\big(i^{-5}\partial_\xi^5\tau\big) &= -\frac{45\lambda}{32}\,|D_x|^{-7/2}\mathcal{H} + O\big(|D_x|^{-11/2}\big).
\end{aligned}$$

Each remainder of the type $O(|D_x|^\alpha)$ in the last five equalities is a Fourier multiplier (independent on (t,x)) that comes from a Taylor expansion in ξ of a derivative of $\tau(\xi)$, and whose symbol $g(\xi)$ satisfies $|g(\xi)| \leq C(1+|\xi|)^\alpha$ for all $\xi \in \mathbb{R}$, for some $C > 0$ depending only on κ, and $g(\xi) = 0$ for $|\xi| \leq 1/3$. We also calculate

$$\begin{aligned}
\partial_x a &= \big\{i|j|^{1/2}p\beta_x + p_x\big\}e^{i|j|^{1/2}\beta},\\
\partial_x^2 a &= \big\{-|j|p\beta_x^2 + i|j|^{1/2}(2p_x\beta_x + p\beta_{xx}) + p_{xx}\big\}e^{i|j|^{1/2}\beta},\\
\partial_x^3 a &= \big\{-i|j|^{3/2}p\beta_x^3 - 3|j|\beta_x(p_x\beta_x + p\beta_{xx})\\
&\qquad + i|j|^{1/2}(3p_{xx}\beta_x + 3p_x\beta_{xx} + p\beta_{xxx}) + p_{xxx}\big\}e^{i|j|^{1/2}\beta},\\
\partial_x^4 a &= \{|j|^2p\beta_x^4 - i|j|^{3/2}(4p_x\beta_x^3 + 6p\beta_x^2\beta_{xx}) + r_1\}e^{i|j|^{1/2}\beta},\\
\partial_x^5 a &= \big\{i|j|^{5/2}p\beta_x^5 + r_2\big\}e^{i|j|^{1/2}\beta},
\end{aligned}$$

where $r_\alpha = r_\alpha(t, x, j)$, $\alpha = 1, 2$, is a function that can be explicitly calculated and satisfies

$$\|r_\alpha(\cdot, \cdot, j)\|_s \leq_s (1 + |j|)^\alpha (\|\beta\|_{s+3} + \|p - 1\|_{s+2}) \quad \forall s \geq 1, \quad \alpha = 1, 2, \quad j \in \mathbb{Z}.$$

The composition $\partial_x A$ is

$$\partial_x A h = \sum_{j \in \mathbb{Z}} \hat{h}_j (ijp + i|j|^{1/2} \beta_x p + p_x) \mathrm{e}^{i\phi(t,x,j)}.$$

Composition formulas for $|D_x|^r A$, $\mathcal{H}|D_x|^r A$ can be obtained by applying Lemma 12.10. In particular, the composition $\mathcal{H}|D_x|^{1/2} A$ is

$$\mathcal{H}|D_x|^{1/2} A h = \sum_{j \neq 0} \hat{h}_j c(t, x, j) \mathrm{e}^{i\phi(t,x,j)} + \mathcal{R}_{A2} h,$$

$$c(t, x, j) = -i \operatorname{sgn}(j) |j|^{1/2} p - i \frac{1}{2} \beta_x p + |j|^{-1/2} \left\{ i \frac{1}{8} \operatorname{sgn}(j) \beta_x^2 p - \frac{1}{2} p_x \right\}$$
$$+ |j|^{-1} \left\{ \frac{1}{8} \operatorname{sgn}(j) \, (2\beta_x p_x + \beta_{xx} p) - i \frac{1}{16} \beta_x^3 p \right\};$$

the composition $|D_x|^{-1/2} A$ is

$$|D_x|^{-1/2} A h = \sum_{j \neq 0} \hat{h}_j c(t, x, j) \mathrm{e}^{i\phi(t,x,j)} + \mathcal{R}_{A3} h,$$
$$c(t, x, j) = |j|^{-1/2} p - |j|^{-1} \frac{1}{2} \operatorname{sgn}(j) \beta_x p;$$

the composition $\mathcal{H}|D_x|^{-1} A$ is

$$\mathcal{H}|D_x|^{-1} A h = \sum_{j \neq 0} \hat{h}_j c(t, x, j) \mathrm{e}^{i\phi(t,x,j)} + \mathcal{R}_{A4} h, \quad c(t, x, j) = -|j|^{-1} i \operatorname{sgn}(j) \, p;$$

and the commutator $[\partial_t, A] = \partial_t A - A \partial_t$ is

$$[\partial_t, A] h = \sum_{j \in \mathbb{Z}} \hat{h}_j c(t, x, j) \mathrm{e}^{i\phi(t,x,j)},$$
$$c(t, x, j) = p_t(t, x, j) + i|j|^{1/2} \beta_t(t, x) p(t, x, j).$$

The remainders $\mathcal{R}_{A2}, \mathcal{R}_{A3}, \mathcal{R}_{A4}$ are all operators of order $-3/2$, and they are estimated in the proof of Lemma 9.3. Using the expansions above, we calculate the difference

$$\mathcal{L}_5 A - A \mathcal{D} = E + \mathcal{R}_{A5}, \tag{9.5}$$

where

$$\mathcal{R}_{A5} := i\mathcal{R}_{A1} + i a_{31} \mathcal{R}_{A2} + i a_{32} \mathcal{R}_{A3} + a_{30} \mathcal{R}_{A4} \tag{9.6}$$

and E is the operator $h \mapsto Eh = \sum_{j \in \mathbb{Z}} h_j(t) \, c(t, x, j) \, \mathrm{e}^{i\phi(t,x,j)}$ having the same phase function ϕ as in (9.1), and amplitude

$$c(t, x, j) = ijp \left(\frac{3}{2} \lambda \beta_x + a_{14} \right) + \sum_{-2 \leq k \leq 1} |j|^{k/2} \, T^{(k)}[p] + r(t, x, j) \tag{9.7}$$

for $j \neq 0$, and $c(t, x, 0) = \omega p_t + a_{29} p$ for $j = 0$. The terms $T^{(k)}$ in (9.7) are the linear differential operators

$$
\begin{aligned}
T^{(1)} &:= v_1^{(1)} \partial_x + v_0^{(1)} - i\lambda_1, \\
T^{(0)} &:= \omega \partial_t + v_1^{(0)} \partial_x + v_0^{(0)}, \\
T^{(-1)} &:= v_2^{(-1)} \partial_{xx} + v_1^{(-1)} \partial_x + v_0^{(-1)} - i\lambda_{-1}, \\
T^{(-2)} &:= v_2^{(-2)} \partial_{xx} + v_1^{(-2)} \partial_x + v_0^{(-2)},
\end{aligned}
$$

with coefficients

$$v_1^{(1)} := \mathrm{sgn}(j) \frac{3\lambda}{2}, \qquad v_0^{(1)} := \mathrm{sgn}(j) a_{31} + i \left(\omega \beta_t + \frac{3\lambda}{8} \beta_x^2 + a_{14} \beta_x \right), \tag{9.8}$$

$$v_1^{(0)} := \frac{3\lambda}{4} \beta_x + a_{14}, v_0^{(0)} := \left(\frac{3\lambda}{8} \beta_{xx} + \frac{1}{2} a_{31} \beta_x + a_{29} \right) - i \mathrm{sgn}(j) \frac{\lambda}{16} \beta_x^3, \tag{9.9}$$

$$v_2^{(-1)} := -i \frac{3\lambda}{8}, \qquad v_1^{(-1)} := -\mathrm{sgn}(j) \frac{3\lambda}{16} \beta_x^2 - i \frac{1}{2} a_{31}, \tag{9.10}$$

$$v_0^{(-1)} := -\mathrm{sgn}(j) \left(\frac{3\lambda}{16} \beta_x \beta_{xx} + \frac{1}{8} a_{31} \beta_x^2 \right) + i \left(\frac{3\lambda}{2^7} \beta_x^4 + a_{32} \right), \tag{9.11}$$

$$v_2^{(-2)} := i \mathrm{sgn}(j) \frac{3\lambda}{16} \beta_x, \qquad v_1^{(-2)} := \frac{3\lambda}{32} \beta_x^3 + i \mathrm{sgn}(j) \left(\frac{3\lambda}{16} \beta_{xx} + \frac{1}{4} a_{31} \beta_x \right), \tag{9.12}$$

$$
\begin{aligned}
v_0^{(-2)} := &\left(\frac{9\lambda}{64} \beta_x^2 \beta_{xx} + \frac{1}{16} a_{31} \beta_x^3 \right) \\
&+ i \mathrm{sgn}(j) \left(-\frac{3\lambda}{2^8} \beta_x^5 + \frac{\lambda}{16} \beta_{xxx} + \frac{1}{8} a_{31} \beta_{xx} - \frac{1}{2} a_{32} \beta_x - a_{30} - \frac{\sqrt{m}}{4\sqrt{\kappa}} \beta_x \right).
\end{aligned} \tag{9.13}
$$

The function $r(t, x, j)$ in (9.7) collects all the terms of order $O(|j|^{-3/2})$ coming from $\mathrm{Op}(\partial_x^n a) \circ \mathrm{Op}(i^{-n} \partial_\xi^n \tau)$, $1 \leq n \leq 5$ in the commutator $[T, A]$, and it satisfies

$$\| r(\cdot, \cdot, j) \|_s \leq_s |j|^{-3/2} (\|\beta\|_{s+3} + \|p - 1\|_{s+2}) \quad \forall s \geq 1, \quad j \in \mathbb{Z} \setminus \{0\}. \tag{9.14}$$

Our goal is to choose $\lambda_1, \lambda_{-1}, \beta, p$ such that the amplitude $c(t, x, j)$ in (9.7) is of order $O(|j|^{-3/2})$. For $j = 0$, we simply fix $p(t, x, 0) := 1$, so that $c(t, x, 0) = a_{29}(t, x)$. For $j \neq 0$, we split the construction into several steps.

Elimination of the order 1. The function a_{14} is odd(t), odd(x), therefore it has zero space-average, and $\partial_x^{-1} a_{14}$, which is the dx-primitive of a_{14} with zero average, is well-posed. We fix

$$\beta(t, x) := \beta_0(t) + \beta_1(t, x), \quad \beta_1 := -\frac{2}{3\lambda} \partial_x^{-1} a_{14}, \tag{9.15}$$

where $\beta_0(t)$ is a periodic function of t only, which will be determined later (see the next step). We have eliminated the terms of order $O(|j|)$ from (9.7). Since a_{14} is odd(t), odd(x), we get $\beta_1 \in Y$.

We seek p under the form

$$p(t, x, j) = \sum_{-3 \leq m \leq 0} |j|^{m/2} p^{(m)}(t, x, j),$$

with all $p^{(m)}$ bounded in j. Then, by linearity, (9.7) becomes

$$c(t,x,j) = \sum_{\substack{-2\leq k\leq 1\\ -3\leq m\leq 0}} |j|^{\frac{k+m}{2}} T^{(k)}[p^{(m)}] + r(t,x,j). \tag{9.16}$$

Elimination of the order 1/2. To eliminate the term of order 1/2 from (9.16), we have to solve the equation

$$T^{(1)}[p^{(0)}] = 0 \tag{9.17}$$

in the unknown $p^{(0)}$. Write $p^{(0)}$ as

$$p^{(0)}(t,x,j) = \exp\big(f(t,x,j)\big), \tag{9.18}$$

so that (9.17) becomes the equation

$$v_1^{(1)} f_x + v_0^{(1)} - i\lambda_1 = 0 \tag{9.19}$$

for the unknown f. The coefficients $v_1^{(1)}$, $v_0^{(1)}$ are given in (9.8). Equation (9.19) has a solution f if and only if

$$\int_{\mathbb{T}} \big(v_0^{(1)}(t,x,j) - i\lambda_1\big)\,\mathrm{d}x = 0 \quad \forall t\in\mathbb{T},\ j\in\mathbb{Z}. \tag{9.20}$$

We look for $\beta_0(t)$, λ_1 such that the (crucial) average condition (9.20) holds. Remember that $\beta = \beta_0 + \beta_1$, where $\beta_1(t,x)$ has already been determined in (9.15) and $\beta_0(t)$ is still free; $\beta_x = (\beta_1)_x$ because β_0 depends only on t. Moreover, $\int_{\mathbb{T}} a_{31}\,\mathrm{d}x = 0$ because a_{31} is odd in x. Therefore (9.20) becomes

$$\omega\partial_t\beta_0(t) - \lambda_1 + \rho(t) = 0,$$

where

$$\rho(t) := \frac{1}{2\pi}\int_0^{2\pi}\bigg(\omega(\beta_1)_t + \frac{3\lambda}{8}(\beta_1)_x^2 + a_{14}(\beta_1)_x\bigg)\,\mathrm{d}x = -\frac{1}{4\pi\lambda}\int_{\mathbb{T}} a_{14}^2\,\mathrm{d}x$$

($(\beta_1)_t$ has zero space-average because $\partial_x^{-1}a_{14}$ has zero space-average). We fix

$$\lambda_1 := \frac{1}{2\pi}\int_0^{2\pi}\rho(t)\,\mathrm{d}t = -\frac{1}{8\pi^2\lambda}\int_{\mathbb{T}^2} a_{14}^2\,\mathrm{d}x\,\mathrm{d}t, \qquad \beta_0 := -\frac{1}{\omega}\,\partial_t^{-1}(\rho-\lambda_1), \tag{9.21}$$

and (9.20) is solved. λ_1 is a negative real number, and β_0 is a real-valued function of t, independent on x, j. Then (9.19) has solutions

$$f(t,x,j) := f_0(t,j) + f_1(t,x,j), \quad f_1 := -\frac{2}{3\lambda}\,\mathrm{sgn}(j)\,\partial_x^{-1}(v_0^{(1)} - i\lambda_1) \tag{9.22}$$

where f_0 does not depend on x and it will be determined in the next step. $p^{(0)} = \exp(f)$ solves (9.17). Since $\beta_1\in Y$, it follows that $\rho\in X$, and therefore $\beta_0\in Y$. Thus $\beta\in Y$, namely $\beta = \beta(t,x)$ is a real-valued function, odd(t), even(x), independent of j. A direct calculation gives

$$f_1 = -\frac{2}{3\lambda}\partial_x^{-1}a_{31} + i\,\mathrm{sgn}(j)\frac{2}{3\lambda}\partial_x^{-1}\bigg(\frac{1}{2\lambda}a_{14}^2 + \rho + \frac{2\omega}{3\lambda}\partial_x^{-1}\partial_t a_{14}\bigg).$$

Remark 9.1. By the parity of a_i, $i \leq 34$, and $\beta \in Y$, it follows that the coefficients $v_m^{(k)}$ have the form

$$\begin{aligned} v_1^{(1)}, v_1^{(-1)} &= \mathrm{sgn}(j)a + ib, \quad a \in X, \quad b = \mathrm{even}(t), \mathrm{odd}(x); \\ v_0^{(1)}, v_2^{(-1)}, v_0^{(-1)} &= \mathrm{sgn}(j)a + ib, \quad a = \mathrm{even}(t), \mathrm{odd}(x), \quad b \in X; \\ v_1^{(0)}, v_1^{(-2)} &= a + i\mathrm{sgn}(j)b, \quad a = \mathrm{odd}(t), \mathrm{odd}(x), \quad b \in Y; \\ v_0^{(0)}, v_2^{(-2)}, v_0^{(-2)} &= a + i\mathrm{sgn}(j)b, \quad a \in Y, \quad b = \mathrm{odd}(t), \mathrm{odd}(x), \end{aligned}$$

where a, b denote (different) real-valued functions of (t, x), independent of j.

By the parity of $v_0^{(1)}$ (see the previous remark), f_1 has the form

$$f_1 = a + i\mathrm{sgn}(j)b, \quad a \in X, \quad b = \mathrm{even}(t), \mathrm{odd}(x), \tag{9.23}$$

with a, b real-valued functions of (t, x), independent of j. In particular, f_1 is even(t).

Elimination of the order 0. The order zero in (9.16) vanishes if

$$T^{(1)}[p^{(-1)}] + T^{(0)}[p^{(0)}] = 0. \tag{9.24}$$

In general, for any function g, one has

$$T^{(1)}[\exp(f)g] = \exp(f)\, v_1^{(1)} g_x \tag{9.25}$$

by (9.19), and

$$T^{(0)}[\exp(f)g] = \exp(f)\, \left(b^{(0)} g + \omega g_t + v_1^{(0)} g_x\right), \quad b^{(0)} := \omega f_t + v_1^{(0)} f_x + v_0^{(0)}. \tag{9.26}$$

In particular, for $g = 1$, we get $T^{(0)}[p^{(0)}] = \exp(f)\, b^{(0)}$. By variation of constants, write $p^{(-1)}$ as

$$p^{(-1)} := p^{(0)} g^{(-1)} = \exp(f) g^{(-1)}.$$

Equation (9.24) becomes

$$v_1^{(1)} g_x^{(-1)} + b^{(0)} = 0 \tag{9.27}$$

in the unknown $g^{(-1)}$. Since $v_1^{(1)}$ is a constant, (9.27) has a solution $g^{(-1)}$ if and only if

$$\int_{\mathbb{T}} b^{(0)}(t, x, j)\, \mathrm{d}x = 0 \quad \forall t \in \mathbb{T}, \ j \in \mathbb{Z}. \tag{9.28}$$

Remember that $f = f_0 + f_1$, where f_1 has already been determined in (9.22), and $f_0 = f_0(t, j)$ is still at our disposal. Thus $b^{(0)} = \omega(f_0)_t + \omega(f_1)_t + v_1^{(0)}(f_1)_x + v_0^{(0)}$. By (9.23) and Remark 9.1,

$$\omega(f_1)_t + v_1^{(0)}(f_1)_x + v_0^{(0)} = a + i\mathrm{sgn}(j)b, \quad a \in Y, \quad b = \mathrm{odd}(t)\mathrm{odd}(x),$$

for some a, b real-valued functions of (t, x), independent of j. Therefore

$$\int_{\mathbb{T}} \left(\omega(f_1)_t + v_1^{(0)}(f_1)_x + v_0^{(0)}\right) \mathrm{d}x = \mathrm{odd}(t)$$

is a real-valued function of t only, independent of x, j, with zero mean (because it is odd). We fix

$$f_0 := -\frac{1}{2\pi\omega}\,\partial_t^{-1}\left\{\int_{\mathbb{T}}\Big(\omega(f_1)_t + v_1^{(0)}(f_1)_x + v_0^{(0)}\Big)\,\mathrm{d}x\right\},$$

and (9.28) is satisfied. f_0 is a real-valued even function of t only, independent of x, j. A direct calculation gives

$$f_0 = -\frac{1}{2\pi\omega}\,\partial_t^{-1}\left\{\int_{\mathbb{T}}\Big(a_{29} - \frac{2}{3\lambda}a_{14}a_{31}\Big)\,\mathrm{d}x\right\}.$$

Remark 9.2. f and $b^{(0)}$ are of the form

$$f = a + i\,\mathrm{sgn}(j)b, \quad a \in X, \quad b = \mathrm{even}(t), \mathrm{odd}(x), \tag{9.29}$$

$$b^{(0)} = a + i\,\mathrm{sgn}(j)b, \quad a \in Y, \quad b = \mathrm{odd}(t), \mathrm{odd}(x), \tag{9.30}$$

where a, b denote (different) real-valued functions of (t, x), independent of j.

We choose

$$g^{(-1)}(t,x,j) := g_0^{(-1)}(t,j) + g_1^{(-1)}(t,x,j), \quad g_1^{(-1)} := -\frac{2}{3\lambda}\,\mathrm{sgn}(j)\,\partial_x^{-1}(b^{(0)}), \tag{9.31}$$

where $g_0^{(-1)}(t, j)$ will be determined at the next step. (9.27) is satisfied. $g_1^{(-1)}$ is of the form

$$g_1^{(-1)} = \mathrm{sgn}(j)a + ib, \quad a = \mathrm{odd}(t), \mathrm{odd}(x), \quad b \in Y, \tag{9.32}$$

for some a, b real-valued functions of (t, x), independent of j.

Elimination of lower orders. Once the first two steps in p are done (i.e. elimination of orders $1/2$ and 0), the algorithm proceeds in a similar way. For the sake of completeness (and to obtain λ_{-1}), we write the calculations for the order $-1/2$ in details, then lower orders will be similar.

Elimination of the order $-1/2$. We have to solve

$$T^{(1)}[p^{(-2)}] + T^{(0)}[p^{(-1)}] + T^{(-1)}[p^{(0)}] = 0 \tag{9.33}$$

in the unknown $p^{(-2)}$ (and also $g_0^{(-1)}$ is still free). By variation of constants, write $p^{(-2)} = \exp(f)g^{(-2)}$. By (9.25) and (9.26),

$$T^{(1)}[p^{(-2)}] = \exp(f)\,v_1^{(1)}g_x^{(-2)},$$
$$T^{(0)}[p^{(-1)}] = \exp(f)\,\Big\{b^{(0)}g^{(-1)} + \omega g_t^{(-1)} + v_1^{(0)}g_x^{(-1)}\Big\}.$$

Recall that $g^{(-1)} = g_0^{(-1)} + g_1^{(-1)}$. Let

$$b^{(-1)} := \exp(-f)\,\Big(T^{(0)}[p^{(-1)}] + T^{(-1)}[p^{(0)}]\Big) = b^{(0)}g_0^{(-1)} + \omega(g_0^{(-1)})_t + r^{(-1)}, \tag{9.34}$$

where

$$r^{(-1)} := b^{(0)}g_1^{(-1)} + \omega(g_1^{(-1)})_t + v_1^{(0)}(g_1^{(-1)})_x + \exp(-f)\,T^{(-1)}[p^{(0)}].$$

Thus (9.33) becomes

$$v_1^{(1)} g_x^{(-2)} + b^{(-1)} = 0. \tag{9.35}$$

If

$$\int_{\mathbb{T}} b^{(-1)}(t, x, j)\, \mathrm{d}x = 0 \quad \forall t \in \mathbb{T},\ j \in \mathbb{Z}, \tag{9.36}$$

then (9.35) has a solution $g^{(-2)}$. By (9.34) and (9.28), the average condition (9.36) becomes

$$2\pi\omega (g_0^{(-1)})_t(t, j) + \int_{\mathbb{T}} r^{(-1)}(t, x, j)\, \mathrm{d}x = 0. \tag{9.37}$$

If

$$\int_{\mathbb{T}^2} r^{(-1)}(t, x, j)\, \mathrm{d}x\, \mathrm{d}t = 0, \tag{9.38}$$

then we can choose $g_0^{(-1)}$ such that (9.37) is satisfied. By (9.32), (9.30) and Remark 9.1, one proves that $r^{(-1)}$ has the form

$$r^{(-1)} = \mathrm{sgn}(j) a + i b - i\lambda_{-1}, \quad a = \mathrm{even}(t), \mathrm{odd}(x), \quad b \in X, \tag{9.39}$$

for some a, b real-valued functions of (t, x), independent of j. We fix

$$\lambda_{-1} := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} b(t, x)\, \mathrm{d}x\, \mathrm{d}t, \tag{9.40}$$

where b in (9.40) is the function b in (9.39), so that (9.38) is satisfied. Note that λ_{-1} is a real number. Then we fix

$$g_0^{(-1)} = -\frac{1}{2\pi\omega} \partial_t^{-1} \Big(\int_{\mathbb{T}} r^{(-1)}(t, x, j)\, \mathrm{d}x \Big),$$

and (9.37) is satisfied. By (9.39) it follows that $g_0^{(-1)}$ is a purely imaginary, odd function of t, independent of x, j, and therefore

$$g^{(-1)} = \mathrm{sgn}(j) a + i b, \quad a = \mathrm{odd}(t), \mathrm{odd}(x), \quad b \in Y, \tag{9.41}$$

for some a, b real-valued functions of (t, x), independent of j. Thus (9.36) is satisfied. We choose

$$g^{(-2)} = g_0^{(-2)}(t, j) + g_1^{(-2)}(t, x, j), \quad g_1^{(-2)} := -\mathrm{sgn}(j) \frac{2}{3\lambda}\, \partial_x^{-1} b^{(-1)}, \tag{9.42}$$

where $g_0^{(-2)}$ is free (it will be fixed in the next step), so that (9.35) is satisfied. $b^{(-1)}$ is of the form

$$b^{(-1)} = \mathrm{sgn}(j) a + i b, \quad a = \mathrm{even}(t), \mathrm{odd}(x), \quad b \in X,$$

therefore $g_1^{(-2)}$ is of the form

$$g_1^{(-2)} = a + i\, \mathrm{sgn}(j) b, \quad a \in X, \quad b = \mathrm{even}(t), \mathrm{odd}(x),$$

where a, b denote (different) real-valued functions of (t, x), independent of j.

Elimination of the order -1. We proceed similarly as in the previous step, with $T^{(-1)}[p^{(-1)}] + T^{(-2)}[p^{(0)}]$ instead of $T^{(-1)}[p^{(0)}]$; $g^{(-3)}$ instead of $g^{(-2)}$; etc. There is no need of leaving $g_0^{(-3)}(t, j)$ free, as this is the last step: so we fix $g_0^{(-3)}(t, j) := 0$, and $g^{(-3)} := g_1^{(-3)}$. Regarding parities, we obtain coefficients of the form

$$\begin{aligned} r^{(-2)}, b^{(-2)} &= a + i\,\mathrm{sgn}(j)b, \quad a \in Y, \quad b = \mathrm{odd}(t), \mathrm{odd}(x), \\ g^{(-2)} &= a + i\,\mathrm{sgn}(j)b, \quad a \in X, \quad b = \mathrm{even}(t), \mathrm{odd}(x), \\ g^{(-3)} &= \mathrm{sgn}(j)a + ib, \quad a = \mathrm{odd}(t), \mathrm{odd}(x), \quad b \in Y, \end{aligned}$$

where a, b denote (different) real-valued functions of (t, x), independent of j.

We have found $\lambda_1, \lambda_{-1}, \beta(t, x), p(t, x, j)$ such that the amplitude $c(t, x, j)$ in (9.7), namely (9.16), is

$$\begin{aligned} c(t, x, j) = {} & |j|^{-\frac{3}{2}}\{T^{(0)}[p^{(-3)}] + T^{(-1)}[p^{(-2)}] + T^{(-2)}[p^{(-1)}]\} \\ & + |j|^{-1}\{T^{(-1)}[p^{(-3)}] + T^{(-2)}[p^{(-2)}]\} \\ & + |j|^{-\frac{5}{2}} T^{(-2)}[p^{(-3)}] + r(t, x, j) \end{aligned} \tag{9.43}$$

for all $j \neq 0$, and therefore the difference $\mathcal{R}_6 := \mathcal{L}_5 A - A\mathcal{D} = E + \mathcal{R}_{A5}$ in (9.5) is of order $-3/2$.

Lemma 9.3. *There exist constants $\sigma, C > 0$ such that, if $u = \bar{u}_\varepsilon + \tilde{u}$, with $\|\tilde{u}\|_{s_0+\sigma} \leq C\varepsilon^{2+\delta}$, $s_0 \geq 5$, $\delta > 0$, then $\mathcal{R}_6 := E + \mathcal{R}_{A5}$ given in* (9.5) *satisfies, for all $s \geq s_0$,*

$$\|\mathcal{R}_6 |D_x|^{3/2} h\|_s \leq_s \varepsilon \|h\|_s + \|\tilde{u}\|_{s+\sigma} \|h\|_{s_0}. \tag{9.44}$$

The transformation $A : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ is invertible. Both A and its inverse A^{-1} satisfy, for all $s \geq s_0$,

$$\|Ah\|_0 + \|A^{-1}h\|_0 \leq C\|h\|_0, \quad \|Ah\|_s + \|A^{-1}h\|_s \leq_s \|h\|_s + \|\tilde{u}\|_{s+\sigma}\|h\|_{s_0}. \tag{9.45}$$

Both A and A^{-1} preserve the parity, namely they map $X + iY$ into itself, and $Y + iX$ into itself.

Proof. By Lemmas 6.3 and 8.2, the functions $a_{14}, a_{29}, a_{30}, a_{31}, a_{32}$ all satisfy $\|a_i\|_s \leq_s \varepsilon + \|\tilde{u}\|_{s+\sigma}$. Therefore, by the construction above, $\|\beta\|_s$, $\|p - 1\|_s$, and $\|T^{(k)}[p^{(m)}]\|_s$, for all $T^{(k)}[p^{(m)}]$ in (9.43), are all bounded by $C(s)(\varepsilon + \|\tilde{u}\|_{s+\sigma})$ (with a possibly larger σ). The function r in (9.43) satisfies (9.14). Hence c in (9.43) satisfies

$$\|c(\cdot, \cdot, j)\|_s \leq_s |j|^{-3/2}(\|\beta\|_{s+3} + \|p - 1\|_{s+2}) \leq_s |j|^{-3/2}(\varepsilon + \|\tilde{u}\|_{s+\sigma}) \quad \forall s \geq 1$$

for all $j \neq 0$ (with a larger σ), while, for $j = 0$, one has $\|c(\cdot, \cdot, 0)\|_s = \|a_{29}\|_s \leq_s \varepsilon + \|\tilde{u}\|_{s+\sigma}$. The composition $E|D_x|^{3/2}$ has amplitude $c(t, x, j)|j|^{3/2}$ and phase $\phi(t, x, j)$, and therefore, by Lemma 12.9 applied with $B = E|D_x|^{3/2}$,

$$\|E|D_x|^{3/2}h\|_s \leq_s \varepsilon\|h\|_s + \|\tilde{u}\|_{s+\sigma}\|h\|_{s_0}.$$

To estimate $\mathcal{R}_{A1}$, we apply Lemma 12.10 with $\Gamma = T$, $g(\xi) = \tau(\xi)$, $r = m = 3/2$, $s_0 = 1$, $N = 9$, and $\mathcal{R}_{A1} = \sum_{\alpha=6,7,8} B_\alpha + R_N$ (in the notation of Lemma 12.10). The operator B_α has some amplitude function $c_\alpha(t, x, j)$ and the same phase function ϕ as A. Each term in the amplitude c_α, $\alpha = 6, 7, 8$, can be explicitly calculated and therefore easily estimated. Hence the estimate for $B_\alpha |D_x|^{3/2}$ follows from Lemma 12.9 applied with $B = B_\alpha |D_x|^{3/2}$. The bound for $R_N |D_x|^{3/2}$, $N = 9$, is obtained by applying Corollary 12.12. Since $\mathcal{R}_{A1} = B_6 + B_7 + B_8 + R_9$, this proves that

$$\|\mathcal{R}_{A1} |D_x|^{3/2} h\|_s \leq_s \varepsilon \|h\|_s + \|\tilde{u}\|_{s+\sigma} \|h\|_{s_0}.$$

The other remainders $\mathcal{R}_{A2}, \mathcal{R}_{A3}, \mathcal{R}_{A4}$ are estimated in the same way as for $\mathcal{R}_{A1}$, applying Lemma 12.10 with $\Gamma = \mathcal{H}|D_x|^{1/2}$ or $|D_x|^{-1/2}$ or $\mathcal{H}|D_x|^{-1}$, with the corresponding order $r = 1/2$ or $-1/2$ or -1, with $m = 3/2$, $s_0 = 1$, and suitable values of N. Then $\mathcal{R}_{A5}$ in (9.6) also satisfies the same estimates as $\mathcal{R}_{A1}$, and the proof of (9.44) is complete.

The invertibility of A and the estimates for A, A^{-1} follow from Lemma 12.9, using that $\|p - 1\|_s + \|\beta\|_s \leq_s \varepsilon + \|\tilde{u}\|_{s+\sigma}$.

A function $h(t, x) = \sum_{k \in \mathbb{Z}} h_k(t) e^{ikx}$ belongs to $X + iY$ if (i) $h_{-k}(t) = h_k(t)$, and (ii) $h_{-k}(-t)$ is the complex conjugate of $h_k(t)$. In the construction of the functions $\beta(t, x)$, $p(t, x, j)$, the parity of each component has been calculated. Using such parities, one can directly check that A maps $X + iY$ into itself. □

To proceed as in the previous sections, we need $\tilde{A} := \mathbb{P}A\mathbb{P}$ to be invertible. Split $L^2(\mathbb{T}^2, \mathbb{C}) = H_{01} \oplus H_2$, with

$$\begin{aligned} H_{01} &= \left\{ h = \sum_{j=0,\pm 1} h_j(t) e^{ijx} \in L^2(\mathbb{T}^2) \right\}, \\ H_2 &= \left\{ h = \sum_{|j| \geq 2} h_j(t) e^{ijx} \in L^2(\mathbb{T}^2) \right\}, \end{aligned}$$

so that $\mathbb{F}$ is the projection onto H_{01} and $\mathbb{P}$ is the one onto H_2.

Lemma 9.4. *There are $\sigma, C > 0$ with the following property. Let $u = \bar{u}_\varepsilon + \tilde{u}$, with $\|\tilde{u}\|_{s_0+\sigma} < C\varepsilon^{2+\delta}$, $5 \leq s_0 \leq s$, $\delta > 0$, $\varepsilon \in (0, \varepsilon_0(s))$ for some $\varepsilon_0(s)$ depending on s.*

Then the operator $\tilde{A} := \mathbb{P}A\mathbb{P} : H_2 \to H_2$ is invertible. Both $\tilde{A}$ and its inverse $\tilde{A}^{-1}$ satisfy the same inequalities (9.45) as A, A^{-1}, and preserve the parity, namely they map $H_2 \cap (X + iY)$ into itself and $H_2 \cap (Y + iX)$ into itself.

Proof. Let $h \in H_{01}$, and write $h = v_0 + v_1 + v_{-1}$, where $v_j := h_j(t) e^{ijx}$. Recalling that the symbol of A is (9.2), we have $(A - I)h = b_0 v_0 + b_1 v_1 + b_{-1} v_{-1}$, where

$$b_j(t, x) := p(t, x, j) e^{i|j|^{1/2} \beta(t,x)} - 1.$$

By construction, $p(t,x,0)=1$, whence $b_0=0$. For $j=\pm1$, we split $pe^{i\beta}-1=(p-1)e^{i\beta}+(e^{i\beta}-1)$ and use the bound $\|p-1\|_s+\|\beta\|_s\leq_s \varepsilon+\|\tilde u\|_{s+\sigma}$ to deduce that $\|b_j\|_{s_0}\leq C\varepsilon$ and $\|b_j\|_s\leq_s \varepsilon+\|\tilde u\|_{s+\sigma}$ for $s\geq s_0$, $j=\pm1$. Hence

$$\|b_jv_j\|_0\leq C\|b_j\|_{s_0}\|v_j\|_0\leq C\varepsilon\|v_j\|_0,\quad \|b_jv_j\|_s\leq_s \varepsilon\|v_j\|_s+\|\tilde u\|_{s+\sigma}\|v_j\|_{s_0}$$

for $j=\pm1$, and therefore, by triangular inequality,

$$\|(A-I)h\|_0\leq C\varepsilon\|h\|_0,\quad \|(A-I)h\|_s\leq_s \varepsilon\|h\|_s+\|\tilde u\|_{s+\sigma}\|h\|_{s_0}\quad \forall h\in H_{01}, \tag{9.46}$$

all $s\geq s_0$. We have used the inequality $\sum_{|j|\leq1}\|v_j\|_s\leq C(\sum_{|j|\leq1}\|v_j\|_s^2)^{1/2}$ (which holds for sums of a finite number of terms) and the formula $(\sum_j\|v_j\|_s^2)^{1/2}=\|h\|_s$ (which holds not only in H_{01}, but, more generally, for all $h\in H^s(\mathbb{T}^2)$, all $s\geq 0$).

Now $\|\mathbb{P}h\|_s\leq\|h\|_s$, $\|\mathbb{F}h\|_s\leq\|h\|_s$ for all $h\in H^s(\mathbb{T}^2)$, and $\mathbb{P}A\mathbb{F}=\mathbb{P}(A-I)\mathbb{F}$. Thus, using (9.46),

$$\|\mathbb{F}(A-I)\mathbb{F}h\|_0\leq C\varepsilon\|h\|_0,\qquad \|\mathbb{F}(A-I)\mathbb{F}h\|_s\leq_s \varepsilon\|h\|_s+\|\tilde u\|_{s+\sigma}\|h\|_{s_0} \tag{9.47}$$

$$\|\mathbb{P}A\mathbb{F}h\|_0\leq C\varepsilon\|h\|_0,\qquad \|\mathbb{P}A\mathbb{F}h\|_s\leq_s \varepsilon\|h\|_s+\|\tilde u\|_{s+\sigma}\|h\|_{s_0} \tag{9.48}$$

for all $h\in L^2(\mathbb{T}^2)$. Since $\mathbb{F}:H_{01}\to H_{01}$ is the identity map of H_{01}, by the first inequality in (9.47) and by Neumann series it follows that $\mathbb{F}A\mathbb{F}=\mathbb{F}+\mathbb{F}(A-I)\mathbb{F}:H_{01}\to H_{01}$ is invertible if ε is small enough. Using also the second inequality in (9.47) and Neumann series, for $\varepsilon\leq\varepsilon_0(s)$ for some $\varepsilon_0(s)$ depending on s we get

$$\|(\mathbb{F}A\mathbb{F})^{-1}h\|_0\leq C\|h\|_0,\quad \|(\mathbb{F}A\mathbb{F})^{-1}h\|_s\leq_s \|h\|_s+\|\tilde u\|_{s+\sigma}\|h\|_{s_0} \tag{9.49}$$

for all $h\in L^2(\mathbb{T}^2)$. We also note that, by Lemma 12.9 (applying directly the bound for A, without estimating $A-I$), we have

$$\|\mathbb{F}A\mathbb{P}h\|_0\leq C\|h\|_0,\quad \|\mathbb{F}A\mathbb{P}h\|_s\leq_s \|h\|_s+\|\tilde u\|_{s+\sigma}\|h\|_{s_0} \tag{9.50}$$

for all $h\in L^2(\mathbb{T}^2)$.

Since A is invertible, for every $f\in L^2$ there is a unique $h\in L^2$ such that $Ah=f$. In particular, for every $f_2\in H_2$ there is a unique $h=h_{01}+h_2\in L^2$, with $h_{01}\in H_{01}$ and $h_2\in H_2$, such that $Ah=f_2$, namely

$$\begin{cases}\mathbb{F}A\mathbb{F}h_{01}+\mathbb{F}A\mathbb{P}h_2=0,\\ \mathbb{P}A\mathbb{F}h_{01}+\mathbb{P}A\mathbb{P}h_2=f_2,\end{cases}\quad\text{i.e.}\quad \begin{cases}h_{01}=-(\mathbb{F}A\mathbb{F})^{-1}\mathbb{F}A\mathbb{P}h_2,\\ [\mathbb{P}A\mathbb{P}-\mathbb{P}A\mathbb{F}(\mathbb{F}A\mathbb{F})^{-1}\mathbb{F}A\mathbb{P}]h_2=f_2.\end{cases}$$

Then for every $f_2\in H_2$ there is a unique $h_2\in H_2$ such that $Mh_2=f_2$, where $M:=\mathbb{P}A\mathbb{P}-\mathbb{P}A\mathbb{F}(\mathbb{F}A\mathbb{F})^{-1}\mathbb{F}A\mathbb{P}$. This means that $M:H_2\to H_2$ is invertible. Its inverse satisfies

$$\|M^{-1}f_2\|_0=\|h_2\|_0\leq\|h\|_0=\|A^{-1}f_2\|_0\leq C\|f_2\|_0 \tag{9.51}$$

for all $f_2\in H_2$ (we have used Lemma 12.9). Moreover, for $f_2\in H_2\cap H^s$,

$$\|M^{-1}f_2\|_s=\|h_2\|_s\leq\|h\|_s=\|A^{-1}f_2\|_s\leq_s \|f_2\|_s+\|\tilde u\|_{s+\sigma}\|f_2\|_{s_0}. \tag{9.52}$$

Let $R := \mathbb{P}A\mathbb{F}(\mathbb{F}A\mathbb{F})^{-1}\mathbb{F}A\mathbb{P}$, so that $\mathbb{P}A\mathbb{P} = M + R$. Using (9.48),…,(9.52) we get

$$\|M^{-1}Rh\|_0 \le C\varepsilon\|h\|_0, \quad \|M^{-1}Rh\|_s \le_s \varepsilon\|h\|_s + \|\tilde{u}\|_{s+\sigma}\|h\|_{s_0}.$$

By Neumann series, it follows that $I + M^{-1}R$ is invertible, with

$$\|(I + M^{-1}R)^{-1}h\|_0 \le C\|h\|_0, \quad \|(I + M^{-1}R)^{-1}h\|_s \le_s \|h\|_s + \|\tilde{u}\|_{s+\sigma}\|h\|_{s_0}.$$

Hence $\mathbb{P}A\mathbb{P} = M(I + M^{-1}R)$ is invertible, with $(\mathbb{P}A\mathbb{P})^{-1} = (I + M^{-1}R)^{-1}M^{-1}$, and the estimates for $\mathbb{P}A\mathbb{P}$ and its inverse follow by composition. □

By the equalities $\mathcal{L}_5 A = A\mathcal{D} + \mathcal{R}_6$ and $I = \mathbb{P} + \mathbb{F}$ we get

$$\tilde{A}^{-1}(\tilde{\mathcal{L}}_5 + \tilde{\mathcal{R}}_5)\tilde{A} = \tilde{\mathcal{D}} + \tilde{\mathcal{R}}_6, \qquad \tilde{\mathcal{D}} := \mathbb{P}\mathcal{D}\mathbb{P} = \mathbb{P}\mathcal{D} = \mathcal{D}\mathbb{P}, \tag{9.53}$$

$$\tilde{\mathcal{R}}_6 := \tilde{A}^{-1}\{\tilde{\mathcal{R}}_5\tilde{A} + \mathbb{P}\mathcal{R}_6\mathbb{P} - \mathbb{P}\mathcal{L}_5\mathbb{F}A\mathbb{P}\}. \tag{9.54}$$

Lemma 9.5. *In the assumptions of Lemma 9.3, the operator $\tilde{\mathcal{R}}_6 : W_2 \to R_2$ defined in (9.54) satisfies, for all $s \ge s_0$, the same estimate (9.44) as $\mathcal{R}_6$.*

Proof. The operators $\tilde{A}$, $\tilde{A}^{-1}$ are estimated in Lemma 9.4, $\tilde{\mathcal{R}}_5$ in Lemma 8.2, and $\mathcal{R}_6$, A in Lemma 9.3. To estimate $\mathbb{P}\mathcal{L}_5\mathbb{F}$, note that $\mathbb{P}\partial_t\mathbb{F} = 0$, $\mathbb{P}T\mathbb{F} = 0$, and $\|\partial_x\mathbb{F}h\|_s \le \|h\|_s$, $\|\mathcal{H}|D_x|^{1/2}\mathbb{F}h\|_s \le \|h\|_s$ for all $h \in H^s(\mathbb{T}^2)$, all $s \ge 0$. □

10. Inversion of the Restricted Linearized Operator

Recall that our goal is the restricted inversion problem in Remark 4.5. The diagonal operator $\tilde{\mathcal{D}} := \mathcal{D}\mathbb{P} : W_2 \to R_2$, where $\mathcal{D}$ is defined in (9.3), has purely imaginary eigenvalues

$$\begin{aligned} &\tilde{\mathcal{D}}[e^{ilt}\cos(jx)] = i(\omega l + \mu_j)\, e^{ilt}\cos(jx), \\ &\mu_j := \lambda_3(j + \kappa j^3)^{1/2} + \lambda_1 j^{1/2} + \lambda_{-1} j^{-1/2} \in \mathbb{R}, \end{aligned} \tag{10.1}$$

with $l \in \mathbb{Z}$, $j \ge 2$, where we denote $\lambda_3 := \sqrt{m}$. Let $\gamma := \varepsilon^{5/6} > 0$, $\tau := 3/2$, and assume that ω satisfies the first-order Melnikov non-resonance condition

$$|\omega l + \mu_j| \ge \frac{\gamma}{|j|^\tau} \quad \forall l \in \mathbb{Z}, \ j \ge 2. \tag{10.2}$$

Then $\tilde{\mathcal{D}}$ has inverse

$$\tilde{\mathcal{D}}^{-1}h(t,x) := \sum_{l\in\mathbb{Z},\, j\ge 2} \frac{h_{lj}e^{ilt}}{i(\omega l + \mu_j)}\cos(jx), \qquad \tilde{\mathcal{D}}^{-1} : R_2 \to W_2,$$

of order $O(|D_x|^{3/2})$ and size $1/\gamma$, namely

$$\||D_x|^{-3/2}\tilde{\mathcal{D}}^{-1}h\|_s \le \gamma^{-1}\|h\|_s, \tag{10.3}$$

because $|\omega l + \mu_j| j^{3/2} \ge \gamma$ for all $l \in \mathbb{Z}$, $j \ge 2$.

By (10.3) and Lemma 9.5, writing explicitly the constant $C(s)$, we have

$$\begin{aligned}\|\tilde{\mathcal{R}}_6\tilde{\mathcal{D}}^{-1}h\|_s &= \|(\tilde{\mathcal{R}}_6|D_x|^{3/2})(|D_x|^{-3/2}\tilde{\mathcal{D}}^{-1})h\|_s \\ &\le C(s)\gamma^{-1}(\varepsilon\|h\|_s + \|\tilde{u}\|_{s+\sigma}\|h\|_{s_0}) \\ &\le \frac{1}{2}\|h\|_s + C(s)\gamma^{-1}\|\tilde{u}\|_{s+\sigma}\|h\|_{s_0},\end{aligned}$$

where the last inequality holds for ε sufficiently small, namely $C(s)\varepsilon^{1/6} \le 1/2$. Therefore, by tame Neumann series (see e.g. Lemma B.2 in [7], Appendix B), $(I_{R_2} + \tilde{\mathcal{R}}_6\tilde{\mathcal{D}}^{-1})$ is invertible on $H^s \cap R_2$, where I_{R_2} is the identity map of R_2, and

$$\|(I_{R_2} + \tilde{\mathcal{R}}_6\tilde{\mathcal{D}}^{-1})^{\pm 1}h\|_s \le 2\|h\|_s + 4C(s)\gamma^{-1}\|\tilde{u}\|_{s+\sigma}\|h\|_{s_0}.$$

As a consequence, $\tilde{\mathcal{D}} + \tilde{\mathcal{R}}_6 = (I_{R_2} + \tilde{\mathcal{R}}_6\tilde{\mathcal{D}}^{-1})\tilde{\mathcal{D}} : W_2 \to R_2$ is invertible, with

$$\|(\tilde{\mathcal{D}} + \tilde{\mathcal{R}}_6)^{-1}h\|_s \le_s \gamma^{-1}(\|h\|_{s+\tau} + \gamma^{-1}\|\tilde{u}\|_{s+\tau+\sigma}\|h\|_{s_0}). \tag{10.4}$$

In Sections 5–9 we have conjugated the restricted operator $\mathcal{L}_2^2 + \mathcal{R}$ (see Remark 4.5) to $\tilde{\mathcal{D}} + \tilde{\mathcal{R}}_6$,

$$\begin{aligned}\mathcal{L}_2^2 + \mathcal{R} &= \Phi_1(\tilde{\mathcal{D}} + \tilde{\mathcal{R}}_6)\Phi_2^{-1}, \\ \Phi_1 := \tilde{\mathcal{Z}}\tilde{\mathcal{B}}\tilde{\mathcal{A}}\tilde{P}\tilde{S}\tilde{M}\tilde{A}, &\quad \Phi_2 := \tilde{\mathcal{Z}}\tilde{\mathcal{B}}\tilde{\mathcal{A}}\tilde{Q}\tilde{S}\tilde{M}\tilde{A}.\end{aligned} \tag{10.5}$$

All these operators have been estimated in the previous sections (they are all bounded, except $\tilde{S}$, which is of order 1/2). By composition, we obtain the following result.

Theorem 10.1. (Inversion of the restricted linearized operator) *There are $\sigma, C > 0$ with the following property. Let $u = \bar{u}_\varepsilon + \tilde{u}$, with $\|\tilde{u}\|_{s_0+\sigma} < C\varepsilon^{2+\delta}$, $5 \le s_0 \le s$, $\delta > 0$, $\varepsilon < \varepsilon_0(s)$ for some $\varepsilon_0(s)$ depending on s. Assume that the first Melnikov conditions (10.2) hold. Then the operator $\mathcal{L}_2^2 + \mathcal{R} : W_2 \to R_2$ is invertible, with*

$$\|(\mathcal{L}_2^2 + \mathcal{R})^{-1}h\|_s \le_s \gamma^{-1}(\|h\|_{s+2} + \gamma^{-1}\|\tilde{u}\|_{s+\sigma}\|h\|_{s_0}). \tag{10.6}$$

By Theorem 10.1 and Lemmas 4.1 and 4.4, we deduce (with a larger σ if necessary)

Corollary 10.2. (Inversion of the linearized operator) *Assume the hypotheses of Theorem 10.1. Then the linearized operator $F'(u) : X \times Y \to Y \times X$ is invertible, with*

$$\|F'(u)^{-1}h\|_s \le_s \varepsilon^{-2}(\|h\|_{s+2} + \gamma^{-1}\|\tilde{u}\|_{s+\sigma}\|h\|_{s_0}).$$

For u, ω, h depending on the parameters (ε, ξ), using Corollary 10.2 we prove a tame estimate for $F'(u)^{-1}$ also in Lipschitz norms (3.1) (with a larger $\sigma \ge 6$).

Lemma 10.3. (Inversion in Lipschitz norms) *Let $5 \le s_0 \le s$, $u = \bar{u}_\varepsilon + \tilde{u}$, where $\bar{u}_\varepsilon$ is defined in (4.27) and $\tilde{u} = \tilde{u}(\varepsilon, \xi)$ is defined for parameters $\varepsilon \in (0, \varepsilon_0)$, $\xi \in \mathcal{G}$, with $\varepsilon_0 = \varepsilon_0(s) < 1$, $\mathcal{G} \subseteq [1, 2]$. Let $\|\tilde{u}\|_{s_0+\sigma}^{\mathrm{Lip}(\varepsilon)} < C\varepsilon^{2+\delta}$, $\delta > 0$. Let ω be given by (4.26). Assume that (10.2) holds for all $\xi \in \mathcal{G}$. Then*

$$\|F'(u)^{-1}h\|_s^{\mathrm{Lip}(\varepsilon)} \le_s \varepsilon^{-2}(\|h\|_{s+6}^{\mathrm{Lip}(\varepsilon)} + \varepsilon^{-2}\|\tilde{u}\|_{s+\sigma}^{\mathrm{Lip}(\varepsilon)}\|h\|_{s_0}^{\mathrm{Lip}(\varepsilon)}). \tag{10.7}$$

Proof. Denote A_i the linearized operator $F'(u)$ when $u = u_i := u(\xi_i), \omega = \omega(\xi_i)$, $i = 1, 2$, and denote $h_i := h(\xi_i)$. Thus

$$A_1^{-1} h_1 - A_2^{-1} h_2 = A_1^{-1}[h_1 - h_2] + A_1^{-1}(A_2 - A_1) A_2^{-1} h_2.$$

A_1^{-1}, A_2^{-1} satisfy (10.6). The difference $A_2 - A_1$ is

$$(A_2 - A_1)h = \big\{\bar\omega_2 \varepsilon^2 (\xi_2 - \xi_1) + \bar\omega_3 \big(\xi_2^{3/2} - \xi_1^{3/2}\big)\big\} \partial_t h \\ + \int_0^1 \mathcal{N}''(u_1 + \vartheta(u_2 - u_1))[u_2 - u_1, h]\, d\vartheta,$$

where $\mathcal{N}(u)$ is the nonlinear part of $F(u)$. Since $\|\bar u_\varepsilon\|_s^{\mathrm{Lip}(\varepsilon)} \leq_s \varepsilon$, we get

$$\begin{aligned} \frac{\|(A_2 - A_1)h\|_s}{|\xi_2 - \xi_1|} &\leq_s \varepsilon \|h\|_{s+2} + (\|\tilde u\|_{s+2}^{\mathrm{lip}} + \|\tilde u\|_{s_0+2}^{\mathrm{lip}} \|\tilde u\|_{s+2}^{\sup}) \|h\|_{s_0+2} \\ &\leq_s \varepsilon \|h\|_{s+2} + \varepsilon^{-1} \|\tilde u\|_{s+2}^{\mathrm{Lip}(\varepsilon)} \|h\|_{s_0+2} \\ &\leq_s \varepsilon \|h\|_{s+2} + \varepsilon^{-1} \|\tilde u\|_{s+4}^{\mathrm{Lip}(\varepsilon)} \|h\|_{s_0}, \end{aligned}$$

and the thesis follows by composition. □

10.1. Dependence of the Eigenvalues on the Parameters

The constants $\lambda_3, \lambda_1, \lambda_{-1}$ in the formula for the eigenvalues μ_j in (10.1) depend on the point $u = (\eta, \psi)$ where the linearization $F'(u)$ takes place, and on ω. In particular, λ_3 depends only on u by formula (6.4), namely

$$\lambda_3 = \lambda_3(u) = \sqrt{m} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 + \eta_x^2}\, \mathrm{d}x \right)^{-3/2} \mathrm{d}t. \qquad (10.8)$$

λ_1 depends on u, ω by the following formula (which is obtained using (9.21) and going back with the changes of variable $\mathcal{A}$, B of Section 6 in the integral)

$$\lambda_1 = \lambda_1(u, \omega) = -\frac{1}{8\pi^2 \sqrt{\kappa} \lambda_3(u)} \int_{\mathbb{T}^2} \frac{1 + \beta_x}{1 + \alpha'(t)} [\omega \beta_t + V(1 + \beta_x)]^2 \,\mathrm{d}t\, \mathrm{d}x, \quad (10.9)$$

where V is defined in (3.3), α, β are defined in Section 6 and satisfy (6.7). To write an explicit formula for λ_{-1} is possible, but more involved and not necessary for our purposes. Anyway, λ_{-1} is the space-time-average of a polynomial function of $a_{14}, a_{29}, a_{31}, a_{32}$ (and their derivatives), $\omega, \omega^{-1}, \lambda_3, \lambda_3^{-1}$ with real coefficients. Hence this is a polynomial function of $a_7, \ldots, a_{13}$ (and their derivatives), $\omega, \omega^{-1}, \lambda_3, \lambda_3^{-1}$ and, going back with the changes of variable $\mathcal{A}$, B of Section 6 in the integral, one obtains for λ_{-1} a similar result as for λ_1. Thus we have:

Lemma 10.4. *$\lambda_3(u), \lambda_1(u, \omega), \lambda_{-1}(u, \omega)$ are C^2 functions of (u, ω) in the domain $\|u\|_{\sigma_0} < \delta$, $\omega \in [\frac{1}{2}\bar\omega, \frac{3}{2}\bar\omega]$, where $\sigma_0 > 0$ is a universal constant and $\delta > 0$ depends only on κ.*

As a consequence, if u_1, u_2 are in the ball $\|u_i\|_{\sigma_0} < \delta$, $i = 1, 2$, then $|\lambda_k(u_1) - \lambda_k(u_2)| \leq C \|u_1 - u_2\|_{\sigma_0}$, $k = 3, 1, -1$, where $C > 0$ depends only on κ.

The number σ_0 in Lemma 10.4 can be explicitly computed by counting how many derivatives of u are involved in the transformation procedure of Sections 5–9.

For (u, ω) depending on the parameters (ε, ξ), we deduce the following expansion for $\lambda_3, \lambda_1, \lambda_{-1}$:

Lemma 10.5. *Let $u = \bar{u}_\varepsilon + \tilde{u}$, where $\bar{u}_\varepsilon$ is defined in (4.27) and $\tilde{u} = \tilde{u}(\varepsilon, \xi)$ is defined for parameters $\varepsilon \in (0, \varepsilon_0)$, $\xi \in \mathcal{G}$, with $\varepsilon_0 < 1$, $\mathcal{G} \subseteq [1, 2]$. Let $\|\tilde{u}\|_{\sigma_0}^{\mathrm{Lip}(\varepsilon)} < C\varepsilon^{2+\delta}$, $\delta > 0$. Let ω be given by (4.26). Then $\lambda_3, \lambda_1, \lambda_{-1}$ depend on ξ in a Lipschitz way, with*

$$\lambda_3 = 1 - \frac{3}{16}\varepsilon^2\xi + r_3, \quad |r_3|^{\mathrm{Lip}(\varepsilon)} \le C\varepsilon^3, \quad |\lambda_1|^{\mathrm{Lip}(\varepsilon)} + |\lambda_{-1}|^{\mathrm{Lip}(\varepsilon)} \le C\varepsilon^2, \quad (10.10)$$

where $r_3 := \lambda_3 - 1 + \frac{3}{16}\varepsilon^2\xi$.

Proof. By (4.27), (4.5), $\eta = \varepsilon\sqrt{\xi}\cos(t)\cos(x) + O(\varepsilon^2)$. Therefore the inequality for r_3 follows easily from formula (10.8). By Lemma 10.4, λ_1, λ_{-1} are functions of (u, ω) of class C^2. Since $u = \varepsilon\sqrt{\xi}\, v_0 + O(\varepsilon^2)$ (v_0 is defined in (4.5)) and $\omega = \bar{\omega} + O(\varepsilon^2)$, one has

$$|\lambda_i(u, \omega) - \lambda_i(\varepsilon\sqrt{\xi}\, v_0, \bar{\omega})|^{\mathrm{Lip}(\varepsilon)} \le C\varepsilon^2, \quad i = 1, -1$$

by the mean value theorem and standard analysis for composition of functions. Thus the inequalities for λ_1, λ_{-1} in (10.10) hold if

$$\lambda_i(\varepsilon\sqrt{\xi}\, v_0, \bar{\omega}) = O(\varepsilon^2), \quad i = 1, -1. \quad (10.11)$$

To prove (10.11), let $u = \varepsilon\sqrt{\xi}\, v_0$, $\omega = \bar{\omega}$. By (3.3), $V = \varepsilon\sqrt{\xi}\bar{\omega}\sin(t)\sin(x) + O(\varepsilon^2)$. By (6.7), $\alpha, \beta = O(\varepsilon^2)$. Therefore, by (10.9), we get (10.11) for λ_1.

To prove (10.11) for λ_{-1}, we compute the order ε of almost all the coefficients in Sections 5–9, namely: a, c, B, V in Section 5; $\beta, \alpha, a_1, \ldots, a_{19}$ in Section 6; a_{25}, a_{27}, a_{28} in Section 7; $a_{29}, \ldots, a_{32}, v_2, v_4, g_3, g_5$ in Section 8; and, in Section 9, $\beta, v_m^{(k)}$ (with $k = 1, 0, -1, -2$; $m = 2, 1, 0$), $p^{(0)}, p^{(-1)}, p^{(-2)}, f, b^{(0)}, b^{(-1)}, g^{(-1)}, g^{(-2)}$, and finally $r^{(-1)}$, which gives λ_{-1} by (9.40). All these coefficients are functions of the form

$$c_0 + c_1\varepsilon\sqrt{\xi}\,\psi(t, x) + O(\varepsilon^2),$$

where c_0, c_1 are real constants, and $\psi = \cos(t)\cos(x)$, or $\psi = \cos(t)\sin(x)$, or $\psi = \sin(t)\cos(x)$, or $\psi = \sin(t)\sin(x)$. We calculate that the term of order $O(1)$ in λ_{-1} is zero, while its term of order $O(\varepsilon)$ is automatically zero because ψ has zero mean. The proof of (10.11) is complete. □

By (10.10) we deduce that the eigenvalues μ_j in (10.1) satisfy

$$|\mu_j - (j + \kappa j^3)^{1/2}| \le C\varepsilon^2 j^{3/2}. \quad (10.12)$$

11. Nash–Moser Iteration and Measure of Parameter Set

Consider the finite-dimensional subspaces $E_n := \{u : u = \Pi_n u\}$, $n \geq 0$, where

$$N_n := N_0^{\chi^n} = (N_0)^{\chi^n}, \quad \chi := \frac{3}{2}, \quad N_0 := \varepsilon^{-\rho_0} := \varepsilon^{-1/\rho_1}, \quad \rho_0 := \frac{1}{\rho_1} > 0, \tag{11.1}$$

and Π_n are the projectors (Fourier truncation)

$$\Pi_n u(t,x) := \sum_{|l|+|j|<N_n} \hat{u}_{lj} e^{i(lt+jx)} \quad \text{where } u(t,x) = \sum_{l,j\in\mathbb{Z}} \hat{u}_{lj} e^{i(lt+jx)}.$$

We denote $\Pi_n^\perp := I - \Pi_n$. The classical smoothing properties also hold for the Lipschitz norms (3.1): for all $\alpha, \beta, s \geq 0$,

$$\|\Pi_n u\|_{s+\alpha}^{\mathrm{Lip}(\varepsilon)} \leq N_n^\alpha \|u\|_s^{\mathrm{Lip}(\varepsilon)}; \quad \|\Pi_n^\perp u\|_s^{\mathrm{Lip}(\varepsilon)} \leq N_n^{-\beta} \|u\|_{s+\beta}^{\mathrm{Lip}(\varepsilon)}. \tag{11.2}$$

Define the following constants:

$$\begin{aligned} &\delta = \frac{1}{2}, \qquad \alpha_0 = 6 + \sigma, \qquad \alpha_1 = \rho_1 = 9\alpha_0, \\ &\kappa_1 = 3(\sigma + 4 + 2\rho_1) + 1, \qquad \beta_1 = 3 + \sigma + \alpha_1 + \frac{2}{3}\kappa_1 + 4\rho_1. \end{aligned} \tag{11.3}$$

All these constants depend only on σ, where $\sigma \geq 6$ is the loss of regularity in (10.7).

Theorem 11.1. (Nash–Moser iteration) *Let $s_0 \geq 5$. There exists $\varepsilon_0 > 0$ such that, if $\varepsilon \in (0, \varepsilon_0]$, then, for all $n \geq 0$:*

$(\mathcal{P}1)_n$ (Convergent sequence). *There exists a function $u_n = \bar{u}_\varepsilon + \tilde{u}_n : \mathcal{G}_n \subseteq [1,2] \to E_n$, $\xi \mapsto u_n(\xi) = (\eta_n(\xi), \psi_n(\xi))$, where $\bar{u}_\varepsilon$ is defined in* (4.27), *and $u_0 := \bar{u}_\varepsilon$, $\tilde{u}_0 = 0$, such that*

$$\|u_n\|_{s_0+\sigma}^{\mathrm{Lip}(\varepsilon)} \leq C_* \varepsilon, \quad \|\tilde{u}_n\|_{s_0+\sigma}^{\mathrm{Lip}(\varepsilon)} \leq C_* \varepsilon^{2+\delta}. \tag{11.4}$$

The function u_n has parity $u_n \in X \times Y$. The sets $\mathcal{G}_n$ are defined inductively by: $\mathcal{G}_0 := [1,2]$,

$$\mathcal{G}_{n+1} := \left\{ \xi \in \mathcal{G}_n \ : \ |\omega l + \mu_j(u_n)| > \frac{\gamma}{j^\tau} \quad \forall l \in \mathbb{Z},\ j \geq 2 \right\}, \tag{11.5}$$

where $\mu_j(u_n) = \mu_j(\xi, u_n(\xi))$ are defined in (10.1). *The difference $h_n := u_n - u_{n-1} = \tilde{u}_n - \tilde{u}_{n-1}$, $n \geq 1$ (with $h_0 := 0$) is defined on $\mathcal{G}_n$, and*

$$\begin{aligned} \|h_n\|_{s_0+\sigma}^{\mathrm{Lip}(\varepsilon)} &\leq C_* \varepsilon^{2+\delta} N_n^{-\alpha_0} = C_* \varepsilon^{2+\delta+\frac{1}{9}\chi^n}, \\ \|F(u_n)\|_{s_0}^{\mathrm{Lip}(\varepsilon)} &\leq C_* \varepsilon^4 N_n^{-\alpha_1} = C_* \varepsilon^{4+\chi^n}. \end{aligned} \tag{11.6}$$

$(\mathcal{P}2)_n$ (High norms). $\|\tilde{u}_n\|_{s_0+\beta_1}^{\mathrm{Lip}(\varepsilon)} \leq C_* N_n^{\kappa_1}$ *and* $\|F(u_n)\|_{s_0+\beta_1}^{\mathrm{Lip}(\varepsilon)} \leq C_* N_n^{\kappa_1}$.

$(\mathcal{P}3)_n$ (Measure). *For all $n \geq 0$, the Lebesgue measure of the set $\mathcal{G}_n \backslash \mathcal{G}_{n+1}$ satisfies*

$$|\mathcal{G}_n \backslash \mathcal{G}_{n+1}| \leq C_* \varepsilon^{1/18} 2^{-n}. \tag{11.7}$$

In Sections 11.1, 11.2 we prove Theorem 11.1.

11.1. Proof of the Nonlinear Iteration

In this section we prove $(\mathcal{P}1,2)_n$ by induction. The proof of $(\mathcal{P}3)_n$ is in Section 11.2. To shorten the notation, in this section we use the following abbreviations: $|\ |_s := \|\ \|_s^{\mathrm{Lip}(\varepsilon)}$, $F_n := F(u_n)$, $L_n := F'(u_n)$, $s_1 := s_0 + \beta_1$.

Proof of $(\mathcal{P}1,2)_0$. $u_0 = \bar{u}_\varepsilon \in E_0$ if $N_0 > 5$, i.e. for ε sufficiently small. By Lemma 4.6, the bounds (11.4), (11.6) hold. To satisfy also $(\mathcal{P}2)_0$, take $C_* = C_*(s_1)$ large enough.

Assume that $(\mathcal{P}1,2)_n$ *hold for some* $n \geq 0$, *and prove* $(\mathcal{P}1,2)_{n+1}$. By (11.4) and Corollary 10.2, $F'(u_n)$ is invertible for all $\xi \in \mathcal{G}_{n+1}$, and the inverse satisfies (10.6), (10.7). For $\xi \in \mathcal{G}_{n+1}$ we define

$$u_{n+1} := u_n + h_{n+1}, \quad h_{n+1} := -\Pi_{n+1} F'(u_n)^{-1} F(u_n). \tag{11.8}$$

Let

$$Q(u_n, h) := F(u_n + h) - F(u_n) - F'(u_n)h, \quad Q_n := Q(u_n, h_{n+1}). \tag{11.9}$$

By the definitions (11.8), (11.9), and splitting $\Pi_{n+1} = I - \Pi_{n+1}^\perp$,

$$\begin{aligned} &F(u_{n+1}) = F(u_n) + F'(u_n)h_{n+1} + Q_n = R_n + Q_n, \\ &R_n := F'(u_n)\Pi_{n+1}^\perp F'(u_n)^{-1} F(u_n). \end{aligned} \tag{11.10}$$

Estimate of Q_n. For all $h \in E_{n+1}$, by (11.2) with $\alpha = 2$,

$$\begin{aligned} |Q(u_n, h)|_s &\leq_s |h|_{s+2}|h|_{s_0+2} + |\tilde{u}_n|_{s+2}|h|_{s_0+2}^2, \\ |Q(u_n, h)|_{s_0} &\leq_{s_0} N_{n+1}^4 |h|_{s_0}^2. \end{aligned} \tag{11.11}$$

By the Definition (11.8) of h_{n+1}, (10.7) and (11.2) with $\alpha = \sigma$, $\alpha = 6$,

$$\begin{aligned} |h_{n+1}|_{s_1} &\leq_{s_1} \varepsilon^{-2} N_{n+1}^\sigma \big(|F_n|_{s_1} + \varepsilon^{-2}|\tilde{u}_n|_{s_1}|F_n|_{s_0}\big), \\ |h_{n+1}|_{s_0} &\leq_{s_0} \varepsilon^{-2} N_{n+1}^6 |F_n|_{s_0}. \end{aligned} \tag{11.12}$$

Then Q_n in (11.9) satisfies

$$\begin{aligned} |Q_n|_{s_1} &\leq_{s_1} \varepsilon^{-4} N_{n+1}^{\sigma+10} |F_n|_{s_0} (|F_n|_{s_1} + \varepsilon^{-2}|\tilde{u}_n|_{s_1}|F_n|_{s_0}), \\ |Q_n|_{s_0} &\leq_{s_0} \varepsilon^{-4} N_{n+1}^{16} |F_n|_{s_0}^2. \end{aligned} \tag{11.13}$$

Estimate of R_n. The linearized operator $L_n = F'(u_n)$ satisfies

$$|L_n h|_s \leq_s |h|_{s+2} + |\tilde{u}_n|_{s+2}|h|_{s_0+2}, \quad |L_n h|_{s_0} \leq_{s_0} |h|_{s_0+2} \quad \forall h. \tag{11.14}$$

Then, by (11.2) with $\beta = \beta_1 - 2 - \sigma$ and (10.7),

$$\begin{aligned} |R_n|_{s_0} &\leq_{s_0} |\Pi_{n+1}^\perp L_n^{-1} F_n|_{s_0+2} \leq_{s_0} N_{n+1}^{-\beta} |L_n^{-1} F_n|_{s_0+2+\beta} \\ &\leq_{s_1} \varepsilon^{-2} N_{n+1}^{-\beta_1+2+\sigma} (|F_n|_{s_1} + \varepsilon^{-2}|\tilde{u}_n|_{s_1}|F_n|_{s_0}). \end{aligned} \tag{11.15}$$

For the high norm, since $\Pi_{n+1}^{\perp} = I - \Pi_{n+1}$, we split $R_n = F_n - L_n \Pi_{n+1} L_n^{-1} F_n$. By (11.14) and (11.2) with $\alpha = 2 + \sigma$, $\alpha = 2$,

$$\begin{aligned} |L_n \Pi_{n+1} L_n^{-1} F_n|_{s_1} &\leq_{s_1} |\Pi_{n+1} L_n^{-1} F_n|_{s_1+2} + |\tilde{u}_n|_{s_1+2} |\Pi_{n+1} L_n^{-1} F_n|_{s_0+2} \\ &\leq_{s_1} N_{n+1}^{2+\sigma} |L_n^{-1} F_n|_{s_1-\sigma} + N_n^2 |\tilde{u}_n|_{s_1} N_{n+1}^2 |L_n^{-1} F_n|_{s_0} \\ &\leq_{s_1} N_{n+1}^{2+\sigma} \varepsilon^{-2} (|F_n|_{s_1-\sigma+6} + \varepsilon^{-2} |\tilde{u}_n|_{s_1} |F_n|_{s_0}) \\ &\quad + N_{n+1}^4 |\tilde{u}_n|_{s_1} \varepsilon^{-2} |F_n|_{s_0+2} \\ &\leq_{s_1} \varepsilon^{-2} N_{n+1}^{4+\sigma} (|F_n|_{s_1} + \varepsilon^{-2} |\tilde{u}_n|_{s_1} |F_n|_{s_0}). \end{aligned} \tag{11.16}$$

In the last inequality we have used the interpolation estimate

$$|\tilde{u}_n|_{s_1} |F_n|_{s_0+2} \leq |\tilde{u}_n|_{s_1+2} |F_n|_{s_0} + |\tilde{u}_n|_{s_0+2} |F_n|_{s_1}$$

and then (11.2) with $\alpha = 2$ for $\tilde{u}_n \in E_n$.

Estimate of F_{n+1}. Since $F_{n+1} = R_n + Q_n$, by (11.16), (11.13),

$$\begin{aligned} |F_{n+1}|_{s_1} &\leq_{s_1} \varepsilon^{-2} N_{n+1}^{\sigma+4} \left\{ 1 + \varepsilon^{-2} N_{n+1}^6 |F_n|_{s_0} \right\} \Big(|F_n|_{s_1} + \varepsilon^{-2} |\tilde{u}_n|_{s_1} |F_n|_{s_0} \Big) \\ &\leq_{s_1} \varepsilon^{-2} N_{n+1}^{\sigma+4} (|F_n|_{s_1} + \varepsilon^{-2} |\tilde{u}_n|_{s_1} |F_n|_{s_0}). \end{aligned} \tag{11.17}$$

Note that $\varepsilon^{-2} N_{n+1}^6 |F_n|_{s_0} \leq 1$ for ε sufficiently small, because $\alpha_1 > 6\chi$. Also, by (11.15), (11.13),

$$|F_{n+1}|_{s_0} \leq_{s_1} \varepsilon^{-2} N_{n+1}^{-\beta_1+2+\sigma} (|F_n|_{s_1} + \varepsilon^{-2} |\tilde{u}_n|_{s_1} |F_n|_{s_0}) + \varepsilon^{-4} N_{n+1}^{16} |F_n|_{s_0}^2. \tag{11.18}$$

Estimate of $\tilde{u}_{n+1}$. By (11.12), and using that $\tilde{u}_{n+1} = \tilde{u}_n + h_{n+1}$, $\varepsilon^{-2} |F_n|_{s_0} \leq 1$, we get

$$|\tilde{u}_{n+1}|_{s_1} \leq_{s_1} \varepsilon^{-2} N_{n+1}^{\sigma} (|\tilde{u}_n|_{s_1} + |F_n|_{s_1}). \tag{11.19}$$

Let $B_n := |\tilde{u}_n|_{s_1} + |F_n|_{s_1}$. From (11.17), (11.19), using that $\varepsilon^{-2} |F_n|_{s_0} \leq 1$, we get

$$B_{n+1} \leq C \varepsilon^{-2} N_{n+1}^{\sigma+4} B_n \leq C N_{n+1}^{\sigma+4+2\rho_1} B_n \quad \forall n \geq 0,$$

for some $C = C(s_1)$ independent of n, because $\varepsilon^{-2} = N_0^{2\rho_1} < N_{n+1}^{2\rho_1}$. Hence, by induction, $B_n \leq C' N_n^{\kappa_1}$ for all $n \geq 0$, for some C', because $\kappa_1 > 3(\sigma + 4 + 2\rho_1)$. Thus $(\mathcal{P}2)_{n+1}$ is proved.

Proof of $(\mathcal{P}1)_{n+1}$. Using (11.18), (11.6), $(\mathcal{P}2)_n$,

$$|F_{n+1}|_{s_0} \leq C_1 \{ \varepsilon^{-2} N_{n+1}^{-\beta_1+2+\sigma} C_* N_n^{\kappa_1} + \varepsilon^{-4} N_{n+1}^{16} (C_* \varepsilon^4 N_n^{-\alpha_1})^2 \}, \tag{11.20}$$

for some $C_1 = C_1(s_1)$. The right-hand side term of (11.20) is $\leq C_* \varepsilon^4 N_{n+1}^{-\alpha_1}$ if

$$2C_1 \varepsilon^{-6} N_{n+1}^{-\beta_1+2+\sigma+\alpha_1} N_n^{\kappa_1} \leq 1, \quad 2C_1 C_* N_{n+1}^{16+\alpha_1} N_n^{-2\alpha_1} \leq 1 \quad \forall n \geq 0. \tag{11.21}$$

Recalling (11.1), (11.3), the inequalities in (11.21) hold taking ε small enough. This gives $|F_{n+1}|_{s_0} \leq C_* \varepsilon^4 N_{n+1}^{-\alpha_1}$. The bound (11.6) for h_{n+1} follows by (11.2) (with $\alpha = \sigma$), (11.12), and the bound (11.6) for F_n, using (11.3), and taking ε small enough.

Finally, using (11.6), the bound (11.4) for $\tilde{u}_{n+1}$ holds because $\tilde{u}_{n+1} = h_1 + \ldots + h_{n+1}$ and $\sum_{k=1}^{\infty} N_k^{-\alpha_0} < 1$ for ε small. The proof of $(\mathcal{P}1, 2)_n$ is concluded.

11.2. Measure Estimates

In this section we prove $(\mathcal{P}3)_n$ for all $n \geq 0$. Let us estimate $[1,2]\backslash\mathcal{G}_1$ first. For $l \in \mathbb{Z}$, $j \geq 2$, define

$$\mathcal{A}_{lj} := \left\{\xi \in [1,2] : |\omega l + \mu_j| < \gamma j^{-\tau}\right\}$$

where $\gamma := \varepsilon^{5/6}$ and the eigenvalues $\mu_j = \mu_j(u_0)$. If $\mathcal{A}_{lj} \neq \emptyset$, then there exists $\xi \in [1,2]$ for which

$$-\frac{\mu_j}{\omega} - \frac{\gamma}{\omega j^\tau} < l < -\frac{\mu_j}{\omega} + \frac{\gamma}{\omega j^\tau}$$

(where μ_j, ω depend on ξ). By the inequality $|\omega^{-1} - \bar{\omega}^{-1}| \leq C\varepsilon^2$, and using (10.12), we deduce that $\mu_j \omega^{-1} = (j + \kappa j^3)^{1/2}\bar{\omega}^{-1} + O(\varepsilon^2 j^{3/2})$, and

$$-\frac{(j+\kappa j^3)^{1/2}}{\bar{\omega}} - C\varepsilon^2 j^{3/2} - \frac{2\gamma}{\bar{\omega} j^\tau} < l < -\frac{(j+\kappa j^3)^{1/2}}{\bar{\omega}} + C\varepsilon^2 j^{3/2} + \frac{2\gamma}{\bar{\omega} j^\tau} \tag{11.22}$$

because $\omega > \bar{\omega}/2$ for ε sufficiently small. Note that all the terms in the inequality (11.22) are independent of ξ. As a consequence, for each fixed $j \geq 2$,

$$\sharp\{l \in \mathbb{Z} : \mathcal{A}_{lj} \neq \emptyset\} < C\varepsilon^2 j^{3/2} + 2 \tag{11.23}$$

for ε sufficiently small, simply because the number of integers in an interval (a,b) is $< b - a + 1$.

Now we study the variation of the eigenvalues with respect to the parameter ξ. By (11.22),

$$l = -\frac{(j+\kappa j^3)^{1/2}}{\bar{\omega}} + O(\varepsilon^2 j^{3/2}) + O(\gamma j^{-\tau}). \tag{11.24}$$

Let $f_{lj}(\xi) := \omega l + \mu_j$, where the dependence on ξ of the eigenvalue is put into evidence. Replacing l by (11.24), and using (10.1), (10.10),

$$\begin{aligned}\frac{f_{lj}(\xi_2) - f_{lj}(\xi_1)}{\xi_2 - \xi_1} &= \left(\varepsilon^2\bar{\omega}_2 + \varepsilon^3\bar{\omega}_3\frac{\xi_2^{3/2} - \xi_1^{3/2}}{\xi_2 - \xi_1}\right) l \\ &\quad + \left(-\frac{3}{16}\varepsilon^2 + \frac{r_3(\xi_2) - r_3(\xi_1)}{\xi_2 - \xi_1}\right)(j + \kappa j^3)^{1/2} \\ &\quad + \frac{\lambda_1(\xi_2) - \lambda_1(\xi_1)}{\xi_2 - \xi_1} j^{1/2} + \frac{\lambda_{-1}(\xi_2) - \lambda_{-1}(\xi_1)}{\xi_2 - \xi_1} j^{-1/2} \\ &= \varepsilon^2\left(-\frac{3}{16} - \frac{\bar{\omega}_2}{\bar{\omega}} + O(\varepsilon)\right)(j + \kappa j^3)^{1/2} + O(\varepsilon^2 j^{1/2}).\end{aligned}$$

Now $-\frac{3}{16} - \frac{\bar{\omega}_2}{\bar{\omega}}$ is nonzero for all $\kappa \geq 0$ (using (4.24), one can check that $|\frac{3}{16} + \frac{\bar{\omega}_2}{\bar{\omega}}| \geq 2$ for all $\kappa \geq 0$). Hence

$$|f_{lj}(\xi_2) - f_{lj}(\xi_1)| \geq \varepsilon^2 c j^{3/2}|\xi_2 - \xi_1| \quad \forall j \geq C, \tag{11.25}$$

where $C, c > 0$ are constants depending only on κ.

Remark 11.2. For $2 \le j < C$ one could impose a finite list of inequalities for κ, and obtain, as a consequence, that (11.25) holds for all $j \ge 2$. However, there is no need of doing in that way: using the cut-off (11.27) below, the low frequencies $j < C$ have not to be studied if ε is small enough.

By (11.25), the measure of the set $\mathcal{A}_{lj}$ is

$$|\mathcal{A}_{lj}| \le \frac{2\gamma}{j^\tau} \frac{1}{\varepsilon^2 c j^{3/2}} = \frac{C\gamma\varepsilon^{-2}}{j^{\tau+(3/2)}}. \tag{11.26}$$

We impose a Diophantine condition on the surface tension coefficient κ: we assume (2.6), namely

$$|\bar{\omega} l + (j + \kappa j^3)^{1/2}| = |\sqrt{1+\kappa}\, l + \sqrt{j + \kappa j^3}| > \frac{\gamma_*}{j^{\tau_*}} \quad \forall l \in \mathbb{Z},\ j \ge 2,$$

for some constant $\gamma_* \in (0, 1/2)$, where we fix $\tau_* = 3/2$. By (2.6), if $\mathcal{A}_{lj} \neq \emptyset$, then

$$\frac{\gamma}{j^\tau} > |\omega l + \mu_j| \ge |\bar{\omega} l + (j + \kappa j^3)^{1/2}| - C\varepsilon^2 j^{3/2} > \frac{\gamma_*}{j^{\tau_*}} - C\varepsilon^2 j^{3/2}$$

for some $C \ge 1$, whence $C\varepsilon^2 j^{3/2} > \gamma_* j^{-\tau_*} - \gamma j^{-\tau} \ge \gamma_* j^{-\tau_*}/2$ if $\gamma \le \gamma_*/2$ (i.e. ε small enough) and $\tau \ge \tau_*$ (we have fixed $\tau = \tau_* = 3/2$). Thus we have found the following "cut-off": $\mathcal{A}_{lj}$ can be nonempty only for

$$j > \left(\frac{\gamma_*}{2C\varepsilon^2} \right)^{\frac{1}{\tau_* + (3/2)}} =: C_0 \varepsilon^{-\alpha}, \quad \alpha := \frac{2}{\tau_* + (3/2)} = \frac{2}{3}. \tag{11.27}$$

Thus, by (11.23), (11.26),

$$\begin{aligned} \Big| \bigcup_{l \in \mathbb{Z},\ j \ge 2} \mathcal{A}_{lj} \Big| &\le \sum_{j > C_0 \varepsilon^{-\alpha}} (C\varepsilon^2 j^{3/2} + 2) \frac{C\gamma\varepsilon^{-2}}{j^{\tau+(3/2)}} \\ &\le C\gamma \sum_{j > C_0\varepsilon^{-\alpha}} \frac{1}{j^\tau} + C\gamma\varepsilon^{-2} \sum_{j > C_0\varepsilon^{-\alpha}} \frac{1}{j^{\tau+(3/2)}} \\ &\le C\gamma (\varepsilon^{-\alpha})^{-\tau+1} + C\gamma\varepsilon^{-2} (\varepsilon^{-\alpha})^{-\tau-\frac{1}{2}} \le C\varepsilon^{1/18}. \end{aligned} \tag{11.28}$$

We have proved that $[1, 2] \backslash \mathcal{G}_1$ has Lebesgue measure $\le C\varepsilon^{1/18}$, which is (11.7) for $n = 0$.

Remark 11.3. The condition (2.6) allows to get a positive measure estimate even if $\gamma = \varepsilon^{5/6} \gg \varepsilon$. The advantage of imposing (2.6) is that, regarding size, $\mathcal{D}^{-1}\mathcal{R} = O(\varepsilon\gamma^{-1}) = O(\varepsilon^{1/6}) \ll 1$, so that $\mathcal{D} + \mathcal{R}$ can be inverted simply by Neumann series.

Without (2.6), in the sum (11.28) the cut-off $j > C\varepsilon^{-2/3}$ disappears, and the second sum becomes $\le C\gamma\varepsilon^{-2}$. Therefore, to get a parameter set of asymptotically full measure, it should be $\gamma = o(\varepsilon^2)$ as $\varepsilon \to 0$. But then $\mathcal{D}^{-1}\mathcal{R} = O(\gamma^{-1})\mathcal{R}$ is small only if $\mathcal{R} = o(\gamma)$. This means that one has to expand $\mathcal{R} = \varepsilon\mathcal{R}_1 + \varepsilon^2\mathcal{R}_2 + o(\varepsilon^2)$, to calculate the precise formula of $\mathcal{R}_1, \mathcal{R}_2$, to invert $\mathcal{D} + \varepsilon\mathcal{R}_1 + \varepsilon^2\mathcal{R}_2$ in a non-perturbative way, and then to invert $\mathcal{D}+\mathcal{R}$ as a perturbation of $\mathcal{D}+\varepsilon\mathcal{R}_1+\varepsilon^2\mathcal{R}_2$. This means, in fact, that one has to calculate the normal form of order 2.

Remark 11.4. We could also fix γ to be independent of ε, taking a larger value of τ. However, the larger is τ, the larger is the number of steps we have to make in Section 9 to reach a sufficiently regularizing remainder $\mathcal{R}$ (it should be $\mathcal{R} = O(|D_x|^{-\tau})$ to obtain $\mathcal{R}\mathcal{D}^{-1}$ bounded in Section 10). Hence it is convenient to keep τ as low as possible, but still sufficiently large to get a positive measure set of parameters.

Now we prove (11.7) for $n \geq 1$. Let $J_n := \varepsilon^{-11/18} 4^n$. Let $\mathcal{A}_{lj}^{n+1} := \{\xi \in \mathcal{G}_n : |\omega l + \mu_j(u_n)| < \gamma j^{-\tau}\}$. For $j > J_n$ we follow exactly the same argument above, and we find

$$\begin{aligned}\Big| \bigcup_{l\in\mathbb{Z},\, j>J_n} \mathcal{A}_{lj}^{n+1} \Big| &\leq \sum_{j>J_n} (C\varepsilon^2 j^{3/2} + 2) \frac{C\gamma\varepsilon^{-2}}{j^3} \\ &\leq C\gamma \sum_{j>J_n} \frac{1}{j^{3/2}} + C\gamma\varepsilon^{-2} \sum_{j>J_n} \frac{1}{j^3} \\ &\leq C\gamma J_n^{-1/2} + C\gamma\varepsilon^{-2} J_n^{-2} \leq C\varepsilon^{1/18} 2^{-n}.\end{aligned}$$

For $j \leq J_n$ we use Lemma 10.4, (11.6), the Lipschitz estimate

$$|\mu_j(u_n) - \mu_j(u_{n-1})| \leq C\|u_n - u_{n-1}\|_{\sigma_0} j^{3/2} = C\|h_n\|_{\sigma_0} j^{3/2} \leq C\varepsilon^{2+\delta+\frac{1}{9}\chi^n} j^{3/2}$$

and the triangular inequality to deduce that, if $\xi \in \mathcal{G}_n$, then

$$\begin{aligned}|\omega l + \mu_j(u_n)| &\geq |\omega l + \mu_j(u_{n-1})| - |\mu_j(u_n) - \mu_j(u_{n-1})| \geq \gamma j^{-\tau} \\ &\quad - C\varepsilon^{2+\delta+\frac{1}{9}\chi^n} j^{3/2}.\end{aligned}$$

On the other hand, if $\xi \in \mathcal{A}_{lj}^{n+1}$, then $|\omega l + \mu_j(u_n)| < \gamma j^{-\tau}$, and therefore $f_{lj}(\xi) := \omega l + \mu_j(u_n)$ is in a region of Lebesgue measure $\leq C\varepsilon^{2+\delta+(1/9)\chi^n} j^{3/2}$. Thus we follow the same argument as above, but with $C\varepsilon^{2+\delta+(1/9)\chi^n} j^{3/2}$ instead of $2\gamma j^{-\tau}$. We get

$$\begin{aligned}\Big| \bigcup_{l\in\mathbb{Z},\, j\leq J_n} \mathcal{A}_{lj}^{n+1} \Big| &\leq \sum_{j\leq J_n} (C\varepsilon^2 j^{3/2} + 2) C\varepsilon^{2+\delta+\frac{1}{9}\chi^n} j^{3/2} \frac{1}{c\varepsilon^2 j^{3/2}} \\ &\leq C\varepsilon^{2+\delta+\frac{1}{9}\chi^n} J_n^{5/2} + C\varepsilon^{\delta+\frac{1}{9}\chi^n} J_n,\end{aligned}$$

which is $\leq C\varepsilon^{1/18} 2^{-n}$ because $2 + \delta + \frac{1}{9}\chi - \frac{5}{2}\frac{11}{18} \geq \frac{1}{18}$ and $\delta + \frac{1}{9}\chi - \frac{11}{18} \geq \frac{1}{18}$. $(\mathcal{P}3)_n$ is proved. □

Proof of Theorem 2.4 concluded. Theorem 11.1 implies that the sequence u_n is well-defined for $\xi \in \mathcal{G}_\infty := \cap_{n\geq 0} \mathcal{G}_n \subset [1, 2]$. By (11.7), the set $\mathcal{G}_\infty$ has positive Lebesgue measure $|\mathcal{G}_\infty| \geq 1 - C\varepsilon^{1/18}$, asymptotically full $|\mathcal{G}_\infty| \to 1$ as $\varepsilon \to 0$. By (11.6), u_n is a Cauchy sequence in $\| \ \|_{s_0+\sigma}$, and therefore it converges to a limit u_∞ in $H^{s_0+\sigma}(\mathbb{T}^2)$. By (11.6), for all $\xi \in \mathcal{G}_\infty$, u_∞ is a solution of $F(u_\infty, \omega) = 0$, with $\|u_\infty - \bar{u}_\varepsilon\|_{s_0+\sigma} \leq C\varepsilon^{2+\delta}$, where $\omega = \omega(\xi)$ is given by (4.26). Renaming $u := u_\infty$, $\mathcal{G}_\varepsilon := \mathcal{G}_\infty$, the proof of Theorem 2.4 is complete. □

Proof of Lemma 2.3. Let $\kappa_0 > 0$, $\tau_* > 1$. Let $\gamma_* \in (0, 1/2)$, $l \in \mathbb{Z}$, $j \geq 2$, and define

$$\mathcal{A}_{l,j}(\gamma_*) := \left\{ \kappa \in [0, \kappa_0] : |f_{lj}(\kappa)| \leq \frac{\gamma_*}{j^{\tau_*}} \right\}, \qquad f_{lj}(\kappa) := l + \frac{\sqrt{j + \kappa j^3}}{\sqrt{1+\kappa}}.$$

If $\mathcal{A}_{lj}(\gamma_*) \neq \emptyset$, then $|l| < C j^{3/2}$ for some constant $C > 0$ depending on κ_0 and independent of τ_*, γ_*, l, j. Therefore for each j there are at most $C j^{3/2}$ indices l such that $\mathcal{A}_{lj}(\gamma_*) \neq \emptyset$. Moreover the derivative of $f_{lj}(\kappa)$ with respect to κ is

$$f'_{lj}(\kappa) = \frac{j^3 - j}{2\sqrt{j + \kappa j^3}(1+\kappa)^{3/2}} \geq \frac{j^3 - j}{2\sqrt{j + \kappa_0 j^3}(1+\kappa_0)^{3/2}} \geq c j^{3/2}$$

for some $c > 0$ (depending on κ_0 and independent of τ_*, γ_*, l, j). Hence the Lebesgue measure of $\mathcal{A}_{lj}(\gamma_*)$ is

$$|\mathcal{A}_{lj}(\gamma_*)| \leq \frac{2\gamma_*}{j^{\tau_*}} \frac{1}{c j^{3/2}} = \frac{C\gamma_*}{j^{\tau_* + (3/2)}}$$

for some $C > 0$. Since $\tau_* > 1$,

$$\left| \bigcup_{l \in \mathbb{Z},\, j \geq 2} \mathcal{A}_{lj}(\gamma_*) \right| \leq \sum_{j \geq 2} \frac{C\gamma_*}{j^{\tau_* + (3/2)}} j^{3/2} \leq C\gamma_*$$

for some C depending on κ_0, τ_*. As a consequence, the set $\tilde{\mathcal{K}}(\gamma_*) := \{\kappa \in [0, \kappa_0] : |f_{lj}(\kappa)| > \gamma_* j^{-\tau_*}\}$ has Lebesgue measure $|\tilde{\mathcal{K}}(\gamma_*)| \geq \kappa_0 - C\gamma_*$. Therefore $\tilde{\mathcal{K}} := \bigcup_{\gamma_* \in (0,1/2)} \tilde{\mathcal{K}}(\gamma_*)$ has full measure $|\tilde{\mathcal{K}}| = \kappa_0$. Finally note that $\tilde{\mathcal{K}} \subset \mathcal{K} \subset [0, \kappa_0]$. The proof of Lemma 2.3 is complete. □

12. Pseudo-differential Operators in the Class $S^0_{1/2,1/2}$

In this section we prove some results on pseudo-differential operators in the class $S^0_{1/2,1/2}$ on the 1-dimensional torus that are used in our existence proof for the water waves problem. These results also hold for a more general class of Fourier integral operators. In Sections 12.1–12.3 we prove the invertibility, composition formulas and tame estimates for operators depending on the space variable $x \in \mathbb{T}$ only, then in Section 12.4 we explain how to include the dependence on the time variable $t \in \mathbb{T}$.

12.1. Invertibility

We consider Fourier integral operators that change e^{ikx} into $e^{i\phi(x,k)}$ for some phase function ϕ. Namely, let $L > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with

$$f(0) = 0, \qquad \|f'\|_{L^\infty} \leq L, \tag{12.1}$$

so that $|f(\xi)-f(\eta)|\le L|\xi-\eta|$ and $|f(\xi)|\le L|\xi|$ for all $\xi,\eta\in\mathbb{R}$. Let $\beta(x)$ be a real-valued periodic function and let

$$\phi(x,\xi):=\xi x+f(\xi)\beta(x),\quad x,\xi\in\mathbb{R}.$$

Denote

$$w_\xi(x):=\mathrm{e}^{i\phi(x,\xi)},\quad e_\xi(x):=\mathrm{e}^{i\xi x},\quad x,\xi\in\mathbb{R}.$$

When $\xi=k$ is an integer, both e_k and w_k are 2π-periodic functions of x. We define the operator A by setting $Ae_\xi=w_\xi$ for $\xi\in\mathbb{R}$. Thus

$$Ag(x)=\int_{\mathbb{R}}\hat g(\xi)\,w_\xi(x)\,\mathrm{d}\xi,\quad g(x)=\int_{\mathbb{R}}\hat g(\xi)\,e_\xi(x)\,\mathrm{d}\xi \tag{12.2}$$

for functions $g:\mathbb{R}\to\mathbb{C}$, where $\hat g$ is the Fourier transform of g on the real line, and, on the torus,

$$Au(x)=\sum_{k\in\mathbb{Z}}\hat u_k\,w_k(x),\quad u(x)=\sum_{k\in\mathbb{Z}}\hat u_k\,e_k(x) \tag{12.3}$$

for periodic functions $u:\mathbb{T}\to\mathbb{C}$, where $\hat u_k$ are the Fourier coefficients of u.

Adjoint operators. Quantitative estimates for A and its inverse are the goal of this section. To obtain these bounds, we shall study A^*A and AA^*.

Consider the scalar product of $L^2(\mathbb{T})$ and that one of $L^2(\mathbb{R})$,

$$(u,v)_{L^2(\mathbb{T})}=\int_{\mathbb{T}}u(x)\,\overline{v(x)}\,\mathrm{d}x,\quad (g,h)_{L^2(\mathbb{R})}=\int_{\mathbb{R}}g(x)\,\overline{h(x)}\,\mathrm{d}x,$$

where $u,v\in L^2(\mathbb{T})$ and $g,h\in L^2(\mathbb{R})$. Denote $A^*_{\mathbb{T}}$, $A^*_{\mathbb{R}}$ the adjoint of A with respect to the scalar product of $L^2(\mathbb{T})$ and $L^2(\mathbb{R})$ respectively, namely

$$A^*_{\mathbb{T}}u(x)=\sum_{k\in\mathbb{Z}}(u,w_k)_{L^2(\mathbb{T})}\,e_k(x),\quad x\in\mathbb{T},$$

and

$$A^*_{\mathbb{R}}g(x)=\int_{\mathbb{R}}(g,w_\xi)_{L^2(\mathbb{R})}\,e_\xi(x)\,\mathrm{d}\xi,\quad x\in\mathbb{R}.$$

Hence

$$A^*_{\mathbb{T}}\,Au(x)=\sum_{k\in\mathbb{Z}}(Au,w_k)_{L^2(\mathbb{T})}\,e_k(x)=\sum_{k,j\in\mathbb{Z}}(w_j,w_k)_{L^2(\mathbb{T})}\,\hat u_j\,e_k(x),$$

namely the operator $M:=A^*_{\mathbb{T}}A$ is represented by the matrix $(M_k^j)_{k,j\in\mathbb{Z}}$ with respect to the exponentials basis $\{e_k\}_{k\in\mathbb{Z}}$, where

$$M_k^j:=(w_j,w_k)_{L^2(\mathbb{T})},\quad k,j\in\mathbb{Z}. \tag{12.4}$$

On the other hand,

$$AA^*_{\mathbb{T}}\,u(x)=\sum_{k\in\mathbb{Z}}(u,w_k)_{L^2(\mathbb{T})}\,w_k(x). \tag{12.5}$$

We shall see that, to prove the invertibility of $AA^*_{\mathbb{T}}$, instead of writing a matrix representation like M above, it is convenient to study

$$AA^*_{\mathbb{R}}\, g(x) = \int_{\mathbb{R}} (g, w_\xi)_{L^2(\mathbb{R})}\, w_\xi(x)\, \mathrm{d}\xi, \quad x \in \mathbb{R} \tag{12.6}$$

and pass from the real line to the torus in a further step.

Let us begin with estimates on $A^*_{\mathbb{T}}A$. Notation: Sobolev norms on the torus are denoted by

$$\|u\|_s = \|u\|_{H^s(\mathbb{T})}, \quad \|u\|_0 = \|u\|_{L^2(\mathbb{T})};$$

other norms are indicated explicitly.

Lemma 12.1. *(*Estimates for $A^*_{\mathbb{T}}A$*) There exist universal constants $C, \delta > 0$, with $C\delta < 1/4$, with the following properties.*

(i) If $L\|\beta\|_3 \le \delta$, then $M = A^*_{\mathbb{T}}A : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is bounded and invertible, with

$$\|(M - I)u\|_0 + \|(M^{-1} - I)u\|_0 \le CL\|\beta\|_3\|u\|_0. \tag{12.7}$$

As a consequence, for δ small enough,

$$\|Mu\|_0 \le 2\|u\|_0, \quad \|M^{-1}u\|_0 \le 2\|u\|_0.$$

(ii) Let $s \ge 1$. If $L\|\beta\|_3 \le \delta$ and $\beta \in H^{s+2}(\mathbb{T})$, then both M and M^{-1} are bounded and invertible from $H^s(\mathbb{T})$ onto $H^s(\mathbb{T})$, with

$$\|(M - I)u\|_s + \|(M^{-1} - I)u\|_s \le CL\|\beta\|_3\|u\|_s + C(s)L\|\beta\|_{s+2}\|u\|_1 \tag{12.8}$$

where C is the universal constant of part (i), and $C(s) > 0$ depends only on s. As a consequence,

$$\|Mu\|_s,\ \|M^{-1}u\|_s \le 2\|u\|_s + C(s)L\|\beta\|_{s+2}\|u\|_1. \tag{12.9}$$

Proof. (i) Fix a universal constant $\delta_0 > 0$ such that if $\|u\|_3 \le \delta_0$, then $\|u'\|_{L^\infty} \le 1/2$ and $\|u\|_2 \le 1$. Thus we can assume that $L\|\beta'\|_{L^\infty} \le 1/2$ and $L\|\beta\|_2 \le 1$.

Using the notation (12.4), on the diagonal $j = k$, one has

$$M_k^k = (w_k, w_k)_{L^2(\mathbb{T})} = 2\pi$$

because $\overline{w_k} = w_k^{-1}$. For $j \ne k$,

$$M_k^j = \int_0^{2\pi} \mathrm{e}^{i\omega(x+p(x))}\, \mathrm{d}x, \quad \omega := j - k, \quad p(x) := \frac{f(j) - f(k)}{j - k}\, \beta(x).$$

By (12.1), $|p'(x)| \le L|\beta'(x)| \le 1/2$ for all x, and $\|p\|_s \le L\|\beta\|_s$ for every $s \ge 0$. In particular, $\|p\|_2 \le 1$. By Lemma 13.6,

$$|M_k^j| \le \frac{C(\alpha)L\|\beta\|_{\alpha+1}}{|k - j|^\alpha}. \tag{12.10}$$

Split $M = I + R$, where I is the identity map and $R_k^j = M_k^j$ for $k \neq j$ and $R_k^k = 0$. By Hölder's inequality and (12.10) applied with $\alpha = 2$,

$$
\begin{aligned}
\|Ru\|_0^2 &= \sum_k \Big| \sum_{j \neq k} M_k^j \hat{u}_j \Big|^2 \leq \sum_k \Big(\sum_{j \neq k} |M_k^j| |\hat{u}_j| \Big)^2 \\
&\leq \sum_k \Big(\sum_{j \neq k} \frac{CL\|\beta\|_3}{|k-j|} \, \frac{|\hat{u}_j|}{|k-j|} \Big)^2 \\
&\leq \sum_k \Big(\sum_{j \neq k} \frac{C^2 L^2 \|\beta\|_3^2}{|k-j|^2} \Big) \Big(\sum_{j \neq k} \frac{|\hat{u}_j|^2}{|k-j|^2} \Big) \\
&= C^2 L^2 \|\beta\|_3^2 \, C_2 \sum_j \Big(\sum_{k \neq j} \frac{1}{|k-j|^2} \Big) |u_j|^2 \\
&\leq C^2 L^2 \, C_2^2 \, \|\beta\|_3^2 \, \|u\|_0^2
\end{aligned}
\tag{12.11}
$$

where $C_2 = \sum_{k \neq 0} |k|^{-2} < \infty$. Thus $\|Ru\|_0 \leq C_0 L \|\beta\|_3 \, \|u\|_0$ for some universal constant $C_0 > 0$. This is the desired estimate (12.7) for $R = M - I$. By the Neumann series, if $C_0 L \|\beta\|_3 \leq 1/2$, then $M : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is invertible, with

$$
\|(M^{-1} - I)u\|_0 \leq \sum_{n=1}^{\infty} \|R^n u\|_0 \leq 2C_0 L \|\beta\|_3 \, \|u\|_0.
$$

(ii) For $s \geq 1$, $k \in \mathbb{Z}$, split $\mathbb{Z}_k := \mathbb{Z} \setminus \{k\}$ into two components,

$$
\mathbb{Z}_k = A \cup B, \quad A = \{ j \in \mathbb{Z}_k : \langle k \rangle^s \leq 2 \langle j \rangle^s \}, \quad B = \mathbb{Z}_k \setminus A,
$$

and write

$$
\|Ru\|_s^2 \leq \sum_k \Big| \sum_{j \neq k} M_k^j \hat{u}_j \Big|^2 \langle k \rangle^{2s} \leq 2(S_A + S_B),
$$

where

$$
S_A = \sum_k \Big(\sum_{j \in A} |M_k^j| |\hat{u}_j| \langle k \rangle^s \Big)^2
$$

and similarly S_B. For S_A we use estimate (12.10) with $\alpha = 2$,

$$
S_A \leq \sum_k \Big(\sum_{j \in A} \frac{CL\|\beta\|_3}{|k-j|^2} |\hat{u}_j| \, 2 \langle j \rangle^s \Big)^2,
$$

then repeat the same calculations as (12.11) with $|\hat{u}_j| \langle j \rangle^s$ instead of $|\hat{u}_j|$, whence

$$
S_A \leq (CL\|\beta\|_3 \|u\|_s)^2
$$

where C is a universal constant. To estimate S_B, apply (12.10) with $\alpha = s+1$, and note that $\langle k \rangle \le c_s |k-j|$ for $j \in B$, so that

$$\begin{aligned} S_B &\le \sum_k \left(\sum_{j \in B} \frac{C(s+1) L \|\beta\|_{s+2}}{|k-j|^{s+1}} \, |\hat{u}_j| \, c_s^s |k-j|^s \right)^2 \\ &\le C(s) \sum_k \left(\sum_{j \in B} \frac{L \|\beta\|_{s+2}}{|k-j|} \, |\hat{u}_j| \, \langle j \rangle \, \frac{1}{\langle j \rangle} \right)^2 \\ &\le C(s) \sum_k \left(\sum_{j \in B} \frac{L^2 \|\beta\|_{s+2}^2}{|k-j|^2} \, |\hat{u}_j|^2 \, \langle j \rangle^2 \right) \left(\sum_{j \in B} \frac{1}{\langle j \rangle^2} \right) \\ &\le C(s) L^2 \|\beta\|_{\alpha+2}^2 \|u\|_1^2. \end{aligned} \tag{12.12}$$

This yields

$$\|Ru\|_s \le C_1 L \|\beta\|_3 \|u\|_s + C(s) L \|\beta\|_{s+2} \|u\|_1 \,, \quad \|Ru\|_1 \le C_1 L \|\beta\|_3 \|u\|_1$$

where C_1 is a universal constant and $C(s)$ depends on s. Hence $Mu \in H^s(\mathbb{T})$ for all $u \in H^s(\mathbb{T})$ together with the estimate for $M - I$ given by (12.8). Now, by induction,

$$\|R^n u\|_s \le (C_1 L \|\beta\|_3)^n \|u\|_s + n (C_1 L \|\beta\|_3)^{n-1} C(s) L \|\beta\|_{s+2} \|u\|_1 \quad \forall n \ge 1,$$

and the desired estimate (12.8) for $M^{-1} - I$ follows from Neumann series. □

As an immediate corollary of the operator norm estimate for $A_{\mathbb{T}}^* A$, we have a bound for A:

Lemma 12.2. (L^2-b*ound for A*) *Let* δ *be the universal constant of Lemma* 12.1. *If* $L\|\beta\|_3 \le \delta$, *then both* A *and* $A_{\mathbb{T}}^* : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ *are bounded, with*

$$\|Au\|_0 \le 2\|u\|_0, \quad \|A_{\mathbb{T}}^* u\|_0 \le 2\|u\|_0 \qquad \forall u \in L^2(\mathbb{T}).$$

Now we consider an operator E with the same phase $\phi(x, \xi)$ as A and, in addition, an amplitude $a(x, \xi)$, namely

$$E : e_\xi \mapsto q_\xi, \quad q_\xi(x) := a(x, \xi) \, w_\xi(x),$$

where $a(x, \xi)$ is a 2π-periodic function of x for every $\xi \in \mathbb{R}$ (or, at least, for every $\xi = k \in \mathbb{Z}$). If $u(x)$ is a periodic function with Fourier coefficients $\hat{u}_k$, then

$$Eu(x) = \sum_{k \in \mathbb{Z}} \hat{u}_k q_k(x) = \sum_{k \in \mathbb{Z}} \hat{u}_k a(x, k) w_k(x).$$

Remark 12.3. The adjoint operator $E_{\mathbb{T}}^*$ of E with respect to the L^2-scalar product on the torus is

$$E_{\mathbb{T}}^* u(x) = \sum_{k\in\mathbb{Z}} (u, q_k)_{L^2(\mathbb{T})}\, e_k(x), \quad x \in \mathbb{T},$$

therefore

$$E_{\mathbb{T}}^*\, Eu(x) = \sum_{k\in\mathbb{Z}} (Eu, q_k)_{L^2(\mathbb{T})}\, e_k(x) = \sum_{k,j\in\mathbb{Z}} (q_j, q_k)_{L^2(\mathbb{T})}\, \hat{u}_j\, e_k(x),$$

namely the operator $G := E_{\mathbb{T}}^*\, E$ is represented by the matrix $(G_k^j)_{k,j\in\mathbb{Z}}$ with respect to the exponentials basis $\{e_k\}_{k\in\mathbb{Z}}$, where

$$G_k^j := (q_j, q_k)_{L^2(\mathbb{T})}, \quad k, j \in \mathbb{Z}.$$

□

Lemma 12.4. (L^2-bound with amplitude) *Let δ be the universal constant of Lemma 12.1, and let $L\|\beta\|_3 \le \delta$. Let*

$$\sigma, \tau, K \in \mathbb{R}, \quad \sigma > 1, \quad \tau, K \ge 0.$$

If $a(\cdot\,, k) \in H^\sigma(\mathbb{T})$ for all $k \in \mathbb{Z}$, with

$$\|a(\cdot\,, k)\|_\sigma \le K\langle k\rangle^\tau \quad \forall k \in \mathbb{Z},$$

then $E : H^\tau(\mathbb{T}) \to L^2(\mathbb{T})$ is bounded, with

$$\|Eu\|_0 \le 2C_\sigma K\|u\|_\tau \quad \forall u \in H^\tau(\mathbb{T}),$$

where $C_\sigma := \sum_{j\in\mathbb{Z}}\langle j\rangle^{-\sigma} < \infty$.

Proof. Develop $a(x,k)$ in a Fourier series in x, $a(x,k) = \sum_{j\in\mathbb{Z}} \hat{a}_j(k)\, e_j(x)$, with $|\hat{a}_j(k)|\langle j\rangle^\sigma \le K\langle k\rangle^\tau$. Write Eu as

$$Eu(x) = \sum_k \hat{u}_k \left(\sum_j \hat{a}_j(k)\, e_j(x)\right) w_k(x) = \sum_j (AF_j u)(x)\, e_j(x) \tag{12.13}$$

where F_j is the Fourier multiplier $F_j : e_k \mapsto \hat{a}_j(k)\, e_k$, satisfying

$$\|F_j u\|_0^2 = \sum_k |\hat{u}_k|^2 |\hat{a}_j(k)|^2 \le \sum_k |\hat{u}_k|^2 \frac{K^2\langle k\rangle^{2\tau}}{\langle j\rangle^{2\sigma}} = \frac{K^2}{\langle j\rangle^{2\sigma}}\, \|u\|_\tau^2.$$

Remembering $\|Au\|_0 \le 2\|u\|_0$ (see Lemma 12.2), we obtain

$$\|Eu\|_0 \le \sum_j \|(AF_j u)e_j\|_0 \le \sum_j \|AF_j u\|_0 \le 2\sum_j \|F_j u\|_0 \le 2KC_\sigma\|u\|_\tau$$

where $C_\sigma = \sum_j \langle j\rangle^{-\sigma}$. □

Now go back to the study of $A : e_k \mapsto w_k$.

Lemma 12.5. (Sobolev bounds) *Let $L\|\beta\|_3 \le \delta$, where $\delta \le 1/2$ is the universal constant of the previous lemmas. Let $\alpha \ge 1$ be an integer. If $\beta \in H^{\alpha+2}(\mathbb{T})$, then A and $A_{\mathbb{T}}^* : H^\alpha(\mathbb{T}) \to H^\alpha(\mathbb{T})$ are bounded, with*

$$\|Au\|_\alpha + \|A_{\mathbb{T}}^* u\|_\alpha \le C(\alpha)\,(\|u\|_\alpha + L\|\beta\|_{\alpha+2}\|u\|_1)$$

for some constant $C(\alpha) > 0$.

Proof. Let us prove the estimate for A first. Since we already proved that $\|Au\|_0 \le 2\|u\|_0 \le 2\|u\|_\alpha$ and since $\|Au\|_\alpha \le C(\alpha)(\|Au\|_0 + \|\partial_x^\alpha Au\|_0)$, it is sufficient to estimate the L^2-norm of $\partial_x^\alpha Au$.

The derivatives of $w_k(x) = \mathrm{e}^{i\phi(x,k)}$ satisfy $\partial_x^\alpha(e^{i\phi(x,k)}) = P_\alpha(x,k)\,\mathrm{e}^{i\phi(x,k)}$ with

$$P_\alpha(x,k) = \sum_{n=1}^{\alpha} \sum_{\nu\in S_{\alpha,n}} C(\nu)\,(\partial_x^{\nu_1}\phi)(x,k)\cdots(\partial_x^{\nu_n}\phi)(x,k), \tag{12.14}$$

where $\nu = (\nu_1,\dots,\nu_n) \in S_{\alpha,n}$ means $1 \le \nu_1 \le \dots \le \nu_n$ and $\nu_1 + \dots + \nu_n = \alpha$. Therefore

$$\partial_x^\alpha Au(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k\, P_\alpha(x,k)\, w_k(x) = \sum_{n=1}^{\alpha} \sum_{\nu\in S_{\alpha,n}} C(\nu)\, E_\nu u(x),$$

where

$$E_\nu u(x) := \sum_{k\in\mathbb{Z}} \hat{u}_k\, a_\nu(x,k)\, w_k(x), \quad a_\nu(x,k) := (\partial_x^{\nu_1}\phi)(x,k)\cdots(\partial_x^{\nu_n}\phi)(x,k).$$

Write ϕ as

$$\phi(x,k) = k\,h(x,k), \quad h(x,k) = x + \frac{f(k)}{k}\,\beta(x), \quad k \ne 0.$$

Since $\nu_i \ge 1$, one may write $\|\partial_x^{\nu_i}\phi(\cdot\,,k)\|_2 \le |k|\,\|h'\|_{\nu_i+1}$ ($h' = \partial_x h$). Therefore

$$\|a_\nu(\cdot\,,k)\|_2 \le |k|^n C^{n-1}\|h'\|_{\nu_1+1}\cdots\|h'\|_{\nu_n+1}$$

where C is the algebra constant of $H^2(\mathbb{T})$ so that $\|uv\|_2 \le C\|u\|_2\|v\|_2$. By Lemma 12.4 (here $\sigma = 2$)

$$\|E_\nu u\|_0 \le C(\alpha)\,\|h'\|_{\nu_1+1}\cdots\|h'\|_{\nu_n+1}\,\|u\|_n$$

for some constant $C(\alpha)$ depending on α. By interpolation in Sobolev class, since $1 \le n \le \alpha$,

$$\|h'\|_{\nu_i+1} \le 2\,\|h'\|_2^{\vartheta_i}\,\|h'\|_{\alpha+1}^{1-\vartheta_i}, \quad \|u\|_n \le 2\,\|u\|_1^{\vartheta_0}\|u\|_\alpha^{1-\vartheta_0},$$

with $\vartheta_0, \vartheta_i \in [0,1]$, $i = 1,\dots,n$, and

$$\nu_i + 1 = 2\vartheta_i + (\alpha+1)(1-\vartheta_i), \quad n = 1\,\vartheta_0 + \alpha(1-\vartheta_0).$$

Hence

$$\prod_{i=1}^{n} \|h'\|_{\nu_i+1}\|u\|_n \leq 2^{n+1}\|h'\|_2^{\vartheta_1+\cdots+\vartheta_n}\|h'\|_{\alpha+1}^{n-(\vartheta_1+\cdots+\vartheta_n)}\|u\|_1^{\vartheta_0}\|u\|_\alpha^{1-\vartheta_0}.$$

Since $\nu_1+\cdots+\nu_n=\alpha$,

$$\vartheta_0+\vartheta_1+\cdots+\vartheta_n=n, \quad (\vartheta_1+\cdots+\vartheta_n)=(n-1)+(1-\vartheta_0),$$

and

$$\begin{aligned}\prod_{i=1}^{n} \|h'\|_{\nu_i+1}\|u\|_n &\leq 2^{n+1}\|h'\|_2^{n-1}\left(\|h'\|_2\|u\|_\alpha\right)^{1-\vartheta_0}\left(\|h'\|_{\alpha+1}\|u\|_1\right)^{\vartheta_0}\\ &\leq 2^{n+1}\|h'\|_2^{n-1}\left(\|h'\|_2\|u\|_\alpha+\|h'\|_{\alpha+1}\|u\|_1\right).\end{aligned}$$

By assumption, $L\|\beta\|_3\leq\delta\leq 1$, and, by (12.1), $|f(\xi)|\leq L|\xi|$ for all ξ, therefore

$$\|h'\|_2=\|1+f(k)k^{-1}\beta'\|_2\leq 1+L\|\beta\|_3\leq 2, \quad \|h'\|_{\alpha+1}\leq 1+L\|\beta\|_{\alpha+2},$$

and

$$\|E_\nu u\|_0\leq C(\alpha)\left(\|u\|_\alpha+(1+L\|\beta\|_{\alpha+2})\|u\|_1\right)\leq C(\alpha)\left(\|u\|_\alpha+L\|\beta\|_{\alpha+2}\|u\|_1\right).$$

Taking the sum over $n=1,\dots,\alpha$, $\nu\in S_{\alpha,n}$, gives the desired estimate for the L^2-norm of $\partial_x^\alpha Au$, which completes the proof of the estimate for A.

Now we prove the same estimate for $A_{\mathbb{T}}^*$. Remember that

$$A_{\mathbb{T}}^* u(x)=\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{T}} u(y)\,\mathrm{e}^{-i\phi(y,k)}\,\mathrm{d}y\right)\mathrm{e}^{ikx}.$$

Write $-\phi(y,k)$ as

$$-\phi(y,k)=(-k)\big(y+p(y,k)\big), \quad p(y,k)=\frac{f(k)}{k}\beta(y).$$

Using Lemma 13.6 together with the notations introduced in its proof, with $\omega=-k$,

$$\begin{aligned}\partial_x^\alpha(A_{\mathbb{T}}^*u)(x)&=\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{T}} u(y)\,\mathrm{e}^{-i\phi(y,k)}\,\mathrm{d}y\right)(ik)^\alpha\,\mathrm{e}^{ikx}\\ &=\sum_{k\in\mathbb{Z}}\left(\frac{i^\alpha}{(-k)^\alpha}\int_{\mathbb{T}} Q_\alpha(y)\,\mathrm{e}^{-i\phi(y,k)}\,\mathrm{d}y\right)(ik)^\alpha\,\mathrm{e}^{ikx}\\ &=\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{T}} Q_\alpha(y)\,\mathrm{e}^{-i\phi(y,k)}\,\mathrm{d}y\right)\mathrm{e}^{ikx}\\ &=\sum_{n=0}^{\alpha}\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{T}}(\partial_y^n u)(y)\,Q_\alpha^{(n)}(y,k)\,\mathrm{e}^{-i\phi(y,k)}\,\mathrm{d}y\right)\mathrm{e}^{ikx}\\ &=\sum_{n=0}^{\alpha}E_n^*\,(\partial_x^n u)(x)\end{aligned}$$

where E_n^* is the $L^2(\mathbb{T})$-adjoint operator of the FIO E_n having phase ϕ and amplitude a_n,

$$E_n v(x) := \sum_{k\in\mathbb{Z}} \hat{v}_k\, a_n(x,k)\, \mathrm{e}^{i\phi(x,k)},$$

$$a_n(x,k) := Q_\alpha^{(n)}(x,k) = \frac{1}{(h')^{2\alpha}} \sum_{\nu\in\mathcal{V}_{\alpha,n}} C(\nu)(\partial^{\nu_1}h)(x,k)\cdots(\partial^{\nu_\alpha}h)(x,k),$$

and $\nu \in \mathcal{V}_{\alpha,n}$ means

$$\nu = (\nu_1,\ldots,\nu_\alpha) \in \mathbb{Z}^\alpha, \quad \nu_i \geq 1, \quad \nu_1 + \cdots + \nu_\alpha = 2\alpha - n.$$

Here, as above, $h'(x,k) = 1 + f(k)k^{-1}\beta'(x)$ so $1/h' - 1 = F(\beta')$ for some smooth function F vanishing at the origin and hence $\|1/h'\|_2 \leq 2$, provided that $\|\beta\|_3$ is small enough. Then, with similar calculations as above, one proves that

$$\|a_n(\cdot\,,k)\|_2 \leq C(\alpha) \sum_{\nu\in\mathcal{V}_{\alpha,n}} \|h'\|_{\nu_1+1}\ldots\|h'\|_{\nu_\alpha+1}.$$

Therefore, by Lemma 12.4, the operator norm $\|E_n\|_{0,0} := \sup\{\|E_n u\|_0 : \|u\|_0 = 1\}$ satisfies

$$\|E_n\|_{0,0} \leq C(\alpha) \sum_{\nu\in\mathcal{V}_{\alpha,n}} \|h'\|_{\nu_1+1}\ldots\|h'\|_{\nu_\alpha+1}.$$

Since $\|E_n^*\|_{0,0} = \|E_n\|_{0,0}$,

$$\|E_n^*\,\partial_x^n u\|_0 \leq C(\alpha) \sum_{\nu\in\mathcal{V}_{\alpha,n}} \|h'\|_{\nu_1+1}\ldots\|h'\|_{\nu_\alpha+1}\,\|u\|_n$$

and we conclude the proof using the interpolation like above and summing for $n = 0,\ldots,\alpha$. $\square$

We have proved that $A_\mathbb{T}^* A$ is invertible, therefore $A_\mathbb{T}^*$ is surjective and A is injective. To prove that A is invertible, we need the invertibility of $AA_\mathbb{T}^*$. We prove it by studying $AA_\mathbb{R}^*$.

Recall that we consider a phase function $\phi(x,\xi) = x\xi + f(\xi)\beta(x)$ where $\beta(x)$ is a smooth real-valued periodic function and f is a C^∞ function such that $f(0) = 0$ and $\|f'\|_{L^\infty} < +\infty$. Hereafter, we make a further assumption.

Assumption 12.6. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that

$$f(\xi) = 0 \quad \forall|\xi| \leq 1/4; \qquad f(\xi) = |\xi|^r \quad \forall|\xi| \geq 1,$$

where $0 < r < 1$ is a real number.

We shall apply the following results with $r = 1/2$.

Lemma 12.7. *Assume that f satisfies the above assumption. There exist constants $C_1, \delta_1 > 0$, with $C_1\delta_1 \le 1/4$, such that, if $\|\beta\|_3 \le \delta_1$, then $AA_{\mathbb{T}}^* : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is invertible, with operator norm*

$$\|AA_{\mathbb{T}}^* u\|_0 \le 2\|u\|_0, \quad \|(AA_{\mathbb{T}}^*)^{-1} u\|_0 \le 2\|u\|_0.$$

More precisely,

$$\|(AA_{\mathbb{T}}^* - I)u\|_0 \le C_1 \|\beta\|_3 \|u\|_0 \le \frac{1}{4}\|u\|_0.$$

Proof. The proof is split into several steps.

STEP 1. Observe the following fact. Let $\beta \in H^{m+1}(\mathbb{T})$ for some integer $m \ge 2$, with $L\|\beta'\|_{L^\infty} \le \delta$ and $\|\beta\|_{m+1} \le K$, $K > 0$. Then, for every $\psi \in C_0^\infty(\mathbb{R})$,

$$|(\psi, w_\xi)_{L^2(\mathbb{R})}| = \left| \int_{\mathbb{R}} \psi(y)\, \mathrm{e}^{-i\phi(y,\xi)}\, \mathrm{d}y \right| \le \frac{C(\psi, K)}{1 + |\xi|^m} \quad \forall \xi \in \mathbb{R} \tag{12.15}$$

for some constant $C(\psi, K)$ which depends on $\|\psi\|_{W^{m,\infty}}$ and K. Indeed, integrating by parts gives

$$\int \psi(y)\, \mathrm{e}^{-i\phi(y,\xi)}\, \mathrm{d}y = \frac{1}{i\xi} \int \mathrm{e}^{-i\phi(y,\xi)}\, \mathcal{L}\psi(y)\, \mathrm{d}y,$$

where

$$\mathcal{L}\psi = \partial_y(v\psi), \quad v = \left(1 + \frac{f(\xi)}{\xi}\beta'(y)\right)^{-1}.$$

To gain a factor ξ^m at the denominator, integrate by parts m times.

STEP 2. To prove the invertibility of $AA_{\mathbb{T}}^*$, it is convenient to study $AA_{\mathbb{R}}^*$ and pass from the real line to the torus in a further step. $AA_{\mathbb{R}}^*$ is given by (12.6), namely

$$AA_{\mathbb{R}}^*\, g(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(y)\, \mathrm{e}^{-i\phi(y,\xi)}\, \mathrm{d}y \right) \mathrm{e}^{i\phi(x,\xi)}\, \mathrm{d}\xi. \tag{12.16}$$

For $g \in C_0^\infty(\mathbb{R})$ and $\beta \in H^3(\mathbb{T})$, with $L\|\beta'\|_{L^\infty} \le 1/2$, the integral is finite by (12.15). Now we want to change the integration variable ξ: this is the reason for which we consider real frequencies $\xi \in \mathbb{R}$ and not only integers $k \in \mathbb{Z}$.

Fix δ_0 small enough so that $|cf'(\xi)| \le 1/4$ for all $\xi \in \mathbb{R}$, provided that $|c| \le \delta_0$. For each $|c| \le \delta_0$, the map

$$\xi \mapsto \gamma(\xi, c) := \xi + cf(\xi)$$

is a diffeomorphism of $\mathbb{R}$ because $3/4 \le \partial_\xi \gamma(\xi, c) \le 5/4$, and γ is C^∞ in both the variables (ξ, c). Therefore, by the implicit function theorem, the inverse map $\mu(\vartheta, c)$,

$$\xi = \mu(\vartheta, c) \quad \Leftrightarrow \quad \vartheta = \gamma(\xi, c),$$

satisfies $4/5 \le \partial_\vartheta \mu(\vartheta, c) \le 4/3$ and is C^∞ in both (ϑ, c). Let

$$h(\vartheta, c) := \mu(\vartheta, c) - \vartheta.$$

Thus $h \in C^\infty$, $|\partial_\vartheta h(\vartheta, c)| \leq 1/3$ for all $\vartheta \in \mathbb{R}$, $|c| \leq \delta_0$,

$$h(\vartheta, c) + cf(\xi) = 0, \quad \partial_\vartheta h(\vartheta, c) = -\frac{cf'(\xi)}{1 + cf'(\xi)} \quad \text{where } \xi = \vartheta + h(\vartheta, c),$$
$$\partial_c h(\vartheta, c) = -\frac{f(\xi)}{1 + cf'(\xi)}, \quad \partial_{\vartheta c} h(\vartheta, c) = \frac{cf(\xi) f''(\xi)}{[1 + cf'(\xi)]^3} - \frac{f'(\xi)}{[1 + cf'(\xi)]^2}.$$

Then one proves that

$$\begin{aligned} &\|\partial_{\vartheta c} h(\cdot, c)\|_{L^\infty(\mathbb{R}, \mathrm{d}\vartheta)} \leq C, \\ &\|\partial_\vartheta^m \partial_c h(\cdot, c)\|_{L^1(\mathbb{R}, \mathrm{d}\vartheta)} \leq C, \quad m = 2, 3, 4, \end{aligned} \tag{12.17}$$

for all $|c| \leq \delta_0$, where the constant $C > 0$ depends only on f. Now let

$$\tilde{\beta}(x, y) := \frac{\beta(x) - \beta(y)}{x - y} \quad \text{for } x \neq y, \qquad \tilde{\beta}(x, x) := \beta'(x),$$

and let ϑ be the new frequency variable,

$$\vartheta = \xi + f(\xi)\tilde{\beta}(x, y), \quad \xi = \vartheta + h(\vartheta, \tilde{\beta}(x, y)) = \mu(\vartheta, \tilde{\beta}(x, y)). \tag{12.18}$$

The order of integration in (12.16) cannot be changed because the double integral does not converge absolutely. We overcome this problem as usual, fixing $\psi \in C_0^\infty(\mathbb{R})$ with $\psi(0) = 1$ and noting that

$$AA_{\mathbb{R}}^* g(x) = \lim_{\varepsilon \to 0} I_\varepsilon(x), \quad I_\varepsilon(x) := \int_{\mathbb{R}} \psi(\varepsilon\xi) \left(\int_{\mathbb{R}} g(y)\, e^{-i\phi(y, \xi)}\, \mathrm{d}y \right) \mathrm{e}^{i\phi(x, \xi)}\, \mathrm{d}\xi$$

by the dominated convergence theorem. It is found that

$$AA_{\mathbb{R}}^* g(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(y)\, \mathrm{e}^{-i\vartheta y}\, (1 + q(x, y, \vartheta))\, \mathrm{d}y \right) \mathrm{e}^{i\vartheta x}\, \mathrm{d}\vartheta$$

with

$$q(x, y, \vartheta) := \partial_\vartheta h(\vartheta, \tilde{\beta}(x, y)).$$

Namely $AA_{\mathbb{R}}^*$ is the sum $(I + Q)$ of the identity map and the pseudo-differential operator Q of compound symbol $q(x, y, \vartheta)$.

STEP 3. First order Taylor's formula in the y variable at $y = x$ gives

$$\begin{aligned} q(x, y, \vartheta) &= q_0(x, \vartheta) + q_1(x, y, \vartheta), \\ q_0(x, \vartheta) &:= q(x, x, \vartheta) = (\partial_\vartheta h)(\vartheta, \beta'(x)), \\ q_1(x, y, \vartheta) &:= \int_0^1 (\partial_y q)\big(x, x + s(y - x), \vartheta\big)\, \mathrm{d}s\, (y - x). \end{aligned}$$

Split $Q = Q_0 + Q_1$ accordingly. Since $q_0(x, \vartheta)$ does not depend on y,

$$Q_0 g(x) = \int_{\mathbb{R}} \hat{g}(\vartheta)\, q_0(x, \vartheta)\, \mathrm{e}^{i\vartheta x}\, \mathrm{d}\vartheta.$$

$q_0(x, \vartheta)$ is 2π-periodic in x, for every $\vartheta \in \mathbb{R}$, because $\beta'(x)$ is periodic. By (12.17),

$$|q_0(x, \vartheta)| \leq C|\beta'(x)|, \quad |\partial_x q_0(x, \vartheta)| \leq C|\beta''(x)|,$$

whence

$$\|q_0(\cdot\,, \vartheta)\|_1 \leq C\|\beta\|_2 \quad \forall \vartheta \in \mathbb{R} \tag{12.19}$$

for some constant $C > 0$ (where, remember, $\|\cdot\|_m$ is the $H^m(\mathbb{T})$ norm).

We next study Q_1. If the order of integration in Q_1 can be changed, then an integration by parts in the ϑ variable gives

$$Q_1 g(x) = -i \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} (\partial_{\vartheta y} q)\big(x, x + s(y - x), \vartheta\big)\, \mathrm{e}^{i\vartheta(x-y)}\, g(y)\, \mathrm{d}\vartheta\, \mathrm{d}y\, \mathrm{d}s.$$

Since

$$|\partial_{\vartheta z} q(x, z, \vartheta)| = |(\partial_{\vartheta\vartheta c} h)(\vartheta, \tilde{\beta}(x, z))|\, |\partial_z \tilde{\beta}(x, z)|,$$

by (12.17),

$$\|\partial_{\vartheta z} q(x, z, \cdot)\|_{L^1(\mathbb{R}, \mathrm{d}\vartheta)} \leq C\|\beta''\|_{L^\infty} \tag{12.20}$$

for all x, z in $\mathbb{R}$. Therefore the triple integral converges absolutely ($g \in C_0^\infty(\mathbb{R})$ by assumption), and one can prove that the order of integration can actually be changed (introduce a cut-off function $\psi(\varepsilon\vartheta)$, with $\psi(0) = 1$, $\psi \in C_0^\infty(\mathbb{R})$, and pass to the limit as $\varepsilon \to 0$ like above). Denote

$$a(x, y, \vartheta) := -i \int_0^1 (\partial_{\vartheta z} q)\big(x, x + s(y - x), \vartheta\big)\, \mathrm{d}s,$$

so that

$$Q_1 g(x) = \int_{\mathbb{R}^2} a(x, y, \vartheta)\, g(y)\, \mathrm{e}^{i\vartheta(x-y)}\, \mathrm{d}y\, \mathrm{d}\vartheta.$$

By (12.20), for all x one has

$$|Q_1 g(x)| \leq \int_{\mathbb{R}} |g(y)| \left(\int_{\mathbb{R}} |a(x, y, \vartheta)|\, \mathrm{d}\vartheta \right) \mathrm{d}y \leq C\|\beta\|_3 \|g\|_{L^1(\mathbb{R})}. \tag{12.21}$$

Denote $Tg(x) := xg(x)$. The commutator $[T, Q_1] = TQ_1 - Q_1 T$ is the same integral as Q_1 with an additional factor $(x - y)$,

$$[T, Q_1]g(x) = \int_{\mathbb{R}^2} a(x, y, \vartheta)\, (x - y)\, \mathrm{e}^{i\vartheta(x-y)}\, g(y)\, \mathrm{d}y\, \mathrm{d}\vartheta.$$

Integrating by parts in ϑ again,

$$[T, Q_1]g(x) = i \int_{\mathbb{R}^2} \partial_\vartheta a(x, y, \vartheta)\, \mathrm{e}^{i\vartheta(x-y)}\, g(y)\, \mathrm{d}y\, \mathrm{d}\vartheta,$$

and $\partial_{\vartheta\vartheta z} q(x, z, \vartheta)$ satisfies the same estimate (12.20) as $\partial_{\vartheta z} q$. Note that no other derivatives in y are involved in this argument, therefore β does not increase its derivation order. Repeat the same integration by parts twice: write

$$x^2 = [(x - y) + y]^2 = (x - y)^2 + 2(x - y)y + y^2,$$

so that

$$x^2 Q_1 g(x) = \int_{\mathbb{R}^2} \Big(i^2 (\partial_\vartheta^2 a)\, g + 2i (\partial_\vartheta a)(Tg) + a(T^2 g) \Big)\, e^{i\vartheta(x-y)}\, dy\, d\vartheta.$$

Every $\partial_\vartheta^m a(x, y, \vartheta)$, $m = 0, 1, 2$, satisfies an estimate like (12.20), namely

$$\|\partial_\vartheta^m a(x, y, \cdot)\|_{L^1(\mathbb{R}, d\vartheta)} \le C \|\beta\|_3, \quad m = 0, 1, 2,$$

for some constant $C > 0$. Now assume that $g(y) = 0$ for all $|y| > 2\pi$. Then, by Hölder's inequality,

$$\int_{\mathbb{R}} |y^m g(y)|\, dy \le C \|g\|_{L^2(\mathbb{R})} \quad \forall m = 0, 1, 2.$$

Thus we have $|x^2 Q_1 g(x)| \le C \|\beta\|_3 \, \|g\|_{L^2(\mathbb{R})}$ and, using also (12.21),

$$|Q_1 g(x)| \le \frac{C \|\beta\|_3 \, \|g\|_{L^2(\mathbb{R})}}{1 + |x|^2}.$$

Hence, provided that $g(y) = 0$ for all $|y| > 2\pi$, both $Q_1 g$ and $T Q_1 g$ are in $L^2(\mathbb{T})$, with

$$\|Q_1 g\|_{L^2(\mathbb{R})} + \|T Q_1 g\|_{L^2(\mathbb{R})} \le 2 \|(1 + x^2)^{1/2} Q_1 g(x)\|_{L^2(\mathbb{R})} \le C \|\beta\|_3 \, \|g\|_{L^2(\mathbb{R})}. \tag{12.22}$$

STEP 4. Let $\mathcal{P}$ be the "periodization" map defined in the Appendix. We observe that

$$\mathcal{P}(A A_{\mathbb{R}}^* \psi) = A A_{\mathbb{T}}^* (\mathcal{P}\psi) \quad \forall \psi \in C_0^\infty(\mathbb{R}). \tag{12.23}$$

To prove (12.23), fix $\psi \in C_0^\infty(\mathbb{R})$ and calculate

$$\begin{aligned} \mathcal{P}(A A_{\mathbb{R}}^* \psi)(x) &= \sum_{j \in \mathbb{Z}} (A A_{\mathbb{R}}^* \psi)(x + 2\pi j) \\ &= \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} (\psi, w_\xi)_{L^2(\mathbb{R})}\, w_\xi(x + 2\pi j) d\xi \\ &= \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} (\psi, w_\xi)_{L^2(\mathbb{R})}\, w_\xi(x)\, e^{i 2\pi j \xi}\, d\xi \end{aligned}$$

because $\phi(x + 2\pi j, \xi) = \phi(x, \xi) + 2\pi j \xi$. For each fixed $x \in \mathbb{R}$, by (12.15), the map $\xi \mapsto g(\xi) := (\psi, w_\xi)_{L^2(\mathbb{R})}\, w_\xi(x)$ satisfies

$$(1 + |\xi|^2)(|g(\xi)| + |g'(\xi)|) \le C \quad \forall \xi \in \mathbb{R},$$

for some constant C. Then, by Lemma 13.8 and (13.19),

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} g(\xi)\, e^{i 2\pi j \xi}\, d\xi &= \sum_{k \in \mathbb{Z}} g(k) \\ &= \sum_{k \in \mathbb{Z}} (\psi, w_k)_{L^2(\mathbb{R})}\, w_k(x) \\ &= \sum_{k \in \mathbb{Z}} (\mathcal{P}\psi, w_k)_{L^2(\mathbb{T})}\, w_k(x) \\ &= A A_{\mathbb{T}}^* (\mathcal{P}\psi)(x). \end{aligned}$$

STEP 5. From the two previous steps, since $\mathcal{P}\mathcal{S} = I$ on $L^2(\mathbb{T})$,

$$AA_{\mathbb{T}}^* = AA_{\mathbb{T}}^* \mathcal{P}\mathcal{S} = \mathcal{P}\, AA_{\mathbb{R}}^* \mathcal{S} = \mathcal{P}\,(I + Q_0 + Q_1)\,\mathcal{S} = I + \mathcal{P}\, Q_0\, \mathcal{S} + \mathcal{P}\, Q_1\, \mathcal{S}.$$

For $u \in C^\infty(\mathbb{T})$, by Lemma 13.8,

$$\begin{aligned}
\mathcal{P}\, Q_0\, \mathcal{S}u(x) &= \sum_{k\in\mathbb{Z}} (Q_0 \mathcal{S}u)(x + 2\pi k) \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{(\mathcal{S}u)}(\xi)\, q_0(x,\xi)\, e^{i\xi(x+2\pi k)}\, d\xi \\
&\overset{(13.17)}{=} \sum_{k\in\mathbb{Z}} \widehat{(\mathcal{S}u)}(k)\, q_0(x,k)\, e^{ikx} \\
&\overset{(13.15)}{=} \sum_{k\in\mathbb{Z}} \hat{u}_k\, q_0(x,k)\, e^{ikx}.
\end{aligned}$$

It is possible to use (13.17) here because $|q_0(x,\xi)| + |\partial_\xi q_0(x,\xi)| \le C$ for all x, ξ, for some $C > 0$, and $\widehat{\mathcal{S}u}(\xi)$ rapidly decreases as $\mathcal{S}u$ is compactly supported. Therefore

$$\|\mathcal{P}\, Q_0\, \mathcal{S}u\|_0 \le C \|\beta\|_2\, \|u\|_0$$

by (12.19) and Lemma 13.1(ii), and, by density, this holds for all $u \in L^2(\mathbb{T})$.

By Lemma 13.9 and (12.22),

$$\begin{aligned}
\|\mathcal{P}\, Q_1 \mathcal{S}u\|_0 &\le \|Q_1 \mathcal{S}u\|_{L^2(\mathbb{R})} + \|T\, Q_1 \mathcal{S}u\|_{L^2(\mathbb{R})} \le C\|\beta\|_3\, \|\mathcal{S}u\|_{L^2(\mathbb{R})} \\
&\le C\|\beta\|_3\, \|u\|_0
\end{aligned}$$

for some constant $C > 0$. We have proved that

$$AA_{\mathbb{T}}^* = I + B, \quad \|Be\|_0 \le C\|\beta\|_3 \|u\|_0,$$

where $B := \mathcal{P}\, Q_0\, \mathcal{S} + \mathcal{P}\, Q_1\, \mathcal{S}$. Therefore, by Neumann series, $AA_{\mathbb{T}}^* : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is invertible, with operator norm ≤ 2, for $\|\beta\|_3 \le \delta_1$, for some constant δ_1. □

Collecting the previous estimates, and taking the worst $\|\beta\|$ among all, we have the following

Lemma 12.8. *There exist universal constants $C, \delta_1 > 0$ such that, if $\beta \in H^3(\mathbb{T})$, $\|\beta\|_3 \le \delta_1$, then $A, A_{\mathbb{T}}^* : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ are invertible operators, with*

$$\|Au\|_0 + \|A^{-1}u\|_0 + \|A_{\mathbb{T}}^* u\|_0 + \|(A_{\mathbb{T}}^*)^{-1}u\|_0 \le C\, \|u\|_0.$$

If, in addition, $\beta \in H^{\alpha+2}(\mathbb{T})$, $\alpha \ge 1$ integer, then $A, A_{\mathbb{T}}^* : H^\alpha(\mathbb{T}) \to H^\alpha(\mathbb{T})$ are invertible, with

$$\|Au\|_\alpha + \|A^{-1}u\|_\alpha + \|A_{\mathbb{T}}^* u\|_\alpha + \|(A_{\mathbb{T}}^*)^{-1}u\|_\alpha \le C(\alpha)\, (\|u\|_\alpha + \|\beta\|_{\alpha+2}\, \|u\|_1),$$

where $C(\alpha) > 0$ is a constant that depends only on α.

Proof. Both $AA_{\mathbb{T}}^*$ and $A_{\mathbb{T}}^* A$ are invertible on $L^2(\mathbb{T})$, therefore A and A^* are also invertible. The estimates for A^{-1} and $(A_{\mathbb{T}}^*)^{-1}$ come from the equalities

$$A^{-1} = (A_{\mathbb{T}}^* A)^{-1} A_{\mathbb{T}}^*, \quad (A_{\mathbb{T}}^*)^{-1} = A(A_{\mathbb{T}}^* A)^{-1}.$$

□

12.2. With Amplitude

Now let E be the operator with phase $\phi(x,k)$ and amplitude $a(x,k)$, namely

$$Eu(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k\, a(x,k) e^{i\phi(x,k)}\,,$$

where $a(x,k)$ is 2π-periodic in x, and ϕ is like above. We are interested to the case when the amplitude is of order zero in k and it is a perturbation of 1,

$$a(x,k) = 1 + b(x,k).$$

Denote $|b|_s := \sup_{k\in\mathbb{Z}} \|b(\cdot\,,k)\|_s$. Let A be the operator with phase $\phi(x,k)$ and amplitude 1, and let B be the operator of phase $\phi(x,k)$ and amplitude $b(x,k)$, so that $E = A + B$.

Lemma 12.9. *There exists a universal constant $\delta > 0$ with the following properties. Let $\beta \in H^3(\mathbb{T})$ and $b(\cdot\,,k) \in H^3(\mathbb{T})$ for all $k \in \mathbb{Z}$. If*

$$\|\beta\|_3 + |b|_2 \le \delta,$$

then $\|Bu\|_0 \le C|b|_2\|u\|_0$, and E and $E^*_{\mathbb{T}}$ are invertible from $L^2(\mathbb{T})$ onto itself, with

$$\|Eu\|_0 + \|E^{-1}u\|_0 + \|E^*_{\mathbb{T}}\, u\|_0 + \|(E^*_{\mathbb{T}})^{-1}u\|_0 \le C\, \|u\|_0\,,$$

where $C > 0$ is a universal constant.

If, in addition, $\alpha \ge 1$ is an integer, $\beta \in H^{\alpha+2}(\mathbb{T})$ and $b(\cdot\,,k) \in H^{\alpha+2}(\mathbb{T})$ for all $k \in \mathbb{Z}$, then

$$\|Bu\|_\alpha + \|B^*_{\mathbb{T}} u\|_\alpha \le C(\alpha)\, (|b|_2\|u\|_\alpha + (|b|_{\alpha+2} + \|\beta\|_{\alpha+2}|b|_2)\, \|u\|_1)$$

and

$$\begin{aligned} &\|Eu\|_\alpha + \|E^{-1}u\|_\alpha + \|E^*_{\mathbb{T}}\, u\|_\alpha + \|(E^*_{\mathbb{T}})^{-1}u\|_\alpha \\ &\le C(\alpha)\, \{\|u\|_\alpha + (|b|_{\alpha+2} + \|\beta\|_{\alpha+2})\|u\|_1\} \end{aligned}$$

where $C(\alpha) > 0$ depends only on α.

Proof. Step 1. By Lemma 12.4, $\|Bu\|_0 \le C|b|_2\|u\|_0$, therefore, using Lemma 12.8, $\|A^{-1}Bu\|_0 \le C|b|_2\|u\|_0$ for some universal constant $C > 0$, provided that δ is small enough. Then $E = A(I + A^{-1}B)$ is invertible in $L^2(\mathbb{T})$ by the Neumann series. Analogous proof for E^*.

Step 2. The matrix $L := E^*_{\mathbb{T}}E$ is given by Remark 12.3, and it is

$$E^*E = A^*A + A^*B + B^*A + B^*B.$$

On the diagonal,

$$L^k_k = \int_{\mathbb{T}} |a(x,k)|^2\, dx = \int_{\mathbb{T}} |1 + b(x,k)|^2\, dx \ge \frac{1}{2}$$

if $|b(x,k)| \leq 1/2$ for all $x \in \mathbb{T}$, $k \in \mathbb{Z}$. Off-diagonal,

$$L_j^k = M_j^k + \int_{\mathbb{T}} \Big(\overline{b(x,k)} + b(x,j) + b(x,j)\, \overline{b(x,k)} \Big) \, \mathrm{e}^{i[\phi(x,j)-\phi(x,k)]} \, \mathrm{d}x,$$

where the matrix M_j^k is defined in (12.4). Using (12.10) for the first term and Lemma 13.6 for the other three terms,

$$|L_j^k| \leq \frac{C(\alpha, K)}{|k-j|^{\alpha}} \big(|b|_{\alpha} + \|\beta\|_{\alpha+1} \big), \quad k \neq j, \tag{12.24}$$

for $\|\beta\|_2 \leq K$ and $|b|_1 \leq K$. Let D be the Fourier multiplier $e_k \mapsto L_k^k e_k$ and R the off-diagonal part $R = L - D$. For $\alpha = 2$ in (12.24),

$$\|D^{-1} R u\|_0 \leq C(K) \, (\|\beta\|_3 + |b|_2) \, \|u\|_0,$$

therefore L is invertible in $L^2(\mathbb{T})$ if $\|\beta\|_3 + |b|_2 \leq \delta$ for some universal $\delta > 0$ (for example, fix $K = 1$ first, then fix δ sufficiently small). For $s \geq 1$ integer, and for $\|\beta\|_3 + |b|_2 \leq \delta$,

$$\|Ru\|_s \leq C\delta \|u\|_s + C(s)(\|\beta\|_{s+2} + |b|_{s+1}) \|u\|_1$$

by (12.24) and usual calculations for off-diagonal matrices. This gives the tame estimate for L, and, by the Neumann series, also for L^{-1}, namely

$$\|E_{\mathbb{T}}^* E u\|_s + \|(E_{\mathbb{T}}^* E)^{-1} u\|_s \leq C\|u\|_s + C(s)(\|\beta\|_{s+2} + |b|_{s+1})\|u\|_1 \tag{12.25}$$

where $C > 0$ is a universal constant and $C(s) > 0$ depends on s.

STEP 3. $Bu(x)$ is given by (12.13) where F_j is the Fourier multiplier $F_j e_k = \hat{b}_j(k)\, e_k$. Integrating by parts, for $j \neq 0$,

$$|\hat{b}_j(k)| = \left| \int_{\mathbb{T}} b(x,k) \, \mathrm{e}^{-ijx} \, \mathrm{d}x \right| = \frac{1}{|j|^m} \left| \int_{\mathbb{T}} \partial_x^m b(x,k) \, \mathrm{e}^{-ijx} \, \mathrm{d}x \right| \leq \frac{C \|\partial_x^m b(\cdot\,,k)\|_0}{|j|^m}$$

for all $k \in \mathbb{Z}$. Hence,

$$\|F_j u\|_{\alpha} \leq \frac{|b|_m}{\langle j \rangle^m} \|u\|_{\alpha}, \qquad \forall j \in \mathbb{Z}. \tag{12.26}$$

Using the tame product rule $\|uv\|_{\alpha} \leq C(\alpha)(\|u\|_{L^\infty} \|v\|_{\alpha} + \|v\|_{L^\infty} \|u\|_{\alpha})$ together with the Sobolev embedding $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$ we deduce that

$$\|Bu\|_{\alpha} \leq \sum_{j \in \mathbb{Z}} \|(AF_j u)\, e_j\|_{\alpha} \leq C(\alpha) \sum_{j \in \mathbb{Z}} \big(\|AF_j u\|_{\alpha} + \|AF_j u\|_1 \, \langle j \rangle^{\alpha} \big).$$

Thus, applying Lemma 12.8 and using (12.26) with either $m = 2$ or $m = \alpha + 3$,

$$\begin{aligned} \|Bu\|_{\alpha} &\leq C(\alpha) \sum_{j \in \mathbb{Z}} \|F_j u\|_{\alpha} + \|\beta\|_{\alpha+2} \|F_j u\|_1 + \|F_j u\|_1 \, \langle j \rangle^{\alpha} \\ &\leq C(\alpha) \sum_{j \in \mathbb{Z}} (\|u\|_{\alpha} + \|\beta\|_{\alpha+2} \|u\|_1) \, \frac{|b|_2}{\langle j \rangle^2} + \|u\|_1 \, \frac{|b|_{\alpha+2}}{\langle j \rangle^2} \\ &\leq C(\alpha) \, (\|u\|_{\alpha} + \|\beta\|_{\alpha+2} \|u\|_1) \; |b|_2 + \|u\|_1 \, |b|_{\alpha+2}. \end{aligned}$$

The sum with the analogous estimate for A gives

$$\|Eu\|_\alpha \le C(\alpha)\ (\|u\|_\alpha + (|b|_{\alpha+2} + \|\beta\|_{\alpha+2})\|u\|_1)\,.$$

For $E^*_{\mathbb{T}}$, note that

$$E^*_{\mathbb{T}} u = \sum_{j\in\mathbb{Z}} F^*_j A^*_{\mathbb{T}}(e_{-j}\, u)$$

and repeat the same argument.

Since $E^{-1} = (E^*_{\mathbb{T}}E)^{-1}E^*_{\mathbb{T}}$, and both $(E^*_{\mathbb{T}}E)^{-1}$ and $E^*_{\mathbb{T}}$ satisfy tame estimates, then the tame estimate for E^{-1} is obtained by composition. Similarly for $(E^*_{\mathbb{T}})^{-1}$. □

12.3. Composition Formula

Consider the periodic FIO of amplitude a and phase function ϕ,

$$Au(x) = \sum_{k\in\mathbb{Z}} \hat u_k\, a(x,k)\, \mathrm{e}^{i\phi(x,k)}, \qquad \phi(x,k) = kx + f(k)\beta(x), \tag{12.27}$$

where $f(k) = |k|^{1/2}$ for all $k \in \mathbb{Z}$.

Lemma 12.10. *Let A be the operator (12.27), with $\|\beta\|_{W^{1,\infty}} \le 1/4$ and $\|\beta\|_2 \le 1/2$. Let $r \in \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that, for every integer $n \ge 0$, the n-ti derivative $g^{(n)}(\xi)$ of g satisfies*

$$|g^{(n)}(\xi)| \le C_{r,n}(1 + |\xi|)^{r-n} \quad \forall \xi \in \mathbb{R},$$

for some constant $C_{r,n} > 0$. Let $\Gamma := \mathrm{Op}(g)$ be the Fourier multiplier with symbol $g(\xi)$. Let

$$m, s_0 \in \mathbb{R}, \quad m \ge 0, \quad s_0 > 1/2, \quad N \in \mathbb{N}, \quad N \ge 2(m + r + 1) + s_0.$$

Then

$$\Gamma A u = \sum_{\alpha=0}^{N-1} B_\alpha u + R_N u,$$

where

$$B_\alpha u(x) = \frac{1}{i^\alpha\, \alpha!} \sum_{k\in\mathbb{Z}} g^{(\alpha)}(k)\partial_x^\alpha \left\{a(x,k)\mathrm{e}^{if(k)\beta(x)}\right\} \hat u_k\, \mathrm{e}^{ikx}.$$

For every $s \ge s_0$, the remainder satisfies

$$\|R_N |D_x|^m u\|_s \le C(s)\left\{\mathcal{K}_{2(m+r+s_0+1)}\, \|u\|_s + \mathcal{K}_{s+N+m+2}\, \|u\|_{s_0}\right\}, \tag{12.28}$$

where $\mathcal{K}_\mu := \|a\|_\mu + \|a\|_1\|\beta\|_{\mu+1}$ for $\mu \ge 0$ and $\|a\|_\mu := \sup_{k\in\mathbb{Z}} \|a(\cdot,k)\|_\mu$.

If, in addition, $a(x,k) = 1$, then (12.28) holds with $\mathcal{K}_\mu := \|\beta\|_{\mu+1}$.

Remark 12.11. In particular, $B_0 = A\Gamma$. Also, B_α is of order $r - (\alpha/2)$.

Proof. Denote by $(\hat{z}_j(k))_{j\in\mathbb{Z}}$ the Fourier coefficients of the periodic function $x \mapsto a(x,k)\,\mathrm{e}^{if(k)\beta(x)}$. If $\hat{u}_k$ are the Fourier coefficients of a periodic function $u(x)$, then

$$\Gamma A u(x) = \sum_{k,j\in\mathbb{Z}} \hat{u}_k\, \hat{z}_j(k)\, g(k+j)\, \mathrm{e}^{i(k+j)x}. \tag{12.29}$$

Taylor's formula gives, for some $t \in [0,1]$,

$$g(k+j) = \sum_{\alpha=0}^{N-1} \frac{1}{\alpha!}\, g^{(\alpha)}(k)\, j^\alpha + r_N(k,j), \qquad r_N(k,j) = \frac{1}{N!}\, g^{(N)}(k+tj)\, j^N.$$

Accordingly, (12.29) is split into $\Gamma A u = \sum_{\alpha=0}^{N-1} B_\alpha u + R_N u$, where

$$\begin{aligned} B_\alpha u(x) &:= \sum_{k,j\in\mathbb{Z}} \hat{u}_k\, \hat{z}_j(k)\, \frac{1}{\alpha!}\, g^{(\alpha)}(k)\, j^\alpha\, \mathrm{e}^{i(k+j)x} \\ &= \frac{1}{i^\alpha \alpha!} \sum_{k\in\mathbb{Z}} g^{(\alpha)}(k) \left(\sum_{j\in\mathbb{Z}} \hat{z}_j(k)\, (ij)^\alpha\, \mathrm{e}^{ijx} \right) \hat{u}_k\, \mathrm{e}^{ikx} \\ &= \frac{1}{i^\alpha \alpha!} \sum_{k\in\mathbb{Z}^*} g^{(\alpha)}(k) \partial_x^\alpha \left\{ a(x,k)\mathrm{e}^{if(k)\beta(x)} \right\} \hat{u}_k\, \mathrm{e}^{ikx}. \end{aligned}$$

It remains to estimate the remainder

$$R_N u(x) := \sum_{k,j\in\mathbb{Z}} \hat{u}_k\, \hat{z}_j(k)\, r_N(k,j)\, \mathrm{e}^{i(k+j)x}.$$

For $N \geq r$, the N-th derivative of g satisfies $|g^{(N)}(\xi)| \leq C_{r,N} \langle \xi \rangle^{r-N} \leq C_{r,N}$ for all ξ in $\mathbb{R}$. In particular, for $|j| \leq \frac{1}{2}|k|$ and $t \in [0,1]$, we have $|k+tj| \geq \frac{1}{2}|k|$ and hence

$$|r_N(k,j)| \leq C_{r,N}\, \langle k \rangle^{r-N} |j|^N \quad \forall |j| \leq \frac{1}{2}|k|, \tag{12.30}$$

and, in general,

$$|r_N(k,j)| \leq C_{r,N}\, |j|^N \quad \forall j \in \mathbb{Z}. \tag{12.31}$$

We split R_N into 2 components, $R_N = \mathcal{R}_1 + \mathcal{R}_2$, $\mathcal{R}_1$ for low frequencies and $\mathcal{R}_2$ for high frequencies:

$$\begin{aligned} \mathcal{R}_1 u(x) &:= \sum_{|j|\leq\frac{1}{2}|k|} \hat{u}_k\, \hat{z}_j(k)\, r_N(k,j)\, \mathrm{e}^{i(k+j)x}, \\ \mathcal{R}_2 u(x) &:= \sum_{|j|>\frac{1}{2}|k|} \hat{u}_k\, \hat{z}_j(k)\, r_N(k,j)\, \mathrm{e}^{i(k+j)x}. \end{aligned}$$

Let us analyze the coefficients $\hat{z}_j(k)$. First of all, for $j \neq 0$, $\alpha \geq 0$,

$$|\hat{z}_j(k)| \leq \frac{1}{|j|^\alpha} \left\| a(\cdot,k)\, \mathrm{e}^{if(k)\beta(\cdot)} \right\|_\alpha.$$

Now, for $\alpha \in \mathbb{N}$,

$$\partial_x^\alpha \left(a(x,k)\mathrm{e}^{if(k)\beta(x)}\right) = \mathrm{e}^{if(k)\beta(x)} \sum_{\alpha_1+\alpha_2=\alpha} C(\alpha_1,\alpha_2)\,(\partial_x^{\alpha_1} a)(x,k)\, P_{\alpha_2}(x,k),$$

where P_α is defined, as above, by $\partial_x^\alpha\left(\mathrm{e}^{if(k)\beta(x)}\right) = \mathrm{e}^{if(k)\beta(x)}\, P_\alpha(x,k)$. Hence it follows from the estimate (13.9) in the Appendix and an interpolation argument that

$$|\hat{z}_j(k)| \le \frac{C_\alpha |k|^{\alpha/2}}{|j|^\alpha}\left(\|a\|_\alpha + \|a\|_1 \|\beta\|_{\alpha+1}\right) \quad \forall j \neq 0, \tag{12.32}$$

where $\|a\|_\alpha := \sup_{k\in\mathbb{Z}} \|a(\cdot,k)\|_\alpha$. Moreover, if $a=1$, then it follows from (13.9) that the estimate (12.32) holds with the factor $(\|a\|_\alpha + \|a\|_1 |\beta|_\alpha)$ replaced by $\|\beta\|_{\alpha+1}$.

For $|j| > \frac{1}{2}|k|$, estimate (12.32) can be improved:

$$\hat{z}_j(k) = \int_{\mathbb{T}} a(x,k)\,\mathrm{e}^{if(k)\beta(x)}\mathrm{e}^{-ijx}\,\mathrm{d}x = \int_{\mathbb{T}} u(x)\mathrm{e}^{i\omega(x+p(x))}\,\mathrm{d}x,$$

with

$$\omega = -j, \quad p(x) = -\frac{f(k)}{j}\,\beta(x), \quad u(x) = a(x,k).$$

Therefore, applying the non-stationary phase argument of Lemma 13.6,

$$|\hat{z}_j(k)| \le \frac{C_\alpha}{|j|^\alpha}\left(\|a\|_\alpha + \|a\|_1\|\beta\|_{\alpha+1}\right) \quad \forall |j| > \frac{1}{2}|k| \tag{12.33}$$

provided that $\|\beta\|_2 \le 1/2$ and $|\beta|_1 \le 1/4$. Moreover, if $a=1$, then $u=1$, and, by Lemma 13.6, the estimate (12.33) holds with the factor $(\|a\|_\alpha + \|a\|_1\|\beta\|_{\alpha+1})$ replaced by $\|\beta\|_{\alpha+1}$.

• *Estimate for* $\mathcal{R}_1$.— We study the composition $\mathcal{R}_1|D_x|^m$, which is the pseudo-differential operator with symbol

$$\rho_1(x,k) := \sum_{|j|\le\frac{1}{2}|k|} |k|^m\,\hat{z}_j(k)\,r_N(k,j)\,\mathrm{e}^{ijx}.$$

By (12.30) and (12.32), for any $\alpha \in \mathbb{N}$,

$$\begin{aligned}\|\rho_1(\cdot,k)\|_{s_0}^2 &= \sum_{|j|\le\frac{1}{2}|k|} |k|^{2m}\,|\hat{z}_j(k)|^2\,|r_N(k,j)|^2\,\langle j\rangle^{2s_0}\\ &\le \sum_{|j|\le\frac{1}{2}|k|} C\langle k\rangle^{2(m+r-N+(\alpha/2))}\,\langle j\rangle^{2(s_0+N-\alpha)}\left(\|a\|_\alpha + \|a\|_1\|\beta\|_{\alpha+1}\right)^2.\end{aligned}$$

Now, assume that

$$s_0 + N - \alpha \ge 0. \tag{12.34}$$

Then

$$\sum_{|j|\le\frac{1}{2}|k|} \langle j\rangle^{2(s_0+N-\alpha)} \le C\langle k\rangle^{2(s_0+N-\alpha)+1}$$

because $\langle j\rangle \le \langle k\rangle$ and the number of terms in the sum is $\le C|k|$. Hence

$$\|\rho_1(\cdot,k)\|_{s_0}^2 \le C\langle k\rangle^{2(m+r-N+(\alpha/2)+s_0+N-\alpha)+1} \left(\|a\|_\alpha + \|a\|_1\|\beta\|_{\alpha+1}\right)^2.$$

The exponent of $\langle k\rangle$ is ≤ 0 if $\alpha \ge 2(m+r+s_0)+1$. Hence fix α to be the integer

$$\begin{aligned} \alpha_0 &:= \min\{\alpha \in \mathbb{N} : \alpha \ge 2(m+r+s_0)+1\} \\ &= 2(m+r+s_0)+1+\delta_0, \quad 0 \le \delta_0 < 1. \end{aligned}$$

By assumption,

$$N \ge 2(m+r+1)+s_0 > 2(m+r+s_0)+1+\delta_0 - s_0 = \alpha_0 - s_0,$$

therefore $s_0+N-\alpha_0 \ge 0$, and (12.34) is satisfied. We get $\|\rho_1(\cdot,k)\|_{s_0} \le C\big(\|a\|_{\alpha_0} + \|a\|_1|\beta|_{\alpha_0+1}\big)$ and so, by Lemma 13.2,

$$\|\mathcal{R}_1|D_x|^m u\|_s \le C(s)\left(\|a\|_{\alpha_0} + \|a\|_1\|\beta\|_{\alpha_0+1}\right)\|u\|_s$$

for all $s_0 > 1/2$, and $s \ge 0$. Moreover $\alpha_0 \le 2(m+r+s_0)+2$, therefore

$$\|\mathcal{R}_1|D_x|^m u\|_s \le C(s)\left(\|a\|_{2(m+r+s_0+1)} + \|a\|_1\|\beta\|_{2(m+r+s_0)+3}\right)\|u\|_s. \quad (12.35)$$

• *Estimate for $\mathcal{R}_2$.*— Now we study the composition $\mathcal{R}_2|D_x|^m$, which is the pseudo-differential operator with symbol

$$\rho_2(x,k) := \sum_{|j|>\frac{1}{2}|k|} |k|^m\,\hat{z}_j(k)\,r_N(k,j)\,\mathrm{e}^{ijx}.$$

By (12.31) and (12.33), for any $\alpha \in \mathbb{N}$,

$$\begin{aligned} \|\rho_2(\cdot,k)\|_s^2 &= \sum_{|j|>\frac{1}{2}|k|} |k|^{2m}\,|\hat{z}_j(k)|^2\,|r_N(k,j)|^2\,\langle j\rangle^{2s} \\ &\le \sum_{|j|>\frac{1}{2}|k|} C|k|^{2m}\,\langle j\rangle^{2(s+N-\alpha)}\left(\|a\|_\alpha + \|a\|_1\|\beta\|_{\alpha+1}\right)^2 \\ &\le C|k|^{2(m+s+N-\alpha)+1}\left(\|a\|_\alpha + \|a\|_1\|\beta\|_{\alpha+1}\right)^2 \end{aligned}$$

because

$$\sum_{|j|>\frac{1}{2}|k|} \langle j\rangle^{2(s+N-\alpha)} \le C\int_{\frac{1}{2}|k|}^{+\infty} t^{2(s+N-\alpha)}\,\mathrm{d}t \le C|k|^{2(s+N-\alpha)+1}$$

for $2(s+N-\alpha)+1 < 0$, namely for $\alpha > s+N+\frac{1}{2}$.

The exponent of $|k|$ is ≤ 0 for $\alpha \ge s+N+m+\frac{1}{2}$. Fix $\alpha_1 := \min\{n \in \mathbb{N} : n \ge s+N+m+1\}$. Thus

$$\|\rho_2(\cdot,k)\|_s \le C\left(\|a\|_{\alpha_1} + \|a\|_1\|\beta\|_{\alpha_1+1}\right).$$

By Lemma 13.3,

$$\|\mathcal{R}_2|D_x|^m u\|_s \le C(s)\left(\|a\|_{\alpha_1} + \|a\|_1\|\beta\|_{\alpha_1+1}\right)\|u\|_{s_0}$$

for all $s_0 > 1/2$, and $s \geq s_0$. Moreover, since $\alpha_1 \leq s + N + m + 2$,

$$\|\mathcal{R}_2 |D_x|^m u\|_s \leq C(s)\ (\|a\|_{s+N+m+2} + \|a\|_1 \|\beta\|_{s+N+m+3})\ \|u\|_{s_0}. \tag{12.36}$$

- *Estimate for R_N.*— The sum of (12.35) and (12.36) gives

$$\|R_N |D_x|^m u\|_s \leq C(s) \Big\{ \Big(\|a\|_{2(m+r+s_0+1)} + \|a\|_1 \|\beta\|_{2(m+r+s_0)+3} \Big) \|u\|_s + \Big(\|a\|_{s+N+m+2} + \|a\|_1 \|\beta\|_{s+N+m+3} \Big) \|u\|_{s_0} \Big\},$$

which is (12.28).

If $a = 1$, then (12.32) and (12.33) become

$$|\hat{z}_j(k)| \leq \frac{C_\alpha |k|^{\alpha/2}}{|j|^\alpha} \|\beta\|_{\alpha+1} \quad \forall j \neq 0; \qquad |\hat{z}_j(k)| \leq \frac{C_\alpha}{|j|^\alpha} \|\beta\|_{\alpha+1} \quad \forall |j| > \frac{1}{2}|k|,$$

so that the estimates (12.35) and (12.36) are modified accordingly, and (12.28) holds with $\mathcal{K}_\mu := \|\beta\|_{\mu+1}$. □

Corollary 12.12. *Assume the hypotheses of Lemma 12.10, and let $a(x,k) = 1 + b(x,k)$, with $\|b(\cdot,k)\|_1 \leq 1$ for all $k \in \mathbb{Z}$.*

Then (12.28) holds with $\mathcal{K}_\mu := \|b\|_\mu + \|\beta\|_{\mu+1}$, where $\|b\|_\mu := \sup_{k\in\mathbb{Z}} \|b(\cdot,k)\|_\mu$.

Proof. Since $a = 1 + b$, split $A = A_0 + B$, where A_0 has phase ϕ and amplitude 1, and B has phase ϕ and amplitude b. Then the remainder R_N is split accordingly into $R_N(A_0) + R_N(B)$. Apply Lemma 12.10 to A_0 (with $\mathcal{K}_\mu = \|\beta\|_{\mu+1}$) and to B (with $\mathcal{K}_\mu = \|b\|_\mu + \|b\|_1 \|\beta\|_{\mu+1}$), and sum. □

12.4. With Dependence on Time

Assume that the operator A in (12.3) depends on time, namely the phase space is

$$\phi(t,x,k) = kx + |k|^{1/2}\beta(t,x),$$

where β is periodic in the time variable $t \in \mathbb{T}$. Then the inequality of the previous sections also hold (with minor changes) in spaces $H^s(\mathbb{T}^2)$. For example:

Lemma 12.13. *If $\|\beta\|_4 \leq \delta$ (where $\delta \in (0,1)$ is a universal constant), then for all integers s*

$$\|Ah\|_s \leq C\|h\|_s \quad \forall s = 0,1; \qquad \|Ah\|_s \leq C(s)(\|h\|_s + \|\beta\|_{s+3}\|h\|_1) \quad \forall s \geq 2, \tag{12.37}$$

for all $h = h(t,x)$, where $\| \ \|_s$ is the norm of $H^s(\mathbb{T}^2)$.

Proof. We have already proved that, without dependence on time (i.e. $h = h(x)$, $\beta = \beta(x)$),

$$\|Ah\|_{L^2_x} \le C\|h\|_{L^2_x}, \quad \|Ah\|_{H^s_x} \le C(s)(\|h\|_{H^s_x} + \|\beta\|_{H^{s+2}_x}\|h\|_{H^1_x}) \tag{12.38}$$

provided $\|\beta\|_{H^3_x} \le \delta$. Now let h, β depend also on time. For each fixed t, $\|\beta(t)\|_{H^3_x} \le \|\beta\|_{L^\infty_t H^3_x} \le \|\beta\|_{H^1_t H^3_x} \le \|\beta\|_4 \le \delta$, and then (12.38) holds at each t. Therefore

$$\|Ah\|^2_{L^2_t L^2_x} = \int_{\mathbb{T}} \|A(t)h(t)\|^2_{L^2_x}\,\mathrm{d}t \le \int_{\mathbb{T}} C^2\|h(t)\|^2_{L^2_x}\,\mathrm{d}t = C^2\|h\|^2_{L^2_t L^2_x},$$

i.e. $\|Ah\|_0 \le C\|h\|_0$. Similarly, for $s \ge 1$, using $\|\beta\|_{L^\infty_t H^{s+2}_x} \le C\|\beta\|_{H^1_t H^{s+2}_x}$,

$$\begin{aligned}
\|Ah\|^2_{L^2_t H^s_x} &= \int_{\mathbb{T}} \|A(t)h(t)\|^2_{H^s_x}\,\mathrm{d}t \\
&\le C(s)\int_{\mathbb{T}} \{\|h(t)\|^2_{H^s_x} + \|\beta(t)\|^2_{H^{s+2}_x}\|h(t)\|^2_{H^1_x}\}\,\mathrm{d}t \\
&\le C(s)(\|h\|^2_{L^2_t H^s_x} + \|\beta\|^2_{H^1_t H^{s+2}_x}\|h\|^2_{L^2_t H^1_x}),
\end{aligned}$$

whence

$$\|Ah\|_{L^2_t H^s_x} \le C(s)(\|h\|_s + \|\beta\|_{s+3}\|h\|_1). \tag{12.39}$$

The norm $\|u\|_s$ of $H^s(\mathbb{T}^2)$ is equivalent to the norm $\|u\|_{L^2_t H^s_x} + \|u\|_{H^s_t L^2_x}$. Then it remains to prove that also

$$\|Ah\|_{H^s_t L^2_x} \le C(s)(\|h\|_s + \|\beta\|_{s+3}\|h\|_1). \tag{12.40}$$

The time derivative of Ah is $\partial_t(Ah) = A(h_t) + \beta_t A(i|D_x|^{1/2}h)$.

• For $s = 1$, using the inequalities $\|u\|_{L^\infty(\mathbb{T}^2)} \le C\|u\|_2$ and $\|Au\|_0 \le C\|u\|_0$, we have

$$\begin{aligned}
\|\partial_t(Ah)\|_0 &\le \|Ah_t\|_0 + \|\beta_t A(i|D_x|^{1/2}h)\|_0 \\
&\le C(\|h_t\|_0 + \|\beta_t\|_{L^\infty(\mathbb{T}^2)}\|A(i|D_x|^{1/2}h)\|_0) \\
&\le C(\|h\|_1 + \|\beta_t\|_2\||D_x|^{1/2}h\|_0) \le C(\|h\|_1 + \|\beta\|_3\|h\|_1)
\end{aligned}$$

(we have rudely worsened $\|h\|_{1/2} \le \|h\|_1$). Therefore (12.40) holds for $s = 1$, and, as a consequence, (12.37) holds for $s = 1$, namely $\|Ah\|_1 \le C\|h\|_1$ (because $\|\beta\|_4 \le 1$).

• For $s = 2$, we use the product estimate $\|uv\|_1 \le C\|u\|_1\|v\|_2$ to deduce that

$$\begin{aligned}
\|\partial_t(Ah)\|_{H^1_t L^2_x} \le \|\partial_t(Ah)\|_1 &\le \|Ah_t\|_1 + \|\beta_t A|D_x|^{1/2}h\|_1 \\
&\le C(\|h_t\|_1 + \|\beta_t\|_2\|A|D_x|^{1/2}h\|_1) \\
&\le C(\|h\|_2 + \|\beta\|_3\||D_x|^{1/2}h\|_1) \le C\|h\|_2.
\end{aligned}$$

Therefore (12.40) holds for $s = 2$, and, as a consequence, (12.37) holds for $s = 2$.

• Now assume that (12.40) holds for some $s \geq 2$; we prove it for $s+1$. The sum of (12.39) and (12.40) implies that (12.37) holds at that s. We estimate

$$\|\partial_t(Ah)\|_{H_t^s L_x^2} \leq \|Ah_t\|_{H_t^s L_x^2} + \|\beta_t A(i|D_x|^{1/2}h)\|_{H_t^s L_x^2}.$$

By (12.40), $\|Ah_t\|_{H_t^s L_x^2} \leq C(s)(\|h_t\|_s + \|\beta\|_{s+3}\|h_t\|_1) \leq C(s)(\|h\|_{s+1} + \|\beta\|_{s+3} \|h\|_2)$ and, by interpolation, this is $\leq C(s)(\|h\|_{s+1} + \|\beta\|_{s+4}\|h\|_1)$ because $\|\beta\|_4 \leq 1$. For the other term, we use the product estimate $\|uv\|_s \leq C(s)(\|u\|_s\|v\|_2 + \|u\|_2\|v\|_s)$ and (12.37) at s to deduce that

$$\begin{aligned}
&\|\beta_t A(|D_x|^{1/2}h)\|_{H_t^s L_x^2} \leq \|\beta_t A(|D_x|^{1/2}h)\|_s \\
&\leq C(s)(\|\beta_t\|_s\|A(|D_x|^{1/2}h)\|_2 + \|\beta_t\|_2\|A(|D_x|^{1/2}h)\|_s) \\
&\leq C(s)(\|\beta\|_{s+1}\|h\|_3 + \|\beta\|_{s+1}\|\beta\|_5\|h\|_2 + \|\beta\|_3\|h\|_{s+1} + \|\beta\|_3\|\beta\|_{s+3}\|h\|_2)
\end{aligned}$$

which is $\leq C(s)(\|h\|_{s+1} + \|\beta\|_{s+4}\|h\|_1)$ by interpolation. Hence (12.40) holds for $s+1$.

By induction, we have proved (12.40) for all $s \geq 1$. The sum of (12.39) and (12.40) gives the thesis. □

To prove a time-dependent version of Lemma 12.10, note that the time derivative of the operator $R_N|D_x|^m$ in (12.28) is

$$\partial_t(R_N|D_x|^m h) = R_N|D_x|^m h_t + i\beta_t R_N|D_x|^{m+(1/2)}h,$$

namely $R_N|D_x|^m$ satisfies the same formula for the time-derivative as the operator A of Lemma 12.13.

All these proofs can be easily adapted to include the amplitude function $a(t, x, k)$.

13. Appendix

In this appendix we gather some classical facts that are used in the proof.

13.1. Classical Tame Estimates for Pseudo-Differential Operators on the Torus

Let

$$Au(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k\, a(x,k)\, e^{ikx}, \quad x \in \mathbb{R}, \tag{13.1}$$

$a(x, k)$ periodic in x, for all $k \in \mathbb{Z}$.

Lemma 13.1. *(*Bounded or regularizing pseudo-differential operators on the torus*)*

1. *Let s, σ, τ, K be real numbers with $s, \sigma, K \geq 0$ and $\tau > 1/2$. Assume that*

$$\|a(\cdot, k)\|_{s+\sigma}\, \langle k\rangle^{\tau-s} \leq K \quad \forall k \in \mathbb{Z}, \tag{13.2}$$

$$\|a(\cdot, k)\|_{\tau}\, \langle k\rangle^{\sigma} \leq K \quad \forall k \in \mathbb{Z}. \tag{13.3}$$

Then A in (13.1) maps $H^s(\mathbb{T})$ into $H^{s+\sigma}(\mathbb{T})$, with

$$\|Au\|_{s+\sigma} \leq CK\|u\|_s \quad \forall u \in H^s(\mathbb{T}),$$

where $C > 0$ depends on $s+\sigma$ and τ.

2. *The same conclusion holds if* (13.2) *is replaced by*

$$\|a(\cdot\,,k)\|_{s+\sigma+\tau}\,\langle k\rangle^{-s} \le K \quad \forall k\in\mathbb{Z}. \tag{13.4}$$

Proof. Develop $a(x,k)$ in Fourier series, $a(x,k)=\sum_{j\in\mathbb{Z}}\hat a_j(k)\,\mathrm{e}^{ijx}$ so that

$$Au(x)=\sum_{n,k}\hat u_k\,\hat a_{n-k}(k)\,\mathrm{e}^{inx}.$$

One has

$$\langle n\rangle^{2(s+\sigma)} \le C_1\left(\langle n-k\rangle^{2(s+\sigma)}+\langle k\rangle^{2(s+\sigma)}\right).$$

Therefore $\|Au\|_{s+\sigma}^2 \le C_1\,(S_1+S_2)$ where

$$S_1 := \sum_n\left(\sum_k|\hat u_k|\,|\hat a_{n-k}(k)|\right)^2\langle n-k\rangle^{2(s+\sigma)},$$

$$S_2 := \sum_n\left(\sum_k|\hat u_k|\,|\hat a_{n-k}(k)|\right)^2\langle k\rangle^{2(s+\sigma)}.$$

By Hölder's inequality, (13.2) implies that

$$\begin{aligned} S_1 &\le \sum_n\left(\sum_k|\hat u_k|^2\,|\hat a_{n-k}(k)|^2\,\langle k\rangle^{2\tau}\right)\left(\sum_k\frac{1}{\langle k\rangle^{2\tau}}\right)\langle n-k\rangle^{2(s+\sigma)}\\ &\le C_2\sum_k|\hat u_k|^2\langle k\rangle^{2\tau}\,\|a(\cdot\,,k)\|_{s+\sigma}^2\\ &\le C_2K^2\|u\|_s^2\end{aligned}$$

where $C_2:=\sum_j\langle j\rangle^{-2\tau}$ is finite because $\tau>1/2$. Similarly, one estimates S_2 using (13.3) and one obtains that $\|Au\|_{s+\sigma}^2 \le 2C_1C_2K^2\|u\|_s^2$.

To prove part (2) of the lemma, it is sufficient to replace $\langle k\rangle^\tau$ with $\langle n-k\rangle^\tau$ when Hölder's inequality is applied to estimate S_1. □

We now consider paradifferential operators, which are pseudo-differential operators with spectrally localized symbols $a(x,k)$ (see [14]). Namely, develop $a(x,k)$ in Fourier series, $a(x,k)=\sum_{j\in\mathbb{Z}}\hat a_j(k)\,\mathrm{e}^{ijx}$. We shall consider low (resp. high) frequencies symbols such that $\hat a_j(k)=0$ for $|j|\ge C|k|$ (resp. $|j|\le C|k|$).

Lemma 13.2. (Low frequencies symbol) *Let A be the pseudo-differential operator*

$$Au(x)=\sum_{k\in\mathbb{Z}}\hat u_k\,a(x,k)\,\mathrm{e}^{ikx},\quad a(x,k)=\sum_{|j|\le C|k|}\hat a_j(k)\,\mathrm{e}^{ijx},$$

where the symbol a is Fourier supported on $\{j\in\mathbb{Z}:|j|\le C|k|\}$ for some constant C. Then

$$\|Au\|_s \le C(s)\,\|a\|_{s_0}\|u\|_s,\quad \|a\|_{s_0}:=\sup_{k\in\mathbb{Z}}\|a(\cdot,k)\|_{s_0},$$

for all $s_0>1/2$, and $s\ge 0$.

Proof. Notice that (13.3) holds with $\tau = s_0 > 1/2$ and $\sigma = 0$. Also, for $s = 0$, (13.4) holds with $\tau = s_0 > 1/2$ and $\sigma = 0$; which in turn implies that (13.4) holds for any $s \geq 0$ since $\|a(\cdot\,, k)\|_{s+s_0} \leq K(s)\|a(\cdot\,, k)\|_{s_0} \langle k\rangle^s$ in view of the spectral localization. □

Lemma 13.3. (High frequencies symbol) *Let A be the pseudo-differential operator*

$$Au(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k\, a(x,k)\, e^{ikx}, \quad a(x,k) = \sum_{|j|>C|k|} \hat{a}_j(k)\, e^{ijx},$$

where the symbol a is Fourier supported on $\{j \in \mathbb{Z} : |j| > C|k|\}$ for some constant C. Then

$$\|Au\|_s \leq C(s)\, \|a\|_s \|u\|_{s_0}, \quad \|a\|_s := \sup_{k\in\mathbb{Z}} \|a(\cdot, k)\|_s,$$

for all $s_0 > 1/2$, and $s \geq s_0$.

Proof. Notice that (13.2) and (13.3) hold with (τ, σ, s) replaced by $(s_0, s - s_0, s_0)$. To see this, notice that the assumption that a is Fourier supported on $\{j \in \mathbb{Z} : |j| > C|k|\}$ implies that $\|a(\cdot\,, k)\|_{s_1} \langle k\rangle^{s_2-s_1} \leq K(s_1, s_2)\|a(\cdot\,, k)\|_{s_2}$ for $s_2 \geq s_1$. □

We also recall the following estimates for the Hilbert transform (see [33] or Lemma B.5 in [7]).

Lemma 13.4. (1) *Let s, m_1, m_2 in $\mathbb{N}$ with $s \geq 2$, $m_1, m_2 \geq 0$, $m = m_1 + m_2$. Let $f \in H^{s+m}(\mathbb{T})$. Then $[f, \mathcal{H}]u = f\mathcal{H}u - \mathcal{H}(fu)$ satisfies*

$$\left\|\partial_x^{m_1}[f, \mathcal{H}]\partial_x^{m_2} u\right\|_s \leq C(s)\left(\|u\|_s \|f\|_{m+2} + \|u\|_2 \|f\|_{m+s}\right).$$

(2) *There exists a universal constant δ in $(0, 1)$ with the following property. Let s, m_1, m_2 in $\mathbb{N}$ and set $m = m_1+m_2$, $\beta \in W^{s+m+1,\infty}(\mathbb{T}, \mathbb{R})$ with $|\beta|_{W^{1,\infty}} \leq \delta$. Let $Bh(x) = h(x + \beta(x))$ for $h \in H^s(\mathbb{T})$. Then*

$$\left\|\partial_x^{m_1}\left(B^{-1}\mathcal{H}B - \mathcal{H}\right)\partial_x^{m_2} u\right\|_s \leq C(s)\left(|\beta|_{W^{m+1,\infty}} \|u\|_s + |\beta|_{W^{s+m+1,\infty}} \|u\|_0\right).$$

13.2. Interpolation Estimates

Recall the interpolation inequality: if $s_1 \leq s \leq s_2$, then

$$|f|_s \leq C(s_1, s_2)|f|_{s_1}^{\lambda} |f|_{s_2}^{1-\lambda}, \quad s = \lambda s_1 + (1 - \lambda)s_2, \quad \lambda \in [0, 1], \tag{13.5}$$

where $|\ |_s$ is either the C^s-norm the H^s-norm (or other norms of a scale with interpolation). As a consequence, one has the following:

Lemma 13.5. (Interpolation) *Let $n \geq 1$ be an integer, let $\delta \geq 0$, and let $\nu_1, \ldots, \nu_n$ be real numbers with*

$$\nu_j \geq \delta \quad \forall j = 1, \ldots, n, \qquad \sum_{j=1}^{n} \nu_j = \alpha.$$

Then

$$\prod_{j=1}^{n} |f|_{\nu_j} \leq C(\alpha)|f|_{\delta}^{n-1} |f|_{\alpha-\delta n+\delta}. \tag{13.6}$$

Proof. Since $\sum_{j=1}^{n} \nu_j = \alpha$, and each ν_j is ≥ 1,

$$\nu_j = \alpha - \sum_{i \neq j} \nu_i \leq \alpha - \delta(n-1).$$

So $\nu_j \in [\delta, \alpha - \delta n + \delta]$. Apply (13.5) with $s_1 = 1$, $s = \nu_j$, $s_2 = \alpha - \delta n + \delta$, and define $\vartheta_j \in [0,1]$ by

$$\nu_j = \delta \vartheta_j + (\alpha - \delta n + \delta)(1 - \vartheta_j), \quad j = 1, \ldots, n.$$

Since $\nu_1 + \cdots + \nu_n = \alpha$, we find

$$\left(\sum_{j=1}^{n} \vartheta_j \right) (\alpha - \delta n) = n(\alpha - \delta n + \delta) - \alpha = (\alpha - \delta n)(n-1).$$

If $\alpha - \delta n \neq 0$, then $\sum_{j=1}^{n} \vartheta_j = n - 1$ and $\sum_{j=1}^{n} (1 - \vartheta_j) = 1$. Thus, using (13.5),

$$\begin{aligned} \prod_{j=1}^{n} |f|_{\nu_j} &\leq C(\alpha) \prod_{j=1}^{n} |f|_{\delta}^{\vartheta_j} |f|_{\alpha - \delta n + \delta}^{1 - \vartheta_j} = C(\alpha) |f|_{\delta}^{(\sum \vartheta_j)} |f|_{\alpha - \delta n + \delta}^{(\sum (1 - \vartheta_j))} \\ &\leq C(\alpha) |f|_{\delta}^{n-1} \, |f|_{\alpha - \delta n + \delta}. \end{aligned}$$

If, instead, $\alpha - \delta n = 0$, then $\nu_j = \delta$ for all j, and the conclusion still holds. □

The previous lemma, which has an interest per se, can be used to estimate the exponentials. Let $v(x)$ be a function. The derivatives of e^{v} are

$$\partial_x^{\alpha} (\mathrm{e}^{v(x)}) = P_{\alpha}(x) \, \mathrm{e}^{v(x)},$$

where $P_{\alpha}(x)$ satisfies $P_0(x) = 1$ and $P_{\alpha+1}(x) = \partial_x P_{\alpha}(x) + P_{\alpha}(x) \partial_x v(x)$. Thus, by induction,

$$P_{\alpha}(x) = \sum_{n=1}^{\alpha} \sum_{\nu \in S_{\alpha,n}} C(\nu) (\partial_x^{\nu_1} v)(x) \cdots (\partial_x^{\nu_n} v)(x), \qquad \alpha \geq 1, \tag{13.7}$$

where $\nu = (\nu_1, \ldots, \nu_n) \in S_{\alpha,n}$ means $\nu_j \geq 1$ and $\nu_1 + \cdots + \nu_n = \alpha$. The previous lemma implies that

$$|(\partial_x^{\nu_1} v)(x) \cdots (\partial_x^{\nu_n} v)(x)| \leq \prod_{j=1}^{n} |v|_{\nu_j} \leq C(\alpha) |v|_1^{n-1} |v|_{\alpha - n + 1}.$$

Then use (13.5) with $s_1 = 1$, $s = \alpha + 1 - n$, $s_2 = \alpha$, namely $|v|_{\alpha+1-n} \leq C(\alpha) |v|_1^{1-\mu} |v|_{\alpha}^{\mu}$ with μ defined by

$$\alpha + 1 - n = (1 - \mu) + \alpha \mu,$$

which is $\mu = (\alpha - n)/(\alpha - 1)$. Since $n - \mu = \alpha(1 - \mu)$, we get

$$\begin{aligned} |v|_1^{n-1} \, |v|_{\alpha+1-n} &\leq C(\alpha)|v|_1^{n-1} \, |v|_1^{1-\mu}|v|_\alpha^\mu = C(\alpha)|v|_1^{n-\mu}|v|_\alpha^\mu \\ &\leq C(\alpha)(|v|_1^\alpha)^{1-\mu}|v|_\alpha^\mu \\ &\leq C(\alpha)\big(|v|_1^\alpha + |v|_\alpha\big). \end{aligned}$$

As a consequence,

$$|P_\alpha(x)| \leq C(\alpha)(|v|_1^\alpha + |v|_\alpha). \tag{13.8}$$

In the case $v(x) = i|k|^{1/2}\beta(x)$, this gives

$$|P_\alpha(x)| \leq C(\alpha)\big(|k|^{\alpha/2}|\beta|_1^\alpha + |k|^{1/2}|\beta|_\alpha\big). \tag{13.9}$$

13.3. Non-stationary Phase

The following lemma is the classical fast oscillation estimate, based on repeated integrations by parts on the torus, in a tame version.

Lemma 13.6. (Non-stationary phase) *Let $p \in H^2(\mathbb{T}, \mathbb{R})$,*

$$\|p\|_2 \leq K, \quad |p'(x)| \leq \frac{1}{2} \quad \forall x \in \mathbb{R}, \tag{13.10}$$

for some constant $K > 0$. Let ω be an integer, $\omega \neq 0$, and let $\alpha \geq 1$ be an integer. If $p \in H^{\alpha+1}(\mathbb{T})$, $u \in H^\alpha(\mathbb{T})$, then

$$\int_{\mathbb{T}} u(x)\, \mathrm{e}^{i\omega(x+p(x))}\, \mathrm{d}x = \left(\frac{i}{\omega}\right)^\alpha \int_{\mathbb{T}} Q_\alpha(x)\, \mathrm{e}^{i\omega(x+p(x))}\, \mathrm{d}x,$$

where $Q_\alpha \in L^2(\mathbb{T})$,

$$\|Q_\alpha\|_0 \leq C(\alpha, K)\left(\|u\|_\alpha + \|p\|_{\alpha+1}\, \|u\|_1\right),$$

and $C(\alpha, K)$ is a positive constant that depends only on α and K. If $u = 1$, then

$$\|Q_\alpha\|_0 \leq C(\alpha, K)\|p\|_{\alpha+1}.$$

If $p = p(x, \omega)$ and $u = u(x, \omega)$ depend on ω, the estimate still holds if $K \geq \|p(\cdot, \omega)\|_2$ and $|\partial_x p(x, \omega)| \leq 1/2$.

Proof. Put $h(x) := x + p(x)$. By induction, integrating by parts α times gives

$$\int_{\mathbb{T}} u(x)\, \mathrm{e}^{i\omega h(x)}\, \mathrm{d}x = \left(\frac{i}{\omega}\right)^\alpha \int_{\mathbb{T}} Q_\alpha(x)\, \mathrm{e}^{i\omega h(x)}\, \mathrm{d}x,$$

where, for $\alpha \geq 1$, Q_α is of the form

$$Q_\alpha = \frac{1}{(h')^{2\alpha}} \sum_{\nu \in S_\alpha} C(\nu)(\partial^{\nu_0} u)(\partial^{\nu_1} h) \ldots (\partial^{\nu_\alpha} h), \tag{13.11}$$

where $\nu = (\nu_0, \nu_1, \dots, \nu_\alpha) \in S_\alpha$ means

$$0 \le \nu_0 \le \alpha, \quad 1 \le \nu_1 \le \cdots \le \nu_\alpha, \quad \nu_0 + \nu_1 + \cdots + \nu_\alpha = 2\alpha. \tag{13.12}$$

Formula (13.11) is proved by induction starting from $Q_\alpha = \partial_x(Q_{\alpha-1}/h')$. If we organize the sum in (13.11) according to the number of indices among $\nu_1, \dots, \nu_\alpha$ that are equal to 1, we obtain

$$Q_\alpha = \sum_{n=0}^{\alpha} \frac{1}{(h')^{\alpha+n}} \sum_{\mu \in T_{\alpha,n}} C(\mu)(\partial^{\mu_0} u)(\partial^{\mu_1} p) \dots (\partial^{\mu_n} p), \tag{13.13}$$

where $\mu = (\mu_0, \mu_1, \dots, \mu_n) \in T_{\alpha,n}$ means

$$0 \le \mu_0 \le \alpha - n, \quad 2 \le \mu_1 \le \cdots \le \mu_n, \quad \mu_0 + \mu_1 + \cdots + \mu_n = \alpha + n.$$

To estimate the products in (13.13), we distinguish three cases.
Case 1: $n = 0$. Then $\mu_0 = \alpha$ and $\|(\partial^{\mu_0} u)(\partial^{\mu_1} p) \dots (\partial^{\mu_n} p)\|_0 = \|\partial^\alpha u\|_0 \le \|u\|_\alpha$.
Case 2: $n \ge 1$ and $\mu_0 = 0$. Then

$$\begin{aligned}\|(\partial^{\mu_0} u)(\partial^{\mu_1} p) \dots (\partial^{\mu_n} p)\|_0 &\le \|u\|_{L^\infty} \|\partial^{\mu_1} p\|_{L^\infty} \dots \|\partial^{\mu_{n-1}} p\|_{L^\infty} \|\partial^{\mu_n} p\|_0 \\ &\le C^{n-1} \|u\|_1 \|p\|_{\mu_1+1} \dots \|p\|_{\mu_{n-1}+1} \|p\|_{\mu_n}\end{aligned}$$

where C is the universal constant of the embedding $\|u\|_{L^\infty} \le C\|u\|_1$. Now it follows from Lemma 13.5 applied with $\delta = 2$ and $|\cdot|_s$ replaced with the Sobolev norms $\|\cdot\|_s$ (which satisfies the interpolation estimate (13.5)) that

$$\|p\|_{\mu_1+1} \dots \|p\|_{\mu_{n-1}+1} \|p\|_{\mu_n} \le C(\alpha) \|p\|_2^{n-1} \|p\|_{\alpha+1}.$$

Case 3: $n, \mu_0 \ge 1$. For any $i \ge 1$, one has $2 \le \mu_i \le \mu_n \le \alpha$, because

$$2(n-1) + \mu_n \le \mu_1 + \cdots + \mu_n = \alpha + n - \mu_0 \le \alpha + n - 1.$$

Therefore $\mu_i + 1 \le \alpha + 1$ for all $i \ge 1$, and one can write

$$\begin{aligned}&\|(\partial^{\mu_0} u)(\partial^{\mu_1} p) \dots (\partial^{\mu_n} p)\|_0 \\ &\quad \le \|\partial^{\mu_0} u\|_0 \|\partial^{\mu_1} p\|_{L^\infty} \dots \|\partial^{\mu_n} p\|_{L^\infty} \\ &\quad \le C(n) \|u\|_{\mu_0} \|p\|_{\mu_1+1} \dots \|p\|_{\mu_n+1} \\ &\quad \le C(n) \|u\|_1^{\vartheta_0} \|u\|_\alpha^{1-\vartheta_0} \|p\|_2^{\vartheta_1+\ldots+\vartheta_n} \|p\|_{\alpha+1}^{n-(\vartheta_1+\ldots+\vartheta_n)}\end{aligned}$$

where $\vartheta_0, \vartheta_1, \dots, \vartheta_n \in [0, 1]$ are defined by

$$\begin{aligned}\mu_i + 1 &= 2\,\vartheta_i + (\alpha+1)(1-\vartheta_i) \quad \forall i = 1, \dots, n, \\ \mu_0 &= 1\,\vartheta_0 + \alpha(1-\vartheta_0).\end{aligned}$$

Taking the sum gives $\vartheta_0 + \vartheta_1 + \cdots + \vartheta_n = n$ because $\mu_0 + \mu_1 + \cdots + \mu_n = \alpha + n$. One deduces that

$$\begin{aligned}&\|u\|_1^{\vartheta_0} \|u\|_\alpha^{1-\vartheta_0} \|p\|_2^{\vartheta_1+\cdots+\vartheta_n} \|p\|_{\alpha+1}^{n-(\vartheta_1+\cdots+\vartheta_n)} \\ &\le \|p\|_2^{n-1} (\|p\|_2 \|u\|_\alpha)^{1-\vartheta_0} (\|p\|_{\alpha+1} \|u\|_1)^{\vartheta_0}\end{aligned}$$

which in turn is smaller than

$$\|p\|_2^{n-1}(\|p\|_2\|u\|_\alpha + \|p\|_{\alpha+1}\|u\|_1)$$

because $a^{1-\vartheta}b^\vartheta \le (1-\vartheta)a + \vartheta b \le a+b$ for all $a, b > 0$, $\vartheta \in [0,1]$.

Since $\|1/h'\|_{L^\infty} \le 2$, collecting all the above cases gives

$$\|Q_\alpha\|_0 \le C(\alpha, K)(\|u\|_\alpha + \|p\|_{\alpha+1}\|u\|_1)$$

for some constant $C(\alpha, K) > 0$, because $\|p\|_2 \le \|\beta\|_2 \le K$. Note that all the calculations above are not affected by a possible dependence of p on ω.

When $u = 1$, only the case 2 gives a nonzero contribution to the sum. □

13.4. $L^2(\mathbb{T})$ as a Subspace of $L^2(\mathbb{R})$

For the sake of completeness, we recall here how to define an isomorphism of $L^2(\mathbb{T})$ onto a subspace of $L^2(\mathbb{R})$, transforming Fourier series $\sum_{k\in\mathbb{Z}}$ into Fourier integrals $\int_{\mathbb{R}} d\xi$.

Consider a C^∞ function $\rho : \mathbb{R} \to \mathbb{R}$ with compact support such that

(i) $0 \le \rho(x) \le 1 \quad \forall x \in \mathbb{R}$,

(ii) $\rho(x) \ge 1/2 \quad \forall x \in [-\pi, \pi]$,

(iii) $\rho(x) = 0 \quad \forall |x| \ge 3\pi/2$,

(iv) $\rho(x) = 1 \quad \forall |x| \le \pi/2$,

(v) $\sum_{k\in\mathbb{Z}} \rho(x + 2\pi k) = 1 \quad \forall x \in \mathbb{R}$.

Let $\mathcal{S}$ be the multiplication operator $(\mathcal{S}u)(x) := \rho(x)u(x)$, $u \in L^2(\mathbb{T})$, and X its range,

$$X := \{\mathcal{S}u : u \in L^2(\mathbb{T})\} \subset L^2(\mathbb{R}).$$

The following properties of $\mathcal{S}$ follow directly from the properties of ρ.

Lemma 13.7. (The isomorphism $\mathcal{S}$) *The map $\mathcal{S} : L^2(\mathbb{T}) \to X$ is bijective, and*

$$\frac{1}{2}\|u\|_{L^2(\mathbb{T})} \le \|\mathcal{S}u\|_{L^2(\mathbb{R})} \le 2\|u\|_{L^2(\mathbb{T})} \quad \forall u \in L^2(\mathbb{T}).$$

If $u, v \in L^2(\mathbb{T})$, then

$$(u, v)_{L^2(\mathbb{T})} = (\mathcal{S}u, v)_{L^2(\mathbb{R})} \tag{13.14}$$

(the integral $(\mathcal{S}u, v)_{L^2(\mathbb{R})}$ is well-defined because $\mathcal{S}u$ has compact support). In particular, for $v = e_k$,

$$\hat{u}_k = (u, e_k)_{L^2(\mathbb{T})} = (\mathcal{S}u, e_k)_{L^2(\mathbb{R})} = \widehat{(\mathcal{S}u)}(k) \quad \forall k \in \mathbb{Z}. \tag{13.15}$$

Therefore

$$\mathcal{S} \,:\, u(x) = \sum_{k\in\mathbb{Z}} \widehat{(\mathcal{S}u)}(k)\, e_k(x) \quad \mapsto \quad \mathcal{S}u(x) = \int_{\mathbb{R}} \widehat{(\mathcal{S}u)}(\xi)\, e_\xi(x)\, d\xi. \tag{13.16}$$

We also recall the following version of the Poisson summation formula.

Lemma 13.8. (P*oisson summation formula*) *Let* $g : \mathbb{R} \to \mathbb{C}$ *of class* C^1*, with*

$$(1+\xi^2)(|g(\xi)| + |g'(\xi)|) \le C \quad \forall \xi \in \mathbb{R},$$

for some constant $C > 0$*. Then for every* $x \in \mathbb{R}$ *the following two convergent numerical series coincide:*

$$\mathcal{P}g(x) := \sum_{k\in\mathbb{Z}} g(x + 2\pi k) = \sum_{k\in\mathbb{Z}} \hat{g}(k)\, \mathrm{e}^{ikx}.$$

At $x = 0$ *this is the Poisson summation formula* $\sum_{k\in\mathbb{Z}} g(2\pi k) = \sum_{k\in\mathbb{Z}} \hat{g}(k)$*. Moreover,*

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{e}^{i2\pi k\xi}\, g(\xi)\, \mathrm{d}\xi = \sum_{k\in\mathbb{Z}} g(k). \tag{13.17}$$

We next show that the operator $\mathcal{P}$ admits a unique continuous extension to the space of functions g such that $(1+|x|^2)^{1/2} g(x) \in L^2(\mathbb{R})$ (which is equivalent to $g = \hat{h}$ for some $h \in H^1(\mathbb{R})$).

Lemma 13.9. *Let* $g \in L^2(\mathbb{R})$ *with* $Tg \in L^2(\mathbb{R})$*, where* $(Tg)(x) := xg(x)$*. Then the sequence* $\{\hat{g}(k)\}_{k\in\mathbb{Z}}$ *is in* $\ell^2(\mathbb{Z})$*, the series* $u(x) := \sum_{k\in\mathbb{Z}} \hat{g}(k)\, e^{ikx}$ *belongs to* $L^2(\mathbb{T})$*, with*

$$\|u\|_{L^2(\mathbb{T})} \le 2(\|g\|_{L^2(\mathbb{R})} + \|Tg\|_{L^2(\mathbb{R})}).$$

If, in addition, g satisfies the hypotheses of the previous lemma, then $u(x) = \mathcal{P}g(x)$ *for every* x*, whence*

$$\|\mathcal{P}g\|_{L^2(\mathbb{T})} \le 2(\|g\|_{L^2(\mathbb{R})} + \|Tg\|_{L^2(\mathbb{R})}).$$

Proof. $\|u\|^2_{L^2(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |\hat{g}(k)|^2$. For every $\xi \in \mathbb{R}$,

$$|\hat{g}(k)|^2 \le \big(|\hat{g}(\xi)| + |\hat{g}(k) - \hat{g}(\xi)|\big)^2 \le 2\big(|\hat{g}(\xi)|^2 + |\hat{g}(k) - \hat{g}(\xi)|^2\big).$$

Let $I_k := [k - 1/2, k + 1/2]$. Then

$$|\hat{g}(k)|^2 = \int_{I_k} |\hat{g}(k)|^2\, \mathrm{d}\xi \le 2\int_{I_k} |\hat{g}(\xi)|^2\, \mathrm{d}\xi + 2\int_{I_k} |\hat{g}(k) - \hat{g}(\xi)|^2\, \mathrm{d}\xi.$$

By Hölder's inequality,

$$\begin{aligned}
\int_{I_k} |\hat{g}(k) - \hat{g}(\xi)|^2\, \mathrm{d}\xi &\le \int_{I_k} \left| \int_k^\xi |\hat{g}'(t)|\, \mathrm{d}t \right|^2 \mathrm{d}\xi \\
&\le \int_{I_k} \left| \int_k^\xi |\hat{g}'(t)|^2\, \mathrm{d}t \right| |k - \xi|\, \mathrm{d}\xi \\
&\le \int_{I_k} \left(\int_{I_k} |\hat{g}'(t)|^2\, \mathrm{d}t \right) \mathrm{d}\xi = \int_{I_k} |\hat{g}'(\xi)|^2\, \mathrm{d}\xi.
\end{aligned}$$

Therefore

$$\sum_{k\in\mathbb{Z}} |\hat{g}(k)|^2 \le 2\sum_{k\in\mathbb{Z}} \int_{I_k} (|\hat{g}(\xi)|^2 + |\hat{g}'(\xi)|^2)\,\mathrm{d}\xi$$
$$= 2\int_{\mathbb{R}} (|\hat{g}(\xi)|^2 + |\hat{g}'(\xi)|^2)\,\mathrm{d}\xi = 2(\|g\|^2_{L^2(\mathbb{R})} + \|Tg\|^2_{L^2(\mathbb{R})}).$$

□

Remark 13.10. Denote by Λ_s is the Fourier multiplier with symbol $\langle\xi\rangle^s = (1 + |\xi|^2)^{s/2}$. Clearly

$$\Lambda_s \mathcal{P} f(x) = \mathcal{P}\Lambda_s f(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\langle k\rangle^s \mathrm{e}^{ikx}$$

for all test function $f \in C_0^\infty(\mathbb{R})$, namely $\Lambda_s\mathcal{P} = \mathcal{P}\Lambda_s$. The previous lemma thus implies that

$$\|\mathcal{P}g\|_{H^s(\mathbb{T})} \le 2\left(\|g\|_{H^s(\mathbb{R})} + \|T\Lambda_s g\|_{L^2(\mathbb{R})}\right).$$

Now one has $T\Lambda_s = \Lambda_s T + s\Lambda_{s-2}\partial_x$, as can be checked directly using the Fourier transform, so that

$$\|\mathcal{P}g\|_{H^s(\mathbb{T})} \le C(s)(\|g\|_{H^s(\mathbb{R})} + \|Tg\|_{H^s(\mathbb{R})}), \tag{13.18}$$

where $C(s) = 2(1 + |s|)$.

Note that, for s integer, $\partial_x^s T = T\partial_x^s + s\partial_x^{s-1}$, and this is useful to calculate $\|Tg\|_{H^s(\mathbb{R})}$ using $\|\partial_x^s Tg\|_{L^2(\mathbb{R})}$.

Observe that $\mathcal{PS} = I$ because, for u periodic,

$$\mathcal{PS}u(x) = \sum_{k\in\mathbb{Z}} (\mathcal{S}u)(x + 2\pi k) = \sum_{k\in\mathbb{Z}} \rho(x + 2\pi k)u(x) = u(x).$$

Moreover,

$$(\psi, v)_{L^2(\mathbb{R})} = (\mathcal{P}\psi, v)_{L^2(\mathbb{T})} \quad \forall\psi \in C_0^\infty(\mathbb{R}),\ v \in L^2(\mathbb{T}). \tag{13.19}$$

Indeed,

$$(\psi, v)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \psi(x)\left(\sum_{k\in\mathbb{Z}} \overline{\hat{v}_k}\,\mathrm{e}^{-ikx}\right)\mathrm{d}x = \sum_{k\in\mathbb{Z}} \hat{\psi}(k)\,\overline{\hat{v}_k},$$

and also

$$(\mathcal{P}\psi, v)_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \left(\sum_{k\in\mathbb{Z}} \hat{\psi}(k)\,\mathrm{e}^{ikx}\right)\left(\sum_{j\in\mathbb{Z}} \overline{\hat{v}_j}\mathrm{e}^{-ijx}\right)\mathrm{d}x = \sum_{k\in\mathbb{Z}} \hat{\psi}(k)\,\overline{\hat{v}_k}.$$

□

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